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GLOBAL ATTRACTOR FOR REACTION-DIFFUSION EQUATIONS WITH SUPERCRITICAL NONLINEARITY IN UNBOUNDED DOMAINS

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Abstract. We consider the existence of global attractor for the inhomogeneous reaction-diffusion equation

$$u_t - \Delta u - V(x)u + |u|^{p-2}u = g, \quad \text{in } \mathbb{R}^n \times \mathbb{R}^+,$$
$$u(0) = u_0 \in L^2(\mathbb{R}^n) \cap L^p(\mathbb{R}^n),$$

where $p > \frac{2n}{n-2}$ is supercritical and $V(x)$ satisfies suitable assumptions. Since $-\Delta$ is not positive definite in $H^1(\mathbb{R}^n)$, the Gronwall inequality can not be derived and the corresponding semigroup does not possess bounded absorbing sets in $L^2(\mathbb{R}^n)$. Thus, by a special method, we prove that the equation has a global attractor in $L^p(\mathbb{R}^n)$, which attracts any bounded subset in $L^2(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$.

1. Introduction

We consider the existence of global attractor for the inhomogeneous reaction-diffusion equation in the whole space:

$$\begin{gathered} u_t - \Delta u - V(x)u + |u|^{p-2}u = g, \quad \text{in } \mathbb{R}^n \times \mathbb{R}^+, \\ u(0) = u_0 \in L^2(\mathbb{R}^n), \end{gathered} \tag{1.1}$$

where $p > \frac{2n}{n-2}$ is a supercritical exponent, $g \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ is given and the function $V(x)$ satisfies

$$V \in L^{n/2}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)\,. \tag{1.2}$$

The long-time behavior of solution for the reaction-diffusion equation

$$u_t - \Delta u + \lambda u + f(u) = g\,, \tag{1.3}$$

in unbounded domain has been studied by many authors, where $\lambda > 0$ and $f(u)$ satisfying some growth condition. In the pioneering work [3], Babain and Vishik proved the existence of global attractor in some weighted space. In paper [11], under some structural assumptions on the nonlinearity f, Wang proved the existence of global attractor in usual space $L^2(\mathbb{R}^n)$ instead of the weighted space. Other

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investigations of global attractor for equation (1.3) in unbounded domain can be found in [1, 7, 9].

In general, Gronwall inequality is utilized to prove the existence of absorbing set in $L^2(\mathbb{R}^n)$ for (1.3) when $\lambda > 0$. However, it will be difficult in the case $\lambda = 0$ or $\lambda < 0$. Zelik [13] considered the existence of global attractor for the real Ginzburg-Landau equation

$$u_t - \Delta u - u + u^3 = g \tag{1.4}$$

in $\mathbb{R}^n$. For this equation, because of the infiniteness of the energy functional, the global attractor can not be obtained in usual spaces, thus Zelik considered the existence of the locally compact global attractors for the semigroup associated with the equation (1.4) in uniformly local spaces. More detailed information can be found in [6, 10].

Arrieta, Cholewa, Dlotko and Rodríguez-Bernal [2] consider the reaction-diffusion equation

$$u_t - \Delta u = f(x, u) + g$$

with $f(x, s)s \le C(x)|s|^2 + D(x)|s|$ in standard Lebesgue space. They prove that for some suitable functions $C(x)$ and $D(x)$, the existence of global solutions can be obtained. Furthermore, if the operator $\Delta + C(x)I$ generates an analytic semigroup which decay exponentially, then this equation has a global attractor.

Motivated by the above works, we consider the existence of a global attractor for (1.1) (which is inhomogeneous type of equation (1.4) but u^3 is replaced by $|u|^{p-2}u$). Following the proof in [2], we can obtain the existence and uniqueness of the solutions. We encounter difficulties when proving the existence of global attractor, since the operator $\Delta + V(x)I$ may not be able to generate an analytic semigroup, and the Gronwall inequality can not be applied to obtain the absorbing set in $L^2(\mathbb{R}^n)$. To overcome the difficulties, we assume that $V(x)$ satisfies some suitable conditions, and use the method of monotonicity of the energy functional to obtain an absorbing set in $D^{1,2}(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$. Furthermore, in order to establish the ω-limit compactness of the corresponding semigroup, we use the Sobolev embeddings in interior, and estimate the L^p-norm of solutions is arbitrarily small uniformly for large time in exterior. Our main result reads as follows.

Theorem 1.1. *Assume $p > \frac{2n}{n-2}$, $g \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, $V(x)$ satisfies conditions* (1.2). *Then the semigroup $\{S(t)\}_{t\ge 0}$ generated by the equation* (1.1) *has a global attractor $\mathcal{A}$ in $L^p(\mathbb{R}^n)$, which attracts any bounded subset in $L^2(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$.*

Remark 1.2. Zelik [13] proved the existence of global attractor for the Ginzburg-Landau equation

$$u_t - \Delta u - u + u^3 = g \tag{1.5}$$

in $\mathbb{R}^n$, and the attractor is only locally compact in a uniformly local phase space. To obtain a global attractor which is compact in usual space $L^2(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$, we assume that the nonlinear term is supercritical growth and the condition (1.2) holds.

2. Preliminaries

In this section, we first review the basic concept about the Kuratowski measure of noncompactness, which will be used to establish the ω-limit compactness of semigroup. See [5, 8, 12] for its some basic properties.

Definition 2.1. Let (M, d) be a metric space and let A be a bounded subset of M. The measure of noncompactness $\kappa(A)$ is defined by

$$\kappa(A) = \inf\{\delta > 0 \mid A \text{ admits a finite cover by sets of diameter} \leq \delta\}.$$

The properties of the measure of noncompactness $\kappa(A)$ are provided in the following lemmas.

Lemma 2.2. *Let (M, d) be a complete metric space and κ be the measure of noncompactness. Then*

- (i) $\kappa(B) = 0$, *if and only if* $\overline{B}$ *is compact;*
- (ii) *if M is a Banach space, then* $\kappa(B_1 + B_2) \leq \kappa(B_1) + \kappa(B_2)$ *;*
- (iii) $\kappa(B_1) \leq \kappa(B_2)$ *whenever* $B_1 \subset B_2$ *;*
- (iv) $\kappa(B_1 \cup B_2) = \max\{\kappa(B_1), \kappa(B_2)\}$ *;*
- (v) $\kappa(B) = \kappa(\overline{B})$ *.*

Lemma 2.3. *Let M be an infinite dimensional Banach space and let $B(\varepsilon)$ be a ball of radius ε. Then $\kappa(B(\varepsilon)) = 2\varepsilon$.*

The concept of ω-limit compactness of a semigroup, which is an important necessary and sufficient condition for the existence of global attractor (see [8]).

Definition 2.4. A semigroup $\{S(t)\}_{t\geq 0}$ in a complete metric space (M, d) is called a C^0 or continuous semigroup if it satisfies:

- $S(0) = I$,
- $S(t)S(s) = S(s)S(t) = S(t + s)$,
- $S(t)x_0$ is continuous in $x_0 \in M$ and $t \in R$.

Definition 2.5. A continuous semigroup $\{S(t)\}_{t\geq 0}$ in a complete metric space (M, d) is called ω-limit compact, if for any bounded subset B and any $\varepsilon > 0$, there exists a time $t^* \geq 0$ such that

$$\kappa\Big(\cup_{t\geq t^*} S(t)B\Big) \leq \varepsilon.$$

Lemma 2.6. *Let $\{S(t)\}_{t\geq 0}$ be a continuous semigroup in a complete metric space (M, d). Then $S(t)$ has a global attractor $\mathscr{A}$ in M if and only if*

1. *there is a bounded absorbing set $B \subset M$, and*
2. *$\{S(t)\}_{t\geq 0}$ is ω-limit compact.*

Now, we give the general existence and uniqueness of solutions which can be obtained as in [2].

Theorem 2.7. *Let $p > 2$, $g \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, $V(x)$ satisfies conditions (1.2). Then for any $u_0 \in L^2(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ and $T > 0$, there exists a unique weak solution $u(x, t)$ of (1.1) satisfies*

$$u \in C([0, T], L^2(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)) \cap L^2(0, T, D^{1,2}(\mathbb{R}^n)).$$

Furthermore, $u_0 \mapsto u(t)$ is continuous on $L^2(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$.

For convenience, here and subsequently, we can assume $|V(x)| \leq l$ since $V \in L^\infty(\mathbb{R}^n)$. In addition, for any $R > 0$, we denote $\Omega_R := \{x \in \mathbb{R}^n : |x| \leq R\}$.

3. Bounded absorbing set in $D^{1,2}(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ and $L^{2p-2}(\mathbb{R}^n)$

By Theorem 2.7, we can define the operator semigroup $\{S(t)\}_{t\geq 0}$ in $L^2(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ as

$$S(t)u = u(t) : L^2(\mathbb{R}^n) \cap L^p(\mathbb{R}^n) \to L^2(\mathbb{R}^n) \cap L^p(\mathbb{R}^n),$$

which is generated by the weak solutions of (1.1) with initial data $u_0 \in L^2(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$.

Theorem 3.1. *There exist constants $\rho_1 > 0$ and $t_1(|u_0|_2)$ such that, for the solution $u(t)$ of* (1.1),

$$\int_{\mathbb{R}^n} |\nabla u(t)|^2 dx + \int_{\mathbb{R}^n} |u(t)|^p dx \leq \rho_1\,, \quad \textit{for all } t \geq t_1\,.$$

Proof. first, we multiply (1.1) by u and integrate over $\mathbb{R}^n$,

$$\frac{1}{2}\frac{d}{dt}|u|_2^2 + \int_{\mathbb{R}^n} |\nabla u|^2 dx - \int_{\mathbb{R}^n} V u^2 dx + \int_{\mathbb{R}^n} |u|^p dx = \int_{\mathbb{R}^n} gu dx\,. \tag{3.1}$$

Applying Hölder inequality and Young inequality, we estimate the right-hand side as

$$\Big|\int_{\mathbb{R}^n} gu\, dx\Big| \leq \frac{1}{4}\int_{\mathbb{R}^n} |u|^p\, dx + C\big(|g|_{\frac{p}{p-1}}\big). \tag{3.2}$$

Then, we divide the third term on the left-hand side into

$$\Big|\int_{\mathbb{R}^n} V u^2 dx\Big| \leq \int_{\Omega_{R_0}} |V| u^2 dx + \int_{\mathrm{R}^n\setminus\Omega_{R_0}} |V| u^2 dx := I_1 + I_2\,,$$

where the constant R_0 is sufficiently large such that

$$\Big(\int_{\mathbb{R}^n\setminus\Omega_{R_0}} |V|^{n/2} dx\Big)^{2/n} \leq \frac{S}{2},$$

and S is the Sobolev constant satisfying $S|u|^2_{\frac{2n}{n-2}} \leq |\nabla u|_2^2$. Therefore, utilizing Hölder and Young inequality, the two terms I_1 and I_2 can be estimated as

$$I_1 \leq \frac{1}{4}\int_{\mathbb{R}^n} |u|^p\, dx + C(p, l, n, R_0), \quad I_2 \leq \frac{S}{2}|u|^2_{\frac{2n}{n-2}} \leq \frac{1}{2}\int_{\mathbb{R}^n} |\nabla u|^2\, dx. \tag{3.3}$$

Combining the estimates (3.1), (3.2) and (3.3) yields

$$\frac{1}{2}\frac{d}{dt}|u|_2^2 + \frac{1}{2}\int_{\mathbb{R}^n} |\nabla u|^2 dx + \frac{1}{2}\int_{\mathbb{R}^n} |u|^p dx \leq C\,. \tag{3.4}$$

Integrating this inequality between 0 and t gives

$$\int_0^t \int_{\mathbb{R}^n} (|\nabla u(s)|^2 + |u(s)|^p)\, dx\, ds \leq tC + |u(0)|_2^2\,,$$

it follows from $|u(0)|_2^2$ is bounded that there exists a sufficiently large time t_1 such that

$$\int_{\mathbb{R}^n} |\nabla u(t_1)|^2 dx + \int_{\mathbb{R}^n} |u(t_1)|^p dx \leq 2C\,. \tag{3.5}$$

Meanwhile, denoting

$$E(u(t)) := \frac{1}{2}\int_{\mathbb{R}^n} |\nabla u(t)|^2 dx - \frac{1}{2}\int_{\mathbb{R}^n} V|u(t)|^2 dx + \frac{1}{p}\int_{\mathbb{R}^n} |u(t)|^p dx - \int_{\mathbb{R}^n} gu(t) dx\,,$$

and multiplying the equation (1.1) by u_t and integrating over $\mathbb{R}^n$, this yields $\frac{d}{dt}E(u(t)) = -|u_t|_2^2 \le 0$, thus

$$E(u(t)) \le E(u(t_1)), \quad \text{for all } t \ge t_1\,. \tag{3.6}$$

Utilizing the similar techniques in (3.2) and (3.3), the following two estimates

$$|\int_{\mathbb{R}^n} gu\,dx| \le \frac{1}{4p}\int_{\mathbb{R}^n} |u|^p\,dx + C\,,$$
$$|\int_{\mathbb{R}^n} Vu^2\,dx| \le \frac{1}{2}\int_{\mathbb{R}^n} |\nabla u|^2\,dx + \frac{1}{4p}\int_{\mathbb{R}^n} |u|^p\,dx + C$$

are also valid, and yield

$$E(u(t)) \ge \frac{1}{4}\int_{\mathbb{R}^n} |\nabla u(t)|^2 dx + \frac{1}{2p}\int_{\mathbb{R}^n} |u(t)|^p dx - 2C\,, \tag{3.7}$$
$$E(u(t_1)) \le \frac{3}{4}\int_{\mathbb{R}^n} |\nabla u(t_1)|^2 dx + \frac{3}{2p}\int_{\mathbb{R}^n} |u(t)|^p dx + 2C\,. \tag{3.8}$$

Combining the estimates (3.5), (3.6), (3.7) and (3.8), it is obvious that there exists $\rho_1 > 0$ such that

$$\frac{1}{4}\int_{\mathbb{R}^n} |\nabla u(t)|^2 dx + \frac{1}{2p}\int_{\mathbb{R}^n} |u(t)|^p dx \le \frac{\rho_1}{2p}\,.$$

The conclusion is consequently obtained. □

Next, we show the semigroup also has an absorbing set in the space $L^{2p-2}(\mathbb{R}^n)$.

Theorem 3.2. *There exist constants ρ_2 and $t_2(|u_0|_2)$ such that, for the solution $u(t)$ of the equation(1.1),*

$$\int_{\mathbb{R}^n} |u(t)|^{2p-2} dx < \rho_2\,, \quad \textit{for all } t \ge t_2\,.$$

Proof. Similar techniques can be used for (3.4), when multiplying (1.1) by $|u|^{p-2}u$ and $|u|^{2p-4}u$ respectively. We have the following two estimates:

$$\frac{1}{p}\frac{d}{dt}|u|_p^p + \frac{1}{2}\int_{\mathbb{R}^n} |u|^{2p-2} dx \le C\,, \tag{3.9}$$
$$\frac{d}{dt}|u|_{2p-2}^{2p-2} \le C\left(1 + |u|_{2p-2}^{2p-2}\right)\,. \tag{3.10}$$

We can integrate (3.9) between t and $t+1$ to obtain

$$\frac{1}{2}\int_t^{t+1}\int_{\mathbb{R}^n} |u(s)|^{2p-2}\,dx\,ds \le C + \frac{1}{p}|u(t)|_p^p\,.$$

Recalling the fact that $|u|_p^p$ is bounded for all $t \ge t_1$, therefore there exists a constant C such that

$$\int_t^{t+1}\int_{\mathbb{R}^n} |u(s)|^{2p-2}\,dx\,ds \le C \quad for\ all\ t \ge t_1\,. \tag{3.11}$$

Now, integrating (3.10) between s and $t+1$ $(t \le s < t+1)$gives

$$|u(t+1)|_{2p-2}^{2p-2} \le C\Big(1 + \int_s^{t+1} |u(\xi)|_{2p-2}^{2p-2} d\xi\Big) + |u(s)|_{2p-2}^{2p-2}\,,$$

then we integrate this equation with respect to s between t and $t+1$, we obtain

$$|u(t+1)|_{2p-2}^{2p-2} \le C + C\int_t^{t+1}\int_{\mathbb{R}^n} |u(s)|^{2p-2}\,dx\,ds\,. \tag{3.12}$$

It follow from (3.11) that there exists a constant $\rho_2 > 0$ such that

$$|u(t+1)|_{2p-2}^{2p-2} \le \rho_2 \quad \forall t > t_1\,,$$

the proof is complete because $t_2 = t_1 + 1$. □

Remark 3.3. We observe that, in the proofs of Theorem 3.1 and Theorem 3.2, we only need $g \in L^{\frac{p}{p-1}}(\mathbb{R}^n)$ and $g \in L^{\frac{3p-4}{p-1}}(\mathbb{R}^n)$ respectively. Actually, we can prove that the semigroup has a bounded absorbing set in $L^q(\mathbb{R}^n)$ for any $q \in [p,\infty)$ when the function $g \in L^1(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)$.

4. ω-limit compactness and global attractor

We define a smooth function $\theta : \mathbb{R}^+ \to [0,1]$, such that

$$\theta(s) = \begin{cases} 0 & s \le 1,\\ 1 & s \ge 2,\end{cases}$$

with $|\theta'(s)| \le 2$. Let $\theta_R(x) = \theta_R(|x|) = \theta\big(\frac{|x|^2}{R^2}\big)$. In this way, any solutio $u(t)$ of equation (1.1) can be decomposed as $u(t) = \theta_R u(t) + (1-\theta_R)u(t)$. Before the proof of ω-limit compactness and global attractor for the corresponding semigroup, we first give the estimate the L^p-norm of solutions are arbitrarily small uniformly on exterior.

Lemma 4.1. *For arbitrary $\varepsilon > 0$, there exist constants t_3 and $R_0 > 0$, such that for the solution $u(t)$,*

$$\int_{\mathbb{R}^n} \theta_{R_0}^2 |u(t)|^p dx < \varepsilon, \quad \text{for all } t \ge t_3\,.$$

Proof. Multiplying (1.1) by $\theta_R^p |u|^{p-2}u$ and integrating over $\mathbb{R}^n$,

$$\begin{aligned} &\frac{1}{p}\frac{d}{dt}\int_{\mathbb{R}^n}\theta_R^p|u|^p dx - \int_{\mathbb{R}^n}\Delta u\cdot\theta_R^p|u|^{(p-2)}u\,dx - \int_{\mathbb{R}^n}\theta_R^p V|u|^p dx\\ &+\int_{\mathbb{R}^n}\theta_R^p|u|^{2p-2}dx\\ &=\int_{\mathbb{R}^n}\theta_R^p|u|^{p-2}ug\,dx\,.\end{aligned} \tag{4.1}$$

We first consider the estimate of the second term in the left-hand side, since

$$\begin{aligned} &-\int_{\mathbb{R}^n}\Delta u\cdot\theta_R^p|u|^{(p-2)}dx\\ &= \frac{4(p-1)}{p^2}\int_{\mathbb{R}^n}\theta_R^p|\nabla u^{\frac{p}{2}}|^2 dx + p\int_{\mathbb{R}^n}\theta_R^{p-1}u^{p-1}\nabla\theta_R\nabla u dx\\ &\ge \frac{4(p-1)}{p^2}\int_{\mathbb{R}^n}\theta_R^p|\nabla u^{\frac{p}{2}}|^2dx - \frac{1}{4}\int_{\mathbb{R}^n}\theta_R^p|u|^{2p-2}dx - p^2\int_{\mathbb{R}^n}|\nabla\theta_R|^2|\nabla u|^2dx\,.\end{aligned}$$

Referring to Theorem 3.1 and assumption of the function θ_R, we have $|\nabla u|_2^2$ is bounded and $|\nabla \theta_R(x)| \leq \frac{4}{R}$. Thus, there exists a constant $C > 0$ such that for all $t \geq t_1$,

$$\begin{aligned} &-\int_{\mathbb{R}^n} \Delta u \cdot \theta_R^p |u|^{p-2} u \, dx \\ &\geq \frac{4(p-1)}{p^2} \int_{\mathbb{R}^n} \theta_R^p |\nabla u^{\frac{p}{2}}|^2 dx - \frac{1}{4} \int_{\mathbb{R}^n} \theta_R^p |u|^{2p-2} \, dx - \frac{C}{R^2} \,. \end{aligned} \tag{4.2}$$

Then, the application of Hölder and Young inequalities gives the following two estimates

$$\begin{aligned} \Big| \int_{\mathbb{R}^n} \theta_R^p V |u|^p dx \Big| &\leq \Big(\int_{\mathbb{R}^n \setminus \Omega_R} |V|^{n/2} dx \Big)^{2/n} \Big(\int_{\mathbb{R}^n} |\theta_R u^{\frac{p}{2}}|^{2 \cdot \frac{n}{n-2}} dx \Big)^{\frac{n-2}{n}} \\ &\leq S \Big(\int_{\mathbb{R}^n \setminus \Omega_R} |V|^{n/2} dx \Big)^{2/n} \int_{\mathbb{R}^n} \theta_R^2 |\nabla u^{\frac{p}{2}}|^2 dx \,, \end{aligned} \tag{4.3}$$

$$\Big| \int_{\mathbb{R}^n} \theta_R^p |u|^{p-2} u g dx \Big| \leq \frac{1}{4} \int_{\mathbb{R}^n} \theta_R^p |u|^{2p-2} dx + \int_{\mathbb{R}^n} \theta_R^p |g|^2 dx \,. \tag{4.4}$$

It is obvious that the terms $\frac{C}{R^2}$, $\big(\int_{\mathbb{R}^n \setminus \Omega_R} |V|^{n/2} dx \big)^{2/n}$ and $\int_{\mathbb{R}^n} \theta_R^p |g|^2 dx$ can be sufficiently small when $R \to \infty$. Therefore, from (4.1)-(4.4) it follows that, for arbitrary $\varepsilon > 0$, there exists R_0, such that for all $t \geq t_1$ and $R \geq R_0$,

$$\frac{d}{dt} \int_{\mathbb{R}^n} \theta_R^p |u|^p dx + \int_{\mathbb{R}^n} \theta_R^p |u|^{2p-2} dx < \frac{1}{2} \varepsilon^{\frac{2p-2}{p(1-\lambda)}} \,, \tag{4.5}$$

where $\lambda \in (0,1)$ satisfies $\frac{1}{p} = \frac{(n-2)\lambda}{2n} + \frac{1-\lambda}{2p-2}$. Similarly to the proof of Theorem 3.1, there exists a time $t_3 \geq t_1$, such that

$$\int_{\mathbb{R}^n} \theta_R^p |u(t_3)|^{2p-2} dx < \varepsilon^{\frac{2p-2}{p(1-\lambda)}} \,. \tag{4.6}$$

Now, combining (4.5) with (4.6), we can prove that if $t \geq t_3$, there exists a constant $C \sim (\rho_1, p)$, such that

$$\int_{\mathbb{R}^n} \theta_R^p |u(t)|^p dx \leq C\varepsilon \,. \tag{4.7}$$

Actually, applying the interpolation inequality and notice that $|u|^2_{\frac{2n}{n-2}} \leq \frac{1}{S} |\nabla u|_2^2 \leq \frac{\rho_1}{S}$, we have

$$\begin{aligned} \Big(\int_{\mathbb{R}^n} \theta_R^p |u|^p dx \Big)^{1/p} &\leq \varepsilon^{1/p} |u|_{\frac{2n}{n-2}} + \big(\varepsilon^{1/p}\big)^{-\frac{\lambda}{1-\lambda}} \Big(\int_{\mathbb{R}^n} \theta_R^p |u|^{2p-2} dx \Big)^{\frac{1}{2p-2}} \\ &\leq \varepsilon^{1/p} \sqrt{\frac{\rho_1}{S}} + \big(\varepsilon^{1/p}\big)^{-\frac{\lambda}{1-\lambda}} \Big(\int_{\mathbb{R}^n} \theta_R^p |u|^{2p-2} dx \Big)^{\frac{1}{2p-2}} \,. \end{aligned}$$

Therefore (4.7) is valid provided that $\int_{\mathbb{R}^n} \theta_R^p |u(t)|^{2p-2} dx < \varepsilon^{\frac{2p-2}{p(1-\lambda)}}$. On the other hand, if $\int_{\mathbb{R}^n} \theta_R^p |u(t)|^{2p-2} dx \geq \varepsilon^{\frac{2p-2}{p(1-\lambda)}}$, then referring to the estimate (4.5), it follows that

$$\frac{d}{dt} \int \theta_R^p |u(t)|^p dx < -\frac{1}{2} \varepsilon^{\frac{2p-2}{p(1-\lambda)}} < 0 \,,$$

which concludes that $\int \theta_R^p |u(t)|^p dx$ is decreasing with respect to variable t. Hence, in any case as $t \geq t_3$, we have

$$\int_{\mathbb{R}^n} \theta_R^p |u(t_3)|^p dx \leq C\varepsilon .$$

□

Now, we prove that the semigroup generated by the solutions of equation(1.1) has a global attractor $\mathcal{A}$, which attracts any bounded subset $B \subset L^2(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ in the topology of $L^p(\mathbb{R}^n)$.

Proof of Theorem 1.1. We only need to verify that the corresponding semigroup is ω-limit compact. For any fixed R, it follows from Theorem 3.1 and Theorem 3.2 that there exists a time t_2 such that

$$\cup_{t \geq t_2} \cup_{u_0 \in B} (1 - \theta_R) S(t) u_0 \text{ is bounded in } H^1(\Omega_{2R}) \text{ and } L^{2p-2}(\mathbb{R}^n) ,$$

then by the compactness of Sobolev embedding $H^1(\Omega_{2R}) \hookrightarrow L^2(\Omega_{2R})$ and interpolation inequality $(2 < p < 2p - 2)$, we obtain that $\cup_{t \geq t_2} \cup_{u_0 \in B} (1 - \theta_R) S(t) u_0$ is compact in $L^p(\Omega_{2R})$, thus

$$\kappa\Big(\cup_{t \geq t_2} \cup_{u_0 \in B} (1 - \theta_R) S(t) u_0 \Big)_{L^p} = 0, \quad \text{for any } R > 0 .$$

On the other hand, from Lemma 4.1, we know for any $\varepsilon > 0$, there exist constants t_3 and $R_0 > 0$ such that

$$\big| \cup_{t \geq t_3} \cup_{u_0 \in B} \theta_{R_0} S(t) u_0 \big|_p^p < \varepsilon ,$$

by Lemma2.3, its measure of noncompactness is less than 2ε, i.e.,

$$\kappa\Big(\cup_{t \geq t_3} \cup_{u_0 \in B} \theta_{R_0} S(t) u_0 \Big)_{L^p} < 2\varepsilon .$$

Thus taking $t^* = \max\{t_2, t_3\}$, we have

$$\begin{aligned} &\kappa\Big(\cup_{t \geq t^*} \cup_{u_0 \in B} S(t) u_0 \Big)_{L^p} \\ &\leq \kappa\Big(\cup_{t \geq t^*} \cup_{u_0 \in B} \theta_{R_0} S(t) u_0 \Big)_{L^p} + \kappa\Big(\cup_{t \geq t^*} \cup_{u_0 \in B} (1 - \theta_{R_0}) S(t) u_0 \Big)_{L^p} < 2\varepsilon , \end{aligned}$$

which concludes that the semigroup $\{S(t)\}_{t \geq 0}$ is ω-limit compact. Therefore, we obtain the existence of global attractor, which attracts any bounded subset $B \subset L^2(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ in the topology of $L^p(\mathbb{R}^n)$. □

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