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POSITIVE SOLUTIONS FOR SECOND-ORDER BOUNDARY-VALUE PROBLEMS WITH ϕ-LAPLACIAN

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Abstract. This article concerns the existence, localization and multiplicity of positive solutions for the boundary-value problem

$$\begin{gathered}\big(\phi(u')\big)' + f(t,u) = 0,\\ u(0) - au'(0) = u'(1) = 0,\end{gathered}$$

where $f : [0,1] \times \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function and $\phi : \mathbb{R} \to (-b,b)$ is an increasing homeomorphism with $\phi(0) = 0$. We obtain existence, localization and multiplicity results of positive solutions using Krasnosel'skiĭ fixed point theorem in cones, and a weak Harnack type inequality. Concerning systems, the localization is established by the vector version of Krasnosel'skiĭ theorem, where the compression-expansion conditions are expressed on components.

1. Introduction

The aim of this article is to present new results regarding the existence, localization and multiplicity of positive solutions for the problem

$$\begin{gathered}(\phi(u'))' + f(t,u) = 0, \quad 0 < t < 1,\\ u(0) - au'(0) = 0, \quad u'(1) = 0,\end{gathered} \tag{1.1}$$

where $a > 0$, ϕ is a homeomorphism from $\mathbb{R}$ to $(-b,b)$ and $0 < b \leq \infty$.

According [3]-[6] and [14], there are two remarkable models in this context:

(1) The p-Laplacian operator, where $b = \infty$ and $\phi(u) = |u|^{p-2}u$, with $p > 1$.
(2) The curvature operator, where $b < \infty$ and
$$\phi(u) = \frac{u}{\sqrt{1+u^2}}.$$

Problem (1.1) can be considered as a particular, $n = 1$ of the corresponding problem for an n-dimensional system

$$\begin{gathered}(\phi_i(u_i'))' + f_i(t,u_1,u_2,\dots,u_n) = 0, \quad 0 < t < 1,\\ u_i(0) - a_iu_i'(0) = 0, u_i'(1) = 0 \quad (i = 1,2,\dots,n),\end{gathered} \tag{1.2}$$

where $a_i > 0$.

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First we shall concentrate on the problem (1.1) for a single equation, and then we shall extend the results to the system (1.2). The study of ϕ-Laplacian equations is a classical topic that has attracted the attention of many experts because of its applications (see for example [1]). These equations, with different boundary conditions have been studied in a large number of papers using fixed point methods, degree theory, upper and lower solution techniques and variational methods. Robin boundary conditions

$$\alpha_1 u(0) - \beta_1 u'(0) = 0 = \alpha_2 u(1) + \beta_2 u'(1),$$

are commonly used in solving Sturm-Liouville problems which appear in many contexts in science and engineering. These problems have been considered in the literature by many authors in order to search the existence of positive solutions (see [7, 8]). Some of them worked with special cases. For example [2, 9, 11] studied the case $\beta_1 = \beta_2 = 0$ and $\alpha_1 = \alpha_2 = 1$, while [12] discussed the case $\alpha_1 = \beta_2 = 0$, $\alpha_2 = 1$ and $\beta_1 = -1$.

In this article, we considered the case $\alpha_1 = \beta_2 = 1$, $\beta_1 = a$, with $a > 0$ and $\alpha_2 = 0$; we are interested not only on the existence of positive solutions to (1.1) and (1.2), but also on their localization and multiplicity. We shall achive this by using a technique based on Krasnosel'skiĭ's fixed point theorem in cones [13]. This result has been extensively employed in the related literature (see for instance [10]-[12], [18, 19]).

Theorem 1.1 (Krasnosel'skiĭ)**.** *Let $(X, |\cdot|)$ be a normed linear space; $K \subset X$ a cone; $r, R \in \mathbb{R}_+$, $0 < r < R$, $K_{r,R} = \{u \in K : r \le |u| \le R\}$, and let $N : K_{r,R} \to K$ be a compact map. Assume that one of the following conditions is satisfied:*

(a) *$N(u) \not< u$ if $|u| = r$, and $N(u) \not> u$ if $|u| = R$;*
(b) *$N(u) \not> u$ if $|u| = r$, and $N(u) \not< u$ if $|u| = R$.*

Then N has a fixed point u in K with $r \le |u| \le R$.

Here for two elements $u, v \in X$, the strict ordering $u < v$ means $v - u \in K \setminus \{0\}$.

The technique based on the application of Krasnosel'skiĭ theorem for completely continuous operators on a Banach space, requires the construction of a suitable cone of positive functions. To this end, in the case of most boundary value problems, the corresponding Green functions and their properties play an important role. Alternatively, for many problems for which Green functions are not known, one can use weak Harnack type inequalities associated to the differential operators and the boundary conditions, as shown in [16] and [18]. In our case, such an inequality will arise as a consequence of the concavity of the positive solutions.

In the case of systems, we shall allow the homeomorphisms ϕ_i have different ranges and we shall be interested to localize each component of a solution $u = (u_1, u_2, \dots, u_n)$ individually. In this respect we shall use the following vector version of Krasnosel'skiĭ theorem given in [15] (see also [17]).

Theorem 1.2. *Let $(X, |\cdot|)$ be a normed linear space; $K_1, K_2, \dots, K_n \subset X$ cones; $K := K_1 \times K_2 \times \dots \times K_n$; $r, R \in \mathbb{R}^n_+$, $r = (r_1, r_2, \dots, r_n)$, $R = (R_1, R_2, \dots, R_n)$ with $0 < r_i < R_i$ for all i, $K_{r,R} = \{u \in K : r_i \le |u_i| \le R_i,\ i = 1, 2, \dots, n\}$ and let $N : K_{r,R} \to K$, $N = (N_1, N_2, \dots, N_n)$ be a compact map. Assume that for each $i = 1, 2, \dots, n$, one of the following conditions are satisfied in $K_{r,R}$:*

(a) *$N_i(u) \not< u_i$ if $|u_i| = r_i$, and $N_i(u) \not> u_i$ if $|u_i| = R_i$;*
(b) *$N_i(u) \not> u_i$ if $|u_i| = r_i$, and $N_i(u) \not< u_i$ if $|u_i| = R_i$.*

Then N has a fixed point $u = (u_1, u_2, \dots, u_n)$ in K with $r_i \le |u_i| \le R_i$ for $i = 1, 2, \dots, n$.

Note that in the previous theorem, the same symbol $<$ is used to denote the strict ordering induced by any of the cones $K_1, K_2, \dots, K_n$.

It deserves to be underlined the fact that asking the compression condition (a) to be satisfied by some indices i, and the expansion condition (b) by the others, it is allowed that the system nonlinearities behave differently one from the other.

2. Positive solutions of ϕ-Laplace equations

In this section, we seek for positive solutions for (1.1) and prove some existence, localization and multiplicity results. For this, we make the following assumptions:

- (A1) $\phi : \mathbb{R} \to (-b, b)$, $0 < b \le \infty$ is an increasing homeomorphism such that $\phi(0) = 0$.
- (A2) $f : [0,1] \times \mathbb{R}_+ \to \mathbb{R}_+$ is continuous, $f(t, .)$ is nondecreasing on $\mathbb{R}_+$ for each $t \in [0,1]$.

First we obtain the equivalent integral equation for problem (1.1) of positive solutions. Integration of the differential equation (1.1) gives

$$-\phi(u'(t)) = -\phi(u'(0)) + \int_0^t f(s, u(s))\, ds.$$

Then

$$u'(t) = \phi^{-1}\Big(\phi(u'(0)) - \int_0^t f(s, u(s))\, ds\Big). \tag{2.1}$$

Integrating from 0 to t we obtain

$$u(t) = u(0) + \int_0^t \phi^{-1}\Big(\phi(u'(0)) - \int_0^\tau f(s, u(s))\, ds\Big)\, d\tau. \tag{2.2}$$

If we denote $x := u(0)$, substituting it into (2.2) and taking into account the first boundary condition, we have

$$u(t) = x + \int_0^t \phi^{-1}\Big(\phi(\frac{x}{a}) - \int_0^\tau f(s, u(s))\, ds\Big)\, d\tau. \tag{2.3}$$

For $t = 1$, (2.1) gives

$$\phi^{-1}\Big(\phi(\frac{x}{a}) - \int_0^1 f(s, u(s))\, ds\Big) = 0,$$

whence

$$x = a\phi^{-1}\Big(\int_0^1 f(s, u(s))\, ds\Big). \tag{2.4}$$

Next, we may define the integral operator $N : C([0,1]; \mathbb{R}_+) \to C([0,1]; \mathbb{R}_+)$ by

$$N(u)(t) = a\phi^{-1}\Big(\int_0^1 f(s, u(s))\, ds\Big) + \int_0^t \phi^{-1}\Big(\int_\tau^1 f(s, u(s))\, ds\Big)\, d\tau, \tag{2.5}$$

and thus, finding positive solutions to (1.1) is equivalent to the fixed point problem for the operator N on $C([0,1]; \mathbb{R}_+)$. Note that by standard arguments, N is completely continuous. Let $|\cdot|_\infty$ denotes the max norm on $C[0,1]$.

To apply Krasnosel'skiĭ's fixed point theorem in cones we need a weak Harnack type inequality for the differential operator $Lu := -(\phi(u'))'$ subjected to the boundary conditions.

Lemma 2.1. *For each $c \in (0,1)$, and any $u \in C^1[0,1] \cap C([0,1];\mathbb{R}_+)$ with $u(0) - au'(0) = u'(1) = 0$, $\phi \circ u' \in W^{1,1}(0,1)$ and $(\phi(u'))' \leq 0$ a.e. on $[0,1]$, one has*

$$u(t) \geq \gamma(t)|u|_\infty, \quad \text{for all } t \in [0,1], \tag{2.6}$$

where

$$\gamma(t) = \begin{cases} \frac{a+c}{a+1}, & \text{for } t \in [c,1] \\ 0, & \text{for } t \in [0,c). \end{cases}$$

Proof. From $(\phi(u'))' \leq 0$ on $[0,1]$, one has that the function $\phi \circ u'$ is noincreasing on $[0,1]$. Then, from $u' = \phi^{-1}(\phi \circ u')$, and ϕ^{-1} increasing, we deduce that u' is nonincreasing on $[0,1]$. Thus u is concave on $[0,1]$. On the other hand, since the function $\phi \circ u'$ vanishes at $t = 1$, $\phi(u'(t)) \geq 0$ for every $t \in [0,1]$. Then $u' \geq 0$ on $[0,1]$, which shows that u is nondecreasing on $[0,1]$. If we have $u(0) < 0$ then $u'(0) = \frac{u(0)}{a} < 0$ which is excluded by $u' \geq 0$ on $[0,1]$. Hence $u(0) \geq 0$ and so u is nonnegative, nondecreasing, concave and $|u|_\infty = u(1)$. Inequality 2.6 being obvious for $t \in [0,c)$, it remains to prove it for $t \in [c,1]$. If $\min_{t\in[c,1]} u(t) = 0$, then the concavity of u implies $u = 0$ on $[0,1]$, and so (2.6) holds. If $\min_{t\in[c,1]} u(t) > 0$, then we may assume without loss of generality that $\min_{t\in[c,1]} u(t) = 1$ (otherwise, multiply (2.6) by a suitable positive constant). Then $u(c) = 1$. Since u is concave, its graph on $[c,1]$ is under the line containing the points $(0,u(0))$ and $(c,1)$ and so at point $t = 1$ we have

$$u(1) \leq \frac{u(0)(c-1)+1}{c}.$$

However, $u'(0) = \frac{u(0)}{a}$ and being the slope of the line,

$$u'(0) \geq \frac{1-u(0)}{c}.$$

Hence $u(0) \geq \frac{a}{c+a}$ and then $u(1) \leq \frac{a+1}{a+c}$. Now, from $|u|_\infty = u(1)$ we have

$$\frac{a+c}{a+1}|u|_\infty \leq 1.$$

Finally, since $1 \leq u(t)$ for $t \in [c,1]$, we obtain

$$u(t) \geq \frac{a+c}{a+1}|u|_\infty, \quad \text{for all } t \in [c,1],$$

as we wished. Notice that a graphical representation would make more clear the above reasoning. $\square$

Our first result is the following theorem.

Theorem 2.2. *Let (A1) and (A2) hold and assume that there exist $\alpha, \beta > 0$ with $\alpha \neq \beta$ such that*

$$\Phi(\alpha) := a\phi^{-1}\Big(\int_0^1 f(s,\gamma(s)\alpha)\,ds\Big) + \int_0^1 \phi^{-1}\Big(\int_\tau^1 f(s,\gamma(s)\alpha)\,ds\Big)\,d\tau > \alpha, \tag{2.7}$$

$$\Psi(\beta) := a\phi^{-1}\Big(\int_0^1 f(s,\beta)\,ds\Big) + \int_0^1 \phi^{-1}\Big(\int_\tau^1 f(s,\beta)\,ds\Big)\,d\tau < \beta. \tag{2.8}$$

Then (1.1) has at least one positive solution u with $r \leq |u|_\infty \leq R$, where $r = \min\{\alpha,\beta\}$ and $R = \max\{\alpha,\beta\}$.

Proof. We shall apply Krasnosel'skiĭ's fixed point theorem in cones. In our case, $X = C[0,1]$, the cone is

$$\begin{aligned} K = \big\{ & u \in C([0,1];\mathbb{R}_+) : u(0) - au'(0) = u'(1) = 0 \text{ and} \\ & u(t) \geq \gamma(t)|u|_\infty \text{ for all } t \in [0,1] \big\}, \end{aligned}$$

and N is the operator given by (2.5).

Note that if $u, v \in C([0,1];\mathbb{R}_+)$ and $v < u$, that is $u - v \in K \setminus \{0\}$, then $(u-v)(1) \geq \gamma(1)|u-v|_\infty > 0$. Hence

$$|u|_\infty \geq u(1) > v(1). \tag{2.9}$$

First we remark that $N(K) \subset K$. Indeed, if $u \in K$ and $v := N(u)$, then $-(\phi(v'))' = f(t,u)$. We have $f(t,u(t)) \geq 0$ for every $t \in [0,1]$, so $(\phi(v'))' \leq 0$ on $[0,1]$. Then Lemma 2.1 guarantees that $v(t) \geq \gamma(t)|v|_\infty$ for $t \in [0,1]$, that is $v \in K$ as desired.

Next we prove that

$$u \not> N(u) \quad \text{for every } u \in K \text{ with } |u|_\infty = \alpha. \tag{2.10}$$

To this end, assume the contrary, i.e. $u > N(u)$ for some $u \in K$ with $|u|_\infty = \alpha$. Then using (2.9), the definition of K, and the monotonicity of f and ϕ, we deduce

$$\begin{aligned} \alpha = |u|_\infty &\geq |N(u)|_\infty \geq N(u)(1) \\ &= a\phi^{-1}\Big(\int_0^1 f(s,u(s))\,ds\Big) + \int_0^1 \phi^{-1}\Big(\int_\tau^1 f(s,u(s))\,ds\Big)\,d\tau \\ &\geq a\phi^{-1}\Big(\int_0^1 f(s,\gamma(s)\alpha)\,ds\Big) + \int_0^1 \phi^{-1}\Big(\int_\tau^1 f(s,\gamma(s)\alpha)\,ds\Big)\,d\tau, \end{aligned}$$

which contradicts (2.7). Thus (2.10) holds. The next step is to prove that

$$u \not< N(u) \quad \text{for every } u \in K \text{ with } |u|_\infty = \beta. \tag{2.11}$$

Assume the contrary, i.e. $u < N(u)$ for some $u \in K$ with $|u|_\infty = \beta$. Then we would obtain

$$\begin{aligned} \beta = |u|_\infty &\leq |N(u)|_\infty = N(u)(1) \\ &= a\phi^{-1}\Big(\int_0^1 f(s,u(s))\,ds\Big) + \int_0^1 \phi^{-1}\Big(\int_\tau^1 f(s,u(s))\,ds\Big)\,d\tau \\ &\leq a\phi^{-1}\Big(\int_0^1 f(s,\beta)\,ds\Big) + \int_0^1 \phi^{-1}\Big(\int_\tau^1 f(s,\beta)\,ds\Big)\,d\tau, \end{aligned}$$

which contradicts (2.8). Thus (2.11) holds. Now Krasnosel'skiĭ theorem applies and yields the result. □

Remark 2.3. The existence and localization result, Theorem 2.2, immediately yields multiplicity results for the problem (1.1), in case that several (finitely many or infinitely many) couples of distinct numbers α, β satisfying (2.7), (2.8) exist such any two of the corresponding intervals (α, β) are disjoint.

The next theorems are about the existence of at least one pair α, β satisfying the conditions (2.7), (2.8), and the existence of a sequence of positive solutions of (1.1), respectively. Their proofs are as in [12]. However, for the readers convenience we reproduce them.

Theorem 2.4. *Let* (A1) *and* (A2) *hold and assume that one of the following conditions is satisfied:*

(i) $\limsup_{\lambda\to\infty} \frac{\Phi(\lambda)}{\lambda} > 1$ *and* $\liminf_{\lambda\to 0} \frac{\Psi(\lambda)}{\lambda} < 1$;
(ii) $\limsup_{\lambda\to 0} \frac{\Phi(\lambda)}{\lambda} > 1$ *and* $\liminf_{\lambda\to \infty} \frac{\Psi(\lambda)}{\lambda} < 1$.

Then (1.1) *has at least one positive solution.*

Proof. To apply Theorem 2.2, we look for two numbers $\alpha, \beta > 0$, $\alpha \neq \beta$ with

$$\Phi(\alpha) > \alpha \quad \text{and} \quad \Psi(\beta) < \beta.$$

In case (i), one can chose α large enough and β small enough; while in case (ii), α is chosen small enough and β is chosen large enough. $\square$

Theorem 2.5. *Let* (A1) *and* (A2) *hold. If the condition*

(iii) $\limsup_{\lambda\to\infty} \frac{\Phi(\lambda)}{\lambda} > 1$ *and* $\liminf_{\lambda\to \infty} \frac{\Psi(\lambda)}{\lambda} < 1$

holds, then (1.1) *has a sequence of positive solutions* $(u_n)_{n\geq 1}$ *such that* $|u_n|_\infty \to \infty$ *as* $n \to \infty$.

If the condition

(iv) $\limsup_{\lambda\to 0} \frac{\Phi(\lambda)}{\lambda} > 1$ *and* $\liminf_{\lambda\to 0} \frac{\Psi(\lambda)}{\lambda} < 1$

holds, then (1.1) *has a sequence of positive solutions* $(u_n)_{n\geq 1}$ *such that* $|u_n|_\infty \to 0$ *as* $n \to \infty$.

Proof. Clearly (iii) guarantees the existence of two sequences $(\alpha_n)_{n\geq 1}, (\beta_n)_{n\geq 1}$ such that

$$0 < \alpha_n < \beta_n < \alpha_{n+1} \quad \text{for every } n \geq 1, \text{ and } \alpha_n \to \infty \text{ as } n \to \infty. \tag{2.12}$$

For each n, Theorem 2.2 yields a positive solution u_n with $\alpha_n \leq |u_n|_\infty \leq \beta_n$. The condition (2.12) implies that these solutions are distinct and that $|u_n|_\infty \to \infty$ as $n \to \infty$. A similar reasoning can be done in case (iv). $\square$

Notice that the conditions (iii) and (iv) show that f is oscillating towards ∞ and zero, respectively.

3. Positive solutions of ϕ-Laplace systems

In this section we extend the above results to the general case (1.2). We shall allow the homeomorphisms ϕ_i have different ranges, namely $\phi_i : \mathbb{R} \to (-b_i, b_i)$, $0 < b_i \leq \infty$, and we shall assume that ϕ_i are increasing with $\phi_i(0) = 0$, and that $f_i : [0,1] \times \mathbb{R}_+^n \to \mathbb{R}_+$ are continuous functions ($i = 1, 2, \dots, n$). Under these assumptions problem (1.2) is equivalent to the integral system

$$u_i(t) = a_i \phi_i^{-1}\Big(\int_0^1 f_i(s, u(s))\,ds\Big) + \int_0^t \phi_i^{-1}\Big(\int_\tau^1 f_i(s, u(s))\,ds\Big)\,d\tau,$$

for $i = 1, 2, \dots, n$ and $u = (u_1, u_2, \dots, u_n)$.

By Lemma 2.1, for each i and any constant $c_i \in (0,1)$, a weak Harnack type inequality holds for the differential operator $L_i v := -(\phi_i(v'))'$ and the boundary conditions $v(0) - av'(0) = v'(1) = 0$. Based on this result we define the cones

$$\begin{aligned} K_i = \big\{& u_i \in C([0,1]; \mathbb{R}_+) : u_i(0) - a u_i'(0) = u_i'(1) = 0 \text{ and} \\ & u_i(t) \geq \gamma_i(t) |u_i|_\infty, \text{for all } t \in [0,1]\big\}, \end{aligned}$$

for $i=1,2,\dots,n$. We note that the functions γ_i are given by Lemma 2.1 for possibly different c_i and a_i. Now we consider the product cone $K:=K_1\times K_2\times\dots\times K_n$ in $C([0,1],\mathbb{R}^n)$.

Let $N:C([0,1];\mathbb{R}_+^n)\to C([0,1];\mathbb{R}_+^n)$, $N=(N_1,N_2,\dots,N_n)$ be defined by

$$N_i(u)(t)=a_i\phi_i^{-1}\Big(\int_0^1 f_i(s,u(s))\,ds\Big)+\int_0^t \phi_i^{-1}\Big(\int_\tau^1 f_i(s,u(s))\,ds\Big)\,d\tau,$$

for $i=1,2,\dots,n$.

If $u_j\in K_j$ for each j, then $f_i(s,u(s))\ge 0$ and from Lemma 2.1, one has $N_i(u)\in K_i$. Thus the cone K is invariant by N. Moreover, the operator N is completely continuous since, by standard arguments, the components N_i are completely continuous.

The following result is a generalization of Theorem 2.2 and guarantees the existence of positive solutions to (1.2) and their component-wise localization. For any index $i\in\{1,2,\dots,n\}$, we shall say that the homeomorphism $\phi_i:\mathbb{R}\to(-b_i,b_i)$ satisfies (A1) if ϕ_i is increasing and $\phi_i(0)=0$, and that the continuous function $f_i:[0,1]\times\mathbb{R}_+^n\to\mathbb{R}_+$ satisfies (A2) if for each $t\in[0,1]$, $f_i(t,x_1,\dots,x_n)$ is nondecreasing on $\mathbb{R}_+$ with respect to any variable x_j, $j=1,2,\dots,n$.

Theorem 3.1. *Let ϕ_i, f_i satisfy* (A1) *and* (A2) *for $i=1,2,\dots,n$. Assume that there exist c_i, α_i, $\beta_i>0$ with $c_i<1$ and $\alpha_i\neq\beta_i$ such that*

$$\begin{aligned}\Phi_i(\alpha):=\ & a_i\phi_i^{-1}\Big(\int_0^1 f_i(s,\gamma_1(s)\alpha_1,\dots,\gamma_n(s)\alpha_n)\,ds\Big)\\ &+\int_0^1\phi_i^{-1}\Big(\int_\tau^1 f_i(s,\gamma_1(s)\alpha_1,\dots,\gamma_n(s)\alpha_n)\,ds\Big)\,d\tau>\alpha_i,\end{aligned}$$

$$\Psi_i(\beta):=a_i\phi_i^{-1}\Big(\int_0^1 f_i(s,\beta)\,ds\Big)+\int_0^1\phi_i^{-1}\Big(\int_\tau^1 f_i(s,\beta)\,ds\Big)\,d\tau<\beta_i,$$

for $i=1,2,\dots,n$, where $\alpha=(\alpha_1,\alpha_2,\dots,\alpha_n)$ and $\beta=(\beta_1,\beta_2,\dots,\beta_n)$. Then (1.2) *has at least one positive solution $u=(u_1,u_2,\dots,u_n)$ with $r_i\le|u_i|_\infty\le R_i$, where $r_i=\min\{\alpha_i,\beta_i\}$, $R_i=\max\{\alpha_i,\beta_i\}$, $i=1,2,\dots,n$.*

The above result is a consequence of the vectorial version of Krasnosel'skiĭ fixed point theorem in cones.

We shall say that for a given index i, the condition (i) from Theorem 2.4 holds if for every $\lambda_1,\lambda_2,\dots,\lambda_{i-1}>0$,

$$\limsup_{\lambda_i\to\infty}\frac{\Phi_i(\lambda)}{\lambda_i}>1\quad\text{and}\quad\liminf_{\lambda_i\to 0}\frac{\Psi_i(\lambda)}{\lambda_i}<1,$$

uniformly with respect to $\lambda_{i+1},\lambda_{i+2},\dots,\lambda_n\in(0,\infty)$. We shall understand the condition (ii) in a similar manner. Therefore, if for each i the condition (i) or (ii) holds, then we obtain pairs (α_i,β_i) satisfying the assumptions of Theorem 3.1.

Analogously, we say that (iii) from Theorem 2.5 holds for some index i, if for every $\lambda_1,\lambda_2,\dots,\lambda_{i-1}>0$,

$$\limsup_{\lambda_i\to\infty}\frac{\Phi_i(\lambda)}{\lambda_i}>1\quad\text{and}\quad\liminf_{\lambda_i\to\infty}\frac{\Psi_i(\lambda)}{\lambda_i}<1,$$

uniformly with respect to $\lambda_{i+1},\lambda_{i+2},\dots,\lambda_n\in(0,\infty)$. Condition (iv) is understood in a similar manner. Under such type of conditions we obtain sequences of solutions for the system (1.2).

Finally, we note that [12, Theorem 3.2] can be applied to our problem (1.2) in order to guarantee the existence of multiple solutions.

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4. Addendum posted on May 19, 2016

The author would like to thank the anonymous reader for his valuable remarks and suggestions, and to make accordingly the following amendments and completions:

(I) Condition (A2) and Theorem 2.2 needs the additional hypothesis on f,

$$f(t,x) < b$$

for all $t \in [0,1]$ and $x \in \mathbb{R}_+$. This assumption guarantees that the integral operator N is well defined on $C([0,1];\mathbb{R}_+)$.

(II) Cases (i) and (iii) from Theorem 2.4 and Theorem 2.5 are possible only if $b=\infty$. Indeed, if $b<\infty$, then

$$\int_0^1 f(s,\gamma(s)\alpha)\,ds = \int_c^1 f\Big(s,\frac{a+c}{a+1}\alpha\Big)\,ds \le (1-c)b.$$

Also, for $0\le\tau\le c$, as above

$$\int_\tau^1 f(s,\gamma(s)\alpha)\,ds = \int_c^1 f\Big(s,\frac{a+c}{a+1}\alpha\Big)\,ds \le (1-c)b,$$

while for $c\le\tau\le 1$,

$$\int_\tau^1 f(s,\gamma(s)\alpha)\,ds = \int_\tau^1 f\Big(s,\frac{a+c}{a+1}\alpha\Big)\,ds \le (1-\tau)b \le (1-c)b.$$

These inequalities show that $\Phi(\lambda)$ given by (2.7) is bounded, namely

$$\Phi(\lambda)\le (a+1)\Phi^{-1}((1-c)b).$$

Then

$$\limsup_{\lambda\to\infty}\frac{\Phi(\lambda)}{\lambda}=0$$

and so (i) and (iii) can not be satisfied.

(III) The above remarks also apply to the systems considered in Section 3.

(IV) Concerning Theorems 2.4 and 2.5, we present the following four examples.

Example 4.1 (in (ii), case $b<\infty$)**.** In (1.1) we let

$$\phi(u)=\frac{u}{\sqrt{1+u^2}} \quad\text{and}\quad f(t,x)=f(x)=\frac{x}{x+1}.$$

In this case, $b=1$ and one can easily check that condition (A2), particularly the inequality $f(t,x)<1$, holds. Direct computations show that

$$\Phi(\lambda)=A\Big(\frac{a+c}{\sqrt{1-A^2}}+\frac{1-c}{1+\sqrt{1-A^2}}\Big),\quad \Psi(\lambda)=B\Big(\frac{a}{\sqrt{1-B^2}}+\frac{1}{1+\sqrt{1-B^2}}\Big),$$

where

$$A=\frac{\lambda(a+c)(1-c)}{\lambda(a+c)+(a+1)} \quad\text{and}\quad B=\frac{\lambda}{\lambda+1}.$$

Now it is easy to see that

$$\lim_{\lambda\to 0}\frac{\Phi(\lambda)}{\lambda}=\frac{(a+c)(1-c)(2a+c+1)}{2(a+1)} \quad\text{and}\quad \lim_{\lambda\to\infty}\frac{\Psi(\lambda)}{\lambda}=0.$$

Hence condition (ii) from Theorem 2.4 is satisfied if

$$C:=\frac{2(a+1)}{(a+c)(1-c)(2a+c+1)}<1,$$

which holds for sufficiently large a. For example, we can choose $a=7$ and $c=0.5$. For this case, Figure 1 shows the behavior of the function f with respect to the line $y=Cx$.

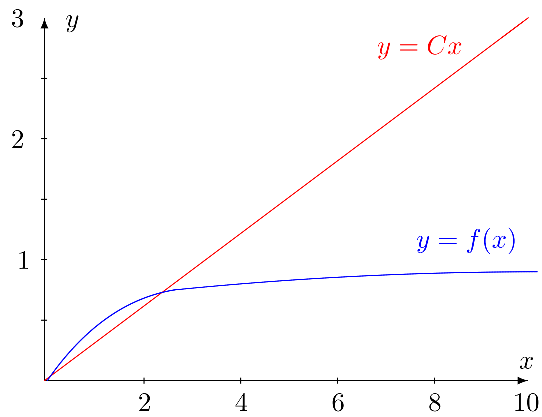

FIGURE 1. Behavior of f with respect to the line $y = Cx$.

Example 4.2 (in (iii), case $b = \infty$)**.** If in (1.1) we let $\phi(u) = u$, then expressions (2.7) and (2.8) become

$$\Phi(\lambda) = f\Big(\frac{a+c}{a+1}\lambda\Big)\Big(\frac{(1-c)(2a+c+1)}{2}\Big), \quad \Psi(\lambda) = f(\lambda)\Big(\frac{2a+1}{2}\Big).$$

Consider $f : [0,1] \times \mathbb{R}_+ \to \mathbb{R}_+$, defined by

$$f(t,x) = f(x) = mx + nx\sin(p\ln(x+1)).$$

In this case $b = \infty$ and condition (A2) holds if

$$m \geq n(p+1). \tag{4.1}$$

Now it is easy to see that

$$\limsup_{\lambda\to\infty} \frac{\Phi(\lambda)}{\lambda} = (m+n)\frac{(a+c)(1-c)(2a+c+1)}{2(a+1)},$$
$$\liminf_{\lambda\to\infty} \frac{\Psi(\lambda)}{\lambda} = (m-n)\frac{2a+1}{2}.$$

Hence condition (iii) from Theorem 2.5 is satisfied if

$$m+n > A \quad \text{and} \quad m-n < B, \tag{4.2}$$

where

$$A = \frac{2(a+1)}{(a+c)(1-c)(2a+c+1)} \quad \text{and} \quad B = \frac{2}{2a+1}.$$

For example, conditions (4.1) and (4.2) hold for

$$a = 2.5, \quad c = 0.3, \quad m = 0.46, \quad n = 0.15, \quad p = 2.$$

For this case, Figure 2 shows the oscillatory behavior of the function f with respect to the lines $y = Ax$ and $y = Bx$.

Example 4.3 (in (iv), case $b = \infty$)**.** We consider $\phi(u) = u$ and the function $f : [0,1] \times \mathbb{R}_+ \to \mathbb{R}_+$ defined by

$$f(t,x) = f(x) = mx + nx\sin(p\ln\frac{1}{x}),$$

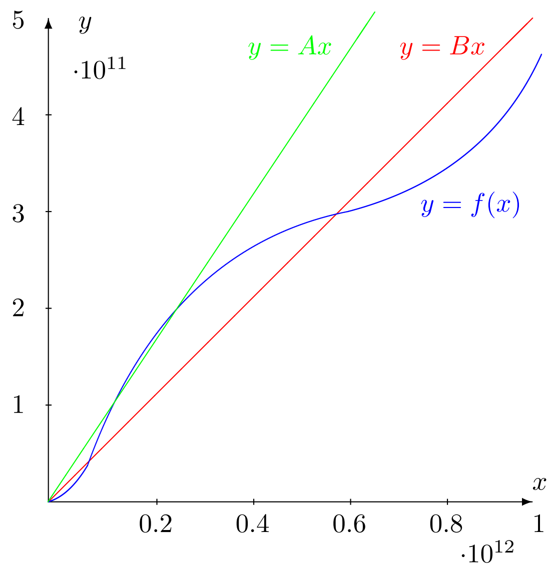

FIGURE 2. Behavior f with respect to the lines $y = Ax$ and $y = Bx$.

for $x > 0$ and $f(0) = 0$. In this case $b = \infty$ and the condition (A2) holds if

$$m \geq n(p+1). \tag{4.3}$$

Now it is easy to see that

$$\limsup_{\lambda\to 0}\frac{\Phi(\lambda)}{\lambda} = (m+n)\frac{(a+c)(1-c)(2a+c+1)}{2(a+1)},$$
$$\liminf_{\lambda\to 0}\frac{\Psi(\lambda)}{\lambda} = (m-n)\frac{2a+1}{2}.$$

Hence condition (iv) from Theorem 2.5 is satisfied if

$$m+n > A \quad \text{and} \quad m-n < B, \tag{4.4}$$

where

$$A = \frac{2(a+1)}{(a+c)(1-c)(2a+c+1)} \quad \text{and} \quad B = \frac{2}{2a+1}.$$

For example, conditions (4.3) and (4.4) hold for

$$a = 2, \quad c = 0.2, \quad m = 0.54, \quad n = 0.16, \quad p = 2.$$

Example 4.4 (in (iv), case $b < \infty$)**.** Consider $\phi(u) = u/\sqrt{1+u^2}$ and the function $f(t,x) = f(x)$ which on a small interval $(0,\epsilon)$ is defined by

$$f(x) = mx + nx\sin\big(p\ln\frac{1}{x}\big).$$

Here $\epsilon > 0$ is chosen such that $f(x) < 1$ on $(0,\epsilon)$. The function f is increasing on $(0,\epsilon)$ if

$$m \geq n(p+1). \tag{4.5}$$

Here

$$\Phi(\lambda) = (a+c)\phi^{-1}\Big((1-c)f\Big(\frac{a+c}{a+1}\lambda\Big)\Big) + \int_c^1 \phi^{-1}\Big((1-\tau)f\Big(\frac{a+c}{a+1}\lambda\Big)\Big)\,d\tau.$$

Since

$$\int_c^1 \phi^{-1}\Big((1-\tau)f\Big(\frac{a+c}{a+1}\lambda\Big)\Big)\,d\tau \geq 0,$$

a sufficient condition for $\Phi(\lambda) > \lambda$ to hold is

$$\phi^{-1}\Big((1-c)f\Big(\frac{a+c}{a+1}\lambda\Big)\Big) > \frac{\lambda}{a+c},$$

or equivalently

$$(1-c)f\Big(\frac{a+c}{a+1}\lambda\Big) > \phi\Big(\frac{\lambda}{a+c}\Big).$$

This gives the condition

$$\frac{f\big(\frac{a+c}{a+1}\lambda\big)}{\frac{a+c}{a+1}\lambda} > \frac{a+1}{(a+c)^2(1-c)\sqrt{1+(\frac{\lambda}{a+c})^2}}.$$

Letting $\lambda \to 0$ yields

$$m+n > \frac{a+1}{(a+c)^2(1-c)}.$$

Also

$$\Psi(\lambda) = a\phi^{-1}(f(\lambda)) + \int_0^1 \phi^{-1}((1-\tau)f(\lambda))\,d\tau,$$

and since

$$\int_0^1 \phi^{-1}((1-\tau)f(\lambda))\,d\tau \leq \phi^{-1}(f(\lambda)),$$

a sufficient condition for $\Psi(\lambda) < \lambda$ to hold is

$$\phi^{-1}(f(\lambda)) < \frac{\lambda}{a+1},$$

or equivalently

$$f(\lambda) < \phi\Big(\frac{\lambda}{a+1}\Big).$$

This gives the condition

$$\frac{f(\lambda)}{\lambda} < \frac{1}{(a+1)\sqrt{1+(\frac{\lambda}{a+1})^2}},$$

which letting $\lambda \to 0$ yields

$$m-n < \frac{1}{a+1}.$$

Hence condition (iv) from Theorem 2.5 is satisfied if

$$m+n > \frac{a+1}{(a+c)^2(1-c)} \quad \text{and} \quad m-n < \frac{1}{a+1}. \tag{4.6}$$

For example, conditions (4.5)) and (4.6) hold for

$$a = 2.5, \quad c = 0.1, \quad m = 0.43, \quad n = 0.17, \quad p = 1.5.$$

End of addendum.

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