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NUMERICAL SOLUTIONS TO HEAT EQUATIONS VIA THE SPECTRAL METHOD

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ABSTRACT. In this article we study a discretized version of the heat equation. For the time semi-discrete problem, we use an implicit Euler's scheme, and for the space discretization we used the spectral method. We estimate for the error between the exact and approximated discrete solutions, and illustrate the features of our method with numerical examples.

1. Introduction

Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ be a bounded connected domain, with boundary $\Gamma = \partial\Omega$ that is assumed Lipschitz continuous, and let T be a positive real number. We consider the heat equation

$$
\begin{gathered}
\frac{\partial u}{\partial t} - \Delta u = f \quad \text{in } \Omega\times]0,T[, \\
u = 0 \quad \text{on } \partial\Omega\times]0,T[, \\
u(\cdot,0) = u_0 \quad \text{in } \Omega,
\end{gathered}
\tag{1.1}
$$

where $f \in L^2(\Omega\times]0,T[)$ and $u_0 \in L^2(\Omega)$ are given, and u is the unknown function.

This model was studied in [1] using the finite element discretization. The aim of this work is to extend this study to the spectral discretization method known for its high precision [4]. The spectral element method is used for the discretization of elliptic equations with discontinuous coefficients. This method is also used for the heat diffusion in an inhomogeneous medium, see [2]. A posteriori analysis of the spectral discretization of the heat equation is presented in [6].

We begin by doing the time discretization of the heat equation using an implicit Euler scheme. The existence and uniqueness of the solution are established, and an error estimate of order 1 in time is presented. Then, we use the spectral method for space discretization and we prove an error estimate.

This article is organized as follows. We begin by presenting in section 2 the variational formulation of the problem and the proof of existence and uniqueness of the solution. Section 3 is devoted to the time and space discretization problems. The error estimates on time and space are proved in section 4. Finally some numerical results are presented in section 5.

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2. Continuous problem

We first study the variational formulation of problem (1.1). Then we present the existence and uniqueness of the solution. In the following, we use the classical Sobolev spaces $H^s(\Omega)$ and $H_0^s(\Omega)$ endowed with the norm $\|\cdot\|_{H^s(\Omega)}$ and the seminorm $|\cdot|_{H^s(\Omega)}$. We use the following notation (see [9, 10]):

- $(\cdot,\cdot)$ is the scalar product defined on $L^2(\Omega)$ and by extension, the duality paring between $H^{-s}(\Omega)$ and $H_0^s(\Omega)$,
- $L_0^2(\Omega)$ is the space of functions in $L^2(\Omega)$ with a null integral,
- $C^0(0,T,X)$ the space of continuous functions from $[0,T]$ with values in X a Banach space with associated norm $\|\cdot\|_X$,
- $L^2(0,T,X)$ the space of square integrable functions from $[0,T]$ with values in X with associated norm

$$\|v\|_{L^2(0,t,X)} = \int_0^t \|v(.,s)\|_X^2 ds.$$

2.1. Variational formulation.

2.1. Variational formulation. Problem (1.1) admits the following variational formulation. Find $u \in C^0(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$ such that for all $t \in]0,T[$,

$$u(\cdot,0) = u_0 \quad \text{in } \Omega; \tag{2.1}$$

$$\int_\Omega \frac{\partial u}{\partial t}(x,t)v(x)dx + \int_\Omega \nabla u(x,t)\nabla v(x)dx = \int_\Omega f(x,t)v(x)dx \quad \forall v \in H_0^1(\Omega). \tag{2.2}$$

It is shown in [8, Theorem 7.2.1], [7], [11] that problem (2.1)-(2.2) has a unique solution $u \in C^0(0,T,L^2(\Omega)) \cap L^2(0,T,H_0^1(\Omega))$.

By choosing $v = u$ in (2.2) and integrating on $]0,t[$, $0 \le t \le T$, we deduce the following stability condition:

$$[u](t) \le C\Big(\|u_0\|_{L^2(\Omega)}^2 + \|f\|_{L^2(0,t,H^{-1}(\Omega))}^2\Big)^{1/2}, \tag{2.3}$$

where $[\cdots]$ is a norm defined for all $w \in L^2(0,t,H_0^1(\Omega))$ by

$$[w](t) = \Big(\|w\|_{L^2(\Omega)}^2 + \|w\|_{L^2(0,t,H_0^1(\Omega))}^2\Big)^{1/2} \tag{2.4}$$

and C is a constant depending only on the domain Ω.

In the same way by taking $v = \frac{\partial u}{\partial t}$ in (2.2) and if $u_0 \in H_0^1(\Omega)$ we obtain

$$|u(t)|_{H_0^1(\Omega)}^2 + \|\frac{\partial u}{\partial t}\|_{L^2(0,T,L^2(\Omega))}^2 \le |u_0|_{H_0^1(\Omega)}^2 + \|f\|_{L^2(\Omega\times(0,T))}^2 \tag{2.5}$$

which implies that u is in $L^2(0,T,H^2(\Omega))$.

3. Discrete problem

3.1. Time discretization. Let T be a fixed positive number and f be a function in $C(0,T,H^{-1}(\Omega))$. We consider the partition of $[0,T]$ into I equal subintervals $[t_{i-1},t_i]$, $1 \le i \le I$ with $t_0 < t_1 < \cdots < t_I = T$ and $\delta t = t_i - t_{i-1} = T/I$.

Let $W_{\delta t}$ be the space of functions $v_{\delta t}$ which are continuous on $[0,T]$ and affine on each subinterval $[(i-1)\delta t, i\delta t]$, $0 \le i \le I$, with values in $L^2(\Omega)$. For each family $v^i \in L^2(\Omega)$, $0 \le i \le I$, we associate the function $v_{\delta t} \in W_{\delta t}$ equal to v^i on $i\delta t$, $0 \le i \le I$. This function is written for $0 \le i \le I$,

$$v_{\delta t} = v^i + \frac{t_i - t}{\delta t}(v^i - v^{i-1}), \quad \forall t \in [t_{i-1},t_i]. \tag{3.1}$$

For simplicity, we denote by $f^i = f(x, t_i)$. Using the implicit Euler scheme, the time discrete problem is written

$$\begin{gathered}\frac{u^i - u^{i-1}}{\delta t} - \Delta u^i = f^i \quad \text{in } \Omega,\ 1 \le i \le I,\\ u^i = 0 \quad \text{on } \partial\Omega,\ 1 \le i \le I,\\ u^0 = u_0 \quad \text{in } \Omega.\end{gathered} \tag{3.2}$$

This problem admits the following equivalent variational formulation. Find u^i in $L^2(\Omega) \times H_0^1(\Omega)^I$ $(1 \le i \le I)$ which satisfies

$$u^0 = u_0 \quad \text{in } \Omega, \tag{3.3}$$

and for each $v \in H_0^1(\Omega)$,

$$\begin{aligned}&\int_\Omega u^i(x)v(x)dx + \delta t \int_\Omega \nabla u^i(x)\nabla v(x)dx\\ &= \int_\Omega u^{i-1}(x)v(x)dx + \delta t \int_\Omega f^i(x)v(x)dx.\end{aligned} \tag{3.4}$$

Using the Lax-Milgram theorem, we deduce that problem (3.3)-(3.4) has a unique solution $u^i \in (H_0^1(\Omega))^I$, $1 \le i \le I$.

Moreover, by taking $v = u^i$ in (3.4) we obtain

$$\|u^i\|^2_{L^2(\Omega)} + \delta t \|\nabla u^i\|^2_{L^2(\Omega)} \le \|u^{i-1}\|^2_{L^2(\Omega)} + \delta t \|f^i\|^2_{H^{-1}(\Omega)}. \tag{3.5}$$

Summing with respect to i, we deduce the following stability condition on the solution u^i, $1 \le i \le I$:

$$\|u^i\|^2_{L^2(\Omega)} + \delta t \sum_{l=1}^{i} \|u^l|^2_{H^1(\Omega)} \le C\Big(\|u^0\|^2_{L^2(\Omega)} + \delta t \sum_{l=1}^{i} \|f^l\|^2_{H^{-1}(\Omega)}\Big), \tag{3.6}$$

where C is a constant independent of δt.

For $v_{\delta t}$ in $W_{\delta t}$ and if $u^i \in (H^1(\Omega))^I$, we define the norm

$$[|v_{\delta t}|](t_i) = \Big(\|v^i\|^2_{L^2(\Omega)} + \delta t \sum_{l=1}^{i} \|v^l|^2_{H^1(\Omega)}\Big)^{1/2}.$$

We remark that

$$[v_{\delta t}](t_i) = \Big(\|v^i\|^2_{L^2(\Omega)} + \int_0^{t_i} |v_{\delta t}(\cdot, s)|^2_{H^1(\Omega)} ds\Big)^{1/2}.$$

Considering that

$$\int_0^{t_i} |v_{\delta t}(., s)|^2_{H^1(\Omega)} ds = \sum_{l=1}^{i} \int_{t_{l-1}}^{t_l} |v_{\delta t}(., s)|^2_{H^1(\Omega)} ds$$

and using (3.1) we obtain

$$\delta t |v^{l-1}|^2_{H^1(\Omega)} \le \int_{t_{l-1}}^{t_l} |v_{\delta t}(., s)|^2_{H^1(\Omega)} ds \le \delta t |v^l|^2_{H^1(\Omega)}.$$

Computing the sum, we show finally that there exist two constants C_1 and C_2 independent of δt such that

$$C_1[v_{\delta t}](t_i) \le [|v_{\delta t}|](t_i) \le C_2[v_{\delta t}](t_i). \tag{3.7}$$

3.1.1. *A priori error estimates.* Let $e^i = u(t_i) - u^i$, $1 \leq i \leq I$ and $e^0 = 0$. By writing problem (2.2) for $t = t_i$ we obtain

$$\int_\Omega \frac{\partial u}{\partial t}(x,t_i)v(x)dx + \int_\Omega \nabla u(x,t_i)\nabla v(x)dx = \int_\Omega f(x,t_i)v(x)dx.$$

Next, observing that

$$\int_\Omega \Big(\int_{t_{i-1}}^{t_i} \frac{\partial u}{\partial t}(x,t)\Big)v(x)dx = \int_\Omega (u(x,t_i) - u(x,t_{i-1}))v(x)dx$$

and using (3.1), we deduce that for all $v \in H_0^1(\Omega)$, the sequence $(e^i)_{1\leq i\leq I}$ is solution of the problem

$$\begin{aligned} &\int_\Omega e^i(x)v(x)dx + \delta t\int_\Omega \nabla e^i(x)\nabla v(x)dx \\ &= \int_\Omega e^{i-1}(x)v(x)dx + \delta t\Big(\frac{1}{\delta t}\int_\Omega \Big(\int_{t_{i-1}}^{t_i} \frac{\partial u}{\partial t}(x,t)dt - \frac{\partial u}{\partial t}(x,t_i)\Big)v(x)dx\Big). \end{aligned} \tag{3.8}$$

Considering (3.6) with $f^i = \frac{1}{\delta t}\Big(\int_{t_{i-1}}^{t_i} \frac{\partial u}{\partial t}(x,t)dt - \frac{\partial u}{\partial t}(x,t_i)\Big)$ and using the fact that $\frac{\partial^2 u}{\partial t^2} \in L^2(0,T,H^{-1}(\Omega))$ we deduce

$$[|u - u_{\delta t}|](t_i) \leq C\delta t\|\frac{\partial^2 u}{\partial t^2}\|^2_{L^2(0,t_i,H^{-1}(\Omega))} \tag{3.9}$$

where C is a constant independent of δt. Using the norm equivalence (3.7), we conclude that there exists a constant C independent of δt such that

$$[u - u_{\delta t}](t_i) \leq C\delta t\|u\|^2_{H^2(0,t_i,H^{-1}(\Omega))}. \tag{3.10}$$

3.2. **Space discretization.** We suppose throughout this part that Ω is a rectangle for $d = 2$ or a parallelepiped for $d = 3$.

For a positive integer N, we denote by $\mathbb{P}_N(\Omega)$ the set of polynomials with d variables and degree $\leq N$ for each variable. Let $\xi_0 = -1$ and $\xi_N = 1$, we define $(N-1)$ nodes ξ_j, $1 \leq j \leq (N-1)$ (which are the zeros of the polynomial L_N' where L_N is the Legendre polynomial) and $(N+1)$ weights ρ_j, $0 \leq j \leq N$ satisfying the Gauss-Lobatto quadrature formula on $]-1,1[$

$$\int_{-1}^1 \psi_N dx = \sum_{j=0}^N \psi_N(\xi_j)\rho_j \quad, \forall \psi_N \in \mathbb{P}_{2N-1}(]-1,1[). \tag{3.11}$$

We recall the following formula (see [5]):

$$\|\psi_N\|^2_{L^2(]-1,1[)} \leq \sum_{j=0}^N \psi_N^2(\xi_j)\rho_j \leq 3\|\psi_N\|^2_{L^2]-1,1[}, \quad \forall \psi_N \in \mathbb{P}_N(]-1,1[). \tag{3.12}$$

Let F be the affine application which transforms $]-1,1[^d$ $(d = 2,3)$ to Ω. We introduce the discrete scalar product

$$(u,v)_N = \begin{cases} \frac{\text{meas}(\Omega)}{4}\sum_{i=0}^N \sum_{j=0}^N u\circ F(\xi_i,\xi_j)v\circ F(\xi_i,\xi_j)\rho_i\rho_j, & d=2 \\ \frac{\text{meas}(\Omega)}{8}\sum_{i=0}^N \sum_{j=0}^N \sum_{k=0}^N u\circ F(\xi_i,\xi_j,\xi_k)v\circ F(\xi_i,\xi_j,\xi_k)\rho_i\rho_j\rho_k, & d=3 \end{cases}$$

We suppose that u_0 and f are respectively continuous on Ω and $\Omega\times]0,T[$. Using the Galerkin method and numerical integration, the space discrete problem of (3.4) is written as follows.

Find $u_N^i \in (\mathbb{P}_N^0(\Omega))^I$, $1 \le i \le I$ such that

$$u_N^0 = \mathcal{I}_N(u_0) \quad \text{in } \Omega \tag{3.13}$$

and for $1 \le i \le I$,

$$(u_N^i, v_N^i)_N + \delta t(\nabla u_N^i, \nabla v_N^i)_N = (u_N^{i-1}, v_N)_N + \delta t(f^i, v_N)_N, \quad \forall v_N \in \mathbb{P}_N^0(\Omega), \tag{3.14}$$

where $\mathcal{I}_N$ is the interpolation operator from $L^2(\Omega)$ to $\mathbb{P}_N(\Omega)$.

Proposition 3.1. *Problem* (3.13)-(3.14) *has a unique solution* $(u_N^0, u_N^i)_{1\le i\le I}$ *in* $\mathbb{P}_N(\Omega) \times (\mathbb{P}_N^0(\Omega))^i$ *and there exists a positive constant* C *independent of* N *such that*

$$\|u_N^i\|_{L^2(\Omega)}^2 + \delta t \sum_{l=1}^{i} |u^l|_{H^1(\Omega)}^2 \le C\Big(\|\mathcal{I}_N(u_0)\|_{L^2(\Omega)}^2 + \delta t \sum_{l=1}^{i} \|\mathcal{I}_N(f^l)\|_{L^2(\Omega)}^2\Big) \tag{3.15}$$

Proof. Using the Lax-Milgram theorem, the Cauchy-Schwarz inequality and inequality (3.12), we show that (3.13)-(3.14) has a unique solution $(u_N^0, u_N^i)_{1\le i\le I} \in \mathbb{P}_N(\Omega) \times (\mathbb{P}_N^0(\Omega))^i$.

To prove (3.15), we begin by choosing $v_N = u_N^i$ in (3.14) and using the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} &(u_N^i, u_N^i)_N + \delta t(\nabla u_N^i, \nabla u_N^i)_N \\ &\le (u_N^{i-1}, u_N^{i-1})_N^{1/2}(u_N^i, u_N^i)_N^{1/2} + \delta t(\mathcal{I}_N f^i, \mathcal{I}_N f^i)_N^{1/2}(u_N^i, u_N^i)_N^{1/2}. \end{aligned} \tag{3.16}$$

Using the inequality $ab \le \frac{a^2}{2} + \frac{b^2}{2}$, the Poincaré-Friedrichs inequality and the inequality (3.12), relation (3.16) becomes

$$\frac{1}{2}\|u_N^i\|_{L^2(\Omega)}^2 + \delta t\|\nabla u_N^i\|_{L^2(\Omega)}^2 \le \frac{\|u_N^{i-1}\|_{L^2(\Omega)}^2}{2} + \delta t\Big(\frac{\|\mathcal{I}_N(f^i)\|_{L^2(\Omega)}^2}{2} + C\|\nabla u_N^i\|_{L^2(\Omega)}^2\Big)$$

where C is the Poincaré-Friedrichs constant. Finally, computing the sum on i, we deduce the inequality (3.15). □

4. Error estimate

We establish now the error estimate between the solution u of the continuous problem (1.1) and the solution $(u_N^i)_{0\le i\le I}$ of the discrete problem (3.14). We define $u_{N\delta t}$ the function in $W_{\delta t}$ equal to u_N^i on $i\delta t$.

Theorem 4.1. *If* f *and* u_0 *are respectively continuous on* $\overline{\Omega} \times [0, T]$ *and* $\overline{\Omega}$, *we have the following error estimates: There exists a constant* C *independent of* δt *and* N *such that*

$$\begin{aligned} &[|u_{\delta t} - u_{N\delta t}|](t_i) \\ &\le C\Big(\inf_{\substack{v_n^l \in \mathbb{P}_N(\Omega)\times\mathbb{P}_{N-1}^0(\Omega) \\ 0\le l\le I}} \Big([|u_{\delta t} - v_{N\delta t}|](t_i) + \|u_0 - v_N^0\|_{L^2(\Omega)}\Big) + \|u_0 - \mathcal{I}_N u_0\|_{L^2(\Omega)} \\ &\quad + \Big(\delta t \sum_{l=1}^{i} \Big[\inf_{f_N^l \in \mathbb{P}_{N-1}(\Omega)} \|f^l - f_N^l\|_{L^2(\Omega)} + \|f^l - \mathcal{I}_N f^l\|_{L^2(\Omega)}\Big]\Big)^{1/2}\Big) \end{aligned} \tag{4.1}$$

Proof. Let $(v_N^i)_{0\leq i\leq N} \in \mathbb{P}_N(\Omega)\times(\mathbb{P}_{N-1}^0(\Omega))^I$. Considering that $(u_N^i)_{0\leq i\leq I}$ and $(u^i)_{0\leq i\leq I}$ are respectively the solutions of (3.14) and (3.4) and taking into account that the quadrature formula (3.11) is exact for polynomials with degree $\leq 2N-1$, we deduce that for all $w_N\in\mathbb{P}_N(\Omega)$

$$\begin{aligned}
&(u_N^i - v_N^i, w_N)_N + \delta t(\nabla(u_N^i - v_N^i), \nabla w_N)_N \\
&= (u_N^{i-1} - v_N^{i-1}, w_N)_N + \int_\Omega (u^i - v_N^i)(x) w_N(x)dx \\
&\quad - \int_\Omega (u^{i-1} - v_N^{i-1})(x) w_N(x)dx + \delta t\int_\Omega \nabla(u^i - v_N^i)(x)\nabla w_N dx \\
&\quad + \delta t(f^i, w_N)_N - \delta t\int_\Omega f^i(x) w_N(x)dx
\end{aligned} \tag{4.2}$$

By introducing f_{N-1}^l in $\mathbb{P}_{N-1}(\Omega)$, then following the exactness of the quadrature formula (3.11), for all $w_N\in\mathbb{P}_N(\Omega)$,

$$\int_\Omega f^l(x)w_N(x)dx - (f^l, w_N)_N = \int_\Omega (f^l - f_{N-1}^l)(x)w_N(x)dx - (\mathcal{I}_N f^l - f_{N-1}^l, w_N)_N \tag{4.3}$$

Summing over i in (4.3) and using the fact that

$$\|u_{\delta t} - u_{N\delta t}\| \leq \|u_{\delta t} - v_{N\delta t}\| + \|u_{N\delta t} - v_{N\delta t}\|$$

we obtain (4.1). $\square$

We define Π_N (respectively $\Pi_N^{1,0}$) the orthogonal projection operator from $L^2(\Omega)^d$ (respectively $H_0^1(\Omega)^d$) onto $\mathbb{P}_N(\Omega)$ [5]. To obtain the order of convergence with respect to N, we take $v_N^l = \Pi_N u_0 \times \Pi_N^{1,0} u_l$ and $f_{N-1}^l = \Pi_{N-1} f^l$; $1\leq l\leq i$ in (4.1) and we use the polynomial approximation and interpolation results [3]. We obtain the following theorem.

Theorem 4.2. *Assume that $f\in C^0([0,T]\times H^\sigma(\Omega))$ and $u_0\in H^\sigma(\Omega)$, $\sigma > \frac{d}{2}$. Let $(u^i)_{0\geq i\leq I}\in H^s(\Omega)$, $s\geq 1$ be the solution of (3.14). Then there exists $C>0$ independent of δt and N such that*

$$\begin{aligned}
\|u_{\delta t} - u_{N\delta t}\|(t_i) \leq C\Big(N^{1-s}\Big(\|u^i\|^2_{H^s(\Omega)} + \delta t\sum_{l=1}^i \|u^l\|^2_{H^s(\Omega)}\Big)^{1/2} \\
+ N^{-\sigma}\Big(\|u_0\|_{H^s(\Omega)} + \|f\|_{C^0([0,T],H^s(\Omega))}\Big)\Big).
\end{aligned}$$

Let u be the solution of (1.1). By writing

$$[|u - u_{N\delta t}|] \leq [|u - u_{\delta t}|] + [|u_{\delta t} - u_{N\delta t}|]$$

and using the property (3.7) we obtain the following property.

Corollary 4.3. *Let $f\in L^2(0,T,H^\sigma(\Omega))$ and $u_0\in L^2(0,T,H^\sigma(\Omega))$. If we assume that the solution u of problem (1.1) is in $L^2(0,T,H^s(\Omega))$ then*

$$\begin{aligned}
&[u - u_{N\delta t}](t_i) \\
&\leq C_1(\delta t + N^{1-s})\|u\|_{L^2(0,t_i,H^s(\Omega))} + C_2(\delta t + N^{-\sigma})(\|f\|_{L^2(0,t_i,H^\sigma(\Omega))} + \|u_0\|_{H^\sigma(\Omega)}).
\end{aligned}$$

where C_1 and C_2 are constants independent of δt and N.

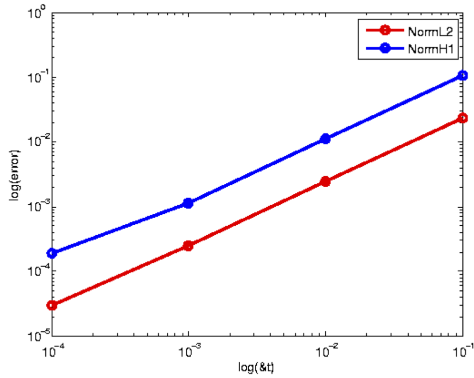

FIGURE 1. Error on time for the solution (5.1)

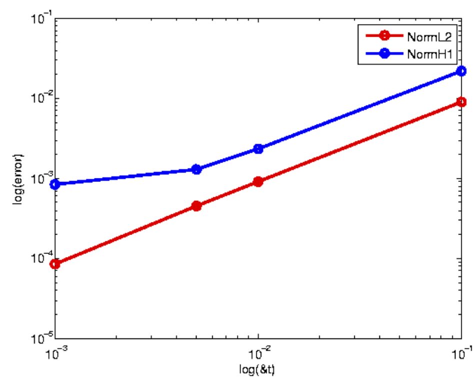

FIGURE 2. Error on time for the solution (5.2)

5. Numerical results

Let h_j be the Lagrange polynomial interpolation defined by

$$h_j \in \mathbb{P}([-1,1]), \quad h_j(\xi_i) = \delta_{ij}, \quad 0 \le i, j \le N$$

where δ_{ij} is the Kronecker symbol.

We have then for each $u_N^i \in \mathbb{P}_N^0(\Omega)^I$ solution of the discrete problem (3.14),

$$u_N^i(x,y) = \sum_{i=1}^{N-1} \sum_{j=1}^{N-1} u_N^i(\xi_i^x, \xi_j^y) h_i^x(x) h_j^y(y)$$

where $(\xi_i^x, \xi_j^y) = F(\xi_i, \xi_j)$ and $h_i^x h_j^y$ verify $h_i^x h_j^y \circ F = h_i h_j$.

If we consider U^i the vector composed by the admissible solutions $u_N^i(\xi_i^x, \xi_j^y)$ then the discrete problem (3.14) can be written as the matrix form

$$(D + \delta t A) U^i = F^i$$

where D is a diagonal matrix of components $\rho_r\rho_s$, $1 \le r,s \le N-1$, A is the matrix with components $(\nabla(h_ih_j);\nabla(h_rh_s))$, $1 \le i,j,r,s \le N-1$ and F^i is a vector with components $(u_N^{i-1}(\xi_r,\xi_s)+\delta t f^i(\xi_r,\xi_s))\rho_r\rho_s$, $1 \le i,r,s \le N-1$.

We note that $D+\delta tA$ is a symmetric, positive definite matrix. The algorithm is then solved using the gradient conjugate method.

We present in the following some numerical tests in order to confirm the theoretical results. The numerical results have been performed in dimension $d=2$.

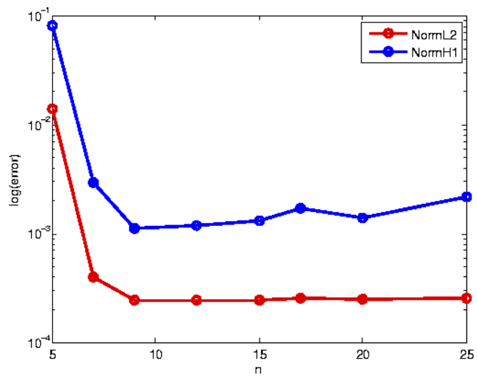

FIGURE 3. Error on space for the solution (5.1)

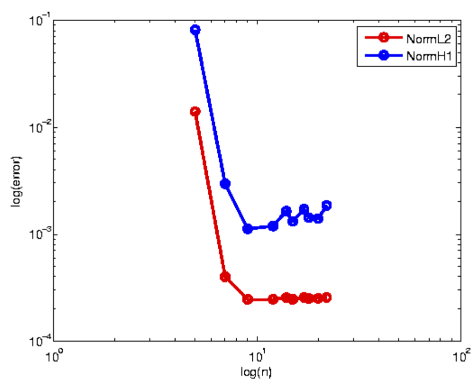

FIGURE 4. Error on space for the solution (5.2)

5.1. **Convergence in time.** We are interested in this case to the time convergence. We consider the domain $\Omega=[-1,1]^2$. Two examples of the exact solution are tested. The first one is

$$u(t,x,y)=\cos(\pi t)\sin(\pi x)\sin(\pi y) \tag{5.1}$$

The domain is discretized with $N=20$. T is taken equal to 1 and $\delta t=10^{-1}$, 10^{-2}, 10^{-3} and 10^{-4}. We present in Figure 1 the quantities $\log_{10}\|u-u_{N\delta t}\|_{H^1(\Omega)^2}$ (in

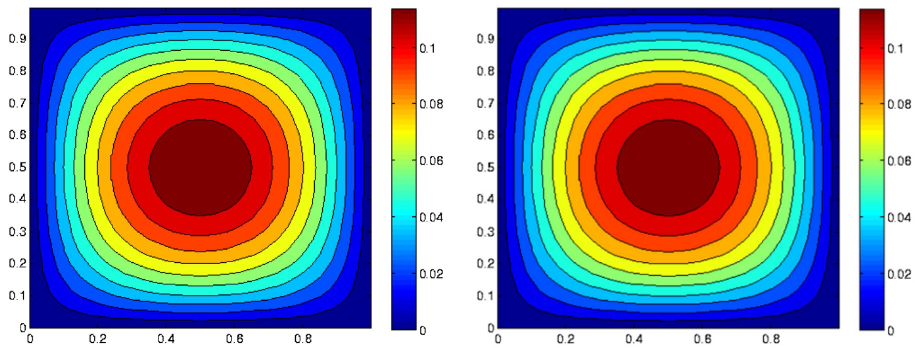

FIGURE 5. Convergence in space: Continuous solution (right), discrete solution (left)

blue) and $\log_{10}\|u-u_{N\delta t}\|_{L^2(\Omega)^2}$ (in red) as a function of $\log_{10}(\delta t)$. For the second test we consider a less regular solution

$$u(t,x,y)=t^{3/2}(1-x^2)^{5/2}(1-y^2)^{5/2} \tag{5.2}$$

and we present in Figure 2 the same quantities as the Figure 1 for $N=20$ a time $T=0.1$ and for time step $\delta t=5\times10^{-2},10^{-2},5\times10^{-3},10^{-4}$. The obtained figures show that the errors decrease, which prove the convergence of the method.

5.2. **Convergence in space.** In this test, we fix the time step $\delta t=0.01$ and we vary N from 5 to 22. We consider

$$u=(1+t)(1-x^2)(1-y^2)$$

and $T=1$. We present in figure 3 (respectively figure 4) the quantities $\log_{10}\|u-u_{N\delta t}\|_{H^1(\Omega)^2}$ (in blue) and $\log_{10}\|u-u_{N\delta t}\|_{L^2(\Omega)^2}$ (in red) as a function of N (respectively $\log_{10}(N)$). We remark that the error norm $\log_{10}\|u-u_{N\delta t}\|_{H^1(\Omega)^2}$ decreases until $N=10$ and becomes sinusoidal for $N>10$. The error $\log_{10}\|u-u_{N\delta t}\|_{L^2(\Omega)^2}$ decreases until $N=10$ and stagned for $N>10$. These results are due to the fact that the convergence order in time is less than that on space and that the time variation polutes the space convergence. The isovalues of the exact and discrete solutions for this case when $\delta t=0.01$ and $N=22$ are presented in Figure 5.

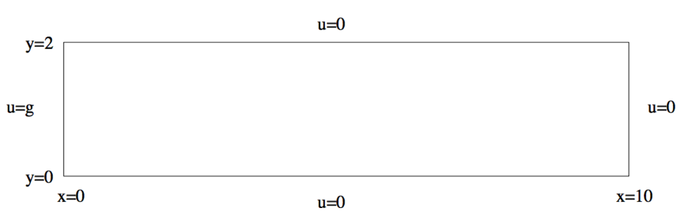

FIGURE 6. Physical test domain

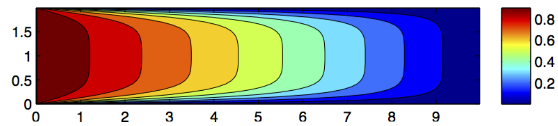

FIGURE 7. Physical test, $g = 1$

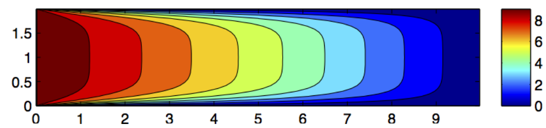

FIGURE 8. Physical test, $g = 10$

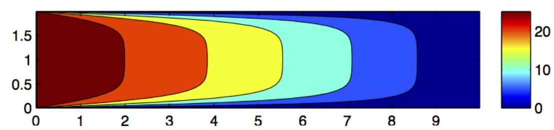

FIGURE 9. Physical test, $g = 30$

5.3. **Physical interpretation.** We consider the test presented in [12] where u represents the temperature. The domain Ω is a rectangle $[0, 10] \times [0, 2]$ (see figure 6). We consider a variable boundary condition $u = g$ at the inlet of the domain and a fixed boundary condition $u = 0$ at the other parts. We consider $T = 1$, $\delta t = 0.01$ and $N = 40$. We present respectively in figures 7, 8 and 9 the isovalues of the discrete solution for g equal respectively to $1, 10$ and 30. We remak the convergence of the solution for the three cases which correspond to the thermic convection in the direction of the x axis.

Remark 5.1. Although the spectral methods are known as highly order method in space, we remark that they have the disadvantage of losing part of this accuracy due to lower order of temporal discretization (often of order 1 or 2).

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