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MULTIPLE HOMOCLINIC SOLUTIONS FOR SUPERQUADRATIC HAMILTONIAN SYSTEMS

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Abstract. In this article we study the existence of infinitely many homoclinic solutions for a class of second-order Hamiltonian systems

$$\ddot{u} - L(t)u + W_u(t,u) = 0, \quad \forall t \in \mathbb{R},$$

where L is not required to be either uniformly positive definite or coercive, and W is superquadratic at infinity in u but does not need to satisfy the Ambrosetti-Rabinowitz superquadratic condition.

1. Introduction and statement of main results

We consider the second-order Hamiltonian system

$$\ddot{u} - L(t)u + W_u(t,u) = 0, \quad \forall t \in \mathbb{R}, \tag{1.1}$$

where $u = (u_1, \dots, u_N) \in \mathbb{R}^N$, $W \in C^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$, $L \in C(\mathbb{R}, \mathbb{R}^{N^2})$ is a symmetric matrix-valued function, and $W_u(t,u)$ denotes the gradient of $W(t,u)$ with respect to u. Here, as usual, we say that a solution u of (1.1) is homoclinic (to 0) if $u \in C^2(\mathbb{R}, \mathbb{R}^N)$, $u(t) \not\equiv 0$, $u(t) \to 0$ and $\dot{u}(t) \to 0$ as $|t| \to \infty$.

As a special case of dynamical systems, Hamiltonian systems are very important in the study of gas dynamics, fluid mechanics, relativistic mechanics and nuclear physics. They also appear in the fields of biology and chemistry (see, e.g.,[13]). It is well known that homoclinic solutions play an important role in analyzing the chaos of Hamiltonian systems. If a system has the transversely intersected homoclinic solutions, then it must be chaotic. If it has the smoothly connected homoclinic solutions, then it cannot stand the perturbation, and its perturbed system probably produces chaotic phenomena. Therefore, it is of practical importance and mathematical significance to consider the existence of homoclinic solutions of Hamiltonian systems emanating from 0.

During the previous decades, the existence and multiplicity of homoclinic solutions for (1.1) have been extensively investigated via variational methods ; see [1, 2, 3, 4, 6, 7, 8, 9, 10, 11, 12, 15, 16, 17, 18, 19, 20, 22, 23, 25, 26, 27, 29, 31, 30, 32, 33, 34, 36, 37] and the references therein. These methods have also been used in many related and similar problems (see, e.g.,[5, 21, 24, 35]). From the beginning, most of them treated the case where L and W are either independent of t

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or periodic in t (see [1, 2, 3, 8, 10, 11, 12, 20, 22, 23, 26]). In this kind of problem, the function L plays an important role. If L is neither a constant nor periodic, the problem is quite different from the ones just described, because of the lack of compactness of the Sobolev embedding. After the work of Rabinowitz and Tanaka [23], many results were obtained for the case where L is neither a constant nor periodic. (see, [4, 6, 7, 9, 11, 15, 16, 17, 18, 19, 25, 26, 27, 29, 31, 30, 32, 33, 34, 36, 37]). However, except for [26], in all these mentioned papers L was always required to satisfy either the uniform positive-definiteness condition:

(A1) there exists $c_0 > 0$ such that

$$L(t)u \cdot u \geq c_0|u|^2, \quad \forall (t, u) \in \mathbb{R} \times \mathbb{R}^N$$

or the coercivity condition:

(A2) the smallest eigenvalue of L tends to ∞ as $|t| \to \infty$, i.e.,

$$l(t) \equiv \inf_{u \in \mathbb{R}^N,\, |u|=1} L(t)u \cdot u \to \infty \quad \text{as } |t| \to \infty,$$

where $\cdot$ and $|\cdot|$ denote the standard inner product and the associated norm in $\mathbb{R}^N$ respectively. Most of these known results were obtained for the case where W is superquadratic at infinity in u and satisfy the usual assumption:

(A3) $\lim_{|u| \to 0} W(t, u)/|u|^2 = 0$ uniformly for $t \in \mathbb{R}$.

In this case, the well-known Ambrosetti-Rabinowitz superquadratic condition was usually assumed on W (see, e.g., [4, 6, 9, 15, 16, 18]).

In this article, we study the existence of infinitely many homoclinic solutions for (1.1) in the case where L is unnecessarily required to be either uniformly positive definite or coercive, and W satisfies some weak superquadratic condition at infinity with respect to u. Before presenting our assumptions, we introduce some notation.

Notation. For two $N \times N$ symmetric matrices M_1 and M_2, we say that $M_1 \geq M_2$ if

$$\min_{u \in \mathbb{R}^N, |u|=1} (M_1 - M_2)u \cdot u \geq 0$$

and that $M_1 \ngeq M_2$ if $M_1 \geq M_2$ does not hold.

We use the following assumptions:

(A4) The smallest eigenvalue of $L(t)$ is bounded from below.

(A5) There exists a constant $r_0 > 0$ such that

$$\lim_{|s| \to \infty} \text{meas}(\{t \in (s - r_0, s + r_0) : L(t) \ngeq MI_N\}) = 0, \quad \forall M > 0,$$

where *meas* denotes the Lebesgue measure in $\mathbb{R}$ and I_N is the identity matrix in $\mathbb{R}^N$.

(A6) $W(t, 0) \equiv 0$, and there exist constants $c_1 > 0$ and $\nu > 2$ such that

$$|W_u(t, u)| \leq c_1(|u| + |u|^{\nu-1}), \quad \forall (t, u) \in \mathbb{R} \times \mathbb{R}^N;$$

(A7) $\lim_{|u| \to \infty} W(t, u)/|u|^2 = \infty$ uniformly for $t \in \mathbb{R}$.

(A8) There exists a constant $\vartheta \geq 1$ such that

$$\vartheta \widetilde{W}(t, u) \geq \widetilde{W}(t, su), \quad \forall (t, u) \in \mathbb{R} \times \mathbb{R}^N \text{ and } s \in [0, 1],$$

where $\widetilde{W}(t, u) := W_u(t, u) \cdot u - 2W(t, u)$.

(A9) $W(t, -u) = W(t, u)$ for all $(t, u) \in \mathbb{R} \times \mathbb{R}^N$.

Our main result reads as follows.

Theorem 1.1. *Suppose that* (A4)*,* (A5) *and* (A6)–(A9) *are satisfied. Then* (1.1) *possesses a sequence of homoclinic solutions* $\{u_k\}$ *satisfying*

$$\frac{1}{2}\int_{\mathbb{R}}(|\dot{u}_k|^2 + L(t)u_k \cdot u_k)dt - \int_{\mathbb{R}} W(t,u_k)dt \to \infty \quad \text{as } k \to \infty.$$

Remark 1.2. It is easy to see that conditions (A4) and (A5) are weaker than the coercivity condition (A2). In our Theorem 1.1, L is unnecessarily uniformly positive definite. Besides, the usual condition (A3) and the well-known Ambrosetti-Rabinowitz superquadratic condition are not required in our Theorem 1.1. There are functions L and W which satisfy all the conditions in our Theorem 1.1 but do not satisfy the corresponding conditions in the aforementioned references for the superquadratic case. For example, let

$$L(t) = (|t|\sin^2 t - 1)I_N,$$
$$W(t,u) = a(t)\big[|u|^2\ln(e+|u|) - \frac{1}{2}|u|^2 + e|u| - e^2(\ln(e+|u|) - 1)\big],$$

where a is a continuous bounded function with positive lower bound, then simple computation shows that they satisfy (A4), (A5) and (A6)–(A9). However, L does not satisfy neither the uniform positive-definiteness condition (A1) nor the coercivity condition (A2). Meanwhile neither the usual assumption (A3) nor the Ambrosetti-Rabinowitz superquadratic assumption holds for W.

2. Variational setting and proof of the main result

To prove our main result via the critical point theory, we need to establish the variational setting for (1.1). Before this, we have the following result.

Remark 2.1. From (A4) and (A6), we know that there exists a positive constant l_0 such that $L(t) + 2l_0 I_N \geq I_N$ for all $t \in \mathbb{R}$ and $W(t,u) + l_0|u|^2 \geq 0$ for all $(t,u) \in \mathbb{R}\times\mathbb{R}^N$. Let $\overline{L}(t) = L(t) + 2l_0 I_N$ and $\overline{W}(t,u) = W(t,u) + l_0|u|^2$. Consider the following Hamiltonian system

$$\ddot{u} - \overline{L}(t)u + \overline{W}_u(t,u) = 0, \quad \forall t \in \mathbb{R}, \tag{2.1}$$

then (2.1) is equivalent to (1.1). Moreover, it is easy to check that the hypotheses (A4), (A5) and (A6)–(A9) still hold for $\overline{L}$ and $\overline{W}$ provided that those hold for L and W. Hence, in what follows, we always assume without loss of generality that $L(t) \geq I_N$ for all $t \in \mathbb{R}$ and $W(t,u) \geq 0$ for all $(t,u) \in \mathbb{R}\times\mathbb{R}^N$.

In view of Remark 2.1, we consider the space $E := \{u \in H^1(\mathbb{R},\mathbb{R}^N) | \int_{\mathbb{R}} L(t)u \cdot u dt < \infty\}$ equipped with the following inner product

$$(u,v) = \int_{\mathbb{R}}(\dot{u}\cdot\dot{v} + L(t)u\cdot v)dt.$$

Then E is a Hilbert space and we denote by $\|\cdot\|$ the associated norm. Moreover, we write E^* for the topological dual of E, and $\langle\cdot,\cdot\rangle : E^*\times E \to \mathbb{R}$ for the dual pairing. Evidently, E is continuously embedded into $H^1(\mathbb{R},\mathbb{R}^N)$ and hence continuously embedded into $L^p \equiv L^p(\mathbb{R},\mathbb{R}^N)$ for $2 \leq p \leq \infty$, i.e., there exists $\tau_p > 0$ such that

$$\|u\|_p \leq \tau_p\|u\|, \quad \forall u \in E, \tag{2.2}$$

where $\|\cdot\|_p$ denotes the usual norm in L^p for all $2 \leq p \leq \infty$. In fact, we further have the following lemma.

Lemma 2.2. *If L satisfies* (A4) *and* (A5)*, then E is compactly embedded into L^p for $2 \le p < \infty$.*

Proof. Let $\{u_n\} \subset E$ be a bounded sequence such that $u_n \rightharpoonup u$ in E. We will show that $u_n \to u$ in L^p for $2 \le p < \infty$. By the interpolation inequality we only need to consider the case $p = 2$. Suppose, without loss of generality, that $u_n \rightharpoonup 0$ in E. The Sobolev embedding theorem implies $u_n \to 0$ in $L^2_{loc}(\mathbb{R}, \mathbb{R}^N)$. Thus it suffices to show that, for any $\epsilon > 0$, there is $r > 0$ such that $\int_{\mathbb{R}\setminus(-r,r)} |u_n|^2 dt < \epsilon$. For any $s \in \mathbb{R}$, we denote by $\mathcal{B}_{r_0}(s)$ the interval in $\mathbb{R}$ centered at s with radius r_0, i.e., $\mathcal{B}_{r_0}(s) := (s - r_0, s + r_0)$, where r_0 is the constant given in (A5). Let $\{s_i\} \subset \mathbb{R}$ be a sequence of points such that $\mathbb{R} = \cup_{i=1}^{\infty} \mathcal{B}_{r_0}(s_i)$ and each $t \in \mathbb{R}$ is contained in at most two such intervals. For any $r > 0$ and $M > 0$, let

$$
\begin{aligned}
\mathcal{C}(r, M) &= \{t \in \mathbb{R} \setminus (-r, r) : L(t) \ge M I_N\}, \\
\mathcal{D}(r, M) &= \{t \in \mathbb{R} \setminus (-r, r) : L(t) \ngeq M I_N\}.
\end{aligned}
$$

Then

$$
\int_{\mathcal{C}(r,M)} |u_n|^2 dt \le \frac{1}{M} \int_{\mathcal{C}(r,M)} L(t) u_n \cdot u_n dt \le \frac{1}{M} \int_{\mathbb{R}} L(t) u_n \cdot u_n dt,
$$

and this can be made arbitrarily small by choosing M large. Also for a fixed $M > 0$,

$$
\begin{aligned}
\int_{\mathcal{D}(r,M)} |u_n|^2 dt &\le \sum_{i=1}^{\infty} \int_{\mathcal{D}(r,M)\cap\mathcal{B}_{r_0}(s_i)} |u_n|^2 dt \\
&\le \sum_{i=1}^{\infty} \Big(\int_{\mathcal{D}(r,M)\cap\mathcal{B}_{r_0}(s_i)} |u_n|^4 dt \Big)^{1/2} (\operatorname{meas}(\mathcal{D}(r, M) \cap \mathcal{B}_{r_0}(s_i)))^{1/2} \\
&\le \varepsilon_r \sum_{i=1}^{\infty} \Big(\int_{\mathcal{B}_{r_0}(s_i)} |u_n|^4 dt \Big)^{1/2} \\
&\le c\varepsilon_r \sum_{i=1}^{\infty} \int_{\mathcal{B}_{r_0}(s_i)} (|\nabla u_n|^2 + |u_n|^2) dt \\
&\le 2c\varepsilon_r \int_{\mathbb{R}} (|\nabla u_n|^2 + |u_n|^2) dt
\end{aligned}
$$

for some constant $c > 0$, where $\varepsilon_r = \sup_{i\in\mathbb{N}} (\operatorname{meas}(\mathcal{D}(r, M) \cap \mathcal{B}_{r_0}(s_i)))^{1/2}$. By (A5), $\varepsilon_r \to 0$ as $r \to \infty$. Noting that $\{u_n\}$ is bounded in E, we can make this term small by choosing r large. This completes the proof. □

For later use, we give the following two technical lemmas.

Lemma 2.3. *Let* (A4), (A5) *and* (A6) *be satisfied. If $u_n \rightharpoonup u$ in E, then*

$$
\int_{\mathbb{R}} |W_u(t, u_n) - W_u(t, u)|^2 dt \to 0 \quad \text{as } n \to \infty.
$$

Proof. Arguing indirectly, we assume by Lemma 2.2 that there exists a subsequence $\{u_{n_k}\}_{k\in\mathbb{N}}$ such that

$$
u_{n_k} \to u \text{ in } L^2 \text{ and } L^{2\nu-2}, \quad \text{and } u_{n_k} \to u \text{ a.e. in } \mathbb{R} \text{ as } k \to \infty; \tag{2.3}
$$

and

$$
\int_{\mathbb{R}} |W_u(t, u_{n_k}) - W_u(t, u)|^2 dt \ge \epsilon_0, \quad \forall k \in \mathbb{N} \tag{2.4}
$$

for some $\epsilon_0 > 0$. Passing to a subsequence if necessary, we may assume by (2.3) that $\sum_{k=1}^{\infty} \|u_{n_k} - u\|_2 < \infty$ and $\sum_{k=1}^{\infty} \|u_{n_k} - u\|_{2\nu-2} < \infty$. Let $w(t) = \sum_{k=1}^{\infty} |u_{n_k}(t) - u(t)|$ for all $t \in \mathbb{R}$, then $w \in L^2 \cap L^{2\nu-2}$. By (A6), for all $k \in \mathbb{N}$ and $t \in \mathbb{R}$, we have

$$\begin{aligned} |W_u(t, u_{n_k}) - W_u(t, u)|^2 &\le c_1^2(|u_{n_k}| + |u| + |u_{n_k}|^{\nu-1} + |u|^{\nu-1})^2 \\ &\le 4c_1^2(|u_{n_k}|^2 + |u|^2 + |u_{n_k}|^{2\nu-2} + |u|^{2\nu-2}) \\ &\le 12c_1^2(|u_{n_k} - u|^2 + |u|^2) + 4^\nu c_1^2(|u_{n_k} - u|^{2\nu-2} + |u|^{2\nu-2}) \\ &\le 12c_1^2(|w|^2 + |u|^2) + 4^\nu c_1^2(|w|^{2\nu-2} + |u|^{2\nu-2}). \end{aligned}$$

Combining this and (2.3), by Lebesgue's Dominated Convergence Theorem, we have

$$\lim_{k\to\infty} \int_{\mathbb{R}} |W_u(t, u_{n_k}) - W_u(t, u)|^2 dt = 0,$$

which contradicts (2.4). The proof is complete. $\square$

Lemma 2.4. *For any finite dimensional subspace $F \subset E$ there exists a constant $\epsilon > 0$ such that*

$$\operatorname{meas}(\{t \in \mathbb{R} : |u(t)| \ge \epsilon\|u\|\}) \ge \epsilon, \quad \forall u \in F \setminus \{0\}.$$

Proof. We argue indirectly. Assume for any $n \in \mathbb{N}$, there exists $u_n \in F \setminus \{0\}$ such that

$$\operatorname{meas}(\{t \in \mathbb{R} : |u_n(t)| \ge \|u_n\|/n\}) < 1/n.$$

Let $v_n = u_n/\|u_n\| \in F$ for each $n \in \mathbb{N}$, then we have $\|v_n\| = 1$ and

$$\operatorname{meas}(\{t \in \mathbb{R} : |v_n(t)| \ge 1/n\}) < 1/n. \tag{2.5}$$

Passing to a subsequence if necessary, we may assume $v_n \to v_0$ in E for some $v_0 \in F$ since F is of finite dimension. Combining this and (2.2), we have

$$\int_{\mathbb{R}} |v_n - v_0|^2 dt \to 0 \quad \text{as } n \to \infty. \tag{2.6}$$

Noting that $\|v_0\| = 1$, there must exists a constant $\delta_0 > 0$ such that

$$\operatorname{meas}(\{t \in \mathbb{R} : |v_0(t)| \ge \delta_0\}) \ge \delta_0. \tag{2.7}$$

Otherwise, for each fixed $n \in \mathbb{N}$, we have

$$\operatorname{meas}\big(\{t \in \mathbb{R} : |v_0(t)| \ge \frac{1}{n}\}\big) \le \operatorname{meas}\big(\{t \in \mathbb{R} : |v_0(t)| \ge \frac{1}{m}\}\big) \le \frac{1}{m}, \quad \forall m \ge n.$$

Lettin $m \to \infty$, we obtain $\operatorname{meas}(\{t \in \mathbb{R} : |v_0(t)| \ge \frac{1}{n}\}) = 0$. Consequently,

$$\begin{aligned} 0 &\le \operatorname{meas}(\{t \in \mathbb{R} : |v_0(t)| \neq 0\}) \\ &= \operatorname{meas}\big(\cup_{n=1}^{\infty} \{t \in \mathbb{R} : |v_0(t)| \ge \frac{1}{n}\}\big) \\ &\le \sum_{n=1}^{\infty} \operatorname{meas}\big(\{t \in \mathbb{R} : |v_0(t)| \ge \frac{1}{n}\}\big) = 0, \end{aligned}$$

which yields $v_0 = 0$, a contradiction to $\|v_0\| = 1$. Thus (2.7) holds. Set $\mathcal{I}_0 = \{t \in \mathbb{R} : |v_0(t)| \ge \delta_0\}$, where δ_0 is the constant given in (2.7). Also, for any $n \in \mathbb{N}$, let

$$\mathcal{I}_n = \{t \in \mathbb{R} : |v_n(t)| < 1/n\} \quad \text{and} \quad \mathcal{I}_n^c = \mathbb{R}^N \setminus \mathcal{I}_n = \{t \in \mathbb{R} : |v_n(t)| \ge 1/n\}.$$

Then for n large enough, by (2.5) and (2.7), we have

$$\operatorname{meas}(\mathcal{I}_n \cap \mathcal{I}_0) \ge \operatorname{meas}(\mathcal{I}_0) - \operatorname{meas}(\mathcal{I}_n^c) \ge \delta_0 - 1/n \ge \delta_0/2.$$

Consequently, for n large enough, there holds

$$\begin{aligned}\int_{\mathbb{R}} |v_n - v_0|^2 dt &\geq \int_{\mathcal{I}_n \cap \mathcal{I}_0} |v_n - v_0|^2 dt \\ &\geq \int_{\mathcal{I}_n \cap \mathcal{I}_0} (|v_0| - |v_n|)^2 dt \\ &\geq (\delta_0 - 1/n)^2 \operatorname{meas}(\mathcal{I}_n \cap \mathcal{I}_0) \\ &\geq \delta_0^3/8 > 0.\end{aligned}$$

This is in contradiction to (2.6). The proof is complete. □

Now we can define the variational functional Φ associated with (1.1) by

$$\begin{aligned}\Phi(u) &= \frac{1}{2}\int_{\mathbb{R}} (|\dot{u}|^2 + L(t)u \cdot u)dt - \int_{\mathbb{R}} W(t,u)dt \\ &= \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}} W(t,u)dt.\end{aligned} \tag{2.8}$$

By (A6), we have

$$|W(t,u)| \leq c_1(|u|^2 + |u|^\nu), \quad \forall (t,u) \in \mathbb{R} \times \mathbb{R}^N. \tag{2.9}$$

This and (2.2) imply that Φ is well defined on E. Furthermore, a standard argument (see, e.g., [23]) shows that $\Phi \in C^1(E,\mathbb{R})$ with the Frechét derivative given by

$$\langle \Phi'(u), v\rangle = (u,v) - \int_{\mathbb{R}} W_u(t,u) \cdot v dt, \quad \forall u, v \in E, \tag{2.10}$$

and nontrivial critical points of Φ are homoclinic solutions of (1.1).

To study the critical points of the variational functional Φ associated with (1.1), we need the following variant fountain theorem established in [38].

Let E be a Banach space with the norm $\|\cdot\|$ and $E = \overline{\oplus_{j\in\mathbb{N}} X_j}$ with $\dim X_j < \infty$ for any $j \in \mathbb{N}$. Set $Y_k = \oplus_{j=1}^k X_j$ and $Z_k = \overline{\oplus_{j=k}^\infty X_j}$. Consider the following C^1-functional $\Phi_\lambda : E \to \mathbb{R}$ defined by

$$\Phi_\lambda(u) := A(u) - \lambda B(u), \quad \lambda \in [1,2].$$

Theorem 2.5 ([38, Theorem 2.1])**.** *Assume that the above functional Φ_λ satisfies*

- (A10) *Φ_λ maps bounded sets to bounded sets for $\lambda \in [1,2]$, and $\Phi_\lambda(-u) = \Phi_\lambda(u)$ for all $(\lambda,u) \in [1,2] \times E$,*
- (A11) *$B(u) \geq 0$ for all $u \in E$, and $A(u) \to \infty$ or $B(u) \to \infty$ as $\|u\| \to \infty$,*
- (A12) *There exist $\rho_k > \sigma_k > 0$ such that*

$$\alpha_k(\lambda) := \inf_{u\in Z_k,\, \|u\|=\sigma_k} \Phi_\lambda(u) > \beta_k(\lambda) := \max_{u\in Y_k,\, \|u\|=\rho_k} \Phi_\lambda(u), \quad \forall \lambda \in [1,2].$$

Then

$$\alpha_k(\lambda) \leq \zeta_k(\lambda) := \inf_{\gamma\in\Gamma_k} \max_{u\in B_k} \Phi_\lambda(\gamma(u)), \quad \forall \lambda \in [1,2],$$

where $B_k = \{u \in Y_k : \|u\| \leq \rho_k\}$ and $\Gamma_k := \{\gamma \in C(B_k, E) : \gamma$ is odd, $\gamma|_{\partial B_k} = id\}$. Moreover, for almost every $\lambda \in [1,2]$, there exists a sequence $\{u_m^k(\lambda)\}_{m=1}^\infty$ such that

$$\sup_m \|u_m^k(\lambda)\| < \infty, \quad \Phi_\lambda'(u_m^k(\lambda)) \to 0, \quad \Phi_\lambda(u_m^k(\lambda)) \to \zeta_k(\lambda) \quad \text{as } m \to \infty.$$

Choose an orthonormal basis $\{e_j : j \in \mathbb{N}\}$ of E and let $X_j = \operatorname{span}\{e_j\}$ for all $j \in \mathbb{N}$. Define the functionals A, B and Φ_λ on our working space E by

$$A(u) = \frac{1}{2}\|u\|^2, \quad B(u) = \Psi(u) = \int_{\mathbb{R}} W(t,u)dt, \tag{2.11}$$

$$\Phi_\lambda(u) = A(u) - \lambda B(u) = \frac{1}{2}\|u\|^2 - \lambda \int_{\mathbb{R}} W(t,u)dt \tag{2.12}$$

for all $u \in E$ and $\lambda \in [1,2]$. Note that $\Phi_1 = \Phi$, where Φ is the functional defined in (2.8). Then we know that $\Phi_\lambda \in C^1(E,\mathbb{R})$ for all $\lambda \in [1,2]$ and

$$\langle \Phi'_\lambda(u), v\rangle = (u,v) - \lambda \int_{\mathbb{R}} W_u(t,u)\cdot v dt, \quad \forall u,v \in E. \tag{2.13}$$

Before applying Theorem 2.5 to prove our main result, we need to establish the following two lemmas.

Lemma 2.6. *Assume* (A4), (A5) *and* (A6) *hold. Then there exists a positive integer k_1 and a sequence $\sigma_k \to \infty$ as $k \to \infty$ such that*

$$\alpha_k(\lambda) := \inf_{u\in Z_k,\, \|u\|=\sigma_k} \Phi_\lambda(u) > 0, \quad \forall k \ge k_1,$$

where $Z_k = \overline{\oplus_{j=k}^\infty X_j} = \overline{\operatorname{span}\{e_k, \dots\}}$ for all $k \in \mathbb{N}$.

Proof. Note first that (2.9) and (2.12) imply

$$\begin{aligned} \Phi_\lambda(u) &\ge \frac{1}{2}\|u\|^2 - 2\int_{\mathbb{R}} W(t,u)dt \\ &\ge \frac{1}{2}\|u\|^2 - 2c_1(\|u\|_2^2 + \|u\|_\nu^\nu), \quad \forall (\lambda,u) \in [1,2]\times E. \end{aligned} \tag{2.14}$$

For each $k \in \mathbb{N}$, let

$$\ell_2(k) = \sup_{u\in Z_k,\, \|u\|=1} \|u\|_2 \quad \text{and} \quad \ell_\nu(k) = \sup_{u\in Z_k,\, \|u\|=1} \|u\|_\nu. \tag{2.15}$$

Since E is compactly embedded into both L^2 and L^ν by Lemma 2.2, then there hold (cf. [28])

$$\ell_2(k) \to 0, \quad \ell_\nu(k) \to 0 \quad \text{as } k \to \infty. \tag{2.16}$$

Combining (2.14) and (2.15), we have

$$\Phi_\lambda(u) \ge \frac{1}{2}\|u\|^2 - 2c_1\ell_2^2(k)\|u\|^2 - 2c_1\ell_\nu^\nu(k)\|u\|^\nu, \quad \forall (\lambda,u) \in [1,2]\times Z_k. \tag{2.17}$$

In view of (2.16), there exists a positive integer k_1 such that

$$2c_1\ell_2^2(k) \le 1/4, \quad \forall k \ge k_1. \tag{2.18}$$

For each $k \ge k_1$, choose

$$\sigma_k := (16c_1\ell_\nu^\nu(k))^{1/(2-\nu)}. \tag{2.19}$$

Then it follows from (2.16) that

$$\sigma_k \to +\infty \quad \text{as } k \to \infty \tag{2.20}$$

since $\nu > 2$. Besides, by (2.17)–(2.19), direct computation shows

$$\alpha_k(\lambda) := \inf_{u\in Z_k,\, \|u\|=\sigma_k} \Phi_\lambda(u) \ge \sigma_k^2/8 > 0, \quad \forall k \ge k_1.$$

The proof is complete. $\square$

Lemma 2.7. *Suppose that* (A4)*,* (A5)*,* (A6) *and* (A7) *are satisfied. Then for the positive integer* k_1 *and the sequence* $\{\sigma_k\}$ *obtained in Lemma 2.6, there exists* $\rho_k > \sigma_k$ *for each* $k \geq k_1$ *such that*

$$\beta_k(\lambda) := \max_{u \in Y_k,\, \|u\| = \rho_k} \Phi_\lambda(u) < 0,$$

where $Y_k = \oplus_{j=1}^k X_j = \operatorname{span}\{e_1, \dots, e_k\}$ *for all* $k \in \mathbb{N}$*.*

Proof. Note that Y_k is finite dimensional for each $k \in \mathbb{N}$. Then by Lemma 2.4, for each $k \in \mathbb{N}$, there exists a constant $\epsilon_k > 0$ such that

$$\operatorname{meas}(\mathcal{I}_u^k) \geq \epsilon_k, \quad \forall u \in Y_k \setminus \{0\}, \tag{2.21}$$

where $\mathcal{I}_u^k := \{t \in \mathbb{R} : |u(t)| \geq \epsilon_k \|u\|\}$ for all $k \in \mathbb{N}$ and $u \in Y_k \setminus \{0\}$. By (A7), for each $k \in \mathbb{N}$, there exists a constant $b_k > 0$ such that

$$W(t,u) \geq u^2/\epsilon_k^3, \quad \forall t \in \mathbb{R} \text{ and } |u| \geq b_k. \tag{2.22}$$

Combining (2.12), (2.21) and (2.22), for any $k \in \mathbb{N}$ and $\lambda \in [1,2]$, we have

$$\begin{aligned}
\Phi_\lambda(u) &\leq \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}} W(t,u)dt &(2.23)\\
&\leq \frac{1}{2}\|u\|^2 - \int_{\mathcal{I}_u^k} (|u|^2/\epsilon_k^3)dt &(2.24)\\
&\leq \frac{1}{2}\|u\|^2 - \epsilon_k^2 \|u\|^2 \operatorname{meas}(\Lambda_u^k)/\epsilon_k^3 &(2.25)\\
&\leq \frac{1}{2}\|u\|^2 - \|u\|^2 = -\frac{1}{2}\|u\|^2 &(2.26)
\end{aligned}$$

for all $u \in Y_k$ with $\|u\| \geq b_k/\epsilon_k$. Here we use the fact that $W(t,u) \geq 0$ for all $(t,u) \in \mathbb{R} \times \mathbb{R}^N$. For each $k \geq k_1$, if we choose $\rho_k > \max\{\sigma_k, b_k/\epsilon_k\}$, then (2.26) implies

$$\beta_k(\lambda) := \max_{u \in Y_k,\, \|u\| = \rho_k} \Phi_\lambda(u) \leq -\rho_k^2/2 < 0.$$

The proof is complete. □

Now we are in a position to give the proof of our main result.

Proof of Theorem 1.1. Firstly, from (2.2), (2.9) and (2.12) it follows that Φ_λ maps bounded sets to bounded sets uniformly for $\lambda \in [1,2]$. Evidently, (A9) implies that $\Phi_\lambda(-u) = \Phi_\lambda(u)$ for all $(\lambda, u) \in [1,2] \times E$. Thus (A10) holds. Next, using again the fact that $W(t,u) \geq 0$ for all $(t,u) \in \mathbb{R} \times \mathbb{R}^N$, we know that (A11) holds by the definition of functional A in (2.11). Finally, Lemma 2.6 and Lemma 2.7 show that (A12) holds for all $k \geq k_1$, where k_1 is given in Lemma 2.6. Therefore, for each $k \geq k_1$, by Theorem 2.5, for almost every $\lambda \in [1,2]$, there exists a sequence $\{u_m^k(\lambda)\}_{m=1}^\infty \subset E$ such that

$$\sup_m \|u_m^k(\lambda)\| < \infty, \ \Phi_\lambda'(u_m^k(\lambda)) \to 0 \quad \text{and} \quad \Phi_\lambda(u_m^k(\lambda)) \to \zeta_k(\lambda) \quad \text{as } m \to \infty, \tag{2.27}$$

where

$$\zeta_k(\lambda) := \inf_{\gamma \in \Gamma_k} \max_{u \in B_k} \Phi_\lambda(\gamma(u)), \quad \forall \lambda \in [1,2]$$

with $B_k = \{u \in Y_k : \|u\| \leq \rho_k\}$ and $\Gamma_k := \{\gamma \in C(B_k, E) : \gamma \text{ is odd}, \gamma|_{\partial B_k} = id\}$. From the proof of Lemma 2.6, we infer that

$$\zeta_k(\lambda) \in \left[\bar{\alpha}_k, \bar{\zeta}_k\right], \quad \forall k \geq k_1 \text{ and } \lambda \in [1,2], \tag{2.28}$$

where $\bar{\zeta}_k := \max_{u\in B_k}\Phi_1(u)$ and $\bar{\alpha}_k := \sigma_k^2/4 \to \infty$ as $k \to \infty$ by (2.20). In view of (2.27), for each $k \geq k_1$, we can choose a sequence $\lambda_n \to 1$ (depending on k) and get the corresponding sequences satisfying

$$\sup_m \|u_m^k(\lambda_n)\| < \infty \quad \text{and} \quad \Phi'_{\lambda_n}(u_m^k(\lambda_n)) \to 0 \quad \text{as } m \to \infty. \tag{2.29}$$

Claim 1. For each λ_n given above, the sequence $\{u_m^k(\lambda_n)\}_{m=1}^\infty$ has a strong convergent subsequence.

For notational simplicity, we will set $u_m = u_m^k(\lambda_n)$ for $m \in \mathbb{N}$ throughout the proof of Claim 1. By (2.29), without loss of generality, we may assume that

$$u_m \rightharpoonup u \quad \text{as } m \to \infty \tag{2.30}$$

for some $u \in E$. Invoking (2.13), we have

$$\begin{aligned} \|u_m - u\|^2 &= \langle \Phi'_{\lambda_n}(u_m), u_m - u\rangle - \langle \Phi'_{\lambda_n}(u), u_m - u\rangle \\ &\quad + \lambda_n \int_{\mathbb{R}} (W_u(t, u_n(t)) - W_u(t, u)) \cdot (u_m - u) dt. \end{aligned} \tag{2.31}$$

By (2.29), we have

$$\langle \Phi'_{\lambda_n}(u_m), u_m - u\rangle \to 0 \quad \text{as } m \to \infty. \tag{2.32}$$

Moreover, (2.30) yields

$$\langle \Phi'_{\lambda_n}(u), u_m - u\rangle \to 0 \quad \text{as } m \to \infty. \tag{2.33}$$

By (2.2), Lemma 2.3 and the Hölder inequality, we have

$$\begin{aligned} &\Big| \int_{\mathbb{R}} (W_u(t, u_n(t)) - W_u(t, u)) \cdot (u_m - u) dt \Big| \\ &\leq \Big(\int_{\mathbb{R}} |W_u(t, u_n) - W_u(t, u)|^2 dt \Big)^{1/2} \|u_m - u\|_2 \\ &\leq c_2 \Big(\int_{\mathbb{R}} |W_u(t, u_n) - W_u(t, u)|^2 dt \Big)^{1/2} \|u_m - u\| \to 0 \quad \text{as } m \to \infty, \end{aligned} \tag{2.34}$$

where c_2 is the constant given in (2.2). Here we use the fact that $\{u_m\}$ is bounded in E. Combining (2.31) and (2.32)–(2.34), we obtain $u_m \to u$ in E. Thus Claim 1 holds.

By Claim 1, without loss of generality, we may assume that

$$\lim_{m\to\infty} u_m^k(\lambda_n) = u_n^k \in E, \quad \forall n \in \mathbb{N} \text{ and } k \geq k_1. \tag{2.35}$$

This, (2.27) and (2.28) imply

$$\Phi'_{\lambda_n}(u_n^k) = 0, \quad \Phi_{\lambda_n}(u_n^k) \in [\bar{\alpha}_k, \bar{\zeta}_k], \quad \forall n \in \mathbb{N} \text{ and } k \geq k_1. \tag{2.36}$$

Claim 2. For each $k \geq k_1$, the sequence $\{u_n^k\}_{n=1}^\infty$ in (2.35) is bounded.

As in the proof of Claim 1, for notational simplicity, we set $u_n = u_n^k$ for all $n \in \mathbb{N}$. We use a indirect argument. If Claim 2 is not true, without loss of generality, we may assume that

$$\|u_n\| \to \infty \quad \text{and} \quad w_n := \frac{u_n}{\|u_n\|} \rightharpoonup w \in E \quad \text{as } n \to \infty. \tag{2.37}$$

By (2.37) and Lemma 2.2, passing to a subsequence if necessary, we have

$$w_n \to w \quad \text{in } L^p \quad \text{for } 2 \leq p < \infty, \tag{2.38}$$

$$w_n(t) \to w(t) \quad \text{a.e. } t \in \mathbb{R}. \tag{2.39}$$

When $w \neq 0$ occurs, $\Theta := \{t \in \mathbb{R} : w(t) \neq 0\}$ has a positive Lebesgue measure. By (2.37), it holds that

$$u_n(t) \to \infty, \quad \forall t \in \Theta. \tag{2.40}$$

Combining (2.12), (2.39), (2.40) and (A7), by Fatou's Lemma, we have

$$\begin{aligned}
\frac{1}{2} - \frac{\Phi_{\lambda_n}(u_n)}{\|u_n\|^2} &= \lambda_n \int_{\mathbb{R}} \frac{W(t,u_n)}{\|u_n\|^2} dt \\
&\geq \int_{\Theta} |w_n|^2 \frac{W(t,u_n)}{|u_n|^2} dt \to +\infty \quad \text{as } n \to \infty,
\end{aligned}$$

a contradiction to (2.36) and (2.37).

When $w = 0$ occurs, as in [14], we choose a sequence $\{s_n\} \subset [0,1]$ such that

$$\Phi_{\lambda_n}(s_n u_n) = \max_{s \in [0,1]} \Phi_{\lambda_n}(s u_n). \tag{2.41}$$

For $M > 0$, let $\widetilde{w}_n := \sqrt{4M} w_n = \frac{\sqrt{4M}}{\|u_n\|} u_n$, then (2.38) yields

$$\widetilde{w}_n \to \sqrt{4M} w = 0 \quad \text{in } L^p \quad \text{for } 2 \leq p < \infty. \tag{2.42}$$

This (2.9) and (2.38) imply

$$\Big| \int_{\mathbb{R}} W(t, \widetilde{w}_n) dt \Big| \leq c_1 \int_{\mathbb{R}} (|\widetilde{w}_n|^2 + |\widetilde{w}_n|^\nu) dt \to 0 \quad \text{as } n \to \infty. \tag{2.43}$$

Note that $0 < \frac{\sqrt{4M}}{\|u_n\|} < 1$ holds by (2.37) for n large enough. Combining this with (2.12) and (2.41), we obtain

$$\begin{aligned}
\Phi_{\lambda_n}(s_n u_n) &\geq \Phi_{\lambda_n}(\widetilde{w}_n) \\
&= \frac{1}{2} \|\widetilde{w}_n\|^2 - \lambda_n \int_{\mathbb{R}} W(t, \widetilde{w}_n) dt \\
&= 2M - \lambda_n \int_{\mathbb{R}} W(t, \widetilde{w}_n) dt \geq M.
\end{aligned}$$

for n large enough. It follows that $\lim_{n\to\infty} \Phi_{\lambda_n}(s_n u_n) = +\infty$. Observing that $\Phi_{\lambda_n}(0) = 0$ and $\Phi_{\lambda_n}(u_n) \in [\bar{\alpha}_k, \bar{\zeta}_k$ in (2.36), we know that $s_n \in (0,1)$ in (2.41) for n large enough. Hence,

$$0 = s_n \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=s_n} \Phi_{\lambda_n}(s u_n) = \langle \Phi'_{\lambda_n}(s_n u_n), s_n u_n \rangle. \tag{2.44}$$

Combining (2.12), (2.13), (2.36), (2.44) and (A8), we have

$$\begin{aligned}
\Phi_{\lambda_n}(u_n) &= \Phi_{\lambda_n}(u_n) - \frac{1}{2} \langle \Phi'_{\lambda_n}(u_n), u_n \rangle \\
&= \frac{\lambda_n}{2} \int_{\mathbb{R}} \widetilde{W}(t, u_n) dt \\
&\geq \frac{\lambda_n}{2\vartheta} \int_{\mathbb{R}} \widetilde{W}(t, s_n u_n) dt \\
&= \frac{1}{\vartheta} \Phi_{\lambda_n}(s_n u_n) - \frac{1}{2\vartheta} \langle \Phi'_{\lambda_n}(s_n u_n), s_n u_n \rangle \\
&= \frac{1}{\vartheta} \Phi_{\lambda_n}(s_n u_n) \to +\infty \quad \text{as } n \to \infty.
\end{aligned}$$

where ϑ is the constant in (A8). This also provides a contradiction to (2.36). Thus Claim 2 is true.

In view of Claim 2 and (2.36), for each $k \geq k_1$, using the similar arguments in the proof of Claim 1, we can also show that the sequence $\{u_n^k\}_{n=1}^{\infty}$ has a strong convergent subsequence with the limit u^k being just a critical point of $\Phi = \Phi_1$. Evidently, $\Phi(u^k) \in \left[\bar{\alpha}_k, \bar{\zeta}_k\right]$ for all $k \geq k_1$. Since $\bar{\alpha}_k \to +\infty$ as $k \to \infty$ in (2.28), we obtain infinitely many nontrivial critical points of Φ. Therefore, (1.1) possesses infinitely many nontrivial solutions. The proof of Theorem 1.1 is complete. $\square$

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