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APPROXIMATE CONTROLLABILITY OF NEUTRAL STOCHASTIC INTEGRO-DIFFERENTIAL SYSTEMS WITH IMPULSIVE EFFECTS

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Abstract. This paper studies the approximate controllability of neutral stochastic integro-differential systems with impulsive effects. Sufficient conditions are formulated and proved for the approximate controllability. The results are obtained by using the Nussbaum fixed point theorem and the theory of analytic resolvent operator. An example is given to show the applications of the proposed results.

1. Introduction

In this article, we study the approximate controllability of the following neutral stochastic integro-differential systems with impulsive effects

$$
\begin{gathered}
d[x(t)+F(t,x_t)] = [-Ax(t)+\int_0^t \gamma(t-s)x(s)\,ds + Bu(t)]dt + G(t,x_t)dw(t), \\
t\in J,\ t\neq t_k, \\
\Delta x(t) = I_k(x(t^-)), \quad t=t_k,\ k=1,2,3,\dots,m, \\
x_0=\phi\in L_2(\Omega,C_\alpha), \quad t\in[-r,0],
\end{gathered} \tag{1.1}
$$

where $J=[0,T]$, ϕ is $\mathfrak{F}_0$-measurable and the linear operator $-A$ generates an analytic semigroup on a separable Hilbert space H with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. $u(\cdot)\in L_2^{\mathfrak{F}_t}(J,U)$ is the control function where U is a Hilbert space. $\gamma(\cdot)$ is a family of closed linear operators to be specified later. B is a bounded linear operator from U into H. Define the Banach space $D(A^\alpha)$ with the norm $\|x\|_\alpha = \|A^\alpha x\|$ for $x\in D(A^\alpha)$, where $D(A^\alpha)$ denotes the domain of the fractional power operator $A^\alpha: H\to H$. Let $H_\alpha := D(A^\alpha)$ and $C_\alpha = C([-r,0],H_\alpha)$ be the space of all continuous functions from $[-r,0]$ into H_α. Define K be an another separable Hilbert space. Suppose $w(t)$ is a given K-valued wiener process with a finite trace nuclear covariance operator $Q\geq 0$. $F: J\times C_\alpha\to H_\alpha$, $G: J\times C_\alpha \to L_2^0(K,H)$ and $I_k: H\to H$, where $L_2^0(K,H)$ is the space of all Q-Hilbert-Schmidt operators from K into H. The collection of all strongly measurable, square integrable, C_α-valued

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random variables denoted by $L_2(\Omega, C_\alpha)$. The histories $x_t : \Omega \to C_\alpha$, $t \in J$, which are defined by setting $x_t(\theta) = x(t+\theta)$, $\theta \in [-r, 0]$. $\Delta x(t)$ denotes the jump of x at t, $\Delta x(t) = x(t^+) - x(t^-) = x(t^+) - x(t)$.

The concept of controllability is an important part of mathematical control theory. Generally speaking, controllability means that it is possible to steer a dynamical control system from an arbitrary initial state to an arbitrary final state using the set of admissible controls. Controllability problems for different kinds of dynamical systems have been studied by several authors, see [4, 8, 9, 15, 18, 19].

Dauer and Mahmudov [4] established sufficient conditions for the controllability of stochastic semi-linear functional differential equations in Hilbert spaces under the assumption that the associated linear part of systems is approximately controllable. They obtained the results by using the Banach fixed point theorem and the fractional power theory. Sakthivel et al [18] considered the approximate controllability issue for nonlinear impulsive differential and neutral functional differential equations in Hilbert spaces. Finally, they applied their results to a control system governed by a heat equation with impulses.

In [19], the authors studied the approximate controllability of stochastic impulsive functional system with infinite delay in abstract space. They obtained some sufficient conditions with no compactness requirement imposed on the semigroup generated by the linear part of the system by using the contraction mapping principle. Then they dropped the restriction of the combination of system parameters with the help of the Nussbaum fixed point theorem.

The theory of integro-differential systems has recently become an important area of investigation, stimulated by their numerous applications to problems from electronics, fluid dynamics, biological models. In many cases, deterministic models often fluctuate due to noise, which is random or at least to be so. So, we have to move from deterministic problems to stochastic ones.

Balachandran et al [1] derived sufficient conditions for the controllability of stochastic integro-differential systems in finite dimensional spaces. Muthukumar and Balasubramaniam [11] investigated the appromimate controllability of mixed stochastic Volterra-Fredholm type integro-differential in Hilbert space by employing the Banach fixed point theorem.

In recent years, the study of impulsive integro-differential systems has received increasing interest, since dynamical systems involving impulsive effects occur in numerous applications: the radiation of electromagnetic waves, population dynamics, biological systems, etc. Subalakshmi and Balachandran [21] studied the approximate controllability properties of nonlinear stochastic impulsive integro-differential and neutral stochastic impulsive integro-differential equations in Hilbert spaces under the assumption that the associated linear part of system is approximately controllable. Moreover, Shen et al. [20] obtained the complete controllability of impulsive stochastic integro-differential systems by using Schaefer's fixed point theorem.

Recently, Mokkedem and Fu [12] studied the approximate controllability of the following semi-linear neutral integro-differential equations with finite delay

$$
\begin{gathered}
\frac{d}{dt}[x(t) + F(t, x_t)] = -Ax(t) + \int_0^t \gamma(t-s)x(s)\,ds + Bu(t) + G(t, x_t), \quad t \in J, \\
x_0 = \phi, \quad t \in [-r, 0].
\end{gathered} \tag{1.2}
$$

They assumed that the linear control system corresponding to system (1.2)

$$\begin{gathered}\frac{d}{dt}x(t) = -Ax(t) + \int_0^t \gamma(t-s)x(s)\,ds + Bu(t), \quad t \in J,\\ x(t) = \phi(t), \quad t \in [-r,0],\end{gathered} \tag{1.3}$$

is approximately controllable on J. With the help of the theory of analytic resolvent operator, the authors defined the mild solution of system (1.2), then they used the Sadovskii fixed point theorem and the fractional power operator theorem to prove the existence of solution. Then the authors obtained the approximate controllability of semi-linear neutral integro-differential systems with finite delay in Hilbert space. However, the authors did not consider the stochastic and impulsive effects. Very recently, Yan and Lu [22] studied the approximate controllability of a class of impulsive partial stochastic functional integro-differential inclusions with infinite delay in Hilbert spaces of the form

$$\begin{gathered}d[x(t) - G(t,x_t)] \in A[x(t) + \int_0^t h(t-s)x(s)ds]dt + Bu(t)dt + F(t,x_t)dw(t),\\ t \in J = [0,b],\ t \neq t_k,\ k = 1,2,\dots,m,\\ x_0 = \varphi \in \mathcal{B},\\ \Delta x(t_k) = I_k(x_{t_k}),\ k = 1,2,\dots,m.\end{gathered} \tag{1.4}$$

They achieved the approximate controllability result for (1.4) by imposing compactness assumption on the resolvent operator $\Phi(t)$, they also assumed that the corresponding linear system of (1.4) is approximately controllable.

The aim of the present work is to study the approximate controllability for the system (1.1) with the aid of the resolvent operator theory and the fractional power theory. The resolvent operator is similar to the semigroup operator for abstract differential equations in Banach spaces. However, the resolvent operator does not satisfy semigroup properties. In many practical models the nonlinear terms involve frequently spacial derivatives, in this case, we can not discuss the problem on the whole space H because the history variables of the functions F and G are only defined on $C([-r,0];H_\alpha)$. In order to study the controllability for system (1.1), we first apply the theory of fractional power operator and α-norm. We also suppose that $(-A, D(-A))$ generates a compact analytic semigroup on H so that the resolvent operator $\Phi(t)$ is analytic. We point out here that we do not require that the resolvent operator be compact which differs greatly from that in [22]. Then with the help of the Nussbaum fixed theorem, some sufficient conditions will be obtained.

This article is organized as follows. In section 2, we give the preliminaries for the paper. In section 3, we consider the existence of mild solutions of system (1.1) and provide the main result. In section 4, an example is given to illustrate the applications of the approximate controllability results.

2. Preliminaries

In this article, the operator $-A$ is the infinitesimal generator of a compact analytic semigroup $(S(t))_{t\geq 0}$. H_α is the space $(D(A^\alpha), \|\cdot\|_\alpha)$, $H_\alpha \subset H$. For each $0 < \alpha \leq 1$, H_α is a Banach space, $H_\alpha \to H_\beta$ for $0 < \beta < \alpha \leq 1$ and the embedding is compact whenever the resolvent operator of A is compact. $\mathcal{L}(H_\alpha; H_\beta)$ is the space of bounded linear operators from H_α into H_β with norm $\|\cdot\|_{\alpha,\beta}$ and $H_0 = H$.

Let $(\Omega, \mathfrak{F}, P)$ be a probability space on which an increasing and right continuous family $\{ \mathfrak{F}_t : t \geq 0\}$ of complete sub-σ-algebras of $\mathfrak{F}$ is defined. The collection of all square integrable and $\mathfrak{F}_t$-adapted processes is denoted by $L_2^{\mathfrak{F}_t}(J, H)$. Let $\beta_n(t)(n = 1, 2, \cdots)$ be a sequence of real valued one dimensional standard Brownian motions mutually independent over $(\Omega, \mathfrak{F}, P)$. We assume there exists a complete orthonormal basis $\{e_n\}$ in K and a bounded sequence of nonnegative real numbers λ_n such that $w(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n}\beta_n(t)e_n$, $t \geq 0$. Let $Q \in L(K, K)$ be an operator defined by $Qe_n = \lambda_n e_n, (n = 1, 2, 3 \dots)$ with finite trace $\operatorname{tr} Q = \sum_{n=1}^{\infty} \lambda_n < \infty$. Then the above K-valued stochastic process $w(t)$ is called a Q-Wiener process. We assume that $\mathfrak{F}_t = \sigma(w(s) : 0 \leq s \leq t)$ is the σ-algebra generated by w and $\mathfrak{F}_t = \mathfrak{F}$. Let $\Psi \in L_2^0(K, H)$ with the norm

$$\|\Psi\|_Q^2 = tr(\Psi Q \Psi^*) = \sum_{n=1}^{\infty} \|\sqrt{\lambda_n}\Psi e_n\|^2.$$

If $\|\Psi\|_Q < \infty$, then Ψ is called a Q-Hilbert-Schmidt operator. Define the space of all $\mathfrak{F}_o$-measurable C_α-valued function $\psi : \Omega \to C_\alpha$ with the norm

$$\mathbb{E}\|\psi\|_{C_\alpha}^2 = \mathbb{E}\{\sup_{\theta\in[-r,0]} \|A^\alpha \psi(\theta)\|^2\} < \infty.$$

Let $L_2(\Omega, \mathfrak{F}, P; H)$ be the space of all $\mathfrak{F}_t$-measurable square integrable random variables with value in H. We assume that: $PC(J_0, L_2(\Omega, \mathfrak{F}, P; H)) = \{x(t) : J_0 = [-r, T] \to L_2(\Omega, \mathfrak{F}, P; H)$ is continuous everywhere except some t_k at which $x(t_k^+)$ and $x(t_k^-)$ exist with $x(t_k) = x(t_k^-)$ satisfying $sup_{s\in J_0} E\|x(s)\|^2 < \infty\}$. Let $PC(J_0, L_2)$ be the closed subspace of $PC(J_0, L_2(\Omega, \mathfrak{F}, P; H))$ consisting of measurable and $\mathfrak{F}_t$-adapted processes and $\mathfrak{F}_0$-adapted processes $y \in L_2(\Omega, \mathfrak{F}_0, P; C_\alpha)$. Let $\|\cdot\|_*$ be a seminorm in $PC(J_0, L_2)$ defined by $\|y\|_* = (\sup_{t\in J} \mathbb{E}\|y_t\|_{C_\alpha}^2)^{1/2} < \infty$.

Definition 2.1 ([5])**.** A family of bounded linear operators $\Phi(t) \in \pounds(H)$ for $t \in J$ is called resolvent operator for

$$\begin{gathered} \frac{d}{dt}x(t) = -Ax(t) + \int_0^t \gamma(t-s)x(s)\,ds, \quad t \in J, \\ x(0) = x_0 \in H, \end{gathered} \tag{2.1}$$

if

- (i) $\Phi(0) = I$ and $\|\Phi(t)\| \leq N_1 e^{\omega t}$ for some $N_1 > 0, \omega \in R$.
- (ii) For all $x \in H$, $\Phi(t)x$ is strongly continuous in t on J.
- (iii) $\Phi(t) \in \pounds(Y)$, for $t \in J$, where Y is the Banach space formed from $D(-A)$ endowed with the graph norm. Moreover for $x \in Y, \Phi(\cdot)x \in C^1(J; H) \cap C(J; Y)$ and, for $t \geq 0$, the following equality holds

$$\frac{d}{dt}\Phi(t)x = -A\Phi(t)x + \int_0^t \gamma(t-s)\Phi(s)x\,ds = -\Phi(t)Ax + \int_0^t \Phi(t-s)\gamma(s)x\,ds.$$

The hypotheses on the operator A and $\gamma(\cdot)$ follow from [12, Hypotheses $(V_1) - (V_3)$]. Then, $\Phi(t)$ is also analytic and there exist $N, N_\alpha > 0$ such that $\|\Phi(t)\| \leq N$ and

$$\|A^\alpha \Phi(t)\| \leq \frac{N_\alpha}{t^\alpha}, 0 < t \leq T, \quad 0 \leq \alpha \leq 1.$$

Lemma 2.2 ([6])**.** *$\Phi(t)$ is continuous for $t > 0$ in the uniform operator topology of $\pounds(H)$.*

Lemma 2.3 ([7]). *$A\Phi(t)$ is continuous for $t > 0$ in the uniform operator topology of $\pounds(H)$.*

To simplify notation, let $A^\alpha\Phi(t)x = \Phi(t)A^\alpha x$, for any $0 \le \alpha \le 1, x \in D(A^\alpha)$. Now we define the mild solution of (1.1) expressed by the resolvent operator $\Phi(t)$.

Definition 2.4. A stochastic process $x(\cdot) \in PC(J_0, L_2(\Omega, \mathfrak{F}, P; H))$ is called a mild solution of (1.1) if the following condition are satisfied:

(1) the initial value $\phi \in L_2(\Omega, C_\alpha)$ and the control $u(\cdot) \in L_2^{\mathfrak{F}_t}(J, U)$.
(2) the function $A\Phi(t-s)F(s, x_s), s \in J$ is integrable and on J_0 it satisfies

$$x(t) = \begin{cases} \phi(t), & t \in [-r, 0], \\ \Phi(t)(\phi(0) + F(0,\phi)) - F(t, x_t) + \int_0^t A\Phi(t-s)F(s, x_s)\, ds \\ + \int_0^t \Phi(t-s)Bu(s)\, ds - \int_0^t \Phi(t-s) \int_0^s \gamma(s-v)F(v, x_v)\, dv\, ds \\ + \int_0^t \Phi(t-s)G(s, x_s)\, dw(s) + \Sigma_{0<t_k<t}\Phi(t-t_k)I_k(x(t_k)), & t \in J. \end{cases} \tag{2.2}$$

Definition 2.5. System (1.1) is said to be approximately controllable on J if

$$\overline{R(T; \phi, u)} = L_2(\Omega, \mathfrak{F}, P; H),$$

where $R(T; \phi, u) = \{x(T; \phi, u), u(\cdot) \in L_2^{\mathfrak{F}_t}(J, U)\}$.

To discuss the approximate controllability of system (1.1), we introduce the following operators.

(1) The controllability Grammian Γ_t^T is defined by

$$\Gamma_t^T = \int_t^T \Phi(T-s)BB^*\Phi^*(T-s)\, ds,$$

where Φ^* denotes the adjoint operator of Φ.

(2) The resolvent operator

$$R(\lambda, \Gamma_t^T) = (\lambda I + \Gamma_t^T)^{-1}.$$

At first, we assume

(H0) $\lambda R(\lambda, \Gamma_t^T) \to 0$, as $\lambda \to 0^+$ in the strong operator topology.

Note that the deterministic linear system corresponding to (1.1) is approximately controllable on $[t, T]$ if and only if (H0) holds , see [13].

We need the Nussbaum fixed-point theorem to prove the existence of mild solutions of system (1.1).

Lemma 2.6 ([16]). *Let S be a closed, bounded convex subset of a Banach space H, and P_1, P_2 be continuous mappings from S into H such that $(P_1 + P_2)S \subset S, \|P_1x - P_1y\| \le k\|x - y\|$ for all $x, y \in S$, where $0 \le k < 1$ is a constant, and P_2S is compact, then the operator $P_1 + P_2$ has a fixed point in S.*

Lemma 2.7 ([14]). *For any $h \in L_2(\Omega, \mathfrak{F}, P; H)$, there exist $\varphi \in L_2^{\mathfrak{F}_t}(J, L_2^0(K, H))$ such that*

$$h = \mathbb{E}h + \int_0^T \varphi(s)\, dw(s).$$

3. Approximate controllability

To study the approximate controllability of system (1.1), we introduce the following hypotheses:

(H1) There exist positive constants M_1, d and d_k, $k = 1, 2, \cdots m$, such that
$$\|B\| \leq M_1, \|I_k(x)\| \leq d_k, d = \sum_{k=1}^{m} d_k.$$

(H2) For arbitrary $\beta, \xi \in C_\alpha, x, y \in H$ and $t \in J$, suppose that there exists constants $d_{1k}, N_1 > 0$, such that
$$\|F(t,\beta) - F(t,\xi)\|_\alpha^2 + \|G(t,\beta) - G(t,\xi)\|_Q^2 \leq N_1 \|\beta - \xi\|_{C_\alpha}^2,$$
$$\|F(t,\xi)\|_\alpha^2 + \|G(t,\xi)\|_Q^2 \leq N_1(1 + \|\xi\|_{C_\alpha}^2),$$
$$\|I_k(x) - I_k(y)\|^2 \leq d_{1k}\|x - y\|^2.$$

(H3) The function $\gamma(t) \in L(H_\alpha, H)$ for each $t \in J$ suppose that there exist a positive constant M_2, such that $\|\gamma(t)\|_{\alpha,0} \leq M_2$.

(H4) The function $F : J \times C_\alpha \to H_\alpha$ and $G : J \times C_\alpha \to L_2^0(K, H)$ are uniformly bounded for $t \in J$, $x_t \in C_\alpha$, there exist a positive constant M_3, such that
$$\|F(t, x_t)\|_\alpha + \|G(t, x_t)\|_Q \leq M_3.$$

For any $\lambda \in (0, 1]$, we define the control function for system (1.1) as:
$$\begin{aligned}
u^\lambda(t,x) &= B^*\Phi^*(T-t)R(\lambda, \Gamma_0^T)[\mathbb{E}h - \Phi(T)(\phi(0) + F(0,\phi)) + F(T, x_T)] \\
&\quad - B^*\Phi^*(T-t)\int_0^t R(\lambda, \Gamma_s^T)A\Phi(T-s)F(s,x_s)\,ds \\
&\quad + B^*\Phi^*(T-t)\int_0^t R(\lambda, \Gamma_s^T)\Phi(T-s)\int_0^s \gamma(s-v)F(v,x_v)\,dvds \\
&\quad - B^*\Phi^*(T-t)\int_0^t R(\lambda, \Gamma_s^T)[\Phi(T-s)G(s,x_s) - \varphi(s)]\,dw(s) \\
&\quad - B^*\Phi^*(T-t)R(\lambda, \Gamma_0^T)\sum_{0<t_k<t}\Phi(T-t_k)I_k(x(t_k)).
\end{aligned}$$
and the operator P^λ on $PC(J_0, L_2(\Omega, \mathfrak{F}, P; H)$ as follows
$$(P^\lambda x)(t) = \begin{cases} \phi(t), & t \in [-r, 0], \\ \Phi(t)(\phi(0) + F(0,\phi)) - F(t,x_t) + \int_0^t A\Phi(t-s)F(s,x_s)\,ds \\ + \int_0^t \Phi(t-s)G(s,x_s)\,dw(s) \\ - \int_0^t \Phi(t-s)\int_0^s \gamma(s-v)F(v,x_v)\,dv\,ds \\ + \int_0^t \Phi(t-s)Bu^\lambda(s,x)\,ds + \sum_{0<t_k<t}\Phi(t-t_k)I_k(x(t_k)), & t \in J. \end{cases} \tag{3.1}$$
We can see the fixed point of P^λ is a mild solution of system (1.1). Now, we prove the following existence theorem.

Theorem 3.1. *Let $\phi \in L_2(\Omega, C_\alpha)$. If the assumptions* (H0)–(H3) *are satisfied then, the operator P^λ has a fixed point provided that*
$$K := 2N_1\Big(\|A^{-\alpha}\|^2 + \frac{N_{1-\alpha}^2 T^{2\alpha}}{\alpha^2}\Big) < 1.$$

Proof. We prove this theorem by using Lemma 2.6. Put

$$Y_r = \{x \in PC(J_0, L_2), \|x\|_* \le r\}.$$

It is obvious that Y_r is a bounded, closed and convex set. We first prove that for arbitrary $0 < \lambda \le 1$, there is a positive constant $r_0 = r_0(\lambda)$ such that $P^\lambda(Y_{r_0}) \subset Y_{r_0}$. For any $x \in Y_{r_0}$, we have

$$\begin{aligned}
&\mathbb{E}\|u^\lambda(t,x)\|^2 \\
&= \mathbb{E}\|B^*\Phi^*(T-t)R(\lambda,\Gamma_0^T)[\mathbb{E}h - \Phi(T)(\phi(0)+F(0,\phi)) + F(T,x_T)] \\
&\quad - B^*\Phi^*(T-t)\int_0^t R(\lambda,\Gamma_s^T)A\Phi(T-s)F(s,x_s)\,ds \\
&\quad + B^*\Phi^*(T-t)\int_0^t R(\lambda,\Gamma_s^T)\Phi(T-s)\int_0^s \gamma(s-v)F(v,x_v)\,dv\,ds \\
&\quad - B^*\Phi^*(T-t)\int_0^t R(\lambda,\Gamma_s^T)[\Phi(T-s)G(s,x_s)-\varphi(s)]\,dw(s) \\
&\quad - B^*\Phi^*(T-t)R(\lambda,\Gamma_0^T)\sum_{0<t_k<t}\Phi(T-t_k)I_k(x(t_k))\|^2 \qquad (3.2)\\
&\le 6\mathbb{E}\|B^*\Phi^*(T-t)R(\lambda,\Gamma_0^T)[\mathbb{E}h - \Phi(T)(\phi(0)+F(0,\phi))]\|^2 \\
&\quad + 6\mathbb{E}\|B^*\Phi^*(T-t)R(\lambda,\Gamma_0^T)A^{-\alpha}A^\alpha F(T,x_T)\|^2 \\
&\quad + 6\mathbb{E}\|B^*\Phi^*(T-t)\int_0^t R(\lambda,\Gamma_s^T)A^{1-\alpha}A^\alpha\Phi(t-s)F(s,x_s)\,ds\|^2 \\
&\quad + 6\mathbb{E}\|B^*\Phi^*(T-t)\int_0^t R(\lambda,\Gamma_s^T)\Phi(T-s)\int_0^s \gamma(s-v)F(v,x_v)\,dvds\|^2 \\
&\quad + 6\mathbb{E}\|B^*\Phi^*(T-t)\int_0^t R(\lambda,\Gamma_s^T)[\Phi(T-s)G(s,x_s)-\varphi(s)]\,dw(s)\|^2 \\
&\quad + 6\mathbb{E}\|B^*\Phi^*(T-t)R(\lambda,\Gamma_0^T)\sum_{0<t_k<t}\Phi(T-t_k)I_k(x(t_k))\|^2 \\
&\le \frac{6M_1^2N^2}{\lambda^2}\mathbb{E}\|\mathbb{E}h - N(\phi(0)+F(0,\phi))\|^2 + \frac{6M_1^2N^2}{\lambda^2}\|A^{-\alpha}\|^2\mathbb{E}\|F(T,x_T)\|_\alpha^2 \\
&\quad + \frac{6M_1^2N^2}{\lambda^2}\frac{N_{1-\alpha}^2T^{2\alpha}}{\alpha^2}\mathbb{E}\|F(s,x_s)\|_\alpha^2 + \frac{6M_1^2N^4M_2^2T^2}{\lambda^2}\mathbb{E}\int_0^T \|F(v,x_v)\|_\alpha^2\,dv \\
&+ \frac{12M_1^2N^4}{\lambda^2}\mathbb{E}\int_0^T \|G(s,x_s)\|_Q^2\,ds + \frac{12M_1^2N^2}{\lambda^2}\mathbb{E}\int_0^T \|\varphi(s)\|_Q^2\,ds + \frac{6M_1^2N^4d^2}{\lambda^2} \\
&\le \frac{6M_1^2N^2}{\lambda^2}\mathbb{E}\{\|\mathbb{E}h - N(\phi(0)+F(0,\phi))\|^2 + \|A^{-\alpha}\|^2N_1(1+\|x_s\|_{C_\alpha}^2) \\
&\quad + \frac{N_{1-\alpha}^2T^{2\alpha}}{\alpha^2}N_1(1+\|x_s\|_{C_\alpha}^2) + N^2(2+M_2^2T^2)\int_0^T N_1(1+\|x_s\|_{C_\alpha}^2)\,ds \\
&\quad + 2\int_0^T \|\varphi(s)\|_Q^2\,ds + N^2d^2\}.
\end{aligned}$$

Here we employ the assumption (H0) that

$$\|R(\lambda,\Gamma_s^T)\| \le \frac{1}{\lambda}, \lambda \in (0,1].$$

Similarly, for $x, y \in PC(J_0, L_2)$, we can also obtain

$$
\begin{aligned}
&\mathbb{E}\|u^\lambda(t,x) - u^\lambda(t,y)\|^2 \\
&\le 5\mathbb{E}\|B^*\Phi^*(T-t)R(\lambda,\Gamma_0^T)A^{-\alpha}A^\alpha[F(T,x_T) - F(T,y_T)]\|^2 \\
&\quad + 5\mathbb{E}\|B^*\Phi^*(T-t)\int_0^t R(\lambda,\Gamma_s^T)A^{1-\alpha}A^\alpha\Phi(T-s)[F(s,x_s) - F(s,y_s)]\,ds\|^2 \\
&\quad + 5\mathbb{E}\|B^*\Phi^*(T-t)\int_0^t R(\lambda,\Gamma_s^T)\Phi(T-s) \\
&\quad \times \int_0^s \gamma(s-v)[F(v,x_v) - F(v,y_v)]\,dvds\|^2 \\
&\quad + 5\mathbb{E}\|B^*\Phi^*(T-t)\int_0^t R(\lambda,\Gamma_s^T)\Phi(T-s)[G(s,x_s) - G(s,y_s)]\,dw(s)\|^2 \\
&\quad + 5\mathbb{E}\|B^*\Phi^*(T-t)R(\lambda,\Gamma_0^T)\sum_{0<t_k<t}\Phi(T-t_k)[I_k(x(t_k)) - I_k(y(t_k))]\|^2 \\
&\le \frac{5M_1^2N^2}{\lambda^2}\|A^{-\alpha}\|^2\mathbb{E}N_1\|x_T - y_T\|_{C_\alpha}^2 \qquad (3.3)\\
&\quad + \frac{5M_1^2N^2}{\lambda^2}\frac{N_{1-\alpha}^2T^{2\alpha}}{\alpha^2}\mathbb{E}N_1\|x_s - y_s\|_{C_\alpha}^2 \\
&\quad + \frac{5M_1^2N^4M_2^2T^2}{\lambda^2}\int_0^T \mathbb{E}N_1\|x_s - y_s\|_{C_\alpha}^2\,ds \\
&\quad + \frac{5M_1^2N^4}{\lambda^2}\int_0^T \mathbb{E}N_1\|x_s - y_s\|_{C_\alpha}^2\,ds \\
&\quad + \frac{5M_1^2N^4}{\lambda^2}\sum_{k=1}^m d_{1k}^2\mathbb{E}\|x_{t_k} - y_{t_k}\|^2 \\
&\le \frac{5M_1^2N^2}{\lambda^2}\{N_1[N^2T(M_2^2T^2+1) + \frac{N_{1-\alpha}^2T^{2\alpha}}{\alpha^2} \\
&\quad + \|A^{-\alpha}\|^2] + N^2\sum_{k=1}^m d_{1k}^2\}\|x-y\|_*^2.
\end{aligned}
$$

Then we have

$$
\begin{aligned}
&E\|(P^\lambda x)(t)\|^2 \\
&= E\|\Phi(t)(\phi(0) + F(0,\phi)) - F(t,x_t) + \int_0^t A\Phi(t-s)F(s,x_s)\,ds \\
&\quad + \int_0^t \Phi(t-s)G(s,x_s)\,dw(s) - \int_0^t \Phi(t-s)\int_0^s \gamma(s-v)F(v,x_v)\,dv\,ds \\
&\quad + \int_0^t \Phi(t-s)Bu^\lambda(s)\,ds + \sum_{0<t_k<t}\Phi(t-t_k)I_k(x(t_k))\|^2 \\
&\le 7E\|\Phi(t)(\phi(0) + F(0,\phi))\|^2 + 7E\|A^{-\alpha}A^\alpha F(t,x_t)\|^2 \\
&\quad + 7E\|\int_0^t A^{1-\alpha}A^\alpha\Phi(t-s)F(s,x_s)\,ds\|^2 + 7E\|\int_0^t \Phi(t-s)Bu^\lambda(s)\,ds\|^2
\end{aligned}
$$

$$
\begin{aligned}
&+ 7E\| \int_0^t \Phi(t-s) \int_0^s \gamma(s-v) F(v, x_v)\, dv ds\|^2 \\
&+ 7E\| \int_0^t \Phi(t-s) G(s, x_s)\, dw(s)\|^2 + 7E\| \sum_{0<t_k<t} \Phi(t-t_k) I_k(x(t_k))\|^2 \\
\le\ & 7E\|N(\phi(0) + F(0,\phi))\|^2 + 7N_1\|A^{-\alpha}\|^2(1 + \|x\|_*^2) \\
&+ 7\frac{N_{1-\alpha}^2 T^{2\alpha}}{\alpha^2} N_1(1 + \|x\|_*^2) + 7T^4N^2M_2^2N_1(1 + \|x\|_*^2) \\
&+ 7N^2TN_1(1 + \|x\|_*^2) + 7N^2M_1^2TE\|u^\lambda(t,s)\|^2 + 7N^2d^2 \\
\le\ & 7N_1(1 + \|x\|_*^2)(\|A^{-\alpha}\|^2 + \frac{N_{1-\alpha}^2 T^{2\alpha}}{\alpha^2} + T^4N^2M_2^2 + N^2T) \\
&+ 7E\|N(\phi(0) + F(0,\phi))\|^2 + 7N^2M_1^2TE\|u^\lambda(t,s)\|^2 + 7N^2d^2.
\end{aligned}
$$

From (3.2) we can imply that $E\|(P^\lambda x)(t)\|^2 < \infty$. So there exist a constant r_0 such that $P^\lambda(Y_{r_0}) \in Y_{r_0}$.

Next we prove that P^λ has a fixed point on Y_{r_0}. To begin, we rewrite P^λ as $P^\lambda = P_1^\lambda + P_2^\lambda$, where

$$
(P_1^\lambda x)(t) = \begin{cases} 0, & t \in [-r, 0], \\ \Phi(t)F(0,\phi) - F(t, x_t) + \int_0^t A\Phi(t-s)F(s, x_s)\, ds, & t \in J. \end{cases}
$$

$$
\begin{aligned}
&(P_2^\lambda x)(t) \\
&= \begin{cases} \phi(t), & t \in [-r, 0], \\ \Phi(t)\phi(0) + \int_0^t \Phi(t-s)G(s, x_s)\, dw(s) \\ - \int_0^t \Phi(t-s) \int_0^s \gamma(s-v)F(v, x_v)\, dv\, ds \\ + \int_0^t \Phi(t-s)Bu^\lambda(s,x)\, ds + \sum_{0<t_k<t} \Phi(t-t_k)I_k(x(t_k)), & t \in J. \end{cases}
\end{aligned}
$$

To prove that P_1^λ is a contraction, we take $x, y \in Y_{r_0}$, then, for each $t \in J$, we can verify

$$
\begin{aligned}
&\mathbb{E}\|(P_1^\lambda x)(t) - (P_1^\lambda y)(t)\|^2 \\
&= \mathbb{E}\|A^{-\alpha}A^\alpha(F(t, x_t) - F(t, y_t)) + \int_0^t A^{1-\alpha}A^\alpha\Phi(t-s)(F(t, x_s) - F(t, y_s))\, ds\|^2 \\
&\le 2\|A^{-\alpha}\|^2N_1\|x - y\|_*^2 + 2\frac{N_{1-\alpha}^2 T^{2\alpha}}{\alpha^2} N_1\|x - y\|_*^2 \\
&= 2N_1(\|A^{-\alpha}\|^2 + \frac{N_{1-\alpha}^2 T^{2\alpha}}{\alpha^2})\|x - y\|_*^2.
\end{aligned}
$$

So we have

$$
\|P_1^\lambda x - P_1^\lambda y\|_*^2 \le K\|x - y\|_*^2.
$$

So P_1^λ is a contraction mapping on Y_{r_0}.

Next we prove that P_2^λ is continuous on Y_{r_0}. Let $\{x^m(\cdot)\} \subseteq Y_{r_0}$ with $x^m(\cdot) \to x(\cdot), (m \to \infty)$. Then we have

$$
\begin{aligned}
&\mathbb{E}\|(P_2^\lambda x^m)(t) - (P_2^\lambda x)(t)\|^2 \\
&\le 4\mathbb{E}\| \int_0^t \Phi(t-s)[G(s, x_s^m) - G(s, x_s)\, dw(s)]\|^2
\end{aligned}
$$

$$
\begin{aligned}
&+ 4\mathbb{E}\| \int_0^t \Phi(t-s)\int_0^s \gamma(s-v)[F(v,x_v^m)-F(v,x_v)]\,dvds\|^2 \\
&+ 4\mathbb{E}\| \int_0^t \Phi(t-s)B[u^\lambda(s,x^m)-u^\lambda(s,x)]\,ds\|^2 \\
&+ 4\mathbb{E}\| \sum_{0<t_k<t} \Phi(t-t_k)[I_k(x^m(t_k))-I_k(x(t_k))]\|^2 \\
&\le 4N^2\mathbb{E}\int_0^T N_1\|x_s^m-x_s\|_{C_\alpha}^2)\,ds \\
&+ 4N^2M_2^2\mathbb{E}\int_0^T\int_0^T N_1\|x_v^m-x_v\|_{C_\alpha}^2\,dv\,ds \\
&+ 4N^2M_1^2\int_0^T \mathbb{E}\|u^\lambda(s,x^m)-u^\lambda(s,x)\|^2\,ds \\
&+ 4N^2\sum_{k=1}^m d_{1k}^2\mathbb{E}\|x_{t_k}^m-x_{t_k}\|^2.
\end{aligned}
$$

From (3.3) and the Lebesgue-dominated convergence theorem, we obtain

$$
\mathbb{E}\|(P_2^\lambda x^m)(t)-(P_2^\lambda x)(t)\|^2 \to 0,
$$

as $m\to\infty$. So P_2^λ is continuous.

We finally prove that the operator P_2^λ maps Y_{r_0} into a relatively compact subset of Y_{r_0}. Denote the set

$$
V(t)=\{(P_2^\lambda x)(t): x\in Y_{r_0}\}.
$$

Step 1. $P_2^\lambda(Y_{r_0})$ is clearly bounded.

Step 2. we have to show that $V(t)$ is equicontinuous on J_0. Let $x\in Y_{r_0}, t_1,t_2\in (0,T]$. Then

$$
\begin{aligned}
&\mathbb{E}\|(P_2^\lambda x)(t_2)-(P_2^\lambda x)(t_1)\|^2 \\
&\le 5\mathbb{E}\|\phi(0)(\Phi(t_2)-\Phi(t_1))\|^2 \\
&+ 5\mathbb{E}\|[\int_0^{t_2}\Phi(t_2-s)G(s,x_s)-\int_0^{t_1}\Phi(t_1-s)G(s,x_s)]\,dw(s)\|^2 \\
&+ 5\mathbb{E}\|\int_0^s \gamma(s-v)F(v,x_v)\Big[\int_0^{t_2}\Phi(t_2-s)-\int_0^{t_1}\Phi(t_1-s)\Big]\,dvds\|^2 \\
&+ 5\mathbb{E}\|[\int_0^{t_2}\Phi(t_2-s)-\int_0^{t_1}\Phi(t_1-s)]Bu^\lambda(s,x)\,ds\|^2 \\
&+ 5\mathbb{E}\|\sum_{0<t_k<t}\big[\Phi(t_2-t_k)-\Phi(t_2-t_k)\big]I_k(x(t_k)\|^2 \\
&= I_1+I_2+I_3+I_4+I_5.
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
I_2 &\le 10\mathbb{E}\|\int_0^{t_1}(\Phi(t_2-s)-\Phi(t_1-s))G(s,x_s)\,dw(s)\|^2 \\
&\quad + 10\mathbb{E}\|\int_{t_1}^{t_2}\Phi(t_2-s)G(s,x_s)\,dw(s)\|^2 \\
&\le 10N^2N_1T(1+\|x\|_*^2)\mathbb{E}\|\Phi(t_2-t_1)-I\|^2 + 10N^2N_1(t_2-t_1)(1+\|x\|_*^2),
\end{aligned}
$$

$$
\begin{aligned}
I_3 &\le 10\mathbb{E}\|\int_0^{t_1}\Phi(t_1-s)(\Phi(t_2-t_1)-I)\int_0^s\gamma(s-v)F(v,x_v)\,dvds\|^2 \\
&\quad + 10\mathbb{E}\|\int_{t_1}^{t_2}\Phi(t_2-s)\int_0^s\gamma(s-v)F(v,x_v)\,dvds\|^2 \\
&\le 10N^2M_2^2T^2N_1(1+\|x\|_*^2)\mathbb{E}\|\Phi(t_2-t_1)-I\|^2 \\
&\quad + 10N^2M_2^2TN_1(1+\|x\|_*^2)(t_2-t_1),
\end{aligned}
$$

$$
\begin{aligned}
I_4 &\le 10\mathbb{E}\|\int_0^{t_1}\Phi(t_1-s)(\Phi(t_2-t_1)-I)Bu^\lambda(s,x)\,ds\|^2 \\
&\quad + 10\mathbb{E}\|\int_{t_1}^{t_2}\Phi(t_2-s)Bu^\lambda(s,x)\,ds\|^2 \\
&\le 10N^2M_1^2T\mathbb{E}\|u^\lambda(s,x)\|^2\|\Phi(t_2-t_1)-I\|^2 + 10N^2M_1^2(t_2-t_1)\mathbb{E}\|u^\lambda(s,x)\|^2.
\end{aligned}
$$

In a similar way, we have

$$
I_5 \le 10N^2d^2\mathbb{E}\|\Phi(t_2-t_1)-I\|^2 + 10N^2d^2(t_2-t_1).
$$

It is easy to see that, as $t_2 \to t_1$ the right-hand side of the above inequality tends to zero, since $\Phi(t)$ is continuous in t in the uniform operator topology by Lemma 2.2. So we obtain the equicontinuity of V.

Step3. We show that for fixed t, the set $V(t)$ is relatively compact. Obviously, $V(t) = \phi(t), t \in [-r,0]$ which is trivially relatively compact. So let $t \in (0,T]$ be fixed, then

$$
V(t) = \Phi(t)\phi(0) + \sum_{k=1}^m \Phi(t-t_k)I_k(x(t_k)) + V_1(t),
$$

where $V_1(t)$ is defined by

$$
\begin{aligned}
V_1(t) := \Big\{\nu(t) = \int_0^t\Phi(t-s)G(s,x_s)\,dw(s) - \int_0^t\Phi(t-s)\int_0^s\gamma(s-v)F(v,x_v)\,dv\,ds \\
+ \int_0^t\Phi(t-s)Bu^\lambda(s,x)\,ds, x\in Y_{r_0}\Big\}.
\end{aligned}
$$

Since $\sum_{k=1}^m\Phi(t-t_k)I_k(x(t_k))$ is uniform bounded and equicontinuous. By the Ascoli-Arzela theorem $\sum_{k=1}^m\Phi(t-t_k)I_k(x(t_k))$ is relatively compact. $\Phi(t)\phi(0)$ is a single point in H. So we just have to show that $V_1(t)$ is relatively compact, let $0<\alpha<\alpha_1<1$, we have

$$
\begin{aligned}
\mathbb{E}\|A^{\alpha_1}\nu(t)\|^2 &\le 3\mathbb{E}\|\int_0^t A^{\alpha_1}\Phi(t-s)G(s,x_s)\,dw(s)\|^2 \\
&\quad + 3\mathbb{E}\|\int_0^t A^{\alpha_1}\Phi(t-s)\int_0^s\gamma(s-v)F(v,x_v)\,dvds\|^2 \\
&\quad + 3\mathbb{E}\|\int_0^t A^{\alpha_1}\Phi(t-s)Bu^\lambda(s,x)\,ds\|^2 \\
&\le 3\frac{N_{\alpha_1}^2T^{2-2\alpha_1}}{(1-\alpha_1)^2}(1+\|x\|_*^2) + 3\frac{N_{\alpha_1}^2T^{4-2\alpha_1}M_2^2}{(1-\alpha_1)^2}(1+\|x\|_*^2) \\
&\quad + 3\frac{N_{\alpha_1}^2T^{2-2\alpha_1}M_1^2}{(1-\alpha_1)^2}\mathbb{E}\|u^\lambda(s,x)\|^2 < \infty,
\end{aligned}
$$

which implies that $A^{\alpha_1}V_1(t)$ is bounded in H. Hence we obtain that $V_1(t)$ is relatively compact in H_α by the compactness of the operator $A^{-\alpha_1}: H \to H_{\alpha_1}$, (noting that the embedding $H_{\alpha_1} \to H_\alpha$ is compact). Therefore, from the Ascoli-Arzela theorem, $P_2^\lambda(Y_{r_0})$ is compact. So, the operator $P^\lambda = P_1^\lambda + P_2^\lambda$ has a fixed point from Lemma 2.6. The proof is complete. □

Theorem 3.2. *Assume that* (H0)–(H4) *are satisfied, then system* (1.1) *is approximately controllable on* J.

Proof. Because the hypotheses of Theorem 3.1 are fulfilled, there is a solution $x^\lambda(\cdot)$ of (1.1) under the control $u^\lambda(t,x)$. Using the stochastic Fubini theorem we can obtain

$$
\begin{aligned}
x^\lambda(T) &= \Phi(T)(\phi(0)+F(0,\phi)) - F(T,x_T^\lambda) + \int_0^T A\Phi(T-s)F(s,x_s^\lambda)\,ds \\
&\quad + \int_0^T \Phi(T-s)G(s,x_s^\lambda)\,dw(s) - \int_0^T \Phi(T-s)\int_0^s \gamma(s-v)F(v,x_v^\lambda)\,dv\,ds \\
&\quad + \sum_{k=1}^m \Phi(T-t_k)I_k(x^\lambda(t_k)) + \Gamma_0^T R(\lambda,\Gamma_0^T)\big[Eh - \Phi(T)(\phi(0)+F(0,\phi)) \\
&\quad + F(T,x_T)\big] \\
&\quad - \int_0^T\int_r^T \Phi(T-s)BB^*\Phi^*(T-s)R(\lambda,\Gamma_r^T)A\Phi(T-r)F(r,x_r^\lambda)\,ds\,dr \\
&\quad + \int_0^T\int_r^T \Phi(T-s)BB^*\Phi^*(T-s)R(\lambda,\Gamma_r^T)\Phi(T-r) \\
&\quad \times \int_0^r \gamma(r-v)F(v,x_v^\lambda)\,dv\,ds\,dr \\
&\quad - \int_0^T\int_r^T \Phi(T-s)BB^*\Phi^*(T-s)R(\lambda,\Gamma_r^T) \\
&\quad \times [\Phi(T-r)G(r,x_r^\lambda) - \varphi(r)]\,ds\,dw(r) \\
&\quad - \Gamma_0^T R(\lambda,\Gamma_0^T)\sum_{k=1}^m \Phi(T-t_k)I_k(x^\lambda(t_k)) \\
&= h - \lambda R(\lambda,\Gamma_0^T)[Eh - \Phi(T)(\phi(0)+F(0,\phi)) + F(T,x_T)] \\
&\quad + \int_0^T \lambda R(\lambda,\Gamma_s^T)A\Phi(T-s)F(s,x_s^\lambda)\,ds \\
&\quad - \int_0^T \lambda R(\lambda,\Gamma_s^T)\Phi(T-s)\int_0^s \gamma(s-v)F(v,x_v^\lambda)\,dv\,ds \\
&\quad + \int_0^T \lambda R(\lambda,\Gamma_s^T)[\Phi(T-s)G(s,x_s^\lambda) - \varphi(s)]\,dw(s) \\
&\quad + \lambda R(\lambda,\Gamma_0^T)\sum_{k=1}^m \Phi(T-t_k)I_k(x^\lambda(t_k)).
\end{aligned}
$$

By (H4),

$$\|F(t,x_t)\|_\alpha + \|G(t,x_t)\|_Q \le M_3.$$

Then there is a subsequence, still denoted by $\{F(s,x_s^\lambda),G(s,x_s^\lambda)\}$, weakly converging to say, $\{F(s,w),G(s,w)\}$ in $H\times L_2^0(K,H)$. By (H0), $\lambda R(\lambda,\Gamma_s^T)\to 0$ as $\lambda\to 0^+$ and $\|\lambda R(\lambda,\Gamma_s^T)\|\le 1$, from which, together with Lebesgue dominated convergence theorem, we have

$$
\begin{aligned}
&\mathbb{E}\|x^\lambda(T)-h\|^2\\
&\le 10\mathbb{E}\|\lambda R(\lambda,\Gamma_0^T)[Eh-\Phi(T)(\phi(0)+F(0,\phi))\|^2\\
&\quad+10\mathbb{E}\|\lambda R(\lambda,\Gamma_0^T)\|^2\|A^{-\alpha}\|^2\|F(T,x_T^\lambda)]\|_\alpha^2\\
&\quad+10\mathbb{E}\Big(\int_0^T\|\lambda R(\lambda,\Gamma_s^T)\|\|A\Phi(T-s)\|\|A^{-\alpha}\|\|F(s,x_s^\lambda)-F(s)\|_\alpha\,ds\Big)^2\\
&\quad+10\mathbb{E}\Big(\int_0^T\|\lambda R(\lambda,\Gamma_s^T)\|\|A\Phi(T-s)\|\|A^{-\alpha}\|\|F(s)\|_\alpha\,ds\Big)^2\\
&\quad+10\mathbb{E}\Big(\int_0^T\|\lambda R(\lambda,\Gamma_s^T)\Phi(T-s)\int_0^s\gamma(s-v)A^{-\alpha}\|\|F(v,x_v^\lambda)-F(v)\|_\alpha\,dv\,ds\Big)^2\\
&\quad+10\mathbb{E}\Big(\int_0^T\|\lambda R(\lambda,\Gamma_s^T)\Phi(T-s)\int_0^s\gamma(s-v)F(v)\|\,dvds\Big)^2\\
&\quad+10\mathbb{E}\int_0^T\|\lambda R(\lambda,\Gamma_s^T)\|^2\|\Phi(T-s)\|^2\|G(s,x_s^\lambda)-G(s)\|_Q^2\,ds\\
&\quad+10\mathbb{E}\int_0^T\|\lambda R(\lambda,\Gamma_s^T)\|^2\|\Phi(T-s)\|^2\|G(s)\|_Q^2\,ds\\
&\quad+10\mathbb{E}\int_0^T\|\lambda R(\lambda,\Gamma_s^T)\|^2\|\varphi(s)\|_Q^2\,ds\\
&\quad+10\mathbb{E}\big\|\lambda R(\lambda,\Gamma_0^T)\sum_{k=1}^m\Phi(T-t_k)I_k(x^\lambda(t_k))\big\|^2\to 0,\quad\text{as }\lambda\to 0^+.
\end{aligned}
$$

This gives the approximate controllability. The proof is complete. □

4. Example

As an application, we consider the neutral stochastic integro-differential system with impulses,

$$
\begin{gathered}
d[z(t,x)+f(t,z(t-r_1(t),x))]\\
=\big[\frac{\partial^2}{\partial x^2}z(t,x)+u(t,x)+\int_0^t b(t-s)\frac{\partial^2}{\partial x^2}z(s,x)ds\big]dt+g(t,z(t-r_2(t),x))d\beta(t),\\
0<t\le T,\quad t\ne t_k,\quad 0\le x\le\pi,\\
z(t,0)=z(t,\pi)=0,\quad 0\le t\le T,\\
z(t,x)=\phi(t,x),\quad -r\le t\le 0,\quad 0\le x\le\pi,\\
\Delta z(t_k,x)=\int_0^\pi K(t_k,x,y)z(t_k,y)\,dy,\quad k=1,2,3,\dots,m,
\end{gathered}
\tag{4.1}
$$

where r_1,r_2 are continuous functions with $0<r_1(t)\le r, 0<r_2(t)\le r$ for all $t\in J$, $\beta(t)$ denotes a one-dimensional standard Brownian motion. The functions f,g,ϕ,K and b will be described below.

System (4.1) arises in the study of heat flow in materials of the so-called retarded type [10, 17]. Here, $z(t,x)$ represents the temperature of the point x at time t. As stated in [3], these problems also arise in systems related to couple oscillators in a noisy environment or in viscoelastic materials under random or stochastic influences. Meanwhile, an impulsive perturbation occurs very often in many practical models. For instance, the system of rigid heat conduction with impulsive effect can be modeled in the form of (4.1).

Let $H = L_2[0,\pi]$, and let $A : H \to H$ be the operator defined by

$$A\xi = -\frac{\partial^2}{\partial x^2}\xi,$$

with domain

$$D(A) = \big\{\xi \in H : \xi, \frac{\partial}{\partial x}\xi \text{ are absolutely continuous}, \frac{\partial^2}{\partial x^2}\xi \in H, \xi(0) = \xi(\pi) = 0\big\}.$$

Then $-A$ is self-adjoint, negative definite and the resolvent operator $R(\lambda, -A) = (\lambda I + A)^{-1}$ is compact when it exist. Moreover, $-A$ generates a strongly semigroup $\{S(t)\}_{t\ge 0}$ which is analytic, compact and self-adjoint. There exists a complete orthonormal set $\{e_n\}$ of eigenvectors of $-A$ with $e_n(x) = \sqrt{\frac{2}{\Pi}}\sin(nx), n = 1, 2, 3, \dots$. Then the following properties hold:

$$A\xi = \sum_{n=1}^{\infty} n^2\langle \xi, e_n\rangle e_n, \quad \xi \in D(A),$$

$$S(t)\xi = \sum_{n=1}^{\infty} \exp(-n^2 t)\langle \xi, e_n\rangle e_n, \quad \xi \in H.$$

Set $H_\alpha = D(A^\alpha)$ and $C_\alpha = C([-r,0], H_\alpha)$. By the classical spectral theorem, we deduce that

$$A^\alpha S(t)\xi = \sum_{n=1}^{\infty} (n^2)^\alpha \exp(-n^2 t)\langle \xi, e_n\rangle e_n.$$

We assume that the following conditions hold:

(i) The functions $f : J \times \mathbb{R} \to \mathbb{R}$ and $g : J \times \mathbb{R} \to \mathbb{R}$ are continuous and global Lipschitz continuous and uniformly bounded.

(ii) The function ϕ is defined by $\phi(\theta)(x) = \phi(\theta, x)$ belongs to C_α.

(iii) $b(t) \in L^1(R^+) \cap C^1(R^+)$ with primitive $B(t) \in L^1_{loc}(R^+)$, $B(t)$ is non-positive, non-decreasing and $B(0) = -1$.

(iiii) $K(t, x, y) : J \to L^2([0,\pi] \times [0,\pi])$ is measurable and continuous, thus bounded. $l_k := \int_0^\pi \int_0^\pi |K(t_k, x, y)|^2 \, dxdy, k = 1, 2, \dots, m$.

Now define the abstract functions F, G, I_k and the operator $\gamma(t)$ respectively by

$$F(t, \psi)(x) = f(t, \psi(-r_1(t))(x), \quad \psi \in C_\alpha,\ x \in [0,\pi],$$

$$G(t, \psi)(x) = g(t, \psi(-r_1(t))(x), \quad \psi \in C_\alpha,\ x \in [0,\pi],$$

$$I_k(\varphi)(x) = \int_0^\pi K(t_k, x, y)\varphi(y)\, dy,\ \varphi \in H,$$

$$\gamma(t) = b(t)A, \quad t \in J.$$

Then system (4.1) is rewritten into the form of (1.1). Moreover, $B = I$, satisfy (H1), and $\gamma(t)$ satisfies condition (H3). From [12], the linear system of (4.1) has an analytic resolvent operator $(W(t))_{t\ge 0}$ which is given by $W(0) = I$.

By condition (i) we have

$$\begin{aligned}\|F(t,\psi)-F(t,\phi)\|_\alpha^2 &= \int_0^\pi |f(t,\psi(-r_1(t))(x))-f(t,\phi(-r_1(t))(x))|^2\,dx\\ &\le N^2\|\psi(-r_1(t))(x)-\phi(-r_1(t))(x)\|^2\\ &= N^2\sum_{n=1}^\infty\langle\psi(-r_1(t))-\phi(-r_1(t)),e_n\rangle^2\\ &\le N^2\sum_{n=1}^\infty n^{4\alpha}\langle\psi(-r_1(t))-\phi(-r_1(t)),e_n\rangle^2\\ &= N^2\|A^\alpha(\psi(-r_1(t))-\phi(-r_1(t))\|^2\\ &\le N^2\|\psi-\phi\|_{C_\alpha}^2,\end{aligned}$$

where N is appropriate constant.

By condition (iiii) we have

$$\begin{aligned}\|I_k(\varphi_1)-I_k(\varphi_2)\|^2 &= \int_0^\pi\Big[\int_0^\pi K(t_k,x,y)\varphi_1(y)\,dy-\int_0^\pi K(t_k,x,y)\varphi_2(y)\,dy\Big]^2\,dx,\\ &= \int_0^\pi\Big[\int_0^\pi K(t_k,x,y)(\varphi_1(y)-\varphi_2(y))\Big]^2\,dx,\\ &\le \int_0^\pi\Big[\int_0^\pi |K(t_k,x,y)|^2\int_0^\pi\|\varphi_1(y)-\varphi_2(y)\|^2\,dy\Big]\,dx,\\ &\le \pi l_k\|\varphi_1-\varphi_2\|^2.\end{aligned}$$

Similarly we can show that F, G and I_k satisfy the assumptions (H2), (H4). From [12, Lemma 4.1], the resolvent operator $W(t)$ of (4.1) is self adjoint. Thus

$$B^*W^*(t)\xi = W(t)\xi,\quad \xi\in H.$$

Let $B^*W^*(t)\xi=0$, for all $t\in J$, thus

$$B^*W^*(t)\xi = W(t)\xi = 0,\quad t\in J.$$

It follows from the fact $W(0)=I$ that $\xi=0$, so by [2, Theorem 4.1.7], the deterministic linear system corresponding to (4.1) is approximately controllable on J. Hence, (H0) holds. Therefore, by Theorem 3.2, system (4.1) is approximately controllable on J.

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