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FRACTIONAL ELLIPTIC EQUATIONS WITH CRITICAL GROWTH AND SINGULAR NONLINEARITIES

TUHINA MUKHERJEE, KONIJETI SREENADH

Abstract. In this article, we study the fractional Laplacian equation with critical growth and singular nonlinearity

$$
\begin{gathered}
(-\Delta)^s u = \lambda a(x) u^{-q} + u^{2^*_s-1}, \quad u>0 \quad \text{in } \Omega,\\
u = 0 \quad \text{in } \mathbb{R}^n \setminus \Omega,
\end{gathered}
$$

where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, $n > 2s$, $s \in (0,1)$, $\lambda > 0$, $0 < q \le 1$, $\theta \le a(x) \in L^\infty(\Omega)$, for some $\theta > 0$ and $2^*_s = \frac{2n}{n-2s}$. We use variational methods to show the existence and multiplicity of positive solutions of the above problem with respect to the parameter λ.

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$, $n > 2s$ and $s \in (0,1)$. We consider the following problem with singular nonlinearity:

$$
\begin{gathered}
(-\Delta)^s u = \lambda a(x) u^{-q} + u^{2^*_s-1}, \quad u>0 \quad \text{in } \Omega,\\
u = 0 \quad \text{in } \mathbb{R}^n \setminus \Omega,
\end{gathered}
\tag{1.1}
$$

where $\lambda > 0$, $0 < q \le 1$, $\theta \le a(x) \in L^\infty(\Omega)$ for some $\theta > 0$, $2^*_s = \frac{2n}{n-2s}$ and $(-\Delta)^s$ is the fractional Laplace operator defined as

$$
(-\Delta)^s u(x) = -\frac{1}{2}\int_{\mathbb{R}^n} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^{n+2s}}\,dy, \quad \text{for all } x \in \mathbb{R}^n.
$$

The fractional power of Laplacian is the infinitesimal generator of Lévy stable diffusion process and arise in anomalous diffusion in plasma, population dynamics, geophysical fluid dynamics, flames propagation, chemical reactions in liquids and American options in finance. For more details, we refer to [2, 11].

Recently, the study of the fractional elliptic equations attracted lot of interest by researchers in nonlinear analysis. There are many works on existence of a solution for fractional elliptic equations with regular nolinearities like $u^q + \lambda u^p$, $p, q > 0$. The sub critical growth problems are studied in [7, 31, 33] and critical exponent problems are studied in [6, 27, 25, 34, 32]. Also, the multiplicity of solutions by the method of Nehari manifold and fibering maps has been investigated in [15, 16, 35, 36]. To the best of our knowledge, there are no works dealing with multiplicity results with

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singular and critical nonlinearities. We also refer [3, 12, 14, 23, 24, 26, 28, 30] for related works with fractional Laplacian, singular nonlinearities, critical growth or critical exponential nonlinearities. In this paper, we attempt to address the multiplicity of positive solutions of problem with singular type nonlinearity $\lambda u^{-q} + u^{2_s^*-1}$, $0 < q \leq 1$.

In the local setting ($s = 1$), the paper by Crandall, Rabinowitz and Tartar [8] is the starting point on semilinear problem with singular nonlinearity. A lot of work has been done related to existence and multiplicity results on singular nonlinearity. Among them we cite the reader to [17, 13, 29, 18, 19, 21, 20] and references therein. In [21], authors studied the critical growth singular problem

$$
\begin{gathered}
-\Delta u = \lambda u^{-q} + u^{2^*-1}, \quad u > 0 \quad \text{in } \Omega, \\
u = 0 \quad \text{on } \partial\Omega,
\end{gathered}
$$

where $0 < q < 1$. Using the variational methods and the geometry of the Nehari manifold, they proved the existence of multiple solutions in a suitable range of λ. Among the works dealing with elliptic equations with singular and critical growth terms, we cite also [1, 9, 17, 20] and references there in, with no attempt to provide a complete list.

The fractional elliptic problem with only singular nonlinear term is studied by Fang [10] where author studied the problem

$$
\begin{gathered}
(-\Delta)^s u = u^{-p}, \quad u > 0 \quad \text{in } \Omega, \\
u = 0 \quad \text{in } \mathbb{R}^n \backslash \Omega,
\end{gathered}
$$

with $0 < p < 1$. Here, authors used the method of sub and super solutions to show the existence of solution. Recently, in [5] the authors considered the problem

$$
\begin{gathered}
(-\Delta)^s u = \lambda \frac{f(x)}{u^\gamma} + M u^p, \quad u > 0 \quad \text{in } \Omega, \\
u = 0 \quad \text{in } \mathbb{R}^n \setminus \Omega,
\end{gathered}
$$

where $n > 2s$, $M \geq 0$, $0 < s < 1$, $\gamma > 0$, $\lambda > 0$, $1 < p < 2_s^* - 1$ and $f \in L^m(\Omega)$, $m \geq 1$ is a nonnegative function. Here, the authors studied the existence of distributional solutions using the uniform estimates of $\{u_n\}$ which are solutions of the regularized problems with singular term $u^{-\gamma}$ replaced by $(u + \frac{1}{n})^{-\gamma}$. They also discussed the multiplicity results when $M > 0$ and for small λ in the sub critical case.

In this article, we study the multiplicity results with convex-concave type critical growth and singular nonlinearity. Here, we follow the approach as in the work of Hirano, Saccon and Shioji [21]. We obtain our results by studying the existence of minimizers that arise out of structure of Nehari manifold. We would like to remark that the results proved here are new even for the case $q = 1$. Also, the multiplicity result is sharp in the sense that we consider the maximal range of λ for which the corresponding fibering maps have two critical points.

The article is organized as follows: In section 2, we present some preliminaries on function spaces required for variational settings. In section 3, we study the corresponding Nehari manifold and properties of minimizers. In section 4 and 5, we show the existence of minimizers and solutions. In section 6, we show some regularity results.

2. Preliminaries and main results

We recall some definitions of function spaces and results that are required in later sections. In [32], Servadei and Valdinoci discussed the Dirichlet boundary value problem in case of fractional Laplacian using the variational techniques. Due to nonlocalness of the fractional Laplacian, they introduced the function space $(X_0, \|\cdot\|_{X_0})$. The space X is defined as

$$X = \big\{u : u : \mathbb{R}^n \to \mathbb{R} \text{ is measurable, } u|_\Omega \in L^2(\Omega) \text{ and } \frac{(u(x)-u(y))}{|x-y|^{\frac{n}{2s}+s}} \in L^2(Q)\big\},$$

where $Q = \mathbb{R}^{2n} \setminus (\mathcal{C}\Omega \times \mathcal{C}\Omega)$ and $\mathcal{C}\Omega := \mathbb{R}^n \setminus \Omega$. The space X is endowed with the norm defined as

$$\|u\|_X = \|u\|_{L^2(\Omega)} + [u]_X , \quad \text{where } [u]_X = \Big(\int_Q \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}} dxdy\Big)^{1/2}.$$

Then we define $X_0 = \{u \in X : u = 0 \text{a.e. in } \mathbb{R}^n \setminus \Omega\}$. Also, there exists a constant $C > 0$ such that $\|u\|_{L^2(\Omega)} \le C[u]_X$, for all $u \in X_0$. Hence, $\|u\| = [u]_X$ is a norm on $(X_0, \|.\|)$ and X_0 is a Hilbert space. Note that the norm $\|\cdot\|$ involves the interaction between Ω and $\mathbb{R}^n \backslash \Omega$. We denote $\|.\|_{L^p(\Omega)}$ as $\|\cdot\|_p$ and $\|\cdot\| = [\cdot]_X$ for the norm in X_0.

Now for each $\alpha \ge 0$, we set

$$C_\alpha = \sup\Big\{\int_\Omega |u|^\alpha dx : \|u\| = 1\Big\}. \tag{2.1}$$

Then $C_0 = |\Omega|$ is the Lebesgue measure of Ω and $\int_\Omega |u|^\alpha dx \le C_\alpha \|u\|^\alpha$, for all $u \in X_0$. In the case of $n > 2s$, we set $2_s^* = \frac{2n}{n-2s}$, $0 < s < 1$. From the embedding results, we know that X_0 is continuously and compactly embedded in $L^r(\Omega)$ when $1 \le r < 2_s^*$ and the embedding is continuous but not compact if $r = 2_s^*$. We define

$$S = \inf_{u \in X_0 \setminus \{0\}} \frac{\int_Q \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}} dx\, dy}{\big(\int_\Omega |u|^{2_s^*}\big)^{2/2_s^*}}.$$

Consider the family of functions $\{U_\epsilon\}$, where U_ϵ is defined as

$$U_\epsilon = \epsilon^{-(n-2s)/2}\, u^*\left(\frac{x}{\epsilon}\right),\ x \in \mathbb{R}^n \text{ , for any } \epsilon > 0,$$

where $u^*(x) = \bar{u}\big(\frac{x}{S^{\frac{1}{2s}}}\big)$, $\bar{u}(x) = \frac{\tilde{u}(x)}{\|u\|_{2_s^*}}$ and $\tilde{u}(x) = \alpha(\beta^2 + |x|^2)^{-\frac{n-2s}{2}}$ with $\alpha \in \mathbb{R} \backslash \{0\}$ and $\beta > 0$ are fixed constants. Then for each $\epsilon > 0$, U_ϵ satisfies

$$(-\Delta)^s u = |u|^{2_s^*-2} u \text{ in } \mathbb{R}^n$$

and verifies the equality

$$\int_{\mathbb{R}^n} \frac{|U_\epsilon(x) - U_\epsilon(y)|^2}{|x-y|^{n+2s}}\, dx\, dy = \int_{\mathbb{R}^n} |U_\epsilon|^{2_s^*} = S^{\frac{n}{2s}}.$$

For a proof, we refer to [34].

Definition 2.1. We say u is a positive weak solution of (1.1) if $u > 0$ in Ω, $u \in X_0$ and

$$\int_Q \frac{(u(x)-u(y))(\psi(x)-\psi(y))}{|x-y|^{n+2s}}\, dx\, dy - \int_\Omega \big(\lambda a(x) u^{-q} - u^{2_*^s - 1}\big)\psi\, dx = 0$$

for all $\psi \in C_c^\infty(\Omega)$.

We define the functional $I_\lambda : X_0 \to (-\infty, \infty]$ by

$$I_\lambda(u) = \frac{1}{2}\int_Q \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}} dx\,dy - \lambda \int_\Omega a(x) G_q(u) dx - \frac{1}{2_s^*}\int_\Omega |u|^{2_s^*} dx,$$

where $G_q : \mathbb{R} \to [-\infty, \infty)$ is the function defined by

$$G_q(x) = \begin{cases} \frac{|x|^{1-q}}{1-q} & \text{if } 0 < q < 1 \\ \ln|x| & \text{if } q = 1 \end{cases}$$

for $x \in \mathbb{R}$. For each $0 < q \le 1$, we set $X_+ = \{u \in X_0 : u \ge 0\}$ and

$$X_{+,q} = \{u \in X_+ : u \not\equiv 0, G_q(u) \in L^1(\Omega)\}.$$

Notice that $X_{+,q} = X_+ \setminus 0$ if $0 < q < 1$ and $X_{+,1} \ne \emptyset$ if $\partial\Omega$ is, for example, of class C^2. We will need the following important lemma.

Lemma 2.2. *For each $w \in X_+$, there exists a sequence $\{w_k\}$ in X_0 such that $w_k \to w$ strongly in X_0, where $0 \le w_1 \le w_2 \le \dots$ and w_k has compact support in Ω, for each k.*

Proof. Let $w \in X_+$ and $\{\psi_k\}$ be sequence in $C_c^\infty(\Omega)$ such that ψ_k is nonnegative and converges strongly to w in X_0. Define $z_k = \min\{\psi_k, w\}$, then $z_k \to w$ strongly to w in X_0. Now, we set $w_1 = z_{r_1}$ where $r_1 > 0$ is such that $\|z_{r_1} - w\| \le 1$. Then $\max\{w_1, z_m\} \to w$ strongly as $m \to \infty$, thus we can find $r_2 > 0$ such that $\|\max\{w_1, z_{r_2}\} - w\| \le 1/2$. We set $w_2 = \max\{w_1, z_{r_2}\}$ and get $\max\{w_2, z_m\} \to w$ strongly as $m \to \infty$. Consequently, by induction we set, $w_{k+1} = \max\{w_k, z_{r_{k+1}}\}$ to obtain the desired sequence, since we can see that $w_k \in X_0$ has compact support, for each k and $\|\max\{w_k, z_{r_{k+1}}\} - w\| \le 1/(k+1)$ which says that $\{w_k\}$ converges strongly to w in X_0 as $k \to \infty$. □

For each $u \in X_{+,q}$ we define the fiber map $\phi_u : \mathbb{R}^+ \to \mathbb{R}$ by $\phi_u(t) = I_\lambda(tu)$. Then we prove the following result.

Theorem 2.3. *Assume $0 < q \le 1$. In case $q = 1$, assume also $X_{+,1} \ne \emptyset$. Let Λ be a constant defined by*

$$\Lambda = \sup\{\lambda > 0 : \text{ for each } u \in X_{+,q}\setminus\{0\},\ \phi_u(t) \text{ has two critical points in } (0,\infty)\}.$$

Then $\Lambda > 0$.

Using the variational methods on the Nehari manifold (see section 3), we will prove the following multiplicity result.

Theorem 2.4. *For each $\lambda \in (0, \Lambda)$, Problem (1.1) has two solutions u_λ and v_λ in $X_{+,q}$.*

3. Nehari manifold and fibering map analysis

We denote $I_\lambda = I$ for simplicity. In this section, we describe the structure of the Nehari manifold associated to the functional I. One can easily verify that the energy functional I is not bounded below on the space X_0. But we will show that I is bounded below on this Nehari manifold and we will extract solutions by minimizing the functional on suitable subsets. The Nehari manifold is defined as

$$\mathcal{N}_\lambda = \{u \in X_{+,q} | \langle I'(u), u\rangle = 0\}.$$

Theorem 3.1. *I is coercive and bounded below on $\mathcal{N}_\lambda$.*

Proof. Case (I) ($0<q<1$): Since $u\in\mathcal{N}_\lambda$, using the embedding of X_0 in $L^{1-q}(\Omega)$, we obtain

$$\begin{aligned} I(u) &= \big(\frac{1}{2}-\frac{1}{2_s^*}\big)\|u\|^2-\lambda\big(\frac{1}{1-q}-\frac{1}{2_s^*}\big)\int_\Omega a(x)|u|^{1-q}dx \\ &\ge c_1\|u\|^2-c_2\|u\|^{1-q} \end{aligned}$$

for some nonnegative constants c_1 and c_2. Thus, I is coercive and bounded below on $\mathcal{N}_\lambda$.

Case (II) ($q=1$): In this case, using the inequality $\ln|u|\le|u|$ and $X_0\hookrightarrow L^1(\Omega)$ we obtain

$$\begin{aligned} I(u) &= \big(\frac{1}{2}-\frac{1}{2_s^*}\big)\|u\|^2-\lambda\Big(\int_\Omega a(x)(\ln|u|-1)\,dx\Big) \\ &\ge \big(\frac{1}{2}-\frac{1}{2_s^*}\big)\|u\|^2-\lambda\Big(\int_\Omega a(x)\ln|u|\,dx\Big) \\ &\ge c_1'\|u\|^2-c_2'\|u\| \end{aligned} \tag{3.1}$$

for some nonnegative constants c_1' and c_2'. This again implies that I is coercive and bounded below on $\mathcal{N}_\lambda$. $\square$

From the definition of fiber map ϕ_u, we have

$$\phi_u(t)=\begin{cases} \frac{t^2}{2}\|u\|^2-\frac{t^{1-q}}{1-q}\int_\Omega a(x)|u|^{1-q}dx-\frac{t^{2_s^*}}{2_s^*}\int_\Omega|u|^{2_s^*}dx & \text{if } 0<q<1 \\ \frac{t^2}{2}\|u\|^2-\frac{\lambda}{1-q}\int_\Omega a(x)\ln(t|u|)dx-\frac{t^{2_s^*}}{2_s^*}\int_\Omega|u|^{2_s^*}dx & \text{if } q=1. \end{cases}$$

which gives

$$\begin{aligned} \phi_u'(t) &= t\|u\|^2-\lambda t^{-q}\int_\Omega a(x)|u|^{1-q}dx-t^{2_s^*-1}\int_\Omega|u|^{2_s^*}dx, \\ \phi_u''(t) &= \|u\|^2+q\lambda t^{-q-1}\int_\Omega a(x)|u|^{1-q}dx-(2_s^*-1)t^{2_s^*-2}\int_\Omega|u|^{2_s^*}dx. \end{aligned}$$

It is easy to see that the points in $\mathcal{N}_\lambda$ are corresponding to critical points of ϕ_u at $t=1$. So, it is natural to divide $\mathcal{N}_\lambda$ in three sets corresponding to local minima, local maxima and points of inflexion. Therefore, we define

$$\begin{aligned} \mathcal{N}_\lambda^+ &= \{u\in\mathcal{N}_\lambda:\phi_u'(1)=0,\phi_u''(1)>0\} \\ &= \{t_0u\in\mathcal{N}_\lambda:t_0>0,\phi_u'(t_0)=0,\phi_u''(t_0)>0\}, \\ \mathcal{N}_\lambda^- &= \{u\in\mathcal{N}_\lambda:\phi_u'(1)=0,\phi_u''(1)<0\} \\ &= \{t_0u\in\mathcal{N}_\lambda:t_0>0,\phi_u'(t_0)=0,\phi_u''(t_0)<0\} \end{aligned}$$

and $\mathcal{N}_\lambda^0=\{u\in\mathcal{N}_\lambda:\phi_u'(1)=0,\phi_u''(1)=0\}$.

Lemma 3.2. *There exist $\lambda_*>0$ such that for each $u\in X_{+,q}\backslash\{0\}$, there is unique $t_{\max},t_1$ and t_2 with the property that $t_1<t_{\max}<t_2$, $t_1u\in\mathcal{N}_\lambda^+$ and $t_2u\in\mathcal{N}_\lambda^-$, for all $\lambda\in(0,\lambda_*)$.*

Proof. Define $A(u)=\int_\Omega a(x)|u|^{1-q}\,dx$ and $B(u)=\int_\Omega|u|^{2_s^*}$. Let $u\in X_{+,q}$ then we have

$$\frac{d}{dt}I(tu)=t\|u\|^2-t^{-q}A(u)-t^{2_s^*-1}B(u)=t^{-q}\left(m_u(t)-\lambda A(u)\right)$$

and we define $m_u(t) := t^{1+q}\|u\|^2 - t^{2_s^*-1+q}B(u)$. Since $\lim_{t\to\infty} m_u(t) = -\infty$, we can easily see that $m_u(t)$ attains its maximum at

$$t_{\max} = \Big[\frac{(1+q)\|u\|^2}{(2_s^*-1+q)B(u)}\Big]^{\frac{1}{2_s^*-2}}$$

and

$$m_u(t_{\max}) = \Big(\frac{2_s^*-2}{2_s^*-1+q}\Big)\Big(\frac{1+q}{2_s^*-1+q}\Big)^{\frac{1+q}{2_s^*-2}}\frac{(\|u\|^2)^{\frac{2_s^*-1+q}{2_s^*-2}}}{(B(u))^{\frac{1+q}{2_s^*-2}}}.$$

Now, $u \in \mathcal{N}_\lambda$ if and only if $m_u(t) = \lambda A(u)$ and we see that

$$\begin{aligned}
&m_u(t_{\max}) - \lambda A(u)dx \\
&\geq m_u(t_{\max}) - \lambda\|a\|_\infty\|u\|_{1-q}^{1-q} \\
&\geq \Big(\frac{2_s^*-2}{2_s^*-1+q}\Big)\Big(\frac{1+q}{2_s^*-1+q}\Big)^{\frac{1+q}{2_s^*-2}}\frac{(\|u\|^2)^{\frac{2_s^*-1+q}{2_s^*-2}}}{B(u)^{\frac{1+q}{2_s^*-2}}} - \lambda\|a\|_\infty\|u\|_{1-q}^{1-q} > 0
\end{aligned}$$

if and only if

$$\lambda < \Big(\frac{2_s^*-2}{2_s^*-1+q}\Big)\Big(\frac{1+q}{2_s^*-1+q}\Big)^{\frac{1+q}{2_s^*-2}}(C_{2_s^*})^{\frac{-1-q}{2_s^*-2}}(\|a\|_\infty C_{1-q})^{-1} = \lambda_*$$

(say), where C_α is defined in (2.1).

Case(I) ($0 < q < 1$): We can also see that $m_u(t) = \lambda\int_\Omega a(x)|u|^{1-q}dx$ if and only if $\phi_u'(t) = 0$. So for $\lambda \in (0, \lambda_*)$, there exists exactly two points $0 < t_1 < t_2$ with $m_u'(t_1) > 0$ and $m_u'(t_2) < 0$ that is, $t_1u \in \mathcal{N}_\lambda^+$ and $t_2u \in \mathcal{N}_\lambda^-$. Thus, ϕ_u has a local minimum at $t = t_1$ and a local maximum at $t = t_2$, that is ϕ_u is decreasing in $(0, t_1)$ and increasing in (t_1, t_2).

Case(II) ($q = 1$): Since $\lim_{t\to 0}\phi_u(t) = \infty$ and $\lim_{t\to\infty}\phi_u(t) = -\infty$ with similar reasoning as above we obtain t_1, t_2. That is, in both cases ϕ_u has exactly two critical points t_1 and t_2 such that $0 < t_1 < t_2$, $\phi_u''(t_1) > 0$ and $\phi_u''(t_2) < 0$ that is $t_1u \in \mathcal{N}_\lambda^+$, $t_2u \in \mathcal{N}_\lambda^-$. □

Corollary 3.3. $\mathcal{N}_\lambda^0 = \{0\}$ *for all* $\lambda \in (0, \Lambda)$.

Proof. Let $u \not\equiv 0 \in \mathcal{N}_\lambda^0$. Then $u \in \mathcal{N}_\lambda^0$ implies $u \in \mathcal{N}_\lambda$ that is, 1 is a critical point of ϕ_u. Using previous results, we say that ϕ_u has critical points corresponding to local minima or local maxima. So, 1 is the critical point corresponding to local minima or local maxima of ϕ_u. Thus, either $u \in \mathcal{N}_\lambda^+$ or $u \in \mathcal{N}_\lambda^-$ which is a contradiction. □

Proof of Theorem 2.3. From lemma 3.2, we see that Λ is positive. If $I_\lambda(tu)$ has two critical points for some $\lambda = \lambda^*$, then $t \mapsto I_\lambda(tu)$ also has two critical points for all $\lambda < \lambda^*$. □

We can show that $\mathcal{N}_\lambda^+$ and $\mathcal{N}_\lambda^-$ are bounded in the following way.

Lemma 3.4. *The following holds:*

(i) $\sup\{\|u\| : u \in \mathcal{N}_\lambda^+\} < \infty$
(ii) $\inf\{\|v\| : v \in \mathcal{N}_\lambda^-\} > 0$ *and* $\sup\{\|v\| : v \in \mathcal{N}_\lambda^-, I(v) \leq M\} < \infty$ *for each* $M > 0$.

Moreover, $\inf I(\mathcal{N}_\lambda^+) > -\infty$ *and* $\inf I(\mathcal{N}_\lambda^-) > -\infty$.

Proof. (i) Let $u \in \mathcal{N}_\lambda^+$. Then we have

$$\begin{aligned}0 < \phi_u''(1) &= (2-2_s^*)\|u\|^2 + \lambda(2_s^*-1+q)\int_\Omega a(x)|u|^{1-q}dx \\ &\le (2-2_s^*)\|u\|^2 + \lambda(2_s^*-1+q)C_{1-q}\|a\|_\infty\|u\|^{1-q}.\end{aligned}$$

Thus we obtain

$$\|u\| \le \Big(\frac{\lambda(2_s^*-1+q)C_{1-q}\|a\|_\infty}{2_s^*-2}\Big)^{\frac{1}{1+q}}.$$

(ii) Let $v \in \mathcal{N}_\lambda^-$. We have

$$\begin{aligned}0 > \phi_v''(1) &= (1+q)\|v\|^2 - (2_s^*-1+q)\int_\Omega |v|^{2_s^*}dx \\ &\ge (1+q)\|v\|^2 - (2_s^*-1+q)C_{2_s^*}\|v\|^{2_s^*}.\end{aligned}$$

Thus, we obtain

$$\|v\| \ge \Big(\frac{1+q}{(2_s^*-1+q)C_{2_s^*}}\Big)^{\frac{1}{2_s^*-2}}$$

which implies that $\inf\{\|v\| : v \in \mathcal{N}_\lambda^-\} > 0$. If $I(v) \le M$, similarly we have for $0 < q < 1$

$$\frac{(2_s^*-2)}{2\times 2_s^*}\|v\|^2 - \lambda\Big(\frac{2_s^*-1+q}{2_s^*(1-q)}\Big)C_{1-q}\|a\|_\infty\|v\|^{1-q} \le M.$$

Now for $q = 1$, using $\ln(|v|) \le |v|$, we obtain

$$M \ge \frac{(2_s^*-2)}{2\times 2_s^*}\|v\|^2 - \lambda\|a\|_\infty C_1\|v\| + \frac{\lambda}{2_s^*}\|a\|_1 \ge \frac{(2_s^*-2)}{2\times 2_s^*}\|v\|^2 - \lambda\|a\|_\infty C_1\|v\|$$

which implies $\sup\{\|v\| : v \in \mathcal{N}_\lambda^-, Iv \le M\} < \infty$, for each $M > 0$. For $u \in \mathcal{N}_\lambda^+$, when $0 < q < 1$ we have

$$I(u) \ge -\frac{(1+q)}{2(1-q)}\|u\|^2 - \frac{(2_s^*-1+q)}{2_s^*(1-q)}C_{2_s^*}\|u\|^{2_s^*}$$

and when $q = 1$, we have

$$I(u) \ge \frac{\|u\|^2}{2} - \lambda\|a\|_\infty|\Omega|^{\frac{2_s^*-1}{2_s^*}}C_{2_s^*}^{\frac{1}{2_s^*}}\|u\| - \frac{C_{2_s^*}}{2_s^*}\|u\|^{2_s^*}.$$

So, using (i) we conclude that $\inf I(\mathcal{N}_\lambda^+) > -\infty$ and similarly, using (ii) we can show that $\inf I(\mathcal{N}_\lambda^-) > -\infty$. □

Lemma 3.5. *Suppose u and v be minimizers of I over $\mathcal{N}_\lambda^+$ and $\mathcal{N}_\lambda^-$ respectively. Then for each $w \in X_+$,*

1. *there exists $\epsilon_0 > 0$ such that $I(u+\epsilon w) \ge I(u)$ for each $\epsilon \in [0,\epsilon_0]$, and*
2. *$t_\epsilon \to 1$ as $\epsilon \to 0^+$, where t_ϵ is the unique positive real number satisfying $t_\epsilon(v+\epsilon w) \in \mathcal{N}_\lambda^-$.*

Proof. (1) Let $w \in X_+$ that is $w \in X_0$ and $w \ge 0$. We set

$$\rho(\epsilon) = \|u+\epsilon w\|^2 + \lambda q\int_\Omega a(x)|u+\epsilon w|^{1-q}\,dx - (2_s^*-1)\int_\Omega |u+\epsilon w|^{2_s^*}$$

for each $\epsilon \ge 0$. Then using continuity of ρ and $\rho(0) = \phi_u''(1) > 0$, since $u \in \mathcal{N}_\lambda^+$, there exist $\epsilon_0 > 0$ such that $\rho(\epsilon) > 0$ for $\epsilon \in [0,\epsilon_0]$. Since for each $\epsilon > 0$, there exists $t'_\epsilon > 0$ such that $t'_\epsilon(u+\epsilon w) \in \mathcal{N}_\lambda^+$, so $t'_\epsilon \to 1$ as $\epsilon \to 0$ and for each $\epsilon \in [0, \epsilon_0]$, we have

$$I(u+\epsilon w) \geq I(t'_\epsilon(u+\epsilon w)) \geq \inf I(\mathcal{N}_\lambda^+) = I(u).$$

(2) We define $h : (0,\infty) \times \mathbb{R}^3 \to \mathbb{R}$ by

$$h(t, l_1, l_2, l_3) = l_1 t - \lambda t^{-q} l_2 - t^{2_s^*-1} l_3$$

for $(t, l_1, l_2, l_3) \in (0,\infty) \times \mathbb{R}^3$. Then, h is a C^∞ function. Also, we have

$$\frac{dh}{dt}(1, \|v\|^2, \int_\Omega a(x)|v|^{1-q}\,dx, \quad \int_\Omega |v|^{2_s^*}) = \phi''_v(1) < 0$$

and for each $\epsilon > 0$, $h(t_\epsilon, \|v+\epsilon w\|^2, \int_\Omega a(x)|v+\epsilon w|^{1-q}\,dx, \int_\Omega |v|^{2_s^*}) = \phi'_{v+\epsilon w}(t_\epsilon) = 0$. Moreover,

$$h\Big(1, \|v\|^2, \int_\Omega a(x)|v|^{1-q}\,dx, \int_\Omega |v|^{2_s^*}\Big) = \phi'_v(1) = 0.$$

Therefore, by implicit function theorem, there exists an open neighborhood $A \subset (0,\infty)$ and $B \subset \mathbb{R}^3$ containing 1 and $(\|v\|^2, \int_\Omega a(x)|v|^{1-q}\,dx, \int_\Omega |v|^{2_s^*})$ respectively such that for all $y \in B$, $h(t,y) = 0$ has a unique solution $t = g(y) \in A$, where $g : B \to A$ is a continuous function. So, $(\|v+\epsilon w\|^2,\ \int_\Omega a(x)|v+\epsilon w|^{1-q}\,dx, \int_\Omega |v+\epsilon w|^{2_s^*}) \in B$ and

$$g\Big(\|v+\epsilon w)\|^2,\ \int_\Omega a(x)|v+\epsilon w|^{1-q}\,dx, \int_\Omega |v+\epsilon w|^{2_s^*}\Big) = t_\epsilon$$

since $h(t_\epsilon, \|v+\epsilon w)\|^2, \int_\Omega a(x)|v+\epsilon w|^{1-q}\,dx, \int_\Omega |v+\epsilon w|^{2_s^*}) = 0$. Thus, by continuity of g, we obtain $t_\epsilon \to 1$ as $\epsilon \to 0^+$. $\square$

Lemma 3.6. *Suppose u and v are minimizers of I on $\mathcal{N}_\lambda^+$ and $\mathcal{N}_\lambda^-$ respectively. Then for each $w \in X_+$, we have $u^{-q}w, v^{-q}w \in L^1(\Omega)$ and*

$$\int_Q \frac{(u(x)-u(y))(w(x)-w(y))}{|x-y|^{n+2s}}\,dx\,dy - \lambda \int_\Omega a(x)u^{-q}w\,dx - \int_\Omega u^{2_s^*-1}w \geq 0, \tag{3.2}$$

$$\int_Q \frac{(v(x)-v(y))(w(x)-w(y))}{|x-y|^{n+2s}}\,dx\,dy - \lambda \int_\Omega a(x)v^{-q}w\,dx - \int_\Omega v^{2_s^*-1}w \geq 0. \tag{3.3}$$

Proof. Let $w \in X_+$. For sufficiently small $\epsilon > 0$, by lemma 3.5,

$$\begin{aligned} 0 &\leq \frac{I(u+\epsilon w) - I(u)}{\epsilon} \\ &= \frac{1}{2\epsilon}(\|u+\epsilon w\|^2 - \|u\|^2) - \frac{\lambda}{\epsilon}\int_\Omega a(x)(G_q(u+\epsilon w) - G_q(u))\,dx \\ &\quad - \frac{1}{\epsilon 2_s^*}\int_\Omega (|u+\epsilon w|^{2_s^*} - |u|^{2_s^*}). \end{aligned} \tag{3.4}$$

We can easily verify that

(i)

$$\frac{(\|u+\epsilon w\|^2 - \|u\|^2)}{\epsilon} \to 2\int_Q \frac{(u(x)-u(y))(w(x)-w(y))}{|x-y|^{n+2s}}\,dx\,dy \quad \text{as } \epsilon \to 0^+,$$

(ii)

$$\int_\Omega \frac{(|u+\epsilon w|^{2_s^*} - |u|^{2_s^*})}{\epsilon} \to 2_s^* \int_\Omega |u|^{2_s^*-1}w \ \text{ as } \epsilon \to 0^+$$

which implies that

$$a(x)\frac{(G_q(u+\epsilon w)-G_q(u))}{\epsilon}\in L^1(\Omega).$$

Also, for each $x\in\Omega$,

$$\frac{G_q(u(x)+\epsilon w(x))-G_q(u(x))}{\epsilon}=\begin{cases}\frac{1}{\epsilon}\big(\frac{|u+\epsilon w|^{1-q}(x)-|u|^{1-q}(x)}{1-q}\big) & \text{if } 0<q<1\\ \frac{1}{\epsilon}\big(\ln(|u+\epsilon w|(x))-ln(|u|(x))\big) & \text{if } q=1\end{cases}$$

which increases monotonically as $\epsilon\downarrow 0$ and

$$\lim_{\epsilon\downarrow 0}\frac{G_q(u(x)+\epsilon w(x))-G_q(u(x))}{\epsilon}=\begin{cases}0 & \text{if } w(x)=0\\ (u(x))^{-q}w(x) & \text{if } w(x)>0, u(x)>0\\ \infty & \text{if } w(x)>0,\ u(x)=0.\end{cases}$$

So using monotone convergence theorem, we obtain $u^{-q}w\in L^1(\Omega)$. Letting $\epsilon\downarrow 0$ in both sides of (3.4), we obtain (3.2). Next, we will show these properties for v. For each $\epsilon>0$, there exists $t_\epsilon>0$ with $t_\epsilon(v+\epsilon w)\in\mathcal{N}_\lambda^-$. By lemma 3.5(2), for sufficiently small $\epsilon>0$, there holds

$$I(t_\epsilon(v+\epsilon w))\ge I(v)\ge I(t_\epsilon v)$$

which implies $I(t_\epsilon(v+\epsilon w))-I(v)\ge 0$ and thus, we have

$$\begin{aligned}&\lambda\int_\Omega a(x)(G_q(|v+\epsilon w|^{1-q})-G_q(|v|^{1-q}))dx\\ &\le\frac{t_\epsilon^q}{2}(\|v+\epsilon w\|^2-\|v\|^2)-\frac{t_\epsilon^{q+2_s^*}}{2_s^*}\int_\Omega(|v+\epsilon w|^{2_s^*}-|v|^{2_s^*}).\end{aligned}$$

As $\epsilon\downarrow 0$, $t_\epsilon\to 1$. Thus, using similar arguments as above, we obtain $v^{-q}w\in L^1(\Omega)$ and (3.3) follows. □

Let $\phi_1>0$ be the eigenfunction of $(-\Delta)^s$ corresponding to the smallest eigenvalue λ_1. Then, $\phi_1\in L^\infty(\Omega)$ (see [34]) and

$$\begin{gathered}(-\Delta)^s\phi_1=\lambda_1\phi_1,\quad u>0\quad\text{in }\Omega,\\ \phi_1=0\quad\text{on }\mathbb{R}^n\setminus\Omega.\end{gathered}$$

For instance, here we assume $\|\phi_1\|_\infty=1$. Let $\eta>0$ be such that $\phi=\eta\phi_1$ satisfies

$$(-\Delta)^s\phi+\lambda a(x)\phi^{-q}+\phi^{2_s^*-1}>0 \tag{3.5}$$

and $\phi^{2_s^*-1+q}(x)\le\lambda a(x)\left(\frac{q}{2_s^*-1}\right)$, for each $x\in\Omega$. Then we have the following Lemma.

Lemma 3.7. *Suppose u and v are minimizers of I on $\mathcal{N}_\lambda^+$ and $\mathcal{N}_\lambda^-$ respectively. Then $u\ge\phi$ and $v\ge\phi$.*

Proof. By lemma 2.2, let $\{w_k\}$ be a sequence in X_0 such that supp(w_k) is compact, $0\le w_k\le(\phi-u)^+$ for each k and $\{w_k\}$ strongly converges to $(\phi-u)^+$ in X_0. Then for each $x\in\Omega$,

$$\frac{d}{dt}(\lambda a(x)t^{-q}+t^{2_s^*-1})=-q\lambda a(x)t^{-q-1}+(2_s^*-1)t^{2_s^*-2}\le 0 \tag{3.6}$$

if and only if $t^{2_s^*-1+q} \leq \lambda a(x)\Big(\frac{q}{2_s^*-1}\Big)$. Using the previous lemma and (3.5), we have

$$
\begin{aligned}
&\Big(\int_Q \frac{(u(x)-u(y))(w_k(x)-w_k(y))}{|x-y|^{n+2s}}\,dx\,dy - \lambda\int_\Omega a(x)u^{-q}w_k\,dx - \int_\Omega u^{2_s^*-1}w_k\Big)\\
&-\Big(\int_Q \frac{(\phi(x)-\phi(y))(w_k(x)-w_k(y))}{|x-y|^{n+2s}}\,dx\,dy - \lambda\int_\Omega a(x)\phi^{-q}w_k\,dx - \int_\Omega \phi^{2_s^*-1}w_k\Big)\\
&\geq 0
\end{aligned}
$$

which implies

$$
\begin{aligned}
&\int_Q \frac{(u(x)-u(y))-(\phi(x)-\phi(y))}{|x-y|^{n+2s}}[w_k(x)-w_k(y)]\,dx\,dy\\
&-\int_\Omega(\lambda a(x)u^{-q}+u^{2_s^*-1})w_k\,dx + \int_\Omega(\lambda a(x)\phi^{-q}+\phi^{2_s^*-1})w_k\,dx \geq 0.
\end{aligned}
$$

Using the strong convergence, we assume $\{w_k\}$ converges to $(\phi-u)^+$ pointwise almost everywhere in Ω and we write $w_k(x) = (\phi-u)^+(x) + o(1)$ as $k\to\infty$. Consider,

$$
\begin{aligned}
&\int_Q \frac{((u-\phi)(x)-(u-\phi)(y))}{|x-y|^{n+2s}}[w_k(x)-w_k(y)]\,dx\,dy\\
&=\int_Q \frac{((u-\phi)(x)-(u-\phi)(y))}{|x-y|^{n+2s}}((\phi-u)^+(x)-(\phi-u)^+(y))\,dx\,dy\\
&\quad+o(1)\int_Q \frac{((u-\phi)(x)-(u-\phi)(y))}{|x-y|^{n+2s}}\,dx\,dy
\end{aligned}
$$

where we can see that

$$
\begin{aligned}
&\int_Q \frac{((u-\phi)(x)-(u-\phi)(y))}{|x-y|^{n+2s}}((\phi-u)^+(x)-(\phi-u)^+(y))\,dx\,dy\\
&=\int_Q \frac{-(\phi-u)(x)+(\phi-u)(y)}{|x-y|^{n+2s}}((\phi-u)^+(x)-(\phi-u)^+(y))\,dx\,dy\\
&=\int_Q \frac{((\phi-u)^+-(\phi-u)^-)(y)-((\phi-u)^+-(\phi-u)^-)(x)}{|x-y|^{n+2s}}\\
&\quad\times\big((\phi-u)^+(x)-(\phi-u)^+(y)\big)\,dx\,dy\\
&\leq -\int_Q \Big(|(\phi-u)^+(y)|^2+|(\phi-u)^+(x)|^2+(\phi-u)^-(y)(\phi-u)^+(x)\\
&\quad+(\phi-u)^-(x)(\phi-u)^+(y)\Big)/|x-y|^{n+2s}\,dx\,dy\\
&=-\int_Q \frac{|(\phi-u)^+(x)-(\phi-u)^+(y)|^2}{|x-y|^{n+2s}}\,dx\,dy\\
&=-\|(\phi-u)^+\|^2.
\end{aligned}
$$

Since $\phi^{2_s^*-1+q}(x) \leq \lambda a(x)(\frac{q}{2_s^*-1})$ for each $x\in\Omega$, using (3.6) we obtain

$$
\begin{aligned}
&\int_\Omega((\lambda a(x)u^{-q}+u^{2_s^*-1})-(\lambda a(x)\phi^{-q}+\phi^{2_s^*-1}))w_k\,dx\\
&=\int_{\Omega\cap\{\phi\geq u\}}((\lambda a(x)u^{-q}+u^{2_s^*-1})-(\lambda a(x)\phi^{-q}+\phi^{2_s^*-1}))(\phi-u)^+(x)\,dx+o(1)
\end{aligned}
$$

$\geq 0.$

This implies

$$
\begin{aligned}
0 &\leq -\|(\phi-u)^+\|^2 - \int_\Omega (\lambda a(x)u^{-q} + u^{2_s^*-1})w_k\,dx \\
&\quad + \int_\Omega (\lambda a(x)\phi^{-q} + \phi^{2_s^*-1})w_k\,dx + o(1) \\
&\leq -\|(\phi-u)^+\|^2 + o(1)
\end{aligned}
$$

and letting $k \to \infty$, we obtain $-\|(\phi-u)^+\|^2 \geq 0$. Thus, we showed $u \geq \phi$. Similarly, we can show $v \geq \phi$. □

4. Existence of minimizer on $\mathcal{N}_\lambda^+$

In this section, we will show that the minimum of I is achieved in $\mathcal{N}_\lambda^+$. Also, we show that this minimizer is also the first solution of (1.1).

Proposition 4.1. *For all $\lambda \in (0, \Lambda)$, there exist $u_\lambda \in \mathcal{N}_\lambda^+$ satisfying $I(u_\lambda) = \inf_{u \in \mathcal{N}_\lambda^+} I(u)$.*

Proof. Assume $0 < q \leq 1$ and $\lambda \in (0, \Lambda)$. Let $\{u_k\} \subset \mathcal{N}_\lambda^+$ be a sequence such that $I(u_k) \to \inf_{u \in \mathcal{N}_\lambda^+} I(u)$ as $k \to \infty$. Using lemma 3.4, we can assume that there exist u_λ such that $u_k \rightharpoonup u_\lambda$ weakly in X_0. First we will show that $\inf_{u \in \mathcal{N}_\lambda^+} I(u) < 0$. Let $u_0 \in \mathcal{N}_\lambda^+$, then we have $\phi''_{u_0}(1) > 0$ which gives

$$
\Big(\frac{1+q}{2_s^*-1+q}\Big)\|u_0\|^2 > \int_\Omega |u_0|^{2_s^*}dx.
$$

Therefore, using $2_s^* - 1 > 1$ we obtain

$$
\begin{aligned}
I(u_0) &= \big(\frac{1}{2} - \frac{1}{1-q}\big)\|u_0\|^2 + \big(\frac{1}{1-q} - \frac{1}{2_s^*}\big)\int_\Omega |u_0|^{2_s^*}dx \\
&\leq -\frac{(1+q)}{2(1-q)}\|u_0\|^2 + \frac{(1+q)}{2_s^*(1-q)}\|u_0\|^2 \\
&= \big(\frac{1}{2_s^*} - \frac{1}{2}\big)\big(\frac{1+q}{1-q}\big)\|u_0\|^2 < 0.
\end{aligned}
$$

We set $w_k := (u_k - u_\lambda)$ and claim that $u_k \to u_\lambda$ strongly in X_0. Suppose $\|w_k\|^2 \to c^2 \neq 0$ and $\int_\Omega |w_k|^{2_s^*}dx \to d^{2_s^*}$ as $k \to \infty$. Since $u_k \in \mathcal{N}_\lambda^+$, using Brezis-Lieb we obtain

$$
0 = \lim_{k\to\infty} \phi'_{u_k}(1) = \phi'_{u_\lambda}(1) + c^2 - d^{2_s^*} \tag{4.1}
$$

which implies

$$
\|u_\lambda\|^2 + c^2 = \lambda \int_\Omega a(x)|u_\lambda|^{1-q}dx + \int_\Omega |u_k|^{2_s^*}dx + d^{2_s^*}.
$$

We claim that $u_\lambda \in X_{+,q}$. Suppose $u_\lambda \equiv 0$. If $0 < q < 1$ and $c = 0$ then $0 > \inf I(\mathcal{N}_\lambda^+) = I(0) = 0$, which is a contradiction and if $c \neq 0$ then

$$
\inf_{u \in \mathcal{N}_\lambda^+} I(u) = I(0) + \frac{c^2}{2} - \frac{d^{2_s^*}}{2_s^*} = \frac{c^2}{2} - \frac{d^{2_s^*}}{2_s^*}. \tag{4.2}
$$

But we have $\|u_k\|_{2_s^*}^2 S \leq \|u_k\|^2$ which gives $c^2 \geq Sd^2$. Also from (4.1), we have $c^2 = d^{2_s^*}$. Then (4.2) implies

$$0 > \inf_{u \in \mathcal{N}_\lambda^+} I(u) = \Big(\frac{1}{2} - \frac{1}{2_s^*}\Big)c^2 \geq \frac{s}{n} S^{\frac{n}{2s}},$$

which is again a contradiction. In the case $q = 1$, the sequence $\{\int_\Omega \ln(|u_k|)\}$ is bounded, since the sequence $\{I(u_k)\}$ and $\{\|u_k\|\}$ is bounded, using Fatou's lemma and for each k, $\ln(|u_k|) \leq u_k$, we obtain

$$-\infty < \overline{\lim_{k\to\infty}} \int_\Omega \ln(|u_k|)dx \leq \int_\Omega \overline{\lim_{k\to\infty}} \ln(|u_k|)dx = \int_\Omega \ln(|u_\lambda|)dx.$$

which implies $u_\lambda \not\equiv 0$. Thus, in both cases we have shown that $u_\lambda \in X_{+,q}$. So, there exists $0 < t_1 < t_2$ such that $\phi'_{u_\lambda}(t_1) = \phi'_{u_\lambda}(t_2) = 0$ and $t_1 u_\lambda \in \mathcal{N}_\lambda^+$. Then, three cases arise:

(i) $t_2 < 1$,

(ii) $t_2 \geq 1$ and $\frac{c^2}{2} - \frac{d^{2_s^*}}{2_s^*} < 0$, and

(iii) $t_2 \geq 1$ and $\frac{c^2}{2} - \frac{d^{2_s^*}}{2_s^*} \geq 0$.

Case (i) Let

$$h(t) = \phi_{u_\lambda}(t) + \frac{c^2 t^2}{2} - \frac{d^{2_s^*} t^{2_s^*}}{2_s^*},$$

$t > 0$. By (4.1), we obtain $h'(1) = \phi'_{u_\lambda}(1) + c^2 - d^{2_s^*} = 0$ and

$$h'(t_2) = \phi'_{u_\lambda}(1) + t_2 c^2 - {t_2}^{2_s^*-1} d^{2_s^*} = t_2(c^2 - {t_2}^{2_s^*-2} d^{2_s^*}) > t_2(c^2 - d^{2_s^*}) > 0$$

which implies that h increases on $[t_2, 1]$. Then we obtain

$$\begin{aligned}
\inf_{u \in \mathcal{N}_\lambda^+} I(u) = \lim I(u_k) &\geq \phi_{u_\lambda}(1) + \frac{c^2}{2} - \frac{d^{2_s^*}}{2_s^*} = h(1) > h(t_2) \\
&= \phi_{u_\lambda}(t_2) + \frac{c^2 t_2^2}{2} - \frac{d^{2_s^*} t_2^{2_s^*}}{2_s^*} \geq \phi_{u_\lambda}(t_2) + \frac{t_2^2}{2}(c^2 - d^{2_s^*}) \\
&> \phi_{u_\lambda}(t_2) > \phi_{u_\lambda}(t_1) \geq \inf_{u \in \mathcal{N}_\lambda^+} I(u),
\end{aligned}$$

which is a contradiction.

Case (ii) In this case, since $\lambda \in (0, \Lambda)$, we have $(c^2/2 - d^{2_s^*}/2_s^*) < 0$ and $Sd^2 \leq c^2$. Also we see that, for each $u_0 \in \mathcal{N}_\lambda^+$

$$\begin{aligned}
0 < \phi''_{u_0}(1) &= \|u_0\|^2 + q\lambda \int_\Omega a(x)|u_0|^{1-q}dx - (2_s^* - 1)\int_\Omega |u_0|^{1+p}dx \\
&= (1+q)\|u_0\|^2 + (-q - 2_s^* + 1)\int_\Omega |u_0|^{2_s^*}dx
\end{aligned}$$

which implies

$$(1+q)\|u_0\|^2 > (q + 2_s^* - 1)\int_\Omega |u_0|^{2_s^*}dx = (q + 2_s^* - 1)\|u_0\|_{2_s^*}^{2_s^*}dx$$

or

$$C_{2_s^*} \leq \Big(\frac{1+q}{q + 2_s^* - 1}\Big)\|u_0\|^{2-2_s^*}, \quad \text{or} \quad \|u_0\|^2 \leq \Big(\frac{1+q}{q + 2_s^* - 1}\Big)^{\frac{2}{2_s^*-2}} S^{\frac{2_s^*}{2_s^*-2}}.$$

Thus, we have

$$\sup\{\|u\|^2 : u \in \mathcal{N}_\lambda^+\} \le \Big(\frac{2}{2_s^*}\Big)^{\frac{2}{2_s^*-2}} S^{\frac{2_s^*}{2_s^*-2}} < c^2 \le \sup\{\|u\|^2 : u \in \mathcal{N}_\lambda^+\},$$

which gives a contradiction. Consequently, in case (iii) we have

$$\inf_{u\in\mathcal{N}_\lambda^+} I(u) = I(u_\lambda) + \frac{c^2}{2} - \frac{d^{2_s^*}}{2_s^*} \ge I(u_\lambda) = \phi_{u_\lambda}(1) \ge \phi_{u_\lambda}(t_1) \ge \inf I(\mathcal{N}_\lambda^+).$$

Clearly, this holds when $t_1 = 1$ and $(c^2/2 - d^{2_s^*}/2_s^*) = 0$ which yields $c = 0$ and $u_\lambda \in \mathcal{N}_\lambda^+$. Thus, $u_k \to u_\lambda$ strongly in X_0 and $I(u_\lambda) = \inf_{u\in\mathcal{N}_\lambda^+} I(u)$. □

Proposition 4.2. *u_λ is a positive weak solution of* (1.1).

Proof. Let $\psi \in C_c^\infty(\Omega)$. By lemma 3.7, since $\phi > 0$, we can find $\alpha > 0$ such that $u_\lambda \ge \alpha$ on support of ψ. Then $u + \epsilon\psi \ge 0$ for small ϵ. With similar reasoning as in the proof of lemma 3.5, $I(u_\lambda + \epsilon\psi) \ge I(u_\lambda)$ for sufficiently small $\epsilon > 0$. Then we have

$$\begin{aligned} 0 &\le \lim_{\epsilon\to 0} \frac{I(u_\lambda + \epsilon\psi) - I(u_\lambda)}{\epsilon} \\ &= \int_Q \frac{(u_\lambda(x) - u_\lambda(y))(\psi(x) - \psi(y))}{|x-y|^{n+2s}}\,dx\,dy - \lambda\int_\Omega a(x)u_\lambda^{-q}\psi dx - \int_\Omega u_\lambda^{2_s^*-1}\psi\,dx. \end{aligned}$$

Since $\psi \in C_c^\infty(\Omega)$ is arbitrary, we conclude that u_λ is a positive weak solution of (1.1). □

As a consequence, we have the following result.

Lemma 4.3. $\Lambda < \infty$.

Proof. Taking ϕ_1 as the test function in (1.1), we obtain

$$\lambda_1 \int_\Omega u\phi_1 = \int_\Omega (\lambda u^{-q} + u^{2_s^*-1})\phi_1.$$

Let $\mu^* > 0$ be such that $\mu^* t^{-q} + t^{2_s^*-1} > (\lambda_1 + \epsilon)t$, for all $t > 0$. Then we obtain $\lambda < \mu^*$ and the proof follows. □

5. Existence of minimizer on $\mathcal{N}_\lambda^-$

In this section, we shall show the existence of second solution by proving the existence of minimizer of I on $\mathcal{N}_\lambda^-$. We need some lemmas to prove this and for instance, we assume $0 \in \Omega$. To put U_ϵ zero outside Ω, we fix $\delta > 0$ such that $B_{4\delta} \subset \Omega$ and let $\zeta \in C_c^\infty(\mathbb{R}^n)$ be such that $0 \le \zeta \le 1$ in $\mathbb{R}^n$, $\zeta \equiv 0$ in $\mathcal{C}B_{2\delta}$ and $\zeta \equiv 1$ in B_δ. For each $\epsilon > 0$ and $x \in \mathbb{R}^n$, we define

$$\Phi_\epsilon(x) := \zeta(x)U_\epsilon(x).$$

Moreover, since u_λ is positive and bounded (see lemma 6.2), we can find $m, M > 0$ such that for each $x \in \Omega$, $m \le u_\lambda(x) \le M$.

Lemma 5.1. $\sup\{I(u_\lambda + t\Phi_\epsilon) : t \ge 0\} < I(u_\lambda) + \frac{s}{n}S^{\frac{n}{2s}}$, *for each sufficiently small* $\epsilon > 0$.

Proof. We assume $\epsilon > 0$ to be sufficiently small. Using [34, Proposition 21], we have

$$\int_Q \frac{|\Phi_\epsilon(x) - \Phi_\epsilon(y)|^2}{|x-y|^{n+2s}} dx\, dy \leq S^{\frac{n}{2s}} + o(\epsilon^{n-2s})$$

which says that we can find $r_1 > 0$ such that

$$\int_Q \frac{|\Phi_\epsilon(x) - \Phi_\epsilon(y)|^2}{|x-y|^{n+2s}} dx\, dy \leq S^{\frac{n}{2s}} + r_1 \epsilon^{n-2s}.$$

Now, we consider

$$\begin{aligned}
\int_\Omega |\Phi_\epsilon|^{2_s^*} dx &= \int_{B_\delta} |U_\epsilon|^{2_s^*} dx + \int_{B_{2\delta}\setminus B_\delta} |\zeta(x) U_\epsilon(x)|^{2_s^*} dx \\
&= \int_{\mathbb{R}^n} |U_\epsilon|^{2_s^*} dx - \int_{\mathbb{R}^n\setminus B_\delta} |U_\epsilon|^{2_s^*} dx + \int_{B_{2\delta}\setminus B_\delta} |\zeta(x) U_\epsilon(x)|^{2_s^*} dx \\
&\geq \int_{\mathbb{R}^n} |U_\epsilon|^{2_s^*} dx - \int_{\mathbb{R}^n\setminus B_\delta} |U_\epsilon|^{2_s^*} dx \\
&= S^{\frac{n}{2s}} - \epsilon^{-n} \int_{\mathbb{R}^n\setminus B_\delta} |u^*(x/\epsilon)|^{2_s^*} dx \\
&\geq S^{\frac{n}{2s}} - r_2 \epsilon^n
\end{aligned}$$

for some constant $r_2 > 0$. We now fix $1 < \rho < \frac{n}{n-2s}$ and we have

$$\begin{aligned}
\int_\Omega |\Phi_\epsilon|^\rho dx &= \epsilon^{-(n-2s)\rho/2} \int_{B_{2\delta}} |\zeta(x) u^*(x/\epsilon)|^\rho dx \\
&\leq r_3{}' \epsilon^{-(n-2s)\rho/2} \int_{B_{2\delta}} \Big(\big|\frac{x}{\epsilon S^{\frac{1}{2s}}}\big|^2 \Big)^{\frac{-(n-2s)\rho}{2}} dx \\
&= r_3 \epsilon^{(n-2s)\rho/2}
\end{aligned}$$

for constants $r_3{}', r_3 > 0$. Now, choosing $\epsilon > \delta / S^{\frac{1}{2s}}$ we see that

$$\begin{aligned}
\int_{B_\delta} |\Phi_\epsilon|^{2_s^*-1} dx &= \alpha^{2_s^*-1} \beta^{-(n+2s)} \epsilon^{(n-2s)/2} \int_{|y| < \frac{\delta}{\epsilon} S^{\frac{1}{2s}}} (1+|y|^2)^{-(n+2s)/2} dy \\
&\geq \alpha^{2_s^*-1} \beta^{-(n+2s)} \frac{\epsilon^{(n-2s)/2}}{2^{(n+2s)/2}} \int_{|y|\leq 1} dy \\
&= r_4 \epsilon^{(n-2s)/2}
\end{aligned}$$

for some constant $r_4 > 0$. We can find appropriate constants $\rho_1, \rho_2 > 0$ such that the following inequalities holds:

$$\begin{gathered}
\lambda \Big(\frac{(c+d)^{1-q}}{1-q} - \frac{c^{1-q}}{1-q} - \frac{d}{c^q} \Big) \geq -\frac{\rho_1 d^\rho}{r_3}, \quad \text{for all } c \geq m, d \geq 0, \\
\Big(\frac{(c+d)^{2_s^*}}{2_s^*} - \frac{c^{2_s^*}}{2_s^*} - c^{2_s^*-1} d \Big) \geq \frac{d^{2_s^*}}{2_s^*}, \quad \text{for all } c, d \geq 0, \\
\frac{(c+d)^{2_s^*}}{2_s^*} - \frac{c^{2_s^*}}{2_s^*} - c^{2_s^*-1} d \geq \frac{d^{2_s^*}}{2_s^*} + \frac{\rho_2 c d^{2_s^*-1}}{r_4 m (2_s^*-1)}, \quad \text{for all } 0 \leq c \leq M,\ d \geq 1.
\end{gathered}$$

Case I: $0 < q < 1$. Since u is a positive weak solution of (1.1), using the above inequalities, we obtain

$$
\begin{aligned}
& I(u_\lambda + t\Phi_\epsilon) - I(u_\lambda) \\
&= I(u_\lambda + t\Phi_\epsilon) - I(u_\lambda) - t\Big(\int_Q \frac{(u_\lambda(x) - u_\lambda(y))(\Phi_\epsilon(x) - \Phi_\epsilon(y))}{|x-y|^{n+2s}}\,dx\,dy \\
&\quad - \lambda \int_\Omega a(x)\Phi_\epsilon(x) u_\lambda^{-q}\,dx - \int_\Omega \Phi_\epsilon(x) u_\lambda^{2_s^*-1}\,dx\Big) \\
&= \frac{t^2}{2}\int_Q \frac{|\Phi_\epsilon(x) - \Phi_\epsilon(y)|^2}{|x-y|^{n+2s}}\,dx\,dy - \int_\Omega (|u_\lambda + t\Phi_\epsilon|^{2_s^*} - |u_\lambda|^{2_s^*})\,dx - t\int_\Omega u_\lambda^{2_s^*-1}\Phi_\epsilon\,dx \\
&\quad - \lambda\Big(\int_\Omega \frac{a(x)(|u_\lambda + t\Phi_\epsilon|^{1-q} - |u_\lambda|^{1-q})}{1-q}\,dx - t\int_\Omega a(x)\Phi_\epsilon u_\lambda^{-q}\,dx\Big) \\
&\le \frac{t^2}{2}(S^{\frac{n}{2s}} + r_1\epsilon^{n-2s}) - \frac{t^{2_s^*}}{2_s^*}\int_\Omega |\Phi_\epsilon|^{2_s^*}dx + \frac{\rho_1 t^\rho}{r_3}\int_\Omega a(x)|\Phi_\epsilon|^\rho dx \\
&\le \frac{t^2}{2}(S^{\frac{n}{2s}} + r_1\epsilon^{n-2s}) - \frac{\rho_2 t^{2_s^*}}{2_s^*}(S^{\frac{n}{2s}} - r_2\epsilon^n) + \|a\|_\infty \rho_1 t^\rho \epsilon^{(n-2s)\rho/2}
\end{aligned}
$$

for $0 \le t \le 1/2$. Since we can assume $t\Phi_\epsilon \ge 1$, for each $t \ge 1/2$ and $|x| \le 2\delta$, we have

$$
\begin{aligned}
& I(u_\lambda + t\Phi_\epsilon) - I(u_\lambda) \\
&\le \frac{t^2}{2}(S^{\frac{n}{2s}} + r_1\epsilon^{n-2s}) - \frac{t^{2_s^*}}{2_s^*}\int_\Omega |\Phi_\epsilon|^{2_s^*}dx - \frac{\rho_2 t^{2_s^*-1}}{r_4(2_s^*-1)}\int_{|x|\le\delta} |\Phi_\epsilon|^{2_s^*-1}dx \\
&\quad + \frac{\rho_1 t^\rho}{r_3}\int_\Omega a(x)|\Phi_\epsilon|^\rho dx \\
&\le \frac{t^2}{2}(S^{\frac{n}{2s}} + r_1\epsilon^{n-2s}) - \frac{t^{2_s^*}}{2_s^*}(S^{\frac{n}{2s}} - r_2\epsilon^n) - \frac{\rho_2 t^{2_s^*-1}}{(2_s^*-1)}\epsilon^{\frac{(n-2s)}{2}} \\
&\quad + \frac{\rho_1 t^\rho}{r_3} + \|a\|_\infty \rho_1 t^\rho \epsilon^{\frac{(n-2s)\rho}{2}}.
\end{aligned}
$$

Now, we define a function $h_\epsilon : [0,\infty) \to \mathbb{R}$ by

$$
h_\epsilon(t) = \begin{cases}
\frac{t^2}{2}(S^{\frac{n}{2s}} + r_1\epsilon^{n-2s}) - \frac{t^{2_s^*}}{2_s^*}(S^{\frac{n}{2s}} - r_2\epsilon^n) + \|a\|_\infty\rho_1 t^\rho \epsilon^{\frac{(n-2s)\rho}{2}} & t \in [0,1/2) \\
\frac{t^2}{2}(S^{\frac{n}{2s}} + r_1\epsilon^{n-2s}) - \frac{t^{2_s^*}}{2_s^*}(S^{\frac{n}{2s}} - r_2\epsilon^n) & \\
-\frac{\rho_2 t^{2_s^*-1}}{(2_s^*-1)}\epsilon^{\frac{(n-2s)}{2}} + \frac{\rho_1 t^\rho}{r_3} + \|a\|_\infty\rho_1 t^\rho\epsilon^{\frac{(n-2s)\rho}{2}} & t \in [1/2,\infty)
\end{cases}
$$

By some computations, it can be checked that h_ϵ attains its maximum at

$$
t = 1 - \frac{\rho_2 \epsilon^{(n-2s)/2}}{(2_s^*-2)S^{\frac{n}{2s}}} + o(\epsilon^{(n-2s)/2}),
$$

so we obtain

$$
\sup\{I(u_\lambda + t\Phi_\epsilon) - I(u_\lambda) : t \ge 0\} \le \frac{s}{n}S^{\frac{n}{2s}} - \frac{\rho_2\epsilon^{(n-2s)/2}}{(2_s^*-1)} + o(\epsilon^{(n-2s)/2}) < \frac{s}{n}S^{\frac{n}{2s}}.
$$

Case II: $q = 1$. Since u_λ is a positive weak solution of (1.1), using previous inequalities, we obtain

$$
I(u_\lambda + t\Phi_\epsilon) - I(u_\lambda)
$$

$$
\begin{aligned}
&= I(u_\lambda + t\Phi_\epsilon) - I(u_\lambda) - t\Big(\int_Q \frac{(u_\lambda(x)-u_\lambda(y))(\Phi_\epsilon(x)-\Phi_\epsilon(y))}{|x-y|^{n+2s}}\,dx\,dy \\
&\quad - \lambda\int_\Omega a(x)\frac{\Phi_\epsilon(x)}{u_\lambda}\,dx - \int_\Omega \Phi_\epsilon(x)u_\lambda^{2_s^*-1}\,dx\Big) \\
&= \frac{t^2}{2}\int_Q \frac{|\Phi_\epsilon(x)-\Phi_\epsilon(y)|^2}{|x-y|^{n+2s}}\,dx\,dy - \int_\Omega (|u_\lambda + t\Phi_\epsilon|^{2_s^*} - |u_\lambda|^{2_s^*})\,dx - t\int_\Omega u_\lambda^{2_s^*-1}\Phi_\epsilon\,dx \\
&\quad - \lambda\Big(\int_\Omega a(x)(\ln|u_\lambda + t\Phi_\epsilon| - \ln|u_\lambda|)\,dx - t\int_\Omega a(x)\frac{\Phi_\epsilon}{u_\lambda}\,dx\Big) \\
&\le \frac{t^2}{2}(S^{\frac{n}{2s}} + r_1\epsilon^{n-2s}) - \frac{t^{2_s^*}}{2_s^*}\int_\Omega |\Phi_\epsilon|^{2_s^*}\,dx - \lambda\int_\Omega a(x)\frac{t^2|u_\epsilon|^2}{2|u_\lambda + tu_\epsilon|^2}\,dx \\
&\le \frac{t^2}{2}(S^{\frac{n}{2s}} + r_1\epsilon^{n-2s}) - \frac{t^{2_s^*}}{2_s^*}\int_\Omega |\Phi_\epsilon|^{2_s^*}\,dx + \frac{\lambda\|a\|_\infty}{2M^2}\int_\Omega t^2|u_\epsilon|^2\,dx \\
&\le \frac{t^2}{2}(S^{\frac{n}{2s}} + r_1\epsilon^{n-2s}) - \frac{t^{2_s^*}}{2_s^*}\int_\Omega |\Phi_\epsilon|^{2_s^*}\,dx + \gamma\|a\|_\infty\epsilon^{(n-2s)}
\end{aligned}
$$

for $0 \le t \le 1/2$, where γ is a constant. Since we can assume $t\Phi_\epsilon \ge 1$, for each $t \ge 1/2$ and $|x| \le 2\delta$, we have

$$
\begin{aligned}
I(u_\lambda + t\Phi_\epsilon) - I(u_\lambda) &\le \frac{t^2}{2}(S^{\frac{n}{2s}} + r_1\epsilon^{n-2s}) - \frac{t^{2_s^*}}{2_s^*}\int_\Omega |\Phi_\epsilon|^{2_s^*}dx \\
&\quad - \frac{\rho_2 t^{2_s^*-1}}{r_4(2_s^*-1)}\int_{|x|\le\delta} |\Phi_\epsilon|^{2_s^*-1}dx - \lambda\int_\Omega a(x)\frac{t^2|u_\epsilon|^2}{2|u_\lambda + tu_\epsilon|^2}dx \\
&\le \frac{t^2}{2}(S^{\frac{n}{2s}} + r_1\epsilon^{n-2s}) - \frac{t^{2_s^*}}{2_s^*}(S^{\frac{n}{2s}} - r_2\epsilon^n) - \frac{\rho_2 t^{2_s^*-1}}{(2_s^*-1)}\epsilon^{\frac{(n-2s)}{2}} \\
&\quad + \frac{\rho_1 t^\rho}{r_3} + \gamma\|a\|_\infty\epsilon^{(n-2s)}
\end{aligned}
$$

for each $t \ge 1/2$. With similar computations as in case (I), it can be shown that

$$
\sup\{I(u_\lambda + t\Phi_\epsilon) : t \ge 0\} < I(u_\lambda) + \frac{s}{n}S^{\frac{n}{2s}}.
$$

$\square$

Lemma 5.2. *There holds* $\inf I(\mathcal{N}_\lambda^-) < I(u_\lambda) + \frac{s}{n}S^{\frac{n}{2s}}$.

Proof. We start by fixing sufficiently small $\epsilon > 0$ as in previous lemma and define functions $\sigma_1, \sigma_2 : [0,\infty) \to \mathbb{R}$ by

$$
\begin{aligned}
\sigma_1(t) &= \int_Q \frac{|(u_\lambda + t\Phi_\epsilon)(x) - (u_\lambda + t\Phi_\epsilon)(y)|^2}{|x-y|^{n+2s}}\,dx\,dy - \lambda\int_\Omega a(x)|u_\lambda + t\Phi_\epsilon|^{1-q}\,dx \\
&\quad - \int_\Omega |u_\lambda + t\Phi_\epsilon|^{2_s^*}\,dx, \\
\sigma_2(t) &= \int_Q \frac{|(u_\lambda + t\Phi_\epsilon)(x) - (u_\lambda + t\Phi_\epsilon)(y)|^2}{|x-y|^{n+2s}}\,dx\,dy \\
&\quad - \lambda q\int_\Omega a(x)|u_\lambda + t\Phi_\epsilon|^{1-q}\,dx - (2_s^*-1)\int_\Omega |u + t\Phi_\epsilon|^{2_s^*}\,dx.
\end{aligned}
$$

Let $t_0 = \sup\{t \ge 0 : \sigma(t) \ge 0\}$, then $\sigma_2(0) = \phi''_{u_\lambda}(1) > 0$ and $\sigma_2(t) \to -\infty$ as $t \to \infty$ which implies $0 < t_0 < \infty$. As $\lambda \in (0,\Lambda)$, we obtain $\sigma_1(t_0) > 0$ and since

$\sigma_1(t) \to -\infty$ as $t \to \infty$, there exists $t' \in (t_0, \infty)$ such that $\sigma_1(t') = 0$. This gives $\phi''_{u_\lambda + t'\Phi_\epsilon}(1) < 0$, because $t' > t_0$ implies $\sigma_2(t') < 0$. Hence, $(u_\lambda + t'\Phi_\epsilon) \in \mathcal{N}_\lambda^-$ and using previous lemma, we obtain the result. $\square$

Proposition 5.3. *There exists $v_\lambda \in \mathcal{N}_\lambda^-$ satisfying $I(v_\lambda) = \inf_{v \in \mathcal{N}_\lambda^-} I(v)$.*

Proof. Let $\{v_k\}$ be sequence in $\mathcal{N}_\lambda^-$ such that $I(v_k) \to \inf I(\mathcal{N}_\lambda^-)$ as $k \to \infty$. Using lemma 3.4, we may assume that there exist v_λ such that $v_k \rightharpoonup v_\lambda$ weakly in X_0. We set $z_k := (v_k - v_\lambda)$ and claim that $v_k \to v_\lambda$ strongly in X_0. Suppose $\|z_k\|^2 \to c^2$ and $\int_\Omega |z_k|^{2_s^*} dx \to d^{2_s^*}$ as $k \to \infty$. Then using Brezis-Lieb lemma, we obtain

$$\|v_\lambda\|^2 + c^2 = \lambda \int_\Omega a(x)|v_\lambda|^{1-q} dx + \int_\Omega |v_\lambda|^{2_s^*} + d^{2_s^*} dx.$$

We claim that $v_\lambda \in X_{+,q}$. Suppose $v_\lambda = 0$, this implies $c \neq 0$ (using lemma 3.4(ii)) and thus

$$\inf I(\mathcal{N}_\lambda^-) = \lim I(v_k) = I(0) + \frac{c^2}{2} - \frac{d^{2_s^*}}{2_s^*} \geq \frac{s}{n} S^{\frac{n}{2s}},$$

but by previous lemma, $(sS^{\frac{n}{2s}})/n \leq \inf I(\mathcal{N}_\lambda^-) < I(u_\lambda) + \frac{s}{n} S^{\frac{n}{2s}}$ implying $I(u_\lambda) > 0$ or we say that $\inf I(\mathcal{N}_\lambda^-) > 0$, which is a contradiction. So $v_\lambda \in X_{+,q}$ and thus, our assumption $\lambda \in (0, \Lambda)$ says that there exists $0 < t_1 < t_2$ such that $\phi'_{v_\lambda}(t_1) = \phi'_{v_\lambda}(t_2) = 0$ and $t_1 v_\lambda \in \mathcal{N}_\lambda^+$, $t_2 v_\lambda \in \mathcal{N}_\lambda^-$. Let us define $f, g : (0, \infty) \to \mathbb{R}$ by

$$f(t) = \frac{c^2 t^2}{2} - \frac{d^{2_s^*} t^{2_s^*}}{2_s^*} \quad \text{and} \quad g(t) = \phi_{v_\lambda}(t) + f(t). \tag{5.1}$$

Then, following three cases arise:

(i) $t_2 < 1$,
(ii) $t_2 \geq 1$ and $d > 0$, and
(iii) $t_2 \geq 1$ and $d = 0$.

Case (i) $t_2 < 1$ implies $g'(1) = \phi'_{v_\lambda}(1) + f'(1) = 0$, using (5.1) and $g'(t_2) = \phi'_{v_\lambda}(t_2) + f'(t_2) = t_2(c^2 - d^{2_s^*} t_2^{2_s^* - 2}) \geq t_2(c^2 - d^{2_s^*}) > 0$. This implies that g is increasing on $[t_2, 1]$ and we have

$$\inf I(\mathcal{N}_\lambda^-) = g(1) > g(t_2) \geq I(t_2 v_\lambda) + \frac{t_2}{2}(c^2 - d^{2_s^*}) > I(t_2 v_\lambda) \geq \inf I(\mathcal{N}_\lambda^-)$$

which is a contradiction.

Case (ii) Let $\underline{t} = (c^2/d^{2_s^*})^{\frac{1}{2_s^* - 2}}$ and we can check that f attains its maximum at $\underline{t}$ and

$$f(\underline{t}) = \frac{c^2 \underline{t}^2}{2} - \frac{d^{2_s^*} \underline{t}^{2_s^*}}{2_s^*} = \Big(\frac{c^2}{d^{2_s^*}}\Big)^{\frac{2_s^*}{2_s^* - 2}} \Big(\frac{1}{2} - \frac{1}{2_s^*}\Big) \geq S^{\frac{2_s^*}{2_s^* - 2}} \frac{s}{n} = \frac{s}{n} S^{\frac{n}{2s}}.$$

Also, $f'(t) = (c^2 - d^{2_s^*} t^{2_s^* - 2})t > 0$ if $0 < t < \underline{t}$ and $f'(t) < 0$ if $t > \underline{t}$. Moreover, we know $g(1) = \max_{t>0}\{g(t)\} \geq g(\underline{t})$ using the assumption $\lambda \in (0, \Lambda)$. If $\underline{t} \leq 1$, then we have

$$\inf I(\mathcal{N}_\lambda^-) = g(1) \geq g(\underline{t}) = I(\underline{t} v_\lambda) + f(\underline{t}) \geq I(t_1 v_\lambda) + \frac{s}{n} S^{\frac{n}{2s}}$$

which contradicts the previous lemma. Thus, we must have $\underline{t} > 1$. Since $g'(t) \leq 0$ for $t \geq 1$, there holds $\phi''_{v_\lambda}(t) \leq -f'(t) \leq 0$ for $1 \leq t \leq \underline{t}$. Then we have $\underline{t} \leq t_1$ or $t_2 = 1$. If $\underline{t} \leq t_1$ then

$$\inf I(\mathcal{N}_\lambda^-) = g(1) \geq g(\underline{t}) = I(\underline{t} v_\lambda) + f(\underline{t}) \geq I(t_1 v_\lambda) + \frac{s}{n} S^{\frac{n}{2s}}$$

which is a contradiction. If $t_2 = 1$ then using $c^2 = d^{2_s^*}$ we obtain

$$\inf I(\mathcal{N}_\lambda^-) = g(1) = I(v_\lambda) + \Big(\frac{c^2}{2} - \frac{d^{2_s^*}}{2_s^*}\Big) \geq I(v_\lambda) + \frac{s}{n} S^{\frac{n}{2s}}$$

which is a contradiction and thus only case (iii) holds. If $c \neq 0$, then $\phi'_{v_\lambda}(1) = -c^2 < 0$ and $\phi''_{v_\lambda}(1) = -c^2 < 0$ which contradicts $t_2 \geq 1$. Thus, $c = 0$ which implies $v_k \to v_\lambda$ strongly in X_0. Consequently, $v_\lambda \in \mathcal{N}_\lambda^-$ and $\inf I(\mathcal{N}_\lambda^-) = I(v_\lambda)$. □

Proposition 5.4. *For $\lambda \in (0, \Lambda)$, v_λ is a positive weak solution of* (1.1).

Proof. Let $\psi \in C_c^\infty(\Omega)$. Using lemma 3.7, since $\phi > 0$ in Ω, we can find $\alpha > 0$ such that $v_\lambda \geq \alpha$ on $supp(\psi)$. Also, $t_\epsilon \to 1$ as $\epsilon \to 0+$, where t_ϵ is the unique positive real number corresponding to $(v_\lambda + \epsilon\psi)$ such that $t_\epsilon(v_\lambda + \epsilon\psi) \in \mathcal{N}_\lambda^-$. Then, by lemma 3.5 we have

$$\begin{aligned}
0 &\leq \lim_{\epsilon\to 0} \frac{I(t_\epsilon(v_\lambda + \epsilon\psi)) - I(v_\lambda)}{\epsilon} \leq \lim_{\epsilon\to 0} \frac{I(t_\epsilon(v_\lambda + \epsilon\psi)) - I(t_\epsilon v_\lambda)}{\epsilon} \\
&= \int_Q \frac{(v_\lambda(x) - v_\lambda(y))(\psi(x) - \psi(y))}{|x-y|^{n+2s}} dx\, dy - \lambda \int_\Omega a(x) v_\lambda^{-q} \psi dx - \int_\Omega v_\lambda^{2_s^*-1} \psi dx.
\end{aligned}$$

Since $\psi \in C_c^\infty(\Omega)$ is arbitrary, we conclude that v_λ is positive weak solution of (1.1). □

Now the proof of Theorem 2.4 follows from proposition 4.2, lemma 4.3 and proposition 5.4.

6. Regularity of weak solutions

In this section, we shall prove some regularity properties of positive weak solutions of (1.1). We begin with the following lemma.

Lemma 6.1. *Suppose u is a weak solution of* (1.1)*, then for each $w \in X_0$, it satisfies*

$$a(x) u^{-q} w \in L^1(\Omega),$$

$$\int_Q \frac{(u(x) - u(y))(w(x) - w(y))}{|x-y|^{n+2s}} dx\, dy - \int_\Omega \Big(\lambda a u^{-q} + u^{2_*^s - 1}\Big) w dx = 0.$$

Proof. Let u be a weak solution of (1.1) and $w \in X_+$. By lemma 2.2, we obtain a sequence $\{w_k\} \in X_0$ such that $\{w_k\} \to w$ strongly in X_0, each w_k has compact support in Ω and $0 \leq w_1 \leq w_2 \leq \dots$. Since each w_k has compact support in Ω and u is a positive weak solution of (1.1), for each k we obtain

$$\lambda \int_\Omega a(x) u^{-q} w_k\, dx = \int_Q \frac{(u(x) - u(y))(w_k(x) - w_k(y))}{|x-y|^{n+2s}} dx\, dy - \int_\Omega u^{2_*^s - 1} w_k\, dx.$$

Using monotone convergence theorem, we obtain $a(x) u^{-q} w \in L^1(\Omega)$ and

$$\lambda \int_\Omega a(x) u^{-q} w\, dx = \int_Q \frac{(u(x) - u(y))(w(x) - w(y))}{|x-y|^{n+2s}} dx\, dy - \int_\Omega u^{2_*^s - 1} w\, dx = 0.$$

If $w \in X_0$, then $w = w^+ - w^-$ and $w^+, w^- \in X_+$. Since we proved the lemma for each $w \in X_+$, we obtain the conclusion. □

Lemma 6.2. *Let u be a weak solution of* (1.1)*. Then $u \in L^\infty(\Omega)$.*

Proof. We follow [4]. We use the inequality known for fractional Laplacian

$$(-\Delta)^s\varphi(u) \le \varphi'(u)(-\Delta)^s u,$$

where φ is a convex and differentiable function. We define

$$\varphi(t) = \varphi_{T,\beta}(t)\begin{cases} 0 & \text{if } t \le 0 \\ t^\beta & \text{if } 0 < t < T \\ \beta T^{\beta-1}(t-T) + T^\beta & \text{if } t \ge T, \end{cases}$$

where $\beta \ge 1$ and $T > 0$ is large. Then φ is Lipschitz with constant $M = \beta T^{\beta-1}$ which gives $\varphi \in X_0$. Thus,

$$\|\varphi(u)\| = \Big(\int_Q \frac{|\varphi(u(x)) - \varphi(u(y))|^2}{|x-y|^{n+2s}}\Big)^{1/2} \le \Big(\int_Q \frac{M^2|u(x)-u(y)|^2}{|x-y|^{n+2s}}\Big)^{1/2} = M^2\|u\|. \tag{6.1}$$

Using $\|\varphi(u)\| = \|(-\Delta)^{s/2}\varphi(u)\|_2$, we obtain

$$\int_\Omega \varphi(u)(-\Delta)^s\varphi(u) = \|\varphi(u)\|^2 \ge S\|\varphi(u)\|^2_{2_s^*},$$

where S is as defined in section 1. Since φ is convex and $\varphi(u)\varphi'(u) \in X_0$, we obtain

$$\begin{aligned}\int_\Omega \varphi(u)(-\Delta)^s\varphi(u) &\le \int_\Omega \varphi(u)\varphi'(u)(-\Delta)^s u \\ &= \int_\Omega \varphi(u)\varphi'(u)(\lambda a(x)u^{-q} + u^{2_s^*-1})dx.\end{aligned} \tag{6.2}$$

Therefore, using (6.1) and (6.2), we obtain

$$\|\varphi(u)\|^2_{2_s^*} \le C\int_\Omega \varphi(u)\varphi'(u)(\lambda a(x)u^{-q} + u^{2_s^*-1})dx, \tag{6.3}$$

for some constant C. We have $u\varphi'(u) \le \beta\varphi(u)$ and $\varphi'(u) \le \beta(1+\varphi(u))$ which gives

$$\begin{aligned}&\int_\Omega \varphi(u)\varphi'(u)(\lambda a(x)u^{-q} + u^{2_s^*-1})dx \\ &= \int_\Omega \Big(\lambda a(x)\varphi(u)\varphi'(u)u^{-q}dx + \varphi(u)\varphi'(u)u^{2_s^*-1}\Big)\,dx \\ &\le \lambda\|a\|_\infty\beta\int_\Omega \varphi(u)u^{-q}(1+\varphi(u)) + \beta\int_\Omega(\varphi(u))^2u^{2_s^*-2}.\end{aligned}$$

Thus from (6.3), we obtain

$$\Big(\int_\Omega |\varphi(u)|^{2_s^*}\Big)^{2/2_s^*} \le C_2\beta\Big(\lambda\|a\|_\infty\int_\Omega(\varphi(u)u^{-q} + (\varphi(u))^2u^{-q}) + \int_\Omega(\varphi(u))^2u^{2_s^*-2}\Big).$$

where $C_2 = C\max\{\lambda\|a\|_\infty, 1\}$. Next we claim that $u \in L^{\beta_1 2_s^*}(\Omega)$, where $\beta_1 = 2_s^*/2$. Fixing some K whose appropriate value is to be determined later and taking $r = \beta_1$, $s = 2_s^*/(2_s^*-2)$, we obtain

$$\begin{aligned}\int_\Omega(\varphi(u))^2u^{2_s^*-2} &= \int_{u\le K}(\varphi(u))^2u^{2_s^*-2} + \int_{u>K}(\varphi(u))^2u^{2_s^*-2} \\ &\le K^{2_s^*-2}\int_{u\le K}(\varphi(u))^2 + \Big(\int_\Omega(\varphi(u))^{2_s^*}\Big)^{2/2_s^*}\Big(\int_{u>K}u^{2_s^*}\Big)^{(2_s^*-2)/2_s^*}.\end{aligned}$$

Using the Monotone Convergence theorem, we choose K such that

$$\Big(\int_{u>K} u^{2_s^*}\Big)^{(2_s^*-2)/2_s^*} \leq \frac{1}{2C_2\beta}$$

and this gives

$$\begin{aligned}&\Big(\int_\Omega (\varphi(u))^{2_s^*}\Big)^{2/2_s^*}\\ &\leq 2C_2\beta\Big(\int_\Omega (\varphi(u))u^{-q} + \int_\Omega (\varphi(u))^2u^{-q} + K^{2_s^*-2}\int_{u\leq K}(\varphi(u))^2\Big).\end{aligned} \tag{6.4}$$

Using $\varphi_{T,\beta_1}(u) \leq u^{\beta_1}$ in left hand side of (6.4) and then letting $T \to \infty$ in right hand side, we obtain

$$\Big(\int_\Omega u^{2_s^*\beta_1}\Big)^{2/2_s^*} \leq 2C_2\beta_1\Big(\int_\Omega u^{\frac{2_s^*}{2}-q} + \int_\Omega u^{2_s^*-q} + K^{2_s^*-2}\int_\Omega u^{2_s^*}\Big)$$

since $2\beta_1 = 2_s^*$. This proves the claim. Again, from (6), using $\varphi_{T,\beta}(u) \leq u^\beta$ in left hand side and then letting $T \to \infty$ in right hand side, we obtain

$$\begin{aligned}\Big(\int_\Omega u^{2_s^*\beta}\Big)^{2/2_s^*} &\leq C_2\beta\Big(\int_\Omega u^{\beta-q} + \int_\Omega u^{2\beta-q} + K^{2_s^*-2}\int_\Omega u^{2\beta+2_s^*-2}\Big)\\ &\leq C_2'\beta\Big(1 + \int_\Omega u^{2\beta-q} + K^{2_s^*-2}\int_\Omega u^{2\beta+2_s^*-2}\Big),\end{aligned} \tag{6.5}$$

where $C_2' > 0$ is a constant. Now we see that

$$\int_\Omega u^{2\beta-q} = \int_{u\geq 1} u^{2\beta-q} + \int_{u<1} u^{2\beta-q} \leq \int_{u\geq 1} u^{2\beta+2_s^*-2} + |\Omega|.$$

Using this in (6.5), with some simplifications, we obtain

$$\Big(1 + \int_\Omega u^{2_s^*\beta}\Big)^{\frac{1}{2_s^*(\beta-1)}} \leq C_\beta^{\frac{1}{2(\beta-1)}}\Big(1 + \int_\Omega u^{2\beta+2_s^*-2}\Big)^{\frac{1}{2(\beta-1)}}, \tag{6.6}$$

where $C_\beta = 4C_2'\beta(1+|\Omega|)$. For $m \geq 1$, let us define β_{m+1} inductively by

$$2\beta_{m+1} + 2_s^* - 2 = 2_s^*\beta_m;$$

that is, $(\beta_{m+1} - 1) = \frac{2_s^*}{2}(\beta_m - 1) = \big(\frac{2_s^*}{2}\big)^m(\beta_1 - 1)$. Hence, from (6.6) it follows that

$$\Big(1 + \int_\Omega u^{2_s^*\beta_{m+1}}\Big)^{\frac{1}{2_s^*(\beta_{m+1}-1)}} \leq C_{\beta_{m+1}}^{\frac{1}{2(\beta_{m+1}-1)}}\Big(1 + \int_\Omega u^{2_s^*\beta_m}\Big)^{\frac{1}{2_s^*(\beta_m-1)}},$$

where $C_{\beta_{m+1}} = 4C_2'\beta_{m+1}(1+|\Omega|)$. Setting $D_{m+1} := \big(1 + \int_\Omega u^{2_s^*\beta_m}\big)^{\frac{1}{2_s^*(\beta_m-1)}}$, we obtain

$$\begin{aligned}D_{m+1} &\leq C_{m+1}^{\frac{1}{2(\beta_{m+1}-1)}} D_m\\ &= \Big(\prod_{i=2}^{m+1} C_k^{\frac{1}{2(\beta_i-1)}}\Big) D_1\{4C_2'(1+|\Omega|)\}^{\sum_{i=2}^{m+1}\frac{1}{2(\beta_i-1)}}\\ &\quad\times\prod_{i=2}^{m+1}\Big(1 + \Big(\frac{2_s^*}{2}\Big)^{i-1}(\beta_1-1)\Big)^{\frac{1}{2((2_s^*/2)^{i-1}(\beta_1-1))}} D_1.\end{aligned}$$

Since

$$\lim_{i\to\infty}\Big(\big(\frac{2_s^*}{2}\big)^{i-1}(\beta_1-1) + 1\Big)^{\frac{1}{2((2_s^*/2)^{i-1}(\beta_1-1))}} = 1,$$

there exists $C_3 > 1$ (independent of β_i) such that

$$D_{m+1} \leq \{4C_2'(1+|\Omega|)\}^{\sum_{i=2}^{m+1} \frac{1}{2(\beta_i-1)}} C_3 D_1.$$

However,

$$\sum_{i=2}^{m+1} \frac{1}{2(\beta_i-1)} = \frac{1}{2(\beta_1-1)} \sum_{i=2}^{m+1} \big(\frac{2_s^*}{2}\big)^i$$

and $\left(\frac{2_s^*}{2}\right) < 1$ implies that there exists a constant $C_4 > 0$ such that $D_{m+1} \leq C_4 D_1$; that is,

$$\Big(1+\int_\Omega u^{2_s^*(\beta_{m+1})}\Big)^{\frac{1}{2_s^*(\beta_{m+1}-1)}} \leq C_4 D_1, \tag{6.7}$$

where $2_s^*\beta_m \to \infty$ as $m \to \infty$. Let us assume $\|u\|_\infty > C_4 D_1$. Then there exists $\eta > 0$ and $\Omega' \subset \Omega$ such that $0 < |\Omega'| < \infty$ and

$$u(x) > C_4 D_1 + \eta, \quad \text{for all } x \in \Omega'.$$

It follows that

$$\liminf_{\beta_m \to \infty} \Big(\int_\Omega |u|^{2_s^*\beta_m} + 1\Big)^{\frac{1}{2_s^*\beta_m - 1}} \geq \liminf_{\beta_m \to \infty} (C_4 D_1 + \eta)^{\frac{\beta_m}{\beta_m - 1}} (|\Omega'|)^{\frac{1}{2_s^*(\beta_m-1)}} = C_4 D_1 + \eta$$

which contradicts (6.7). Hence, $\|u\|_\infty \leq C_4 D_1$ that is $u \in L^\infty(\Omega)$. $\square$

Theorem 6.3. *Let u be a positive solution of* (1.1). *Then there exist $\alpha \in (0, s]$ such that $u \in C^\alpha_{loc}(\Omega')$, for all $\Omega' \Subset \Omega$.*

Proof. Let $\Omega' \Subset \Omega$. Then using lemma 3.7 and above regularity result, for any $\psi \in C_c^\infty(\Omega)$ we obtain

$$\lambda \int_{\Omega'} u^{-q}\psi dx + \int_{\Omega'} u^{2_s^*-1}\psi dx \leq \lambda \int_{\Omega'} \phi_1^{-q}\psi dx + \|u\|_\infty^{2_s^*-1} \int_{\Omega'} \psi dx \leq C \int_{\Omega'} \psi dx$$

for some constant $C > 0$, since we can find $k > 0$ such that $\phi_1 > k$ on Ω'. Thus we have $|(-\Delta)^s u| \leq C$ weakly on Ω'. So, using [22, Theorem 4.4] and applying a covering argument on inequality in [22, Corollary 5.5], we can prove that there exist $\alpha \in (0, s]$ such that $u \in C^\alpha_{loc}(\Omega')$, for all $\Omega' \Subset \Omega$. $\square$

References

[1] Adimurthi, J. Giacomoni; *Multiplicity of positive solutions for a singular and critical elliptic problem in* $\mathbb{R}^2$, Communications in Contemporary Mathematics **8** (2006), 621–656.

[2] D. Applebaum; *Lévy process-from probability to finance and quantum groups*, Notices Amer. Math. Soc. **51** (2004), 1336–1347.

[3] G. Autuori, P. Pucci; *Elliptic problems involving the fractional Laplacian in* $\mathbb{R}^n$, J. Differential Equations **255** (2013), 2340–2362.

[4] B. Barrios, E. Coloradoc, R. Servadei, F. Soriaa; *A critical fractional equation with concave-convex power nonlinearities*, Ann. I. H. Poincaré Analyse Non Linéaire **32** (2015), 875–900.

[5] B. Barrios, I. De Bonis, M. Mara, I. Peral; *Semilinear problems for the fractional laplacian with a singular nonlinearity*, Open Math. **13** (2015), 390–407.

[6] B. Brändle, E. Colorado, A. De Pablo, U. Sánchez; *A concave-convex elliptic problem involving the fractional Laplacian*, Proc. Roy. Soc. Edinburgh. Sect. A **143** (2013), 39–71.

[7] X. Cabré, J.G. Tan; *Positive solutions of nonlinear problem involving the square root of the Laplacian*, Adv. Math. **224** (2) (2010), 2052–2093.

[8] M. G. Crandall, P. H. Rabinowitz, L. Tartar; *On a Dirichlet problem with a singular nonlinearity*, Communications in Partial Differential Equations **2** (1977), 193–222.

[9] R. Dhanya, J. Giacomoni, S. Prashanth, K. Saoudi; *Global bifurcation and local multiplicity results for elliptic equations with singular nonlinearity of super exponential growth in* $\mathbb{R}^2$, Advances in Differential Equations **17** (3-4) (2012), 369–400.

[10] Y. Fang; *Existence, uniqueness of positive solution to a fractional laplacians with singular non linearity* , preprint (2014), http://arxiv.org/pdf/1403.3149.pdf.

[11] A. Garroni, S. Müller; *Γ-limit of a phase-field model of dislocations*, SIAM J. Math. Anal. **36** (2005), 1943–1964.

[12] M. Ghergu, V. Rădulescu; *Singular Elliptic Problems. Bifurcation and Asymptotic Analysis*, Oxford Lecture Series in Mathematics and Its Applications, Oxford University Press, 37 (2008).

[13] M. Ghergu and V. Rădulescu; *Singular elliptic problems with lack of compactness*, Ann. Mat. Pura Appl. **185**(1) (2006), 63-79

[14] J. Giacomoni, P. Kumar Mishra, K. Sreenadh; *Fractional elliptic equations with critical exponential nonlinearity*, Adv. Nonlinear Anal. **5**(1) (2016), 57–74.

[15] S. Goyal, K. Sreenadh; *Existence of multiple solutions of p-fractional Laplace operator with sign-changing weight function*, Adv. Nonlinear Anal. **4**(1) (2015), 37–58.

[16] S. Goyal, K. Sreenadh; *The Nehari manifold for non-local elliptic operator with concave-convex nonlinearities and sign-changing weight functions*, Proc. Indian Acad. Sci. Math. Sci., PMSC-D-13-00460.

[17] Y. Haitao; *Multiplicity and asymptotic behavior of positive solutions for a singular semilinear elliptic problem*, J. Differential Equations **189** (2003), 487–512.

[18] J. Hernández, J. Francisco. J. M. Vega; *Positive solutions for singular nonlinear elliptic equations*, Proc. Roy. Soc. Edinburgh Sect. A **137** (1) (2007), 41-62.

[19] J. Hernández, J. Kartson, P. L. Simon; *Multiplicity for semilinear elliptic equations involving singular nonlinearity*, Nonlinear Anal. **65** (2) (2006), 265-283.

[20] N. Hirano, C. Saccon, N. Shioji; *Brezis-Nirenberg type theorems and multiplicity of positive solutions for a singular elliptic problem*, J. Differential Equations **245** (2008), 1997–2037.

[21] N. Hirano, C. Saccon, N. Shioji; *Existence of multiple positive solutions for singular elliptic problems with concave and convex nonlinearities*, Advances in Differential Equations **9** (1-2) (2004), 197–220.

[22] A. Iannizzotto, S. Mosconi, M. Squassina; *Global Hölder regularity for the fractional p-Laplacian*, Preprint(2015), http://arxiv.org/pdf/1411.2956.pdf

[23] G. Molica Bisci, D. Repovš; *Existence and localization of solutions for nonlocal fractional equations*, Asymptot. Anal. **90** (3-4) (2014), 367–378.

[24] G. Molica Bisci, D. Repovš; *On doubly nonlocal fractional elliptic equations*, Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl. **26** (2) (2015), 161–176.

[25] G. Molica Bisci, R. Servadei; *A Brezis-Nirenberg spitting approach for nonlocal fractional equations*, Nonlinear Anal. **119** (2015), 341–353.

[26] G. Molica Bisci, V. Rădulescu; *Ground state solutions of scalar field fractional Schrödinger equations*, Calc. Var. Partial Differential Equations **54** (3) (2015), 2985–3008.

[27] G. Molica Bisci, R. Servadei; *Lower semicontinuity of functionals of fractional type and applications to nonlocal equations with critical Sobolev exponent*, Advances in Differential Equations **20** (2015), 635–660.

[28] G. Molica Bisci, V. Rădulescu, R. Servadei; *Variational Methods for Nonlocal Fractional Problems*, Encyclopedia of Mathematics, Cambridge University Press, Cambridge, (2016).

[29] N.S. Papageorgiou, V. Rădulescu; *Combined effects of singular and sublinear nonlinearities in some elliptic problems*, Nonlinear Anal. **109** (2014), 236–244.

[30] P. Pucci, M. Xiang, B. Zhang; *Existence and multiplicity of entire solutions for fractional p-Kirchhoff equations*, Adv. Nonlinear Anal. **5** (1) (2016), 27–55.

[31] R. Servadei, E. Valdinoci; *Mountain pass solutions for nonlocal elliptic operators*, J. Math. Anal. Appl. **389** (2012), 887–898.

[32] R. Servadei, E. Valdinoci; *The Brezis-Nirenberg result for the fractional laplacian*, Trans. Amer. Math. Soc. **367** (2015), 67–102.

[33] R. Servadei, E. Valdinoci; *Variational methods for nonlocal operators of elliptic type*, Discrete Contin. Dyn. Syst. **33** (5) (2013), 2105–2137.

[34] R. Servadei, E. Valdinoci; *A Brezis-Nirenberg result for nonlocal critical equations in low dimension*, Commun. Pure Appl. Anal. **12** (6) (2013), 2445–2464.

[35] X. Yu; *The Nehari manifold for elliptic equation involving the square root of the Laplacian*, J. Differential Equations **252** (2012), 1283–1308.

[36] J. Zhang, X. Liu, H. Jiao; *Multiplicity of positive solutions for a fractional Laplacian equations involving critical nonlinearity*, preprint (2015), http://arxiv.org/abs/1502.02222.

Tuhina Mukherjee
Department of Mathematics, Indian Institute of Technology Delhi, Hauz Khaz, New Delhi-16, India
E-mail address: tulimukh@gmail.com

Konijeti Sreenadh
Department of Mathematics, Indian Institute of Technology Delhi, Hauz Khaz, New Delhi-16, India
E-mail address: sreenadh@gmail.com