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PSEUDO ALMOST PERIODIC SOLUTIONS FOR A LASOTA-WAZEWSKA MODEL

SAMIRA RIHANI, AMOR KESSAB, FAROUK CHÉRIF

Abstract. In this work, we consider a new model describing the survival of red blood cells in animals. Specifically, we study a class of Lasota-Wazewska equation with pseudo almost periodic varying environment and mixed delays. By using the Banach fixed point theorem and some inequality analysis, we find sufficient conditions for the existence, uniqueness and stability of solutions. We generalize some results known for one type of delay and for the Lasota-Wazewska model with almost periodic and periodic coefficients. An example illustrates the proposed model.

1. Introduction

In 1976 Wazewska and Lasota [26] proposed the delay logistic equation with one constant concentrated delay

$$N'(t) = -\mu N(t) + pe^{-rN(t-\tau)}$$

to describe the survival of red blood cells in an animal, where $N(t)$ denotes the number of red blood cells at time t, μ is the probability of death of a red blood cell p and r are positive constants related to the production of red blood cells per unit time and τ is the time required to produce a red blood cell. See also [16, 17].

Under some additional assumptions, Gopalsamy and Trofimchuk [13] obtained that the Lasota-Wazewska model with one discrete delay

$$x'(t) = -\alpha(t)x(t) + \beta(t)e^{-\nu x(t-\tau)}$$

has a globally attractive almost periodic solution. In [23], the existence the oscillations and the global attractivity of the unique positive periodic solution of the following equation

$$x'(t) = -\alpha(t)x(t) + \beta(t)e^{-ax(t-nT)}$$

were discussed. In particular, by applying Mawhin's continuation theorem of coincidence degree [12] several sufficient conditions were given ensuring the existence of the periodic solution. Here $a > 0, \alpha(\cdot), \beta(\cdot)$ are positive periodic functions of a fixed period T and n is a positive integer. The authors investigated several results regarding the oscillations and the global attractivity of existence of the periodic solution. Besides, in the work [15] by Huang et al the following delay differential

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equation with multiple time-varying delays and almost periodic coefficients was considered

$$x'(t) = -\alpha(t)x(t) + \sum_{j=1}^{m} \beta_j(t)e^{-\gamma_j(t)x(t-\tau_j(t))}.$$

The authors employed the contraction mapping principle to obtain a positive almost periodic solution.

Recently, Zhou et al [32] studied the problem of positive almost periodic solutions for the generalized Lasota-Wazewska model with infinite delays

$$x'(t) = -\alpha(t)x(t) + \sum_{j=1}^{m} a_j(t)e^{-\omega_j(t)\int_{-\infty}^{0} K_j(s)x(t+s)ds}.$$

Under proper assumptions, the authors obtained a unique positive almost periodic solution of the above model which is exponential stable by using the method of a fixed point theorem in cones. Hence, the stability analysis problem of the Lasota–Wazewska model with time delay has been attracted a large amount of research interest and many sufficient conditions have been proposed to guarantee the asymptotic or exponential stability for the equation with various type of time delays: one discrete or time-varying or distributed (see, for example, [16, 25, 24, 20, 19, 28]). As far as we know, in most published papers, the analysis of the Lasota-Wazewska model has been treated with only one kind of delays. Therefore, it is important and challenging to get some useful results with both multiple time-varying delays and distributed delays.

As we all know, many phenomena in nature have oscillatory character and their mathematical models have led to the introduction of certain classes of functions to describe them. Such a class form pseudo almost periodic functions which a natural generalization of the concept of almost periodicity (in Bochner's sense). These are functions on the real numbers set that can be represented uniquely in the form $f = h + \varphi$, where h (the principal term) is an almost periodic function and φ (the ergodic perturbation) a continuous function whose mean vanishes at infinity. For more on the concepts of almost periodicity and/or pseudo almost periodicity and related issues, we refer the reader to [10, 11, 18, 29, 30, 31].

The aim here is to study the existence, uniqueness and stability of a generalized Lasota-Wazewska model with pseudo almost periodic coefficients and with mixed delays. Roughly speaking, let us consider the following differential equation

$$\begin{aligned} x'(t) = -\alpha(t)x(t) &+ \sum_{j=1}^{m} a_j(t)e^{-\omega_j(t)\int_{-\infty}^{t} K_j(t-s)x(s)ds} \\ &+ \sum_{i=1}^{n} b_i(t)e^{-\beta_i(t)x(t-\tau_i)} \end{aligned} \tag{1.1}$$

where $t \in \mathbb{R}$. The method consists to reduce the existence of the unique solution for the Lasota-Wazewska model (1.1) to the search for the existence of the unique fixed point of an appropriate operator on the Banach space $PAP(\mathbb{R}, \mathbb{R})$.

Hence, the main purpose of this paper is to study the existence and the dynamics of the generalized Lasota-Wazewska model with mixed delays and pseudo almost periodic coefficients. However, to the author's best knowledge, there are no publications considering the pseudo almost periodic solutions for Lasota-Wazewska model with mixed delays. Furthermore the model discussed in this paper is more general than the one in [15, 16, 19, 20, 24, 25, 28, 32], since most of them study the Lasota-Wazewska model with almost periodic coefficients or one kind of delays.

The remainder of this paper is organized as follows: In Section 2, we will introduce some necessary notations, definitions and fundamental properties of the space $PAP(\mathbb{R},\mathbb{R}^+)$ which will be used in the paper. In Section 3, based on different methods and analysis techniques and provides several sufficient conditions ensuring the existence and uniqueness of the pseudo almost periodic solution for the considered system. Section 4 is devoted to the stability of the pseudo almost periodic solution. In section 5, based on suitable Lyapunov function and Dini derivative, we give some sufficient conditions to ensure that all solutions converge exponentially to the positive pseudo almost periodic solution of the equation (1.1). At last, an illustrative example is given.

2. Problem formulation and preliminaries

We introduce notations, definitions and theorems which are used throughout this paper. Let $BC(\mathbb{R},\mathbb{R})$ be the set of bounded continued functions from $\mathbb{R}$ to $\mathbb{R}$. Note that $(BC(\mathbb{R},\mathbb{R}),|\cdot|_\infty)$ is a Banach space where $|\cdot|_\infty$ denotes the sup norm

$$|f|_\infty := \sup_{t\in\mathbb{R}} |f(t)|.$$

Throughout this paper, given a bounded continuous function f defined on $\mathbb{R}$, let $\overline{f}$ and $\underline{f}$ be defined as

$$\overline{f}(t) = \sup_{t\in\mathbb{R}} f(t), \quad \underline{f}(t) = \inf_{t\in\mathbb{R}} f(t),$$

- (H1) The function $\alpha(\cdot)$ is almost periodic and for all $t\in\mathbb{R}$, $\alpha(t)\ge 0$.
- (H2) For all $1\le j\le m$ and $1\le i\le n$, the functions $a_j, b_i, \beta_i, \omega_j : \mathbb{R}\to\mathbb{R}^+$ are pseudo almost periodic.
- (H3) $r = \frac{\sum_{i=1}^n \overline{b_i\beta_i} + \sum_{j=1}^m \overline{a_j\omega_j}}{\underline{\alpha}} < 1$.
- (H4) For all $1\le j\le m$, the delay kernels $K_j : [0,+\infty)\to\mathbb{R}^+$ are continuous, integrable and
$$\int_0^\infty K_j(u) = 1, \quad \int_0^\infty K_j(u)e^{\lambda u}du < +\infty,$$
where λ is a sufficiently non negative small constant. Note that
$$\rho = \max_{1\le j\le m} \int_0^\infty K_j(u)e^{\lambda u}du.$$

Let $\mu = \max_{1\le j\le m}\tau_j$. Denote by $BC(]-\mu,0],\mathbb{R}^+)$ the set of bounded continuous functions from $]-\mu,0]$ to $\mathbb{R}^+$. Notice that we restrict our selves to $\mathbb{R}^+$-valued functions since only non-negative solutions of (2.1) are biologically meaningful. The initial condition associated with system (1.1) is of the form

$$x(s) = \varphi(s), \quad \varphi\in BC(]-\mu,0],\mathbb{R}^+). \tag{2.1}$$

Definition 2.1. A continuous function $f:\mathbb{R}\to\mathbb{R}$ is said to be *almost periodic* (Bohr a.p.) if for each $\epsilon>0$, the set

$$T(f,\epsilon) = \{\tau\in\mathbb{R}, |f(t+\tau)-f(\tau)|<\epsilon \text{ for all } t\in\mathbb{R}\}$$

is relatively dense in R. In other words, there exists $l_\epsilon>0$ such that every interval of length l_ϵ contains at least one point of $T(f,\epsilon)$.

The number τ above is called an ϵ-translation number of the function f and the collection of all such functions will be a Banach space under the sup norm which we denote $AP(\mathbb{R},\mathbb{R})$. We refer the reader to [2] and [6] for the basic theory of almost periodic functions and their applications. Define the class of functions $PAP_0(\mathbb{R},\mathbb{R})$ as follows:

$$\Big\{f \in BC(\mathbb{R},\mathbb{R}) : \lim_{T\to+\infty} \frac{1}{2T}\int_{-T}^{T} |f(t)|dt = 0\Big\}.$$

A function $f \in BC(\mathbb{R},\mathbb{R})$ is called pseudo almost periodic if it can be expressed as

$$f = h + \varphi,$$

where $h \in AP(\mathbb{R},\mathbb{R})$ and $\varphi \in PAP_0(\mathbb{R},\mathbb{R})$. The collection of such functions will be denoted by $PAP(\mathbb{R},\mathbb{R})$.

The functions h and φ in above definition are respectively called the almost periodic component and the ergodic perturbation of the pseudo almost periodic function f. The decomposition given in definition above is unique.

Remark 2.2. Observe that $(PAP(\mathbb{R},\mathbb{R}), |\cdot|_\infty)$ is a Banach space and $AP(\mathbb{R},\mathbb{R})$ is a proper subspace of $PAP(\mathbb{R},\mathbb{R})$ since the function $\phi(t) = \sin^2 \pi t + \sin^2 \sqrt{5}t + e^{-t^t \cos^2 t}$ is pseudo almost periodic function but not almost periodic [7].

3. Existence and uniqueness of pseudo almost periodic solution

As pointed out in the introduction, we shall give here sufficient conditions which ensures existence and uniqueness of pseudo almost periodic solution of (2.1). In order to prove this result, we will state the following lemmas.

Lemma 3.1. *For all $x(\cdot) \in PAP(\mathbb{R},\mathbb{R}^+)$, then the function $x(\cdot+\kappa) \in PAP(\mathbb{R},\mathbb{R}^+)$ for all $\kappa \in \mathbb{R}$.*

The proof of the above lemma can be done in same as in [29, 30, 31]

Lemma 3.2. *If $\varphi, \psi \in PAP(\mathbb{R},\mathbb{R}^+)$, then $\varphi \times \psi \in PAP(\mathbb{R},\mathbb{R}^+)$*

For a proof of the above lemma, see [29, 30, 31].

Lemma 3.3. *For all $x(\cdot) \in PAP(\mathbb{R},\mathbb{R}^+)$ and all $1 \le j \le m$, the function $\phi_j : t \mapsto e^{-\omega_j(t)\int_{-\infty}^{t} K_j(t-s)x(s)ds}$ belongs to $PAP(\mathbb{R},\mathbb{R}^+)$.*

Proof. First, by [4, theorem 1], the function

$$t \mapsto \int_{-\infty}^{t} K_j(t-s)x(s)ds$$

is pseudo almost periodic for all $1 \le j \le m$. So by lemma 3.2 the function

$$t \mapsto \omega_j(t)\int_{-\infty}^{t} K_j(t-s)x(s)ds$$

is also pseudo almost periodic for all $1 \le j \le m$. Also for all $x, y \in \mathbb{R}^+$ one has

$$|e^{-x} - e^{-y}| \le |x - y|.$$

Now, using the fact that the function $(x \mapsto e^{-x})$ is Lipschitzian and Lemma 3.1 and the composition theorem of pseudo-almost periodic functions [3], it is clear that the function

$$\phi_j : t \mapsto e^{-\omega_j(t)\int_{-\infty}^{t} K_j(t-s)x(s)ds}$$

belongs to $PAP(\mathbb{R},\mathbb{R}^+)$ whenever $x \in PAP(\mathbb{R},\mathbb{R}^+)$. $\square$

The same approach gives the following result.

Lemma 3.4. *For all $x(\cdot) \in PAP(\mathbb{R},\mathbb{R}^+)$, the function $\psi_i : t \mapsto e^{-\omega_j(t)x(t-\tau_i)}$ belongs to $PAP(\mathbb{R},\mathbb{R}^+)$ for all $1 \le i \le n$.*

Theorem 3.5. *Suppose that* (H1), (H2) *satisfied. Define the nonlinear operator Γ for each $x \in PAP(\mathbb{R},\mathbb{R}^+)$ by*

$$\begin{aligned}(\Gamma x)(t) = \int_{-\infty}^{t} e^{-\int_s^t \alpha(\xi)d\xi}\Big[\sum_{i=1}^{n} b_i(s)e^{-\beta_i(s)x(s-\tau_i)} \\ + \sum_{j=1}^{m} a_j(s)e^{-\omega_j(s)\int_{-\infty}^{s} K_j(s-\sigma)x(\sigma)d\sigma}\Big]ds\end{aligned}$$

Then Γ maps $PAP(\mathbb{R},\mathbb{R}^+)$ into itself.

Proof. First, let us check that Γ is well defined. Indeed, by Lemma 3.1, for all $\varphi(\cdot) \in PAP(\mathbb{R},\mathbb{R}^+)$ the function $T_h(x) = x(\cdot - h) \in PAP(\mathbb{R},\mathbb{R}^+)$ since $PAP(\mathbb{R},\mathbb{R}^+)$ is a translation invariant closed subspace of $BC(\mathbb{R},\mathbb{R}^+)$. Further, by the composition theorem of pseudo almost periodic functions (see for example [3]) $\xi \mapsto x(s+\xi)e^{-x(\xi+s)}$ is in $PAP(\mathbb{R},\mathbb{R}^+)$. So, the function

$$\chi(s) = \Big[\sum_{i=1}^{n} b_i(s)e^{-\beta_i(s)x(s-\tau_i)} + \sum_{j=1}^{m} a_j(s)e^{-\omega_j(s)\int_{-\infty}^{s} K_j(t-\sigma)x(\sigma)d\sigma}\Big]ds$$

belongs to $PAP(\mathbb{R},\mathbb{R}^+)$. Consequently we can write $\chi = \chi_1 + \chi_2$, where $\chi_1 \in AP(\mathbb{R},\mathbb{R}^+)$ and $\chi_2 \in PAP_0(\mathbb{R},\mathbb{R}^+)$. So, one can write

$$\begin{aligned}(\Gamma\chi)(t) :&= \int_{-\infty}^{t} e^{-\int_s^t \alpha(\xi)d\xi}\chi(s)ds \\ &= \int_{-\infty}^{t} e^{-\int_s^t \alpha(\xi)d\xi}\chi_1(s)ds + \int_{-\infty}^{t} e^{-\int_s^t \alpha(\xi)d\xi}\chi_2(s)ds \\ &= (\Gamma\chi_1)(t) + (\Gamma\chi_2)(t)\end{aligned}$$

Let us prove that $t \to (\Gamma\chi_1)(t) := \int_{-\infty}^{t} e^{-\int_s^t \alpha(\xi)d\xi}\chi_1(s)ds$ is almost periodic. Let us consider, in view of the almost periodicity of the functions α and χ_1, a number l_ϵ such that in any interval $[\delta, \delta + l_\epsilon]$ one finds a number h, such that

$$\sup_{\xi\in\mathbb{R}} |\alpha(\xi+h) - \alpha(\xi)| < \epsilon \quad \text{and} \quad \sup_{\xi\in\mathbb{R}} |\chi_1(\xi+h) - \chi_1(\xi)| < \epsilon.$$

$$\begin{aligned}&(\Gamma\chi_1)(t+h) - (\Gamma\chi_1)(t) \\ &= \int_{-\infty}^{t+h} e^{-\int_s^{t+h} \alpha(\xi)d\xi}\chi_1(s)ds - \int_{-\infty}^{t} e^{-\int_s^t \alpha(\xi)d\xi}\chi_1(s)ds \\ &= \int_{-\infty}^{t+h} e^{-\int_{s-h}^{t} \alpha(\xi+h)d\xi}\chi_1(s)ds - \int_{-\infty}^{t} e^{-\int_s^t \alpha(\xi)d\xi}\chi_1(s)ds \\ &= \int_{-\infty}^{t} e^{-\int_u^t \alpha(\xi+h)d\xi}\chi_1(u+h)du - \int_{-\infty}^{t} e^{-\int_s^t \alpha(\xi)d\xi}\chi_1(s)ds \\ &= \int_{-\infty}^{t} e^{-\int_s^t \alpha(\xi+h)d\xi}\chi_1(s+h)ds - \int_{-\infty}^{t} e^{-\int_s^t \alpha(\xi)d\xi}\chi_1(s+h)ds\end{aligned}$$

$$+\int_{-\infty}^{t} e^{-\int_s^t \alpha(\xi)d\xi}\chi_1(s+h)ds - \int_{-\infty}^{t} e^{-\int_s^t \alpha(\xi)d\xi}\chi_1(s)ds$$

So there exists $\theta \in]0,1[$ such that

$$\begin{aligned}
&|(\Gamma\chi_1)(t+h) - (\Gamma\chi_1)(t)| \\
&\le |\chi_1|_\infty \int_{-\infty}^{t} \big|e^{-\int_s^t \alpha(\xi+h)d\xi} - e^{-\int_s^t \alpha(\xi)d\xi}\big| ds \\
&\quad + \int_{-\infty}^{t} e^{-\int_s^t \alpha(\xi)d\xi}|\chi_1(s+h) - \chi_1(s)|ds \\
&\le |\chi_1|_\infty \int_{-\infty}^{t} \Big\{ e^{-\big[\int_s^t \alpha(\xi+h)d\xi + \theta(\int_s^t \alpha(\xi)d\xi - \int_s^t \alpha(\xi+h)d\xi)\big]} \\
&\quad \times \Big| \int_s^t \alpha(\xi+h)d\xi - \int_s^t \alpha(\xi)d\xi \Big| ds \Big\} + \epsilon \int_{-\infty}^{t} e^{-(t-s)\underline{\alpha}} ds \\
&\le |\chi_1|_\infty \int_{-\infty}^{t} \Big\{ e^{-\int_s^t \alpha(\xi+h)d\xi} e^{-\theta(\int_s^t \alpha(\xi)d\xi - \int_s^t \alpha(\xi+h)d\xi)} \Big| \int_s^t |\alpha(\xi+h) - \alpha(\xi)\big| d\xi | ds \Big\} \\
&\quad + \epsilon \int_{-\infty}^{t} e^{-(t-s)\underline{\alpha}} ds \\
&\le |\chi_1|_\infty \int_{-\infty}^{t} \big[e^{-(t-s)\underline{\alpha}} e^{-\theta\epsilon(t-s)} \epsilon(t-s) \big] ds + \epsilon \int_{-\infty}^{t} e^{-(t-s)\underline{\alpha}} ds \\
&\le \epsilon |\chi_1|_\infty \int_{-\infty}^{t} \big[\epsilon e^{-(t-s)\underline{\alpha}}(t-s) \big] ds + \epsilon \int_{-\infty}^{t} e^{-(t-s)\underline{\alpha}} ds \\
&\le \frac{\epsilon|\chi_1|_\infty}{\underline{\alpha}^2} + \epsilon \int_{-\infty}^{t} e^{-(t-s)\underline{\alpha}} ds \\
&\le \frac{\epsilon|\chi_1|_\infty}{\underline{\alpha}^2} + \frac{\epsilon}{\underline{\alpha}} = \Big(\frac{|\chi_1|_\infty}{\underline{\alpha}^2} + \frac{1}{\underline{\alpha}} \Big)\epsilon.
\end{aligned}$$

Consequently, the function $(\Gamma\chi_1)$ belongs to $AP(\mathbb{R},\mathbb{R}^+)$. Now, let us show that $(\Gamma\chi_2)$ belongs to $PAP_0(\mathbb{R},\mathbb{R}^+)$.

$$\begin{aligned}
&\lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} \Big| \int_{-\infty}^{t} e^{-\int_s^t \alpha(\xi)d\xi}\chi_2(s)ds \Big| dt \\
&\le \lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} \int_{-\infty}^{t} e^{-\int_s^t \alpha(\xi)d\xi}|\chi_2(s)|dsdt \\
&\le \lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} \Big(\int_{-\infty}^{t} e^{-(t-s)\underline{\alpha}}|\chi_2(s)|ds \Big) dt \\
&\le I_1 + I_2
\end{aligned}$$

where

$$\begin{gathered}
I_1 = \lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} \Big(\int_{-T}^{t} e^{-(t-s)\underline{\alpha}}|\chi_2(s)|ds \Big) dt, \\
I_2 = \lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} \Big(\int_{-\infty}^{-T} e^{-(t-s)\underline{\alpha}}|\chi_2(s)|ds \Big)\, dt.
\end{gathered}$$

Now, we shall prove that $I_1 = I_2 = 0$

$$\begin{aligned}
\frac{1}{2T}\int_{-T}^{T}\Big(\int_{-T}^{t}|e^{-(t-s)\underline{\alpha}}\chi_2(s)|ds\Big)dt &= \frac{1}{2T}\int_{-T}^{T}\Big(\int_{-T}^{t}e^{-(t-s)\underline{\alpha}}|\chi_2(s)|ds\Big)dt\\
&\le \frac{1}{2T}\int_{-T}^{T}\Big(\int_{0}^{+\infty}e^{-\underline{\alpha}\xi}|\chi_2(t-\xi)|d\xi\Big)dt\\
&= \int_{0}^{+\infty}e^{-\underline{\alpha}\xi}\Big(\frac{1}{2T}\int_{-T}^{T}|\chi_2(t-\xi)|dt\Big)d\xi\\
&\le \int_{0}^{+\infty}e^{-\underline{\alpha}\xi}\Big(\frac{1}{2T}\int_{-T-\xi}^{T-\xi}|\chi_2(u)|du\Big)d\xi\\
&\le \int_{0}^{+\infty}e^{-\underline{\alpha}\xi}\Big(\frac{1}{2T}\int_{-T-\xi}^{T+\xi}|\chi_2(u)|du\Big)d\xi
\end{aligned}$$

Since the function $\chi_2(\cdot) \in PAP_0(\mathbb{R}, \mathbb{R}^+)$, the function ϕ_T defined by

$$\phi_T(\xi) = \frac{T+\xi}{T}\frac{1}{2(T+\xi)}\int_{-T-\xi}^{T+\xi}|\chi_2(u)|du$$

is bounded and satisfy $\lim_{T\to+\infty}\phi_T(\xi) = 0$. Consequently, by the Lebesgue dominated convergence theorem, we obtain

$$I_1 = \lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}\Big(\int_{-T}^{t}|e^{-(t-s)\underline{\alpha}}\chi_2(s)|ds\Big)dt = 0.$$

On the other hand, notice that $|\chi_2|_\infty = \sup_{t\in\mathbb{R}}|\chi_2(t)| < \infty$, then

$$\begin{aligned}
I_2 &= \lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}\Big(\int_{-\infty}^{-T}|e^{-(t-s)\underline{\alpha}}\chi_2(s)|ds\Big)dt\\
&= \lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}\Big(\int_{-\infty}^{-T}e^{-(t-s)\underline{\alpha}}|\chi_2(s)|ds\Big)dt\\
&\le \lim_{T\to+\infty}\frac{\sup_{t\in\mathbb{R}}|\chi_2(t)|}{2T}\int_{-T}^{T}\Big(\int_{t+T}^{+\infty}e^{-\underline{\alpha}\xi}d\xi\Big)dt\\
&= \lim_{T\to+\infty}\frac{\sup_{t\in\mathbb{R}}|\chi_2(t)|}{2T}\frac{1}{\underline{\alpha}}e^{-\underline{\alpha}T}\int_{-T}^{T}e^{-\underline{\alpha}\,t}dt\\
&= \lim_{T\to+\infty}\frac{\sup_{t\in\mathbb{R}}|\chi_2(t)|}{2T}\frac{1}{\underline{\alpha}^2}e^{-\underline{\alpha}T}[-e^{-\underline{\alpha}T}+e^{\underline{\alpha}T}]\\
&\le \lim_{T\to+\infty}\frac{\sup_{t\in\mathbb{R}}|\chi_2(t)|}{2T}\frac{1}{\underline{\alpha}^2}[1-e^{-2\underline{\alpha}T}] = 0
\end{aligned}$$

Consequently, $(\Gamma\chi_2)$ belongs to $PAP_0(\mathbb{R}, \mathbb{R}^+)$. □

Theorem 3.6. *Suppose that* (H1)–(H4) *hold then the Lasota-Wazewska model with mixed delays* (1.1) *possess a unique pseudo almost periodic solution in the region*

$$\mathcal{B} = \{\psi \in PAP(\mathbb{R}, \mathbb{R}^+), R_1 \le |\psi| \le R_2\},$$

where

$$R_2 = \frac{\sum_{i=1}^{n}\overline{b_i} + \sum_{j=1}^{m}\overline{a_j}}{\underline{\alpha}},$$

$$R_1 = \frac{\sum_{i=1}^n \underline{b_i} e^{-\overline{\beta_i} R_2} + \sum_{j=1}^m \underline{a_j} e^{-\overline{\omega_j} R_2}}{\overline{\alpha}}.$$

Proof. First, let us prove that the operator Γ is a mapping from $\mathcal{B}$ to $\mathcal{B}$. In fact,

$$\begin{aligned}
|(\Gamma x)(t)| &= \int_{-\infty}^t e^{-\int_s^t \alpha(\xi)d\xi} \Big[\sum_{i=1}^n b_i(s) e^{-\beta_i(s) x(s-\tau_i)} \\
&\quad + \sum_{j=1}^m a_j(s) e^{-\omega_j(s) \int_{-\infty}^s K_j(t-\sigma)x(\sigma)d\sigma} \Big] ds \\
&\le \int_{-\infty}^t e^{-\int_s^t \alpha(\xi)d\xi} \Big(\sum_{i=1}^n b_i(s) + \sum_{j=1}^m a_j(s) \Big) ds \\
&\le \frac{\sum_{i=1}^n \overline{b_i} + \sum_{j=1}^m \overline{a_j}}{\underline{\alpha}}
\end{aligned}$$

and

$$\begin{aligned}
|(\Gamma x)(t)| &= \int_{-\infty}^t e^{-\int_s^t \alpha(\xi)d\xi} \Big[\sum_{i=1}^n b_i(s) e^{-\beta_i(s) x(s-\tau_i)} \\
&\quad + \sum_{j=1}^m a_j(s) e^{-\omega_j(s) \int_{-\infty}^s K_j(t-\sigma)x(\sigma)d\sigma} \Big] ds \\
&\ge \int_{-\infty}^t e^{-\int_s^t \alpha(\xi)d\xi} \Big(\sum_{i=1}^n \underline{b_i} e^{-\overline{\beta_i} R_2} + \sum_{j=1}^m \underline{a_j} e^{-\overline{\omega_j} R_2} \Big) ds \\
&\ge \frac{\sum_{i=1}^n \underline{b_i} e^{-\overline{\beta_i} R_2} + \sum_{j=1}^m \underline{a_j} e^{-\overline{\omega_j} R_2}}{\overline{\alpha}},
\end{aligned}$$

which implies that the operator Γ is a mapping from $\mathcal{B}$ to $\mathcal{B}$. To end the proof it suffice to prove that Γ is a contraction mapping. Obviously, for $u, v \in [0, +\infty[$

$$|e^{-u} - e^{-v}| < |u - v|$$

Let $x, y \in \mathcal{B}$. Then

$$\begin{aligned}
&|(\Gamma x)(t) - (\Gamma y)(t)| \\
&= \Big| \int_{-\infty}^t e^{-\int_s^t \alpha(\xi)d\xi} \Big[\sum_{i=1}^n b_i(s) e^{-\beta_i(s) x(s-\tau_i)} \\
&\quad + \sum_{j=1}^m a_j(s) e^{-\omega_j(s) \int_{-\infty}^s K_j(s-\sigma)x(\sigma)d\sigma} \Big] ds \\
&\quad - \int_{-\infty}^t e^{-\int_s^t \alpha(\xi)d\xi} \Big[\sum_{i=1}^n b_i(s) e^{-\beta_i(s) y(s-\tau_i)} \\
&\quad - \sum_{j=1}^m a_j(s) e^{-\omega_j(s) \int_{-\infty}^s K_j(s-\sigma)x(\sigma)d\sigma} \Big] ds \Big| \\
&\le \sup_{t\in\mathbb{R}} \int_{-\infty}^t e^{-\int_s^t \alpha(\xi)d\xi} \Big[\sum_{i=1}^n |b_i(s)| \big| e^{-\beta_i(s) x(s-\tau_i)} - e^{-\beta_i(s) y(s-\tau_i)} \big|
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{j=1}^{m}|a_j(s)|\Big|e^{-\omega_j(s)\int_{-\infty}^{s}K_j(s-\sigma)x(\sigma)d\sigma}-e^{-\omega_j(s)\int_{-\infty}^{s}K_j(s-\sigma)y(\sigma)d\sigma}\Big|\Big]\\
&\le \sup_{t\in\mathbb{R}}\int_{-\infty}^{t}e^{-(t-s)\underline{\alpha}}\Big[\sum_{i=1}^{n}|b_i(s)|\|\beta_i(s)\||x-y|_\infty\\
&+\sum_{j=1}^{m}|a_j(s)||\omega_j(s)||\int_{-\infty}^{s}K_j(s-\sigma)(x(\sigma)-y(\sigma))d\sigma|\Big]ds\\
&\le \sup_{t\in\mathbb{R}}\int_{-\infty}^{t}e^{-(t-s)\underline{\alpha}}\Big[\sum_{i=1}^{n}|b_i(s)|\|\beta_i(s)\|+\sum_{j=1}^{m}|a_j(s)|\,|\omega_j(s)|\Big]ds|x-y|_\infty\\
&\le \Big[\frac{\sum_{i=1}^{n}\overline{b_i\beta_i}+\sum_{j=1}^{m}\overline{a_j\omega_j}}{\underline{\alpha}}\Big]|x-y|_\infty
\end{aligned}
$$

which implies that the mapping Γ is a contraction mapping of $\mathcal{B}$. Consequently, Γ possess a unique fixed point $x^* \in \mathcal{B}$ that is $\Gamma(x^*) = x^*$. Hence, x^* is the unique pseudo almost periodic solution of (1.1) in $\mathcal{B}$. □

4. Global attractivity of the pseudo almost periodic solution

Let $x^*(\cdot)$ the pseudo almost periodic solution in Theorem 3.6 and $x(\cdot)$ be an arbitrary solution of (1.1). So, one has

$$
\begin{aligned}
x^{*\prime}(t) &= -\alpha(t)x^*(t)+\sum_{j=1}^{m}a_j(t)e^{-\omega_j(t)\int_{-\infty}^{t}K_j(t-s)x^*(s)ds}\\
&\quad+\sum_{i=1}^{n}b_i(t)e^{-\beta_i(t)x^*(t-\tau_i)}
\end{aligned}
\tag{4.1}
$$

and

$$
x'(t) = -\alpha(t)x(t)+\sum_{j=1}^{m}a_j(t)e^{-\omega_j(t)\int_{-\infty}^{t}K_j(t-s)x(s)ds}+\sum_{i=1}^{n}b_i(t)e^{-\beta_i(t)x(t-\tau_i)}
$$

Let us set, $z(\cdot) = x(\cdot) - x^*(\cdot)$. Consequently, we obtain

$$
\begin{aligned}
z'(t) &= -\alpha(t)z(t)+\sum_{i=1}^{n}b_i(t)[e^{-\beta_i(t)x(t-\tau_i)}-e^{-\beta_i(t)x^*(t-\tau_i)}]\\
&\quad+\sum_{j=1}^{m}a_j(t)\big[e^{-\omega_j(t)\int_{-\infty}^{t}K_j(t-s)x(s)ds}-e^{-\omega_j(t)\int_{-\infty}^{t}K_j(t-s)x^*(s)ds}\big]
\end{aligned}
\tag{4.2}
$$

Clearly, the pseudo almost periodic solution $x^*(\cdot)$ of system (1.1) is global attractivity if and only if the equilibrium point O of system (4.2) is global attractive. So let us study the global attractivity of the equilibrium point O for system (4.2).

Theorem 4.1. *Suppose that assumptions* (H1)–(H4) *hold. Then the equilibrium point O of the nonlinear system* (4.2) *is global attractive.*

Proof. First, let us prove that the solution of system (4.2) are uniformly bounded. In other words, there exists $M > 0$ such that for all $t \ge 0$ one has $|z(t)| \le M$. By assumption (H3), $1-r>0$. So for any given continuous function $\theta(\cdot)$, there exists a large number $M>0$, such that

$$|\theta|<M \quad \text{and} \quad (1-r)M>0.$$

Let κ a real number, $\kappa<1$. We shall prove that for all $t\geq 0$, $|z(t)|\leq \kappa M$. Suppose the contrary, then there must be some $t'>0$, such that

$$|z(t')|=\kappa M$$
$$|z(t)|<\kappa M, \quad 0\leq t\leq t'$$

In view of (H3), (H4) and the equation (1.1), we have

$$\begin{aligned}
|z(t')| &\leq \Big\{|\theta(0)|e^{-\int_0^{t'}\alpha(u)du}+\int_0^{t'} e^{-\int_s^{t'}\alpha(u)du}\Big(\sum_{i=1}^n |b_i(s)||\beta_i||z|_\infty \\
&\quad +\sum_{j=1}^m |a_j(s)||\omega_j||z|_\infty\Big)ds\Big\} \\
&\leq |\theta(0)|e^{-\underline{\alpha}t'}+|z(s)|_\infty\int_0^{t'} e^{-(t'-s)\underline{\alpha}}\Big(\sum_{i=1}^n \overline{b_i\beta_i}+\sum_{j=1}^m \overline{a_j\omega_j}\Big)\,ds \\
&\leq \kappa M\int_0^{t'} e^{-(t'-s)\underline{\alpha}}\Big(\sum_{i=1}^n \overline{b_i\beta_i}+\sum_{j=1}^m \overline{a_j\omega_j}\Big)ds+\kappa M e^{-\underline{\alpha}t'} \\
&\leq \kappa M\Big\{e^{-\underline{\alpha}t'}+\frac{1}{\underline{\alpha}}\Big[\sum_{i=1}^n \overline{b_i\beta_i}+\sum_{j=1}^m \overline{a_j\omega_j}\Big](1-e^{-\underline{\alpha}t'})\Big\} \\
&\leq \kappa M\Big\{e^{-\underline{\alpha}t'}+\frac{1}{\underline{\alpha}}\Big[\sum_{i=1}^n \overline{b_i\beta_i}+\sum_{j=1}^m \overline{a_j\omega_j}\Big]\Big\} \\
&< \kappa M,
\end{aligned}$$

which gives a contradiction. Consequently, for all $t\geq 0$, $|z(t)|\leq \kappa M$. Let us take $\kappa\to 1$, then for all $t\geq 0$, $|z(t)|\leq M$. Thus, there is a constant $\beta\geq 0$, such that

$$\limsup_{t\to+\infty}|z(t)|=\beta.$$

It follows that

$$\forall\epsilon>0, \exists t_2<0, \forall t,\ (t\geq t_2 \Rightarrow |z(t)|\leq (1+\epsilon)\beta).$$

$$\begin{aligned}
\dot{z}(t)+\alpha(t)z(t) &= \sum_{i=1}^n b_i(t)\big[e^{-\beta_i(t)x(t-\tau_i)}-e^{-\beta_i(t)x^*(t-\tau_i)}\big] \\
&\quad +\sum_{j=1}^m a_j(t)\big[e^{-\omega_j(t)\int_{-\infty}^t K_j(t-s)x(s)ds}-e^{-\omega_j(t)\int_{-\infty}^t K_j(t-s)x^*(s)ds}\big] \\
&\leq \sum_{i=1}^n |\beta_i(t)|\overline{b_i}|z(t-\tau_i)|+\sum_{j=1}^m \overline{a_j\omega_j}|z(t)|_\infty \\
&\leq \Big(\sum_{i=1}^n \overline{\beta_i b_i}+\sum_{j=1}^m \overline{a_j\omega_j}\Big)|z(t)|_\infty
\end{aligned}$$

$$\le \Big(\sum_{i=1}^{n}\overline{\beta_i b_i}+\sum_{j=1}^{m}\overline{a_j\omega_j}\Big)(1+\epsilon)\beta.$$

So, through integration, we obtain the inequality

$$\begin{aligned}
&|z(t)|\\
&\le \Big\{\Big(\sum_{i=1}^{n}\overline{\beta_i b_i}+\sum_{j=1}^{m}\overline{a_j\omega_j}\Big)(1+\epsilon)\beta\Big\}\int_0^t e^{-\int_s^t \alpha(u)du}ds+|\theta(0)|e^{-\int_0^t \alpha(u)du}\\
&\le \Big\{\Big(\sum_{i=1}^{n}\overline{\beta_i b_i}+\sum_{j=1}^{m}\overline{a_j\omega_j}\Big)(1+\epsilon)\beta\Big\}\int_0^t e^{-\underline{\alpha}(t-s)}ds+|\theta|_\infty e^{-\underline{\alpha}t}\\
&\le |\theta|_\infty e^{-\underline{\alpha}t}+\Big(\frac{\sum_{i=1}^{n}\overline{\beta_i b_i}+\sum_{j=1}^{m}\overline{a_j\omega_j}}{\underline{\alpha_i}}\Big)(1+\epsilon)\beta(1-e^{-\underline{\alpha}t}).
\end{aligned}$$

Hence,

$$|z(t)|\le \max_{1\le i\le n}\Big[|\theta|_\infty e^{-\underline{\alpha}t}+\Big(\frac{\sum_{i=1}^{n}\overline{\beta_i b_i}+\sum_{j=1}^{m}\overline{a_j\omega_j}}{\underline{\alpha_i}}\Big)(1+\epsilon)\beta(1-e^{-\underline{\alpha}t})\Big].$$

In particular, by passing to the limit superior we obtain

$$\limsup_{t\to+\infty}|z(t)|\le[r(1+\epsilon)\beta]$$

In other words, $\beta\le r(1+\epsilon)\beta$ Passing to limit when $\epsilon\to 0$, we obtain

$$\beta(1-r)\le 0$$

By condition (H4), we obtain $\beta=0$ which imply that

$$\lim_{t\to+\infty}|z(t)|=\lim_{t\to+\infty}|x(t)-x^*(t)|=0$$

and consequently the proof complete. $\square$

5. Exponential stability of the pseudo almost periodic solution

Next, we give some sufficient conditions to ensure that all solutions converge exponentially to the positive pseudo almost periodic solution x^* of the equation (1.1).

Definition 5.1 ([14])**.** Let $V:\mathbb{R}\to\mathbb{R}$ be a continuous function. Then

$$\frac{D^+V(t)}{dt}=\limsup_{h\to 0^+}\frac{V(t+h)-V(t)}{h}$$

Remark 5.2. The upper-right Dini derivative of $|V(t)|$ is

$$\frac{D^+V|y(t)|}{dt}=\operatorname{sign}(V(t))\frac{dV(t)}{dt}$$

where $\operatorname{sign}(\cdot)$ is the signum function.

Theorem 5.3. *Let*

$$\underline{\alpha}-e^{\lambda\mu}\sum_{i=1}^{n}\overline{b_i\beta_i}-\rho\sum_{j=1}^{m}\overline{a_j\omega_j}>0. \tag{5.1}$$

Suppose that all conditions of Theorem 3.6 are satisfied. Then (1.1) *has exactly one pseudo almost periodic solution x^* in $\mathcal{B}$. Moreover, $x^*(\cdot)$ is locally exponentially stable, the domain of attraction of $x^*(\cdot)$ is the set*

$$\mathcal{D}(x^*) = \big\{\varphi \in BC([-\mu, 0], \mathbb{R}),\ |\varphi - x^*|_1 := \sup_{-\mu \le s \le 0} |\varphi(s) - x^*_\mu(s)| < 1\big\}.$$

Namely, there exists a constant $\lambda > 0$ and $M > 1$ such that for any solution $x(\cdot)$ of (1.1) *in $\mathcal{B}$ with initial value $\varphi \in \mathcal{D}(x^*)$ and for all $t > 0$ we have*

$$|x(t) - x^*(t)| \le M \sup_{-\mu \le s \le 0} |\varphi(s) - x^*_\mu(s)| e^{-\lambda t},$$

where $x^*_\mu(s) = x^*(s)$ for all $s \in [-\mu, 0]$.

Proof. From Theorem 3.6, system (1.1) has exactly one pseudo almost periodic solution $x^* \cdot \mathcal{B}$. Let $x(\cdot)$ be an arbitrary solution of (1.1) with initial value φ. Let $y(\cdot) = x(t) - x^*(t)$, then

$$\begin{aligned} y'(t) &= \frac{d(x(t) - x^*(t))}{dt} \\ &= -\alpha(t)(x(t) - x^*(t)) + \sum_{i=1}^n b_i(t)\big[e^{-\beta_i(t)x(t-\tau_i)} - e^{-\beta_i(t)x^*(t-\tau_i)}\big] \\ &\quad + \sum_{j=1}^m a_j(t)\big[e^{-\omega_j(t)\int_{-\infty}^t K_j(t-s)x(s)ds} - e^{-\omega_j(t)\int_{-\infty}^t K_j(t-s)x^*(s)ds}\big]. \end{aligned} \tag{5.2}$$

Define a continuous function g by setting

$$g_\rho(\xi) = -(\underline{\alpha} - \xi) + e^{\lambda\mu} \sum_{i=1}^n \overline{b_i\beta_i} + \rho \sum_{j=1}^m \overline{a_j\omega_j},\ \xi \in [0, 1]$$

By (H5) one has $g_\rho(0) < 0$ which implies that we can choose a positive constant $\lambda \in]0, 1]$ such that

$$g_\rho(\lambda) = -(\underline{\alpha} - \lambda) + e^{\lambda\mu} \sum_{i=1}^n \overline{b_i\beta_i} + \rho \sum_{j=1}^m \overline{a_j\omega_j} < 0$$

We consider the Lyapunov functional $V : \mathbb{R} \to BC(\mathbb{R}, \mathbb{R}^+)$

$$V(t)t = y(t)e^{\lambda t} = |x(t) - x^*(t)|e^{\lambda t}$$

Let us calculate the upper right Dini derivative D^+V of V along the solution of the equation (5.2) with the initial value $\widetilde{\varphi} = \varphi - x^*_\mu$. Then for all $t > t_0$,

$$\begin{aligned} D^+V(t) &\le -\alpha(t)|y(t)|e^{\lambda t} + \lambda|y(t)|e^{\lambda t} + \sum_{i=1}^n b_i(t)|e^{-\beta_i(t)x(t-\tau_i)} - e^{-\beta_i(t)x^*(t-\tau_i)}|e^{\lambda t} \\ &\quad + \sum_{j=1}^m a_j(t)\big|e^{-\omega_j(t)\int_{-\infty}^t K_j(t-s)x(s)ds} - e^{-\omega_j(t)\int_{-\infty}^t K_j(t-s)x^*(s)ds}\big|e^{\lambda t} \\ &\le (-\alpha(t) + \lambda)|z(t)|e^{\lambda t} + \sum_{i=1}^n b_i(t)|e^{-\beta_i(t)x(t-\tau_i)} - e^{-\beta_i(t)x^*(t-\tau_i)}|e^{\lambda t} \\ &\quad + \sum_{j=1}^m a_j(t)\big|e^{-\omega_j(t)\int_{-\infty}^t K_j(t-s)x(s)ds} - e^{-\omega_j(t)\int_{-\infty}^t K_j(t-s)x^*(s)ds}\big|e^{\lambda t} \end{aligned}$$

Set

$$|\varphi - x^*|_1 = \sup_{-\mu \le s \le 0} |\varphi(s) - x^*_\mu(s)| > 0.$$

Since $|\varphi - x^*|_1 < 1$, one can choose a positive constant $M > 1$ such that

$$M|\varphi - x^*|_1 < 1,$$

consequently,

$$(M|\varphi - x^*|_1)^2 < M|\varphi - x^*|_1$$

It follows from the definition of the Lyapunov function that for all $t \in [-\mu, 0]$,

$$V(t) = |y(t)|e^{\lambda t} < M|\varphi - x^*|_1.$$

Let us prove that for all $t > 0$

$$V(t) = |y(t)|e^{\lambda t} < M|\varphi - x^*|_1.$$

We shall give a proof by contradiction. Suppose the contrary. There exists $t' > 0$ such that

$$V(t') = M|\varphi - x^*|_1$$
$$V(t) < M|\varphi - x^*|_1, \quad -\infty < t < t'$$

Consequently, one can write

$$\begin{aligned}
0 &\le D^+(V(t') - M|\varphi - x^*|_1) = D^+(V(t')) \\
&\le (-\alpha(t') + \lambda)|y(t')|e^{\lambda t'} + \sum_{i=1}^n b_i(t')|e^{-\beta_i(t')x(t'-\tau_i)} - e^{-\beta_i(t')x^*(t'-\tau_i)}|e^{\lambda t'} \\
&\quad + \sum_{j=1}^m a_j(t')\Big|e^{-\omega_j(t')\int_{-\infty}^{t'} K_j(t'-s)x(s)ds} - e^{-\omega_j(t')\int_{-\infty}^{t'} K_j(t'-s)x^*(s)ds}\Big|e^{\lambda t'} \\
&\le (-\alpha(t') + \lambda)|y(t')|e^{\lambda t'} + e^{\lambda\tau_i}\sum_{i=1}^n \overline{b_i\beta_i}|y(t'-\tau_i)|e^{\lambda(t'-\tau_i)} \\
&\quad + \sum_{j=1}^m \overline{a_j\omega_j}|\int_{-\infty}^0 K_j(s)|y(t'+s)|e^{\lambda(s+t')}e^{-\lambda s}ds| \\
&\le (-\underline{\alpha} + \lambda)V(t') + e^{\lambda\mu}\sum_{i=1}^n \overline{b_i\beta_i}V(t'-\tau_i) \\
&\quad + \sum_{j=1}^m \overline{a_j\omega_j}|\int_{-\infty}^0 K_j(s)|V(t'+s)|e^{-\lambda s}ds| \\
&\le (-\underline{\alpha} + \lambda)V(t') + Me^{\lambda\tau_i}\sum_{i=1}^n \overline{b_i\beta_i} + M\rho\sum_{j=1}^m \overline{a_j\omega_j} \\
&= \Big((-\underline{\alpha} + \lambda) + e^{\lambda\mu}\sum_{i=1}^n \overline{b_i\beta_i} + \rho\sum_{j=1}^m \overline{a_j\omega_j}\Big)M|\varphi - x^*|_1
\end{aligned}$$

Thus, we obtain

$$(-\underline{\alpha} + \lambda) + e^{\lambda\mu}\sum_{i=1}^n \overline{b_i\beta_i} + \rho\sum_{j=1}^m \overline{a_j\omega_j} > 0$$

which contradicts (H5) that for all $t > 0$,

$$V(t) = |y(t)|e^{\lambda t} < M|\varphi - x^*|_1.$$

and consequently for all $t > 0$ we have

$$|x(t) - x^*(t)| \le M \sup_{-\mu \le s \le 0} |\varphi(s) - x^*(s)|e^{-\lambda t}.$$

□

6. Discussions and applications

The most universal methods to periodic Lasota-Wazewska models with or without impulsives are Mawhin's continuous theorem [12].

Until now, most articles investigated Lasota-Wazewska model with the almost periodically varying coefficients and constant delay by using some well known fixed point theorems. There are rarely articles considering Lasota-Wazewska model with varying delays. Nevertheless, the use of a time-dependent delay has some constraints, in particular, for Lasota-Wazewska model since the mapping constructed in the proof may be not self-mapping. The main difficulty, is that if $f(\cdot)$ is pseudo almost periodic functions then the function $g(\cdot - f(\cdot))$ may be not an pseudo almost periodic.

Now, let us compare our results with previous works. When we let

$$a_j(\cdot) = 0 \text{ and } \tau_j = \tau_j(t) \quad \text{for all } 1 \le j \le m,$$

the model (1.1) is the one investigated in [15] and recently by Wang et al [25]. Also, when for all $1 \le i \le n$, $b_i(\cdot) = 0$ system (1.1) can be reduced to the model of the recent paper by [32]. Stamov [24] also analyzed the existence and uniqueness of almost periodic solution for impulsive Lasota-Wazewska model with only one constant delay. Hence, our results can be see as a generalization and improvement of [15, 25, 32] since in the cited papers the authors considered the periodic case and the almost periodic case. Further, to our best knowledge, there are no publications considering the pseudo almost periodic solutions for Lasota-Wazewska model. Notice that the pseudo almost periodicity is without importance in the proof of the above theorems; in particular Theorems 4.1 and 5.3. because of the difference in the methods discussed, the results in this paper and those in the above references are different. In this paper, the delays τ_j, $1 \le j \le m$ are constant functions.

The main advantages of the present work include:

- (i) it deals with pseudo almost periodic functions which contains strictly the set of almost periodic functions;
- (ii) it considers both infinite delays [32] and multiple time-varying delays [15].

Let us remark that our analysis is still applied without difficulty to the space of pseudo almost automorphic functions. Consequently, one can establish easily the analogue of the main results of this paper (Theorems 3.5 and 3.6). Notice that pseudo almost automorphic functions [21, 27] arise particularly in the study of the long-term behavior of solutions of evolution equations. These are functions on the real numbers set that can be represented uniquely in the form $f = h + \varphi$, where h (the principal term) is an almost automorphic function and φ is the ergodic perturbation.

It should be mentioned that, several discrete Lasota-Wazewska models have been studied by many authors, see [5, 22].

In order to illustrate some feature of our main results, we will apply them to some special systems and demonstrate the efficiencies of our criteria.

Example. Let us consider the following Lasota-Wazewska model with pseudo almost periodic coefficients and mixed delays

$$x'(t) = -\alpha(t)x(t) + \sum_{j=1}^{3} a_j(t)e^{-\omega_j(t)\int_{-\infty}^{t} K_j(t-s)x(s)ds} + \sum_{i=1}^{3} b_i(t)e^{-\beta_i(t)x(t-\tau_i)} \tag{6.1}$$

where $\alpha(t) = 8 + \cos^2\sqrt{5}t + \cos^2 t$,

$$\begin{pmatrix} a_1(t)\\ a_2(t)\\ a_3(t)\end{pmatrix} = \begin{pmatrix} 1 + 0.25\cos^2\sqrt{2}t + 0.25\cos^2\pi t + \frac{0.5}{1+t^2}\\ 0.5 + 0.25\cos^2\sqrt{3}t + 0.25\cos^2\pi t + \frac{1}{1+t^2}\\ 0.5 + 0.25\cos^2\sqrt{5}t + 0.25\cos^2\sqrt{2}t + e^{-t^2\cos^2 t}\end{pmatrix},$$

$$\begin{pmatrix} \omega_1(t)\\ \omega_2(t)\\ \omega_3(t)\end{pmatrix} = \begin{pmatrix} 0.125\cos^2\sqrt{2}t + 0.125\cos^2\pi t + \frac{0.25}{1+t^2}\\ 0.125\cos^2\sqrt{2}t + 0.125\cos^2\pi t + \frac{0.25}{1+t^2}\\ 0.125\cos^2\sqrt{2}t + 0.125\cos^2\sqrt{2}t + 0.25e^{-t^2\cos^2 t}\end{pmatrix},$$

$$\begin{pmatrix} b_1(t)\\ b_2(t)\\ b_3(t)\end{pmatrix} = \begin{pmatrix} 1 + 0.25\cos^2\sqrt{5}t + 0.25\cos^2\pi t + 0.5e^{-t^2\cos^2 t}\\ 1 + 0.25\cos^2\sqrt{5}t + 0.25\cos^2\pi t + 0.5e^{-t^2\cos^2 t}\\ 1 + 0.25\cos^2\sqrt{5}t + 0.25\cos^2 t + 0.5e^{-t^2\cos^2 t}\end{pmatrix},$$

$$\begin{pmatrix} \beta_1(t)\\ \beta_2(t)\\ \beta_3(t)\end{pmatrix} = \begin{pmatrix} 0.125\cos^2\sqrt{2}t + 0.125\cos^2\pi t + \frac{0.25}{1+t^2}\\ 0.125\cos^2\sqrt{2}t + 0.125\cos^2\pi t + \frac{0.25}{1+t^2}\\ 0.125\cos^2\sqrt{2}t + 0.125\cos^2\sqrt{2}t + 0.25e^{-t^2\cos^2 t}\end{pmatrix},$$

$\tau_1 = 1$, $\tau_2 = 1$, $\tau_3 = 1$ and $K_j(t) = e^{-t}$. Then

$$r = \frac{\sum_{i=1}^{n}\overline{b_i\beta_i} + \sum_{j=1}^{m}\overline{a_j\omega_j}}{\underline{\alpha}} = \frac{3}{4}.$$

Therefore, all conditions of the previous results are satisfied, then Lasta-Wazewska model with a mixed delays (6.1) has a unique pseudo almost periodic solution in the region

$$\mathcal{B} = \big\{x \in PAP(\mathbb{R}, \mathbb{R}^+), R_1 < |x| < R_2\big\}.$$

where

$$R_2 = \frac{\sum_{i=1}^{n}\overline{b_i} + \sum_{j=1}^{m}\overline{a_j}}{\underline{\alpha}} = \frac{12}{8} = \frac{3}{2}$$

and

$$\begin{aligned} R_1 &= \frac{\sum_{i=1}^{n}\underline{b_i}e^{-\overline{\beta_i}R_2} + \sum_{j=1}^{m}\underline{a_j}e^{-\overline{\omega_j}R_2}}{\overline{\alpha}}\\ &= \frac{\underline{a_1}e^{-\overline{\omega_1}R_2} + \underline{a_2}e^{-\overline{\omega_2}R_2} + \underline{a_3}e^{-\overline{\omega_3}R_2} + \underline{b_1}e^{-\overline{\beta_1}R_2} + \underline{b_2}e^{-\overline{\beta_1}R_2} + \underline{b_3}e^{-\overline{\beta_3}R_2}}{\overline{\alpha}}\\ &\le \frac{e^{-\frac{1}{2}\frac{3}{2}} + 0.5e^{-\frac{1}{2}\frac{3}{2}} + 0.5e^{-\frac{1}{2}\frac{3}{2}} + e^{-\frac{1}{2}\frac{3}{2}} + e^{-\frac{1}{2}\frac{3}{2}} + e^{-\frac{1}{2}\frac{3}{2}}}{10}\\ &= 5\frac{e^{-\frac{1}{2}\frac{3}{2}}}{10} = \frac{e^{-\frac{3}{4}}}{2}. \end{aligned}$$

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Samira Rihani
Department of Mathematics, University Haouari Boumediene, 16 111 Bab-Ezzouar, Algeria
E-mail address: maths_samdz@yahoo.fr

Amor Kessab
Department of Mathematics, University Haouari Boumediene, 16 111 Bab-Ezzouar, Algeria
E-mail address: amorkes@yahoo.fr

Farouk Chérif
University of Sousse, 4002 Sousse, Tunisia
E-mail address: faroukcheriff@yahoo.fr