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# EXISTENCE OF POSITIVE RADIAL SOLUTIONS FOR QUASILINEAR ELLIPTIC EQUATIONS AND SYSTEMS

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Abstract. Under simple conditions on $f$ and $g$, we show that existence of positive radial solutions for the quasilinear elliptic equation

$$\operatorname{div}(\phi_1(|\nabla u|)\nabla u) = a(|x|)f(u) \quad x \in \mathbb{R}^N,$$

and for the system

$$\operatorname{div}(\phi_1(|\nabla u|)\nabla u) = a(|x|)f(v) \quad x \in \mathbb{R}^N,$$
$$\operatorname{div}(\phi_2(|\nabla v|)\nabla v) = b(|x|)g(u) \quad x \in \mathbb{R}^N .$$

## 1. Introduction

The purpose of this article is to study the existence of positive radial solutions to the quasilinear elliptic equation

$$\Delta_{\phi_1} u := \operatorname{div}(\phi_1(|\nabla u|)\nabla u) = a(|x|)f(u), \quad x \in \mathbb{R}^N, \tag{1.1}$$

and for the system

$$\begin{aligned} \operatorname{div}(\phi_1(|\nabla u|)\nabla u) &= a(|x|)f(v), \quad x \in \mathbb{R}^N, \\ \operatorname{div}(\phi_2(|\nabla v|)\nabla v) &= b(|x|)g(u), \quad x \in \mathbb{R}^N. \end{aligned} \tag{1.2}$$

In this article by a solution we mean a solution on the entire domain, as opposed to a local solution. To emphasize this property some authors use entire solution, while others use global solution. We assume the following assumptions:

- (A1) $a, b : \mathbb{R}^N \to [0, \infty)$ are continuous;
- (A2) $f, g : [0, \infty) \to [0, \infty)$ are continuous and increasing,
- (A3) $\phi_i \in C^1((0, \infty), (0, \infty))$ $(i = 1, 2)$ satisfy $(t\phi_i(t))' > 0$, for all $t > 0$;
- (A4) there exist $p_i, q_i > 1$ such that
$$p_i \le \frac{t\Psi_i'(t)}{\Psi_i(t)} \le q_i, \quad \forall t > 0,$$
where $\Psi_i(t) = \int_0^t s\phi_i(s)ds$, $t > 0$;
- (A5) there exist $k_i, l_i > 0$ such that
$$k_i \le \frac{t\Psi_i''(t)}{\Psi_i'(t)} \le l_i, \quad \forall t > 0.$$

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The function $\phi_1$ appears in mathematical models in nonlinear elasticity, plasticity, generalized Newtonian fluids, and in quantum physics, see e.g., Benci, Fortunato and Pisani [8], Cencelj, Repovš and Virk [9], Fuchs and Li [13], Fuchs and Osmolovski [14], Fukagai and Narukawa [15], Rădulescu [28] and [29], Rădulescu and Repovš [30], Repovš [31], Zhang and Yuan [39] and Fukagai and Narukawa [16].

Positive solutions to (1.1) were first considered by Santos, Zhou and Santos [32]. Some classical examples of $\phi_1$-Laplacian functions are:

(1) when $\phi_1(t) \equiv 2$, $\Psi_1(t) = t^2$, $t > 0$, $\Delta_{\phi_1} u = \Delta u$ is the Laplacian operator. In this case, $p_1 = q_1 = 2$ in (A4), and $k_1 = l_1 = 1$ in (A5);

(2) when $\phi_1(t) = pt^{p-2}$, $\Psi_1(t) = t^p$, $t > 0$, $p > 1$, $\Delta_{\phi_1} u = \Delta_p u$ is the $p$-Laplacian operator. In this case, $p_1 = q_1 = p$ in (A4), and $k_1 = l_1 = p-1$ in (A5);

(3) when $\phi_1(t) = pt^{p-2} + qt^{q-2}$, $\Psi_1(t) = t^p + t^q$, $t > 0$, $1 < p < q$, $\Delta_{\phi_1} u = \Delta_p u + \Delta_q u$ is called as the $(p+q)$-Laplacian operator, $p_1 = p$, $q_1 = q$ in (A4), and $k_1 = p-1$, $l_1 = q-1$ in (A5);

(4) when $\phi_1(t) = 2p(1+t^2)^{p-1}$, $\Psi_1(t) = (1+t^2)^p - 1$, $t > 0$, $p > 1/2$, $p_1 = \min\{2, 2p\}$, $q_1 = \max\{2, 2p\}$ in (A4), and $k_1 = \min\{1, 2p-1\}$, $l_1 = \max\{1, 2p-1\}$ in (A5);

(5) when $\phi_1(t) = \frac{p(\sqrt{1+t^2}-1)^{p-1}}{\sqrt{1+t^2}}$, $\Psi_1(t) = (\sqrt{1+t^2}-1)^p$, $t > 0$, $p > 1$, $p_1 = p$, $q_1 = 2p$ in (A4), and $k_1 = p-1$, $l_1 = 2p-1$ in (A5);

(6) when $\phi_1(t) = pt^{p-2}(\ln(1+t))^q + \frac{qt^{p-1}(\ln(1+t))^{q-1}}{1+t}$, $\Psi_1(t) = t^p(\ln(1+t))^q$, $t > 0$, $p > 1$, $q > 0$, $p_1 = p$, $q_1 = p+q$ in (A4), and $k_1 = p-1$, $l_1 = p+q-1$ in (A5).

We say that $u \in C^1(\mathbb{R}^N)$ is a solution of (1.1) if

$$\int_{\mathbb{R}^N} \phi_1(|\nabla u|)\nabla u \nabla \psi dx = -\int_{\mathbb{R}^N} a(x) f(u)\psi dx, \quad \forall \psi \in C_0^\infty(\mathbb{R}^N).$$

When $\lim_{|x|\to\infty} u(x) = +\infty$, we say that $u$ is a large solution to equation (1.1).

For convenience, we denote by

$$h_i^{-1} \text{ the inverses of } h_i(t) = t\phi_i(t), \quad t > 0; \tag{1.3}$$

$$I_{i\rho}(\infty) := \lim_{r\to\infty} I_{i\rho}(r), \quad I_{i\rho}(r) := \int_0^r h_i^{-1}(\Lambda_\rho(t))dt, \quad r \geq 0, \tag{1.4}$$

where $\rho \in C([0,\infty), [0,\infty))$ and

$$\Lambda_\rho(t) := t^{1-N} \int_0^t s^{N-1}\rho(s)ds, \quad t > 0; \tag{1.5}$$

$$\theta_i(t) := \min\{t^{p_i}, t^{q_i}\}, \quad \Theta_i(t) := \max\{t^{p_i}, t^{q_i}\}, \quad t \geq 0; \tag{1.6}$$

$$\theta_i^{-1}(t) := \min\{t^{1/p_i}, t^{1/q_i}\}, \quad \Theta_i^{-1}(t) := \max\{t^{1/p_i}, t^{1/q_i}\}, \quad t \geq 0; \tag{1.7}$$

and, for an arbitrary $\alpha > 0$ and $t \geq \alpha$,

$$H_{1\alpha}(\infty) := \lim_{t\to\infty} H_{1\alpha}(t), \quad H_{1\alpha}(t) := \int_\alpha^t \frac{d\tau}{\Theta_1^{-1}(f(\tau))}; \tag{1.8}$$

$$H_{2\alpha}(\infty) := \lim_{t\to\infty} H_{2\alpha}(t), \quad H_{2\alpha}(t) := \int_\alpha^t \frac{d\tau}{\Theta_1^{-1}(f(\tau)) + \Theta_2^{-1}(g(\tau))}. \tag{1.9}$$

We see that for $t > \alpha$,

$$H'_{1\alpha}(t) = \frac{1}{\Theta_1^{-1}(f(t))} > 0,$$
$$H'_{2\alpha}(t) = \frac{1}{\Theta_1^{-1}(f(t)) + \Theta_2^{-1}(g(t))} > 0,$$

and that $H_{1\alpha}, H_{2\alpha}$ have the inverse functions $H_{1\alpha}^{-1}$ and $H_{2\alpha}^{-1}$ on $[0, H_{1\alpha}(\infty))$ and $[0, H_{2\alpha}(\infty))$, respectively.

First, let us review the model

$$\Delta u = a(|x|)f(u), \quad x \in \mathbb{R}^N. \tag{1.10}$$

For $a(x) \equiv 1$ on $\mathbb{R}^N$: when $f$ satisfies (A2), Keller [18] and Osserman [27] supplied a necessary and sufficient condition

$$\int_1^\infty \frac{dt}{\sqrt{2F(t)}} = \infty, \quad F(t) = \int_0^t f(s)ds, \tag{1.11}$$

for the existence of positive radial large solutions to (1.10).

For $N \geq 3$, $f(u) = u^\gamma$, $\gamma \in (0, 1]$, and $a$ satisfies (A1) with $a(x) = a(|x|)$, Lair and Wood [19] first showed that equation (1.10) has infinitely many positive radial large solutions if and only if

$$\int_0^\infty ra(r)dr = \infty. \tag{1.12}$$

The above results have been extended by many authors and in many contexts, see, for instance, [2, 4, 5, 11, 12, 20, 23, 33, 35, 36] and the references therein.

Next we review the system

$$\begin{aligned} \Delta u &= a(x)f(v), \quad x \in \mathbb{R}^N, \\ \Delta v &= b(x)g(u), \quad x \in \mathbb{R}^N. \end{aligned} \tag{1.13}$$

When $N \geq 3$, $f(v) = v^{\gamma_1}$, $g(u) = u^{\gamma_2}$, $0 < \gamma_1 \leq \gamma_2$, and $a(x) = a(|x|)$, $b(x) = b(|x|)$, Lair and Wood [21] have considered the existence and nonexistence of positive radial solutions to system (1.13). For further results, see for instance, [1, 3, 6, 7, 10, 17, 24, 25, 26, 34, 37, 38] and the references therein.

Now we return to equation (1.1). Recently, Santos, Zhou and Santos [32] considered the existence of positive radial and nonradial large solutions to equation

$$\operatorname{div}(\phi_1(|\nabla u|)\nabla u) = a(x)f(u), \quad x \in \mathbb{R}^N.$$

A basic result read as follows.

**Lemma 1.1** ([32, Corollary 1.2]). *Let* (A3)–(A5) *hold, $f$ satisfy* (A2)*, and $a$ satisfy* (A1) *with $a(x) = a(|x|)$ for $x \in \mathbb{R}^N$. If*

$$I_{1a}(\infty) = \infty,$$

*then* (1.1) *admits a sequence of symmetric radial large solutions $u_m(|x|) \in C^1(\mathbb{R}^N)$ with $u_m(0) \to \infty$ as $m \to \infty$ if and only if $f$ satisfies*

$$\int_1^\infty \frac{dt}{\Psi_1^{-1}(F(t))} = \infty,$$

*where $\Psi_1^{-1}$ is the inverse of $\Psi_1$ which is given in* (A4).

Inspired by the above works, by using a monotone iterative method and Arzela-Ascoli theorem, we show existence of positive radial solutions to equation (1.1) and system (1.2) under simple conditions on $f$ and $g$. Our main results for equation (1.1) read as follows.

**Theorem 1.2.** *Let* (A1)–(A5) *hold. If*

(A6) $H_{1\alpha}(\infty) = \infty$,

*then* (1.1) *has a positive radial solution* $u \in C^1(\mathbb{R}^N)$. *Moreover, if* $I_{1a}(\infty) < \infty$, *then* $u$ *is bounded, and* $\lim_{r\to\infty} u(r) = \infty$ *provided* $I_{1a}(\infty) = \infty$.

**Theorem 1.3.** *Under assumptions* (A1)–(A5) *and*

(A7) $I_{1a}(\infty) < H_{1\alpha}(\infty) < \infty$,

*equation* (1.1) *has a positive radial bounded solution* $u \in C^1(\mathbb{R}^N)$ *satisfying*

$$\alpha + \theta_1^{-1}(f(\alpha))I_{1a}(r) \le u(r) \le H_{1\alpha}^{-1}(I_{1a}(r)), \quad \forall r \ge 0,$$

*where* $\theta_1^{-1}$ *is given in* (1.7).

**Remark 1.4.** When $\int_0^1 \frac{d\tau}{\Theta_1^{-1}(f(\tau))} = \infty$, there exists $\alpha > 0$ sufficiently small such that (A7) holds provided $I_{1a}(\infty) < \infty$ and $H_{1\alpha}(\infty) < \infty$.

**Remark 1.5.** For $f(s) = s^{\gamma_1}$ with $s \ge 0$, $\gamma_1 > 0$, since $\Theta_1^{-1}(t) = \frac{1}{p_1}$, $t \ge 1$, one can see that when $\gamma_1 > p_1$, $H_{1\alpha}(\infty) < \infty$, and $H_{1\alpha}(\infty) = \infty$ provided $\gamma_1 \le p_1$, where $p_1$ is given as in (A4).

**Remark 1.6.** For $f(s) = (1+s)^{\gamma_1}(\ln(1+s))^{\mu_1}$ with $s \ge 0$, $\mu_1, \gamma_1 > 0$, one can see that when $\gamma_1 > p_1$ or $\gamma_1 = p_1$ and $\mu_1 > p_1$, $H_{1\alpha}(\infty) < \infty$, and $H_{1\alpha}(\infty) = \infty$ provided $\gamma_1 < p_1$ or $\gamma_1 = p_1$ and $\mu_1 \le p_1$.

**Remark 1.7.** For $f(s) = \exp(c_1 s)$, $s \ge 0$, $c_1 > 0$, one can see that $H_{1\alpha}(\infty) < \infty$.

Our main results for system (1.2) are as follows.

**Theorem 1.8.** *Let* (A1)–(A5) *hold. If*

(A8) $H_{2\alpha}(\infty) = \infty$,

*then* (1.2) *has a positive radial solution* $(u, v)$ *in* $C^1(\mathbb{R}^N) \times C^1(\mathbb{R}^N)$. *Moreover, when* $I_{1a}(\infty) + I_{2b}(\infty) < \infty$, $u$ *and* $v$ *are bounded; when* $I_a(\infty) = I_b(\infty) = \infty$, $\lim_{r\to\infty} u(r) = \lim_{r\to\infty} v(r) = \infty$.

**Theorem 1.9.** *Under hypotheses* (A1)–(A5) *and*

(A9)
$$I_{1a}(\infty) + I_{2b}(\infty) < H_{2\alpha}(\infty) < \infty,$$

*system* (1.2) *has a positive radial bounded solution* $(u, v)$ *in* $C^1(\mathbb{R}^N) \times C^1(\mathbb{R}^N)$ *satisfying*

$$\alpha/2 + \theta_1^{-1}(f(\alpha/2))I_{1a}(r) \le u(r) \le H_{2\alpha}^{-1}(I_{1a}(r) + I_{2b}(r)), \quad \forall r \ge 0;$$
$$\alpha/2 + \theta_2^{-1}(g(\alpha/2))I_{2b}(r) \le v(r) \le H_{2\alpha}^{-1}(I_{1a}(r) + I_{2b}(r)), \quad \forall r \ge 0.$$

**Remark 1.10.** By a similar proof, we can see extend Theorems 1.8 and 1.9 to the more general system

$$\begin{gathered}\operatorname{div}(\phi_1(|\nabla u|)\nabla u) = a(|x|)f_1(v)f_2(u), \quad x \in \mathbb{R}^N, \\ \operatorname{div}(\phi_2(|\nabla v|)\nabla v) = b(|x|)g_1(v)g_2(u), \quad x \in \mathbb{R}^N,\end{gathered} \tag{1.14}$$

where $f_i, g_i$ $(i = 1, 2)$ satisfy (A2).

**Remark 1.11.** For $f(s) = s^{\gamma_1}$, $g(s) = s^{\gamma_2}$, $s \ge 0$, $\gamma_1, \gamma_2 > 0$, when $\gamma_1 > p_1$ or $\gamma_2 > p_2$, $H_{2\alpha}(\infty) < \infty$, and $H_{2\alpha}(\infty) = \infty$ provided $\gamma_1 \le p_1$ and $\gamma_2 \le p_2$, where $p_1$ and $p_2$ are given as in (A4).

**Remark 1.12.** For $f(s) = (1+s)^{\gamma_1}(\ln(1+s))^{\mu_1}$, $g(s) = (1+s)^{\gamma_2}(\ln(1+s))^{\mu_2}$, $s \ge 0$, $\gamma_i, \mu_i > 0$ $(i = 1, 2)$, when $\gamma_1 > p_1$ or $\gamma_2 > p_2$; or $\gamma_1 = p_1$ and $\eta_1 > p_1$; or $\gamma_2 = p_2$ and $\eta_2 > p_2$, $H_{2\alpha}(\infty) < \infty$, and $H_{2\alpha}(\infty) = \infty$ provided $\gamma_1 < p_1$ and $\gamma_2 < p_2$; or $\gamma_1 = p_1$, $\eta_1 \le p_1$ and $\gamma_2 = p_2$, $\eta_2 \le p_2$.

**Remark 1.13.** For $f(s) = \exp(c_1 s)$ or $g(s) = \exp(c_2 s)$, $s \ge 0$, $c_1, c_2 > 0$, one can see that $H_{2\alpha}(\infty) < \infty$.

## 2. Proof of Theorems 1.2 and 1.3

**Lemma 2.1** ([32, Lemma 2.2])**.** *Let* (A3)–(A5) *hold,* $\theta_i, \Theta_i$ *and* $\theta_i^{-1}, \Theta_i^{-1}$ *(*$i = 1, 2$*) be given as in* (1.6) *and* (1.7)*. We have*

(i) $\theta_i$, $\Theta_i$, $\theta_i^{-1}$ *and* $\Theta_i^{-1}$ *are strictly increasing on* $(0, \infty)$*;*
(ii) $\theta_i^{-1}(\beta) h_i^{-1}(t) \le h_i^{-1}(\beta t) \le \Theta_i^{-1}(\beta) h_i^{-1}(t)$*, for all* $\beta, t > 0$.

Let us consider the initial value problem

$$
\begin{gathered}
\big(r^{N-1}\phi_1(u'(r))u'(r)\big)' = r^{N-1}a(r)f(u), \quad r > 0, \\
u(0) = \alpha, \quad u'(0) = 0,
\end{gathered} \tag{2.1}
$$

by a simple calculation,

$$
u'(r) = h_1^{-1}\big(r^{1-N}\int_0^r s^{N-1}a(s)f(u(s))ds\big), \quad r > 0, \quad u(0) = \alpha, \tag{2.2}
$$

and thus

$$
u(r) = \alpha + \int_0^r h_1^{-1}\big(t^{1-N}\int_0^t s^{N-1}a(s)f(u(s))ds\big)dt, \quad r \ge 0. \tag{2.3}
$$

Note that solutions in $C[0, \infty)$ to problem (2.3) are solutions in $C^1[0, \infty)$ to problem (2.1).

Let $\{u_m\}_{m\ge1}$ be the sequence of positive continuous functions defined on $[0, \infty)$ by

$$
\begin{gathered}
u_0(r) = \alpha, \\
u_m(r) = \alpha + \int_0^r h_1^{-1}\big(t^{1-N}\int_0^t s^{N-1}a(s)f(u_{m-1}(s))ds\big)dt, \quad r \ge 0.
\end{gathered} \tag{2.4}
$$

Obviously,

$$
u_m'(r) = h_1^{-1}\big(r^{1-N}\int_0^r s^{N-1}a(s)f(u_{m-1}(s))ds\big), \quad r > 0, \tag{2.5}
$$

and, for all $r \ge 0$ and $m \in \mathbb{N}$, $u_m(r) \ge \alpha$, and $u_0 \le u_1$. Then (A1)–(A3) and Lemma 2.1 yield $u_1(r) \le u_2(r)$ for all $r \ge 0$. Continuing this line of reasoning, we obtain that the sequence $\{u_m\}$ is non-decreasing on $[0, \infty)$. Moreover, we obtain by (A1)–(A3) and Lemma 2.1 that for each $r > 0$,

$$
\begin{aligned}
u_m'(r) &= h_1^{-1}\big(r^{1-N}\int_0^r s^{N-1}a(s)f(u_{m-1}(s))ds\big) \\
&\le h_1^{-1}\big(f(u_m(r))r^{1-N}\int_0^r s^{N-1}a(s)ds\big)
\end{aligned}
$$

$$\le \Theta_1^{-1}(f(u_m(r)))h_1^{-1}\big(r^{1-N}\int_0^r s^{N-1}a(s)ds\big),$$

and

$$\int_a^{u_m(r)} \frac{d\tau}{\Theta_1^{-1}(f(\tau))} \le I_{1a}(r).$$

Consequently, for an arbitrary $R > 0$,

$$H_{1\alpha}(u_m(r)) \le I_{1a}(r) \le I_{1a}(R), \quad \forall r \in [0, R]. \tag{2.6}$$

(i) When (A6) holds, we see that

$$H_{1\alpha}^{-1}(\infty) = \infty, \quad u_m(r) \le H_{1\alpha}^{-1}(I_{1a}(r)) \le H_{1\alpha}^{-1}(I_{1a}(R)), \quad \forall r \in [0, R], \tag{2.7}$$

i.e., the sequence $\{u_m\}$ is bounded on $[0, R]$ for an arbitrary $R > 0$.

It follows from (2.5) that $\{u'_m\}$ is bounded on $[0, R]$. By the Arzela-Ascoli theorem, $\{u_m\}$ has a subsequence converging uniformly to $u$ on $[0, R]$. Since $\{u_m\}$ is non-decreasing on $[0, \infty)$, we see that $\{u_m\}$ itself converges uniformly to $u$ on $[0, R]$. By the arbitrariness of $R$, we see that $u$ is a positive radial solution to equation (1.1). Moreover, when $I_{1a}(\infty) < \infty$, we see by (2.7) that

$$u(r) \le H_{1\alpha}^{-1}(I_{1a}(\infty)), \quad \forall r \ge 0;$$

when $I_{1a}(\infty) = \infty$, we see by (A2) and Lemma 2.1 that

$$u(r) \ge \alpha + \theta_1^{-1}(f(\alpha))I_{1a}(r), \quad \forall r \ge 0.$$

Thus $\lim_{r\to\infty} u(r) = \infty$.

(ii) When (A7) holds, we see by (2.6) that

$$H_{1\alpha}(u_m(r)) \le I_{1a}(\infty) < H_{1\alpha}(\infty) < \infty. \tag{2.8}$$

Since $H_{1\alpha}^{-1}$ is strictly increasing on $[0, H_{1\alpha}(\infty))$, we have

$$u_m(r) \le H_{1\alpha}^{-1}(I_{1a}(\infty)) < \infty, \quad \forall r \ge 0. \tag{2.9}$$

The rest of the proof follows from (i).

## 3. Proof of Theorems 1.8 and 1.9

Let us consider the initial value problem

$$\begin{gathered}
\big(r^{N-1}\phi_1(u'(r))u'(r)\big)' = r^{N-1}a(r)f(v), \quad r > 0,\\
\big(r^{N-1}\phi_2(v'(r))v'(r)\big)' = r^{N-1}b(r)g(u), \quad r > 0,\\
u(0) = v(0) = \alpha/2, \quad u'(0) = v'(0) = 0,
\end{gathered}$$

which is equivalent to

$$\begin{gathered}
u(r) = \alpha/2 + \int_0^r h_1^{-1}\big(t^{1-N}\int_0^t s^{N-1}a(s)f(v(s))ds\big)dt, \quad r \ge 0,\\
v(r) = \alpha/2 + \int_0^r h_2^{-1}\big(t^{1-N}\int_0^t s^{N-1}b(s)g(u(s))ds\big)dt, \quad r \ge 0.
\end{gathered}$$

Let $\{u_m\}_{m\ge1}$ and $\{v_m\}_{m\ge0}$ be the sequences of positive continuous functions defined on $[0, \infty)$ by

$$\begin{gathered}
v_0(r) = \alpha/2,\\
u_m(r) = \alpha/2 + \int_0^r h_1^{-1}\big(t^{1-N}\int_0^t s^{N-1}a(s)f(v_{m-1}(s))ds\big)dt, \quad r \ge 0,
\end{gathered}$$

$$v_m(r) = \alpha/2 + \int_0^r h_2^{-1}\big(t^{1-N}\int_0^t s^{N-1}b(s)g(u_m(s))ds\big)dt, \quad r \geq 0.$$

Obviously, for all $r \geq 0$ and $m \in \mathbb{N}$, $u_m(r) \geq \alpha/2$, $v_m(r) \geq \alpha/2$ and $v_0 \leq v_1$. Assumptions (A1)–(A3) and Lemma 2.1 yield $u_1(r) \leq u_2(r)$, for all $r \geq 0$, then $v_1(r) \leq v_2(r)$, for all $r \geq 0$. Continuing this line of reasoning, we obtain that the sequences $\{u_m\}$ and $\{v_m\}$ are increasing on $[0, \infty)$. Moreover, by (A1)-(A3) and Lemma 2.1 that for each $r > 0$, we obtain

$$\begin{aligned}
u_m'(r) &= h_1^{-1}\big(r^{1-N}\int_0^r s^{N-1}a(s)f(v_{m-1}(s))ds\big) \\
&\leq h_1^{-1}\big(f(v_m(r))r^{1-N}\int_0^r s^{N-1}a(s)ds\big) \\
&\leq \Theta_1^{-1}(f(v_m(r)))h_1^{-1}\big(r^{1-N}\int_0^r s^{N-1}a(s)ds\big) \\
&\leq \Theta_1^{-1}(f(u_m(r)+v_m(r)))(h_1^{-1}(\Lambda_a(r)) + h_2^{-1}(\Lambda_b(r)));
\end{aligned}$$

and

$$\begin{aligned}
v_m'(r) &= h_2^{-1}\big(r^{1-N}\int_0^r s^{N-1}b(s)g(u_m(s))ds\big) \\
&\leq \Theta_2^{-1}(g(u_m(r)))h_2^{-1}\big(r^{1-N}\int_0^r s^{N-1}b(s)ds\big) \\
&\leq \Theta_2^{-1}(g(u_m(r)+v_m(r)))\big(h_1^{-1}(\Lambda_a(r)) + h_2^{-1}(\Lambda_b(r))\big).
\end{aligned}$$

Consequently,

$$\begin{aligned}
u_m'(r) + v_m'(r) &\leq \big(\Theta_1^{-1}(f(v_m(r)+u_m(r))) \\
&\quad + \Theta_2^{-1}(g(v_m(r)+u_m(r)))\big)\big(h_1^{-1}(\Lambda_a(r)) + h_2^{-1}(\Lambda_b(r))\big), \quad r > 0,
\end{aligned}$$

and

$$\begin{gathered}
\int_a^{u_m(r)+v_m(r)} \frac{d\tau}{\Theta_1^{-1}(f(\tau)) + \Theta_2^{-1}(g(\tau))} \leq I_{1a}(r) + I_{2b}(r), \quad r > 0, \\
H_{2\alpha}(u_m(r)+v_m(r)) \leq I_{1a}(r) + I_{2b}(r), \quad \forall r \geq 0.
\end{gathered} \tag{3.1}$$

The remaining proofs are similar to that for Theorems 1.2 and 1.3. Here we omit their proof.

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