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INFINITELY MANY SIGN-CHANGING SOLUTIONS FOR KIRCHHOFF-TYPE EQUATIONS WITH POWER NONLINEARITY

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Abstract. In this article we consider the Kirchhoff-type elliptic problem

$$\begin{gathered}-(a+b\int_\Omega |\nabla u|^2dx)\Delta u = |u|^{p-2}u, \quad \text{in } \Omega,\\ u=0, \quad \text{on } \partial\Omega,\end{gathered}$$

where $\Omega \subset \mathbb{R}^N$ and $p \in (2, 2^*)$ with $2^* = \frac{2N}{N-2}$ if $N \geq 3$, and $2^* = +\infty$ otherwise. We show that the problem possesses infinitely many sign-changing solutions by using combination of invariant sets of descent flow and the Ljusternik-Schnirelman type minimax method.

1. Introduction

We are concerned with the existence of sign-changing solutions to the Kirchhoff-type elliptic problem

$$\begin{gathered}-(a+b\int_\Omega |\nabla u|^2dx)\Delta u = |u|^{p-2}u, \quad \text{in } \Omega,\\ u=0, \quad \text{on } \partial\Omega,\end{gathered} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary and $p \in (2, 2^*)$ with $2^* = \frac{2N}{N-2}$ if $N \geq 3$ and $2^* = +\infty$ otherwise.

Kirchhoff-type problems are often referred as being nonlocal because of the presence of the integral terms, which makes it difficult to be solved. While this motivates the researcher's interest to study it.

Over the past two decades, a great deal of effort has been devoted to the study of existence of solutions to Kirchhoff-type equations such as (1.1) with more general nonlinearities. And there have been interesting results in the existence of various kind of solutions, and just a few in sign-changing (nodal) solutions to the Kirchhoff-type problem.

In recent years, several scholars have studied the existence of sign-changing solutions to the Kirchhoff-type problem with general nonlinearity in bounded domains or in the whole space. Zhang [17] used variational methods and invariant sets of descent flow to obtain a sign-changing solution for (1.1) with general nonlinearity

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$f(x, u)$ in three cases: sublinear, asymptotically-linear, and superlinear at infinity. Mao and Zhang [11] employed the same methods as [10, 17] to show the existence of one sign-changing solution. In [5, 6, 14, 16], the authors studied Kirchhoff-type problem with some different assumptions and showed that there exists one least energy sign-changing solution by variational methods and the quantitative deformation lemma. Batkam [3] obtained infinitely many sign-changing solutions to (1.1) with general nonlinearity, by applying a new version of the symmetric mountain pass theorem. For the existence of sign-changing solutions related problems we refer the reader to [1, 7, 8, 9, 15, 18].

To the best of our knowledge, there is no result in the literature on the existence of sign-changing to problem (1.1) for $p \in (2, 4]$. We apply the approach used in [3, 5, 6, 11, 14, 16] where their results are valid only for $f(x, u) = |u|^{p-2}u$ with $p \in (4, 2^*)$. We give the existence of sign-changing solution to (1.1) for $p \in (2, 4]$.

In this article, $E := H_0^1(\Omega)$ with norm $\|u\| = (\int_\Omega |\nabla u|^2 dx)^{1/2}$. While $L^q(\Omega)$ for $q \in (1, \infty)$ is the usual Lebesgue space with the norm $|u|_p = (\int_\Omega |u|^p dx)^{1/p}$. We use the letter C to denote various positive constants and allow it to be difference from line to line.

Our main results read as follows.

Theorem 1.1. *For each $p \in (2, 2^*)$, problem* (1.1) *has a sequence of sign-changing solutions $\{u_k\}$ such that $I(u_k) \to \infty$ as $k \to \infty$.*

The remainder of this paper is organized as follows. In Section 2, we present some preliminary results; and in Section 3, we prove Theorem 1.1.

2. Preliminaries

First we define the energy functional associated with (1.1),

$$I(u) = \frac{a}{2} \int_\Omega |\nabla u|^2 dx + \frac{b}{4} \Big(\int_\Omega |\nabla u|^2 dx \Big)^2 - \frac{1}{p} \int_\Omega |u|^p dx.$$

Clearly, $I \in C^1(E, \mathbb{R})$. It is well-known that solutions of (1.1) are critical points of the functional I and that

$$\langle I'(u), v\rangle = (a + b \int_\Omega |\nabla u|^2 dx) \int_\Omega \nabla u \nabla v \, dx - \int_\Omega |u|^{p-2} uv \, dx,$$

for every $v \in E$. Hence, if $u \in E$ is a critical point of I, then u is a solution of equations (1.1). Then the gradient of I has the form $\frac{\nabla I}{a+b\|u\|^2} = \mathrm{id} - A$ (see [11, 17]), where $\langle \nabla I(u), v\rangle = \langle I'(u), v\rangle$ for all $v \in E$ and $A : E \to E$ given by

$$A(u) := (-\Delta)^{-1} \frac{|u|^{p-2}u}{a + b\|u\|^2}.$$

Thus we note that following three statement are equivalent: u is a solution of (1.1), u is a critical point of I, and u is a fixed point of A. Then we consider the initial-value problem

$$\begin{gathered} \frac{d}{dt}\varphi(t, u) = -W(\varphi(t, u)), \quad t \ge 0, \\ \varphi(0, u) = u. \end{gathered} \tag{2.1}$$

where

$$W(\varphi) = \frac{\nabla I(\varphi)}{a + b\|\varphi\|^2} = \varphi - A\varphi.$$

It is easy to see that W is locally Lipschitz continuous in E. Thus, for (2.1), there exists a unique solution in some maximal existence interval $[0, T)$, where $T = T(u) \le +\infty$. Then

$$\frac{d}{dt}(I(\varphi(t,u))) = \langle \nabla I(\varphi), \frac{d\varphi}{dt} \rangle = -\frac{\|\nabla I(\varphi(t,u))\|^2}{a + b\|u\|^2} \le 0.$$

Therefore, I is decreasing along the orbits; that is, decreasing in $t \in [0, T)$.

To obtain sign-changing solutions, we use cones of the positive and negative functions as in many reference such as [1, 2, 7, 8, 9]. Precisely, define

$$P^+ := \{u \in E : u \ge 0\} \quad \text{and} \quad P^- := \{u \in E : u \le 0\}.$$

For $\varepsilon > 0$ denote

$$P_\varepsilon^+ := \{u \in E : \operatorname{dist}(u, P^+) < \varepsilon\} \quad \text{and} \quad P_\varepsilon^- := \{u \in E : \operatorname{dist}(u, P^-) < \varepsilon\}.$$

Obviously, $P_\varepsilon^+ = -P_\varepsilon^-$. Set $W := \overline{P_\varepsilon^+} \cup \overline{P_\varepsilon^-}$. Then W is a symmetric subset of E and $Q := E \backslash W$ contains only sign-changing functions. Recall that a subset $D \subset E$ is an invariant set with respect to φ if $\varphi(t,u) \in D$ for any $u \in D$ and $t \in [0, T)$. On the other hand, the next lemma shows that, for ε small, $\overline{P_\varepsilon^+}$ and $\overline{P_\varepsilon^-}$ are invariant set with respect to φ and $\varphi(t, \partial W) \subset int(W)$ for $t \in [0, T)$. Then all sign-changing solutions of equations (1.1) are contained in $Q = E \backslash W$.

Lemma 2.1. *There exists ε_0 such that for any $\varepsilon \in (0, \varepsilon_0)$, the following results hold*

$$A(\partial P_\varepsilon^+) \subset P_\varepsilon^+, \quad A(\partial P_\varepsilon^-) \subset P_\varepsilon^-,$$
$$\varphi(t,u) \in P_\varepsilon^\pm \quad \text{for all } t > 0 \text{ and } u \in \overline{P_\varepsilon^\pm}.$$

Furthermore, every nontrivial solutions $u \in P_\varepsilon^+$ and $u \in P_\varepsilon^-$ of equation (1.1) are positive and negative, respectively.

The proof is similar to the proofs of [2, Lemma 3.1 and Proposition 3.2] and [4, Lemma 2], we omit it.

Lemma 2.2. *Functional I satisfies the Palais-Smale condition.*

Proof. When $p \in [4, 2^*)$, it is trivial to see that functional I satisfies the Palais-Smale condition. While in case $p \in (2, 4)$, let $\{u_n\}$ be a Palais-Smale sequence for I, that is,

$$I(u_n) \to c, \quad I'(u_n) \to 0 \quad \text{as } n \to \infty.$$

Then by computations,

$$\begin{aligned} I(u_n) &= \frac{a}{2}\int_\Omega |\nabla u|^2 dx + \frac{b}{4}\Big(\int_\Omega |\nabla u|^2 dx\Big)^2 - \frac{1}{p}\int_\Omega |u|^p dx \\ &\ge \frac{a}{2}\|u_n\|^2 + \frac{b}{4}\|u_n\|^4 - C\|u_n\|^p, \end{aligned} \tag{2.2}$$

which implies that I is coercive, it follows that $\{u_n\}$ is bounded in E. Then, going if necessary to a subsequence, we may assume that there exists $u \in E$ such that

$$\begin{gathered} u_n \rightharpoonup u \quad \text{weakly in } E, \\ u_n \to u \quad \text{strongly in } L^q(\Omega) \text{ for } q \in (2, 2^*). \end{gathered} \tag{2.3}$$

Since $\langle I'(u_n), (u_n - u)\rangle \to 0$, $u_n \to u$ strongly in E by (2.3). This completes the proof. $\square$

Before stating next lemma, we need some preparation. Denote $I^c := \{u \in E : I(u) \le c\}$, $K_c := \{u \in E : I(u) = c \text{ and } I'(u) = 0\}$, $K_c^w := K_c \cap W$, $K_c^q := K_c \cap Q$ and $K_{c,\rho}^w := \{u \in E : \operatorname{dist}(u, K_c^w) < \rho\}$, $K_{c,\rho}^q := \{u \in E : \operatorname{dist}(u, K_c^q) < \rho\}$ and $B_r := \{u \in E : \|u\| \le r\}$. Because I satisfies the Palais-Smale condition, we have the following deforming lemma [9, Lemma 5.1].

Lemma 2.3. *Let $\rho > 0$ be such that $K_{c,\rho}^w \subset W$. Then there exists ε_0 such that for any $\varepsilon \in (0, \varepsilon_0)$, there is an $\eta \in C([0,1] \times E, E)$ satisfying:*

(1) *$\eta(t,u) = u$ if $t = 0$ or $u \notin I^{-1}([c-\varepsilon_0, c+\varepsilon_0]) \backslash K_{c,\rho}^q$.*
(2) *$\eta(1, I^{c+\varepsilon} \cup W \backslash K_{c,3\rho}^q) \subset I^{c-\varepsilon} \cup W$ and $\eta(1, I^{c+\varepsilon} \cup W) \subset I^{c-\varepsilon} \cup W$ if $K_c^q = \emptyset$.*
(3) *$\eta(t, \cdot)$ is odd and an homeomorphism of E for any $t \in [0,1]$.*
(4) *$\eta(t, W) \subset W$ for any $t \in [0,1]$.*
(5) *$I(\eta(\cdot, u))$ is non-increasing.*
(6) *$\|\eta(t,u) - u\| \le \rho$ for any $(t,u) \in [0,1] \times E$.*

3. Proof of Theorem 1.1

To prove the result, we first need to construct a class of sets for Ljusternik-Schnirelman type minimax process. Set $R = R(E_m)$, where E_m is a m-dimensional subspace of E. Let

$$G_m := \{h \in C(B_R \cap E_m, E) : h \text{ is odd and } h = \text{id on } \partial B_R \cap E_m\}.$$

Observe that $\text{id} \in G_m$ for all $m \in \mathbb{N}$ so $G_m \neq \emptyset$. Define for all $k \ge 2$

$$\Gamma_k := \{h(\overline{B_R \cap E_m \backslash Y}) : h \in G_m, m \ge k, Y = -Y \text{ is close and } \gamma(Y) \le m-k\}.$$

Then, according to [13, Proposition 9.18], we have the following results:

(1) $\Gamma_k \neq \emptyset$ and $\Gamma_{k+1} \subset \Gamma_k$ for all $k \ge 2$;
(2) If $\phi \in C(E,E)$ is odd and $\phi = \text{id}$ on $\partial B_R \cap E_m$ for all $m \ge k$, then $\phi : \Gamma_k \to \Gamma_k$;
(3) If $B \in \Gamma_k$, $Z = -Z$ is close and $\gamma(Z) \le s < k$, then $\overline{B \backslash Z} \in \Gamma_{k-s}$.

Proof of Theorem 1.1. We split it into three steps.

Step 1. Assume ρ small such that $\partial B_\rho \subset \mathcal{O}$. Define for $k \ge 2$,

$$c_k := \inf_{B \in \Gamma_k} \sup_{u \in B \cap Q} I(u).$$

We first need to prove that c_k is well-defined for each $k \ge 2$. It suffices to show that for any $B \in \Gamma_k$, $B \cap Q \neq \emptyset$ and $c_k > -\infty$. To see it, we first consider the attracting domains of 0 in E:

$$\mathcal{O} := \{u \in E : \varphi(t,u) \to 0, \text{ as } t \to \infty\}.$$

Because 0 is a local minimum of the functional I, then we observe that $\mathcal{O}$ is open by the continuous dependence of ODE on initial data. Moreover, $\partial\mathcal{O}$ is an invariant set with respect to φ and $\overline{P_\varepsilon^+} \cap \overline{P_\varepsilon^-} \subset \mathcal{O}$ (see [2, Lemma 3.4]). In particular, there holds $I(u) > 0$ for $u \in \overline{P_\varepsilon^+} \cap \overline{P_\varepsilon^-} \backslash \{0\}$. We now claim that for any $B \in \Gamma_k$ with $k \ge 2$, it holds that

$$B \cap Q \cap \partial\mathcal{O} \neq \emptyset. \tag{3.1}$$

By the assumption of ρ above, then we have $\sup_{B \cap Q} I \ge \inf_{\partial\mathcal{O}} I \ge \inf_{\partial B_\rho} I \ge C > 0$. To see (3.1), take $B = h(\overline{B_R \cap E_m \backslash Y})$ with $\gamma(Y) \le m-k$ and $k \ge 2$. Define

$$\Theta := \{u \in B_R \cap E_m : h(u) \in \mathcal{O}\}.$$

Then note Θ is a bounded open symmetric set with $0 \in \Theta$ and $\overline{\Theta} \subset B_R \cap E_m$. Therefore, due to Borsuk-Ulam Theorem, there is $\gamma(\partial\Theta) = m$ and we conclude that $h(\partial\Theta) \subset \partial\mathcal{O}$ by the continuity of h. Consequently, $h(\partial\Theta\backslash Y) \subset B \cap \partial\mathcal{O}$ and

$$\gamma(B \cap \partial\mathcal{O}) \geq \gamma(h(\partial\Theta\backslash Y)) \geq \gamma(\partial\Theta\backslash Y) \geq \gamma(\partial\Theta) - \gamma(Y) \geq k$$

by [13, Proposition 7.5]. From $\overline{P_\varepsilon^+} \cap \overline{P_\varepsilon^-} \subset \mathcal{O}$, we have $\overline{P_\varepsilon^+} \cap \overline{P_\varepsilon^-} \cap \partial\mathcal{O} = \emptyset$. We define a continuous and odd function $\varphi : W \cap \partial\mathcal{O} \to S^0 := \{1, -1\}$ given by

$$\varphi(u) = \begin{cases} 1 & \text{if } u \in \overline{P_\varepsilon^+} \cap \partial\mathcal{O}; \\ -1 & \text{if } u \in \overline{P_\varepsilon^-} \cap \partial\mathcal{O}. \end{cases}$$

Then, according to the definition of genus γ in so-called Liusternik-Schnirelmann category theory, we can easily get $\gamma(W \cap \partial\mathcal{O}) = 1$. Hence for $k \geq 2$, we deduce that

$$\gamma(B \cap Q \cap \partial\mathcal{O}) \geq \gamma(B \cap \partial\mathcal{O}) - \gamma(W \cap \partial\mathcal{O}) \geq k - 1 \geq 1,$$

which implies that (3.1) holds. So for each $k \geq 2$, c_k is well-defined and increasing with respect to k by the item (1).

Step 2. Next we show that for every $k \geq 2$ equation (1.1) possesses some sign-changing solutions at level c_k. Claim first that $K_{c_k} \cap Q \neq \emptyset$ for every $k \geq 2$, which implies that there exist some sign-changing critical points u_k such that $I(u_k) = c_k$ and conclusion follows. To see the claim, arguing by contradiction, we may suppose $K_{c_k} \cap Q = \emptyset$ for some $k \geq 2$. From the foregoing discussions, we know that $c_k \geq C > 0$ for all $k \geq 2$. Owing to the deformation lemma above, there exist $\varepsilon > 0$ and $\eta \in C([0,1] \times E, E)$ such that $\eta(1, \cdot)$ is odd, $\eta(1, u) = u$ for $u \in I^{c_k - 2\varepsilon}$ and

$$\eta(1, I^{c_k+\varepsilon} \cup W) \subset I^{c_k-\varepsilon} \cup W. \tag{3.2}$$

Then, thanks to the definition of c_k, there is $B \in \Gamma_k$ such that $\sup_{B\cap Q} I \leq c_k + \varepsilon$. Setting $D = \eta(1, B)$, by (3.2), we know that $\sup_{D\cap Q} I \leq c_k - \varepsilon$. On the other hand, we can obtain $I(u) \leq c_k - 2\varepsilon$ for $u \in \partial B_R \cap E_m$ by choosing R small. Then, gain $D \in \Gamma_k$ by the item (2). Consequently, $c_k \leq c_k - \varepsilon$, this is absurd.

Step 3. We prove that $c_k \to \infty$ as $k \to \infty$. Indeed, we may assume that $c_k \to \overline{c} < \infty$ as $k \to \infty$. Because I satisfies Palais-Smale condition, $K_{\overline{c}} \neq \emptyset$ and is compact. Moreover, we note that $K_{\overline{c}}^q \neq \emptyset$. We take a sequence of sign-changing solutions $\{u_k\}$ to equation (1.1) with $I(u_k) = c_k$. By the Sobolev embedding inequality, we obtain $\|u_k^\pm\| \geq c > 0$. Since I satisfies the Palais-Smale condition and the mapping $u \mapsto u^\pm$ is continuous in E, up to a subsequence, the limit $\overline{u}$ of $\{u_k\}$ is still sign-changing and $\overline{u} \in K_{\overline{c}}^q$, where $u^\pm := \min\{\pm u \geq 0\}$.

Suppose $\gamma(K_{\overline{c}}^q) = s$. By Palais-Smale condition again and statement above, $K_{\overline{c}}^q$ is compact. And there exists a neighborhood N of $K_{\overline{c}}^q$ with $K_{\overline{c}}^q \subset N$ such that $\gamma(N) = s$, owing to the "continuous" property of the genus (cf. [13, Proposition 7.5]).

Then, by the deformation lemma again, there exist $\varepsilon > 0$ and $\eta \in C([0,1] \times E, E)$ such that $\eta(1, \cdot)$ is odd, $\eta(1, u) = u$ for $u \in I^{\overline{c}-2\varepsilon}$ and

$$\eta(1, I^{\overline{c}+\varepsilon} \cup W\backslash N) \subset I^{\overline{c}-\varepsilon} \cup W. \tag{3.3}$$

In view of the assumption that $c_k \to \overline{c}$ as $k \to \infty$ and monotonicity of c_k, $c_{k+s} \geq c_k \geq \overline{c} - \frac{1}{2}\varepsilon$ for k enough large. By virtue of the definition of c_{k+s}, we can find a $B \in \Gamma_{k+s}$ such that

$$I(u) \leq c_{k+s} + \varepsilon \leq \overline{c} + \varepsilon, \quad \text{for all } u \in B \cap Q.$$

Then this derives $B \subset I^{\bar{c}+\varepsilon} \cup W$, and by (3.3),

$$\eta(1, B\backslash N) \subset I^{\bar{c}-\varepsilon} \cup W.$$

Selecting R small such that $I(u) < \bar{c} - 2\varepsilon$ for all $u \in \partial B_R \cap E_m$, it follows that $\eta(1, B\backslash N) \in \Gamma_k$ and

$$c_k \leq \sup_{\eta(1,B\backslash N)\cap Q} I \leq \bar{c} - \varepsilon,$$

which is a contradiction with $c_k \geq \bar{c} - \frac{1}{2}\varepsilon$. Therefore it holds that $c_k \to \infty$ as $k \to \infty$.

From Step 2, we know that for any $k \geq 2$, equations (1.1) possesses some sign-changing solutions at level c_k. By arbitrariness of $k \geq 2$ and $c_k \to \infty$ as $k \to \infty$, we obtain that equations (1.1) possesses infinity many sign-changing solutions. The proof is complete. □

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References

[1] T. Bartsch, Z. L. Liu, T. Weth; *Nodal solutions of a p-Laplcain equations*, Proc. London Math. Soc. 91 (2005), 129-152.

[2] T. Bartsch, Z. L. Liu, T. Weth; *Sign changing solutions of superlinear Schrödinger equations*, Commun. Partial Differential Equations 29 (2004), 25-42.

[3] C. J. Batkam; *Multiple sign-changing solutions to a class of Kirchhoff type problems*, arXiv:1501.05733v1.

[4] M. Clapp, T. Weth; *Multiple solutions for the Brézis-Nirenberg problem*, Adv. Differential Equations 10 (2005), 4363-480.

[5] G. M. Figueiredo, R. G. Nascimento; *Existence of a nodal solution with minimal energy for a Kirchhoff equation*, Math. Nachr. 288 (2014), 48-60.

[6] Y. Huang, Z. Liu; *On a class of Kirchhoff type problems*, Arch. Math. 102 (2014), 127-139.

[7] J. Liu, X. Liu, Z.-Q. Wang; *Multiple mixed states of nodal solutions for nonlinear Schrödinger systems*, Calc. Var. Partial Differential Equations 52 (2015), 565-586.

[8] Z. Liu, Z.-Q. Wang, J. Zhang; *Infinitely many sign-changing solutions for the nonlinear Schrödinger-Poisson system*, Ann. Mat. Pura Appl. DOI 10.1007/s10231-015-0489-8.

[9] Z. Liu, F. A. van Heerden, Z.-Q. Wang; *Nodal type bound states of Schrödinger equations via invariant set and minimax methods*, J. Differential Equations 214 (2005), 358-390.

[10] A. Mao, S. Yuan; *Sign-changing solutions of a class of nonlocal quasilinear elliptic boundary value problems*, J. Math. Anal. Appl. 383 (2011), 239-243.

[11] A. Mao, Z. Zhang; *Sign-changing amd multiple solutinos of Kirchhoff type problem without the P.S. condition*, Nonlinear Anal. 70 (2009), 1275-1287.

[12] R. Palais; *Lusternik-Schnirelman theory on Banach manifolds*, Topology 5 (1966), 115-132.

[13] P. H. Rabinowitz; *Minimax methods in critical point theory with applications to differential equations*, CBMS Regional Conf. Ser. in Math., no. 65, Amer. Math. Soc.,Providence, R.I., 1986.

[14] W. Shuai; *Sign-changing solutions for a class of Kirchhoff-type problem in bounded domains*, J. Differential Equations 259 (2015), 1256-1274.

[15] J. Sun, S. Ma; *Infinitely many sign-changing solutions for the Brézis-Nirenberg problem,* Commun. Pure Appl. Anal. 13 (2014), 2317-2330.

[16] H. Ye; *The existence of least energy nodal solutions for some class of Kirchhoff equations and Choquard equations in* $\mathbb{R}^N$, J. Math. Anal.Appl. 431 (2015), 935-954.

[17] Z. Zhang, K. Perera; *Sign changing solutions of Kirchhoff type problems via invariant sets of descent flow*, J. Math. Anal. Appl. 317 (2006) 456-463.

[18] S. J. Li, Z.-Q. Wang; *Ljusternik-Schnirelman theory in partially ordered Hilbert spaces*, Trans. Am. Math. Soc. 354 (2002), 3207-3227.

4. Addendum posted on April 27, 2017

In response to a reader's suggestion, the authors want to introduce the additional conditions $p \in (4, 2^*)$ and $N = 1, 2, 3$ for Theorem 1.1. First we present a lemma whose proof is standard and is omitted here.

Lemma 4.1. *For each $m \geq 1$, there exists $R = R(E_m)$ such that*

$$\sup_{u \in B_R^c \cap E_m} I(u) < 0,$$

where $B_R^c = E \setminus B_R$.

Now we replace Theorem 1.1 with the following theorem.

Theorem 4.2. *Assume that $N = 1, 2, 3$. For each $p \in (4, 2^*)$, problem 1.1 has a sequence of sign-changing solutions $\{u_k\}$ such that $I(u_k) \to \infty$ as $k \to \infty$.*

Proof. **Step 1.** Assume that ρ is small such that $\partial B_\rho \subset \mathcal{O}$. For $k \geq 2$, we define

$$c_k := \inf_{B \in \Gamma_k} \sup_{u \in B \cap Q} I(u).$$

We first need to prove that c_k is well-defined. It suffices to show that for any $B \in \Gamma_k$, $B \cap Q \neq \emptyset$ and $c_k > -\infty$. To see this, we first consider the attracting domain of 0 in E:

$$\mathcal{O} := \{u \in B_{R/2} \cap E_m : \varphi(t, u) \to 0, \text{ as } t \to \infty\}.$$

Because 0 is a local minimum of the functional I, we observe that $\mathcal{O}$ is open, by the continuous dependence of ODE on initial data. Moreover, $\partial\mathcal{O}$ is an invariant set with respect to φ and $\overline{P_\varepsilon^+} \cap \overline{P_\varepsilon^-} \subset \mathcal{O}$. In particular, $I(u) > 0$ for $u \in \overline{P_\varepsilon^+} \cap \overline{P_\varepsilon^-} \setminus \{0\}$.

We now claim that for any $B \in \Gamma_k$ with $k \geq 2$, it holds

$$B \cap Q \cap \partial\mathcal{O} \neq \emptyset. \tag{4.1}$$

Since $\partial\mathcal{O} \subset B_R$, we have $\inf_{\partial\mathcal{O}} I > 0$ by Lemma 4.1. Then according to the selection of ρ, we can see $\inf_{\partial\mathcal{O}} I \geq \inf_{\partial B_\rho} I \geq C > 0$. Therefore,

$$\sup_{B \cap Q} I \geq \inf_{\partial\mathcal{O}} I \geq \inf_{\partial B_\rho} I \geq C > 0.$$

To obtain (4.1), we take $B = h(\overline{B_R \cap E_m \setminus Y})$ with $\gamma(Y) \leq m - k$ and $k \geq 2$. Define

$$\Theta := \{u \in B_R \cap E_m : h(u) \in \mathcal{O}\}.$$

Then note that Θ is a bounded, open and symmetric set with $0 \in \Theta$ and $\overline{\Theta} \subset B_R \cap E_m$. Therefore, by Borsuk-Ulam Theorem, $\gamma(\partial\Theta) = m$ and we conclude that $h(\partial\Theta) \subset \partial\mathcal{O}$ by the continuity of h. Consequently, $h(\partial\Theta \setminus Y) \subset B \cap \partial\mathcal{O}$ and

$$\gamma(B \cap \partial\mathcal{O}) \geq \gamma(h(\partial\Theta \setminus Y)) \geq \gamma(\partial\Theta \setminus Y) \geq \gamma(\partial\Theta) - \gamma(Y) \geq k.$$

From $\overline{P_\varepsilon^+} \cap \overline{P_\varepsilon^-} \subset \mathcal{O}$, we have $\overline{P_\varepsilon^+} \cap \overline{P_\varepsilon^-} \cap \partial\mathcal{O} = \emptyset$. We define a continuous and odd function $\varphi : W \cap \partial\mathcal{O} \to S^0 := \{1, -1\}$ by

$$\varphi(u) = \begin{cases} 1 & \text{if } u \in \overline{P_\varepsilon^+} \cap \partial\mathcal{O}, \\ -1 & \text{if } u \in \overline{P_\varepsilon^-} \cap \partial\mathcal{O}. \end{cases}$$

Then, according to the definition of genus γ in the so-called Liusternik-Schnirelmann category theory, we can easily get $\gamma(W \cap \partial\mathcal{O}) = 1$. Hence for $k \geq 2$, we deduce that

$$\gamma(B \cap Q \cap \partial\mathcal{O}) \geq \gamma(B \cap \partial\mathcal{O}) - \gamma(W \cap \partial\mathcal{O}) \geq k - 1 \geq 1,$$

which implies that (4.1) holds. So for each $k \geq 2$, c_k is well-defined and increasing with respect to k, by item (1) just above the proof of Theorem 1.1.

Step 2. Next we show that for every $k \geq 2$, the problem possesses sign-changing solutions at level c_k. First we claim that $K_{c_k} \cap Q \neq \emptyset$ for every $k \geq 2$, which implies that there exist some sign-changing critical points u_k such that $I(u_k) = c_k$ and the conclusion follows. To see the claim, arguing by contradiction, we assume that $K_{c_k} \cap Q = \emptyset$ for some $k \geq 2$. From the above discussions, we know that $c_k \geq C > 0$ for all $k \geq 2$. Owing to the deformation lemma above, there exist $\varepsilon > 0$ and $\eta \in C([0,1] \times E, E)$ such that $\eta(1,\cdot)$ is odd, $\eta(1,u) = u$ for $u \in I^{c_k - 2\varepsilon}$ and

$$\eta(1, I^{c_k+\varepsilon} \cup W) \subset I^{c_k-\varepsilon} \cup W. \tag{4.2}$$

Then, thanks to the definition of c_k, there is $B \in \Gamma_k$ such that $\sup_{B\cap Q} I \leq c_k + \varepsilon$. Setting $D = \eta(1,B)$, by (4.2), we know that $\sup_{D\cap Q} I \leq c_k - \varepsilon$. On the other hand, we can obtain $I(u) \leq c_k - 2\varepsilon$ for $u \in \partial B_R \cap E_m$ by Lemma 4.1 Then, gain $D \in \Gamma_k$ by item (2), just above the proof of Theorem 1.1. Consequently, $c_k \leq c_k - \varepsilon$, this is absurd.

Step 3. We prove that $c_k \to \infty$ as $k \to \infty$. Indeed, we may assume that $c_k \to \overline{c} < \infty$ as $k \to \infty$. Because I satisfies Palais-Smale condition, $K_{\overline{c}} \neq \emptyset$ and is compact. Moreover, we note that $K_{\overline{c}}^q \neq \emptyset$. We take a sequence of sign-changing solutions $\{u_k\}$ to the problem with $I(u_k) = c_k$. By the Sobolev embedding inequality, we obtain $\|u_k^{\pm}\| \geq c > 0$. Since I satisfies the Palais-Smale condition and the mapping $u \mapsto u^{\pm}$ is continuous on E, up to a subsequence, the limit $\overline{u}$ of $\{u_k\}$ is still sign-changing and $\overline{u} \in K_{\overline{c}}^q$, where $u^{\pm} := \min\{\pm u \geq 0\}$.

Suppose $\gamma(K_{\overline{c}}^q) = s$. By Palais-Smale condition again and the statement above, $K_{\overline{c}}^q$ is compact. And there exists a neighborhood N of $K_{\overline{c}}^q$ with $K_{\overline{c}}^q \subset N$ such that $\gamma(N) = s$, owing to the "continuous" property of the genus.

Then, by the deformation lemma again, there exist $\varepsilon > 0$ and $\eta \in C([0,1]\times E, E)$ such that $\eta(1,\cdot)$ is odd, $\eta(1,u) = u$ for $u \in I^{\overline{c}-2\varepsilon}$ and

$$\eta(1, I^{\overline{c}+\varepsilon} \cup W \backslash N) \subset I^{\overline{c}-\varepsilon} \cup W. \tag{4.3}$$

In view of the assumption that $c_k \to \overline{c}$ as $k \to \infty$ and monotonicity of c_k, $c_{k+s} \geq c_k \geq \overline{c} - \frac{1}{2}\varepsilon$ for k enough large. By the definition of c_{k+s}, we can find a $B \in \Gamma_{k+s}$ such that

$$I(u) \leq c_{k+s} + \varepsilon \leq \overline{c} + \varepsilon, \quad \text{for all } u \in B \cap Q.$$

Then this implies $B \subset I^{\overline{c}+\varepsilon} \cup W$, and by (3.3),

$$\eta(1, B\backslash N) \subset I^{\overline{c}-\varepsilon} \cup W.$$

From Lemma 4.1, it is easy to see $I(u) < \overline{c} - 2\varepsilon$ for all $u \in \partial B_R \cap E_m$, it follows that $\eta(1, B\backslash N) \in \Gamma_k$ and

$$c_k \leq \sup_{\eta(1,B\backslash N)\cap Q} I \leq \overline{c} - \varepsilon,$$

which contradicts $c_k \geq \overline{c} - \frac{1}{2}\varepsilon$. Therefore $c_k \to \infty$ as $k \to \infty$.

From Step 2, we know that for any $k \geq 2$, the problem possesses sign-changing solutions at level c_k. By the arbitrariness of $k \geq 2$ and $c_k \to \infty$ as $k \to \infty$, we obtain that the problem possesses infinity many sign-changing solutions. The proof is complete. □

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