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BLOW-UP CRITERION FOR THE 2D EULER-BOUSSINESQ SYSTEM IN TERMS OF TEMPERATURE

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Abstract. In this article, we study the blow-up slutions for the 2D Euler-Boussinesq equation. In particular, it is shown that if

$$\int_0^{T^*} \sup_{r\geq 2} \frac{\|\Lambda^{1-\alpha}\theta(t)\|_{L^r}}{\sqrt{r\log r}}\,\mathrm{d}t<\infty \quad \text{or} \quad \int_0^{T^*} \|\Lambda^{1-\alpha}\theta\|_{\dot{B}^0_{\infty,\infty}}\,\mathrm{d}t<\infty,$$

then the local solution can be continued to the global one. This is an improvement of classical Lipschitz-type blow-up criterion ($\|\nabla\theta\|_{L^1_tL^\infty}$) in terms of the temperature θ.

1. Introduction

The 2D incompressible generalized Boussinesq equations with the fractional Laplacian dissipation is of the type

$$\begin{gathered}
\partial_t v+(v\cdot\nabla)v+\nu\Lambda^\beta v+\nabla\pi=\theta \mathrm{e}_2,\quad (x,t)\in\mathbb{R}^2\times(0,\infty),\\
\partial_t\theta+(v\cdot\nabla)\theta+\kappa\Lambda^\alpha\theta=0,\\
\nabla\cdot v=0,\\
v(x,0)=v^0,\quad \theta(x,0)=\theta^0,
\end{gathered}\tag{1.1}$$

where $v=v(x,t)=(v_1(x,t),v_2(x,t))$, $\pi=\pi(x,t)$ and $\theta=\theta(x,t)$ stand for, respectively, the velocity vector field, the pressure and the temperature. Here, the constants $\nu\geq 0$ and $\kappa\geq 0$ denote viscosity coefficient and thermal diffusivity coefficient respectively, and $\mathrm{e}_2=(0,1)$. $\Lambda=\sqrt{-\Delta}$ is the Zygmund operator, and Λ^α is defined by the Fourier transform,

$$\widehat{\Lambda^\alpha f}(\xi)=|\xi|^\alpha\widehat{f}(\xi),\quad \widehat{f}(\xi)=\int_{\mathbb{R}^2}e^{-ix\cdot\xi}f(x)\,\mathrm{d}x.$$

The study of the standard 2D incompressible Boussinesq system (with $\nu>0,\kappa>0$ and $\alpha=\beta=2$ in (1.1)) can be traced to 1980s (see [5]). Later, there are many works considering the global well-posedness problem for the standard 2D Boussinesq system without the viscosity ($\nu=0$, $\kappa>0$) or without the thermal diffusivity ($\nu>0$, $\kappa=0$), see [1, 7, 14, 15, 19].

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Recently, the 2D incompressible Boussinesq system with fractional Laplacian generalizations have attracted considerable attention. For example, Hmidi and Zerguine obtained the global well-posedness of Euler-Boussinesq system (1.1) with $1 < \alpha \le 2$ in [17]. Hmidi, Keraani and Rousset obtained the global well-posedness for the Euler-Boussinesq system with critical dissipation (namely, $\alpha = 1$) in [16]. There are many other related results to the Boussinesq equations system (1.1), we refer the reader to [2, 3, 6, 9, 11, 18, 12, 13, 21]. From above mentioned result, we see that it is difficult for us to get the global well-posedness for the system (1.1) with super-critical dissipation ($\alpha + \beta < 1$). Therefore, people may turn to the blow-up criteria in terms of velocity (v) and temperature (θ). As for the blow-up criterion for Boussinesq equations in terms of θ, we refer readers to [8] (see also [7]), in which the authors Chae and Nam obtained the blow-up criterion $\|\nabla\theta\|_{L^1_t L^\infty}$ in the framework of L^2. The purposes of this article is to establish some blow-up criteria better than $\|\nabla\theta\|_{L^1_t L^\infty}$.

In view of [17], one can establish the local well-posedness results for the system (1.1) with $0 < \alpha \le 2, \kappa > 0$ and

$$v^0 \in B^s_{p,1}(\mathbb{R}^2) \quad \text{with} \quad \operatorname{div} v^0 = 0, \quad \theta^0 \in B^{s-\alpha}_{p,1}(\mathbb{R}^2), \tag{1.2}$$

where $s \ge 1 + \frac{2}{p}$ with $p \in]1, \infty[$. Here, we define the function space of solutions as follows

$$\mathcal{X}^{s,p}_T := \mathcal{C}([0,T]; B^s_{p,1}(\mathbb{R}^2)) \times \big(\mathcal{C}([0,T]; B^{s-\alpha}_{p,1}(\mathbb{R}^2)) \cap L^1([0,T]; B^s_{p,1}(\mathbb{R}^2)\big), \tag{1.3}$$

where $B^{s-\alpha}_{p,1}(\mathbb{R}^2)$ and $B^s_{p,1}(\mathbb{R}^2)$ are Besov spaces (see Section 2). Now, we give the main results of this article.

Theorem 1.1. *Let $1/2 < \alpha \le 1$ and $(v, \theta) \in \chi^{s,p}_{T^*}$ be the local unique solution of (1.1) with initial data satisfying (1.2), where T^* is the maximal existence time. If*

$$\int_0^{T^*} \sup_{r \ge 2} \frac{\|\Lambda^{1-\alpha}\theta(t)\|_{L^r}}{\sqrt{r \log r}} \, dt < \infty, \tag{1.4}$$

or

$$\int_0^{T^*} \|\Lambda^{1-\alpha}\theta\|_{\dot B^0_{\infty,\infty}} \, dt < \infty, \tag{1.5}$$

then the solution (v, θ) can be continued beyond T^*.

2. Notation and preliminaries

We begin this section with dyadic decomposition. Let $\mathcal{S}$ be the Schwartz class of rapidly decreasing functions. Let functions $\chi, \varphi \in \mathcal{S}(\mathbb{R}^d)$ supported in $\mathfrak{B} = \{\xi \in \mathbb{R}^d : |\xi| \le 4/3\}$ and $\mathfrak{C} = \{\xi \in \mathbb{R}^d : 3/4 \le |\xi| \le 8/3\}$ respectively, such that

$$\sum_{j \in \mathbb{Z}} \varphi(2^{-j}\xi) = 1, \forall \xi \in \mathbb{R}^d \backslash \{0\},$$
$$\chi(\xi) + \sum_{j \ge 0} \varphi(2^{-j}\xi) = 1, \forall \xi \in \mathbb{R}^d.$$

For $u \in \mathcal{S}'$, we set

$$\varDelta_{-1} u = \chi(D) u \quad \forall q \in \mathbb{N}, \quad \varDelta_q u = \varphi(2^{-q}D) u \quad \forall q \in \mathbb{Z},$$
$$\dot\varDelta_q u = \varphi(2^{-q}D) u.$$

The following low-frequency cut-off will be also used:

$$\mathcal{S}_q u = \sum_{-1\le j\le q-1} \Delta_j u \quad \text{and} \quad \dot{\mathcal{S}}_q u = \sum_{j\le q-1} \dot{\Delta}_j u$$

We now recall the definitions of Besov spaces through the dyadic decomposition. Let $s \in \mathbb{R}$, $p,q \in [1,\infty]$, the inhomogeneous Besov space $B^s_{p,q}(\mathbb{R}^d)$ is the set of tempered distribution u such that

$$\|u\|_{B^s_{p,q}} := \big(2^{js}\|\Delta_j u\|_{L^p}\big)_{\ell^q} < \infty.$$

To define the homogeneous Besov spaces, we first denote by $\mathcal{S}'/\mathcal{P}$ the space of tempered distributions modulo polynomials. Thus we define the space $\dot{B}^s_{p,q}(\mathbb{R}^d)$ as the set of distribution $u \in \mathcal{S}'/\mathcal{P}$ such that

$$\|u\|_{\dot{B}^s_{p,q}} := \big(2^{js}\|\dot{\Delta}_j u\|_{L^p}\big)_{\ell^q} < \infty.$$

We point out that if $s > 0$ then we have $B^s_{p,q}(\mathbb{R}^d) = \dot{B}^s_{p,q}(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ and

$$\|u\|_{B^s_{p,q}} \approx \|u\|_{\dot{B}^s_{p,q}} + \|u\|_{L^p}.$$

In our next study we require two kinds of coupled space-time Besov spaces. The first is defined in the following manner: for $T > 0$ and $q \ge 1$, we denote by $L^r_T\dot{B}^s_{p,q}$ the set of all tempered distribution u satisfying

$$\|u\|_{L^r_T\dot{B}^s_{p,q}} := \|\Big(2^{qs}\|\dot{\Delta}_q u\|_{L^p}\Big)_{\ell^q}\|_{L^r_T} < \infty.$$

The second mixed space is $\widetilde{L}^r_T\dot{B}^s_{p,q}$ which is the set of tempered distribution u satisfying

$$\|u\|_{\widetilde{L}^r_T\dot{B}^s_{p,q}} := \Big(2^{qs}\|\dot{\Delta}_q u\|_{L^r_T L^p}\Big)_{\ell^q} < \infty.$$

3. Proof of main results

In this section we use Φ_k to denote function of the form

$$\Phi_k(t) = C_0 \underbrace{\exp(\dots\exp}_{k \text{ times}}(C_0 t)\dots),$$

where C_0 depends on the involved norms of the initial data and its value may vary from line to line up to some absolute constants. We will make an intensive use (without mentioning it) of the following trivial facts

$$\int_0^t \Phi_k(\tau)d\tau \le \Phi_k(t) \quad \text{and} \quad \exp\Big(\int_0^t \Phi_k(\tau)d\tau\Big) \le \Phi_{k+1}(t).$$

Firstly, we introduce a pseudo-differential operator $\mathcal{R}_\alpha$ defined by $\mathcal{R}_\alpha := \Lambda^{-\alpha}\partial_1 = \Lambda^{1-\alpha}\mathcal{R}, 0 < \alpha < 1$, where $\mathcal{R} := \frac{\partial_1}{\Lambda}$ is the usual Riesz transform. For (1.1), the vorticity equation is

$$\partial_t\omega + v\cdot\nabla\omega = \partial_1\theta, \tag{3.1}$$

and the acting of $\mathcal{R}_\alpha$ on the temperature equation, we have

$$\partial_t\mathcal{R}_\alpha\theta + v\cdot\nabla\mathcal{R}_\alpha\theta + \kappa\Lambda^\alpha\mathcal{R}_\alpha\theta = -[\mathcal{R}_\alpha, v\cdot\nabla]\theta. \tag{3.2}$$

Denote $F = \omega + \mathcal{R}_\alpha\theta$. Thus we obtain

$$\partial_t F + v\cdot\nabla F = -[\mathcal{R}_\alpha, v\cdot\nabla]\theta. \tag{3.3}$$

Firstly, we give the following crucial Lemmas which are useful for us to proof the main results.

Lemma 3.1 ([4]). *Assume that v is divergence-free and that f satisfies the transport equation on $\mathbb{R}^d$,*

$$\begin{gathered}\partial_t f + v\cdot\nabla f = g,\\ f|_{t=0} = f_0.\end{gathered}\tag{3.4}$$

There exists a constant C, depending only on d, such that for all $1\le p,r\le\infty$ and $t\in\mathbb{R}_+$, we have

$$\|f\|_{\widetilde{L}^\infty_t(B^0_{p,r})} \le C\big(\|f_0\|_{B^0_{p,r}} + \|g\|_{\widetilde{L}^1_t(B^0_{p,r})}\big)\Big(1+\int_0^t \|\nabla v(\tau)\|_{L^\infty}d\tau\Big).\tag{3.5}$$

Lemma 3.2 ([17]). *Let v be a solution of the incompressible Euler system on $\mathbb{R}^2$*

$$\begin{gathered}\partial_t v + v\cdot\nabla v + \nabla\pi = f,\\ v(x,0) = v^0,\\ \operatorname{div} v = 0.\end{gathered}\tag{3.6}$$

Then for $s>-1$, $(p,r)\in(1,\infty)\times[1,\infty]$ we have

$$\|v\|_{\widetilde{L}^\infty_t(B^s_{p,r})} \le C\mathrm{e}^{CV(t)}\Big(\|v^0\|_{B^s_{p,r}} + \int_0^t \mathrm{e}^{-CV(\tau)}\|f(\tau)\|_{B^s_{p,r}}d\tau\Big),\tag{3.7}$$

with $V(t) := \int_0^t \|\nabla v(\tau)\|_{L^\infty}d\tau$.

Lemma 3.3. *Let $\alpha\in(0,1)$, v be a smooth divergence-free vector field.*

(i) *for every $(s,p,r)\in(-1,\alpha)\times[2,\infty]\times[1,\infty]$, there exists a constant $C>0$ such that*

$$\|[\mathcal{R}_\alpha, v\cdot\nabla]\theta\|_{B^s_{p,r}} \le C\|\nabla v\|_{L^p}\left(\|\theta\|_{B^{s+1-\alpha}_{\infty,r}} + \|\theta\|_{L^p}\right),\tag{3.8}$$

(ii) *for every $(r,\varrho)\in[1,\infty]\times(1,\infty)$ and $\epsilon>0$, there exists a constant $C>0$ such that*

$$\|[\mathcal{R}_\alpha, v\cdot\nabla]\theta\|_{B^0_{\infty,r}} \le C(\|\omega\|_{L^\infty} + \|\omega\|_{L^\varrho})\left(\|\theta\|_{B^{\epsilon+1-\alpha}_{\infty,r}} + \|\theta\|_{L^\varrho}\right).\tag{3.9}$$

Proof. Part (i) can be found in [21, Proposition 3.3]. For the second part, we can imitate the proof of [16, Theorem 3.3(2)] to get (ii). We omit the detail here. □

Lemma 3.4. *Let $\alpha\in[0,2]$, $\kappa>0$ and v be a smooth divergence-free vector-field of $\mathbb{R}^2$. Let θ be a smooth solution of*

$$\begin{gathered}\partial_t\theta + v\cdot\nabla\theta + \kappa\Lambda^\alpha\theta = f,\\ \theta(x,0) = \theta^0.\end{gathered}\tag{3.10}$$

Then (i) for every $\rho\in[1,\infty]$, $p\in[1,\infty]$, $s>-1$ one has (see [17])

$$\|\theta\|_{\widetilde{L}^\rho_t(B^{s+\frac{\alpha}{\rho}}_{p,1})} \le C\mathrm{e}^{CV(t)}\Big(\|\theta^0\|_{B^s_{p,1}}(1+t^{\frac{1}{\rho}}) + \int_0^t \mathcal{T}_s(\tau)d\tau + \|f\|_{L^1_t(B^s_{p,1})}\Big)\tag{3.11}$$

with

$$V(t) := \int_0^t \|\nabla v(\tau)\|_{L^\infty}d\tau, \quad \mathcal{T}_s(t) := \|\nabla\theta(t)\|_{L^\infty}\|v(t)\|_{B^s_{p,1}}1_{[1,\infty)}(s).$$

(ii) for every $p\in[1,\infty]$, we have the L^p estimates (see [10])

$$\|\theta\|_{L^p} \le \|\theta^0\|_{L^p} + \int_0^t \|f(\tau)\|_{L^p}d\tau.\tag{3.12}$$

(iii) for every $p \in [1,\infty]$ (see [16])

$$\|\theta\|_{\widetilde{L}^\infty_t B^0_{p,1}} \le C\big(\|\theta^0\|_{B^0_{p,1}} + \|f\|_{L^1_t B^0_{p,1}}\big)\Big(1 + \int_0^t \|\nabla v(\tau)\|_{L^\infty} d\tau\Big). \tag{3.13}$$

(iv) for $p \in (1,\infty)$, $\rho \in [1,\infty]$ and $f = 0$, there exists a constant C such that (see [17])

$$\sup_{q\in\mathbb{N}} 2^{q\frac{\alpha}{\rho}} \|\Delta_q \theta\|_{L^\rho_t L^p} \le C\|\theta^0\|_{L^p} + C\|\theta^0\|_{L^\infty}\|\omega\|_{L^1_t L^p}. \tag{3.14}$$

Proof of Theorem 1.1. By using the above Lemmas and the equation (3.3), we first estimate $\|\nabla v(t)\|_{L^\infty}$. We do this in several steps.

Step 1: Estimation of $\|\omega(t)\|_{L^a}$, for some $\max\{2,p\} < a < \infty$. From (3.3), for any $t < T^*$, by Lemma 3.4 (ii) we have

$$\|\digamma(t)\|_{L^a} \le \|\digamma(0)\|_{L^a} + \int_0^t \|[\mathcal{R}_\alpha, v\cdot\nabla]\theta(\tau)\|_{L^a} d\tau. \tag{3.15}$$

Note that (see [10, p 516]) $\int_0^t \Lambda^\alpha \mathcal{R}_\alpha\theta|\mathcal{R}_\alpha\theta|^{a-2}\mathcal{R}_\alpha\theta d\tau \ge 0$, by (3.2) we have

$$\|\mathcal{R}_\alpha\theta(t)\|_{L^a} \le \|\mathcal{R}_\alpha\theta^0\|_{L^a} + \int_0^t \|[\mathcal{R}_\alpha, v\cdot\nabla]\theta(\tau)\|_{L^a} d\tau. \tag{3.16}$$

Since $v^0 \in B^s_{p,1}$, $s \ge 1 + 2/p$, we see that $\omega^0 \in L^p \cap L^\infty$, and for $\theta^0 \in B^{s-\alpha}_{p,1}$ with $0 < \alpha \le 1$, we have $\theta^0 \in L^p \cap L^\infty$ note that $p < a < \infty$, the embedding $B^{s-\alpha}_{p,1} \hookrightarrow B^{s-\alpha-\frac{2}{p}+\frac{2}{a}}_{a,1} \hookrightarrow B^{1-\alpha+\frac{2}{a}}_{a,1} \hookrightarrow B^{1-\alpha}_{a,1}$, we have $\mathcal{R}_\alpha\theta^0 \in L^a$. From (3.15) and (3.16) we obtain

$$\begin{aligned}\|\omega\|_{L^a} &\le \|\digamma(t)\|_{L^a} + \|\mathcal{R}_\alpha\theta(t)\|_{L^a}\\ &\le C\big(\|\omega^0\|_{L^p\cap L^\infty} + \|\theta^0\|_{B^{s-\alpha}_{p,1}}\big) + C\int_0^t \|[\mathcal{R}_\alpha, v\cdot\nabla]\theta(\tau)\|_{L^a} d\tau.\end{aligned} \tag{3.17}$$

Using the classical embedding $B^0_{a,1} \hookrightarrow L^a$, and Lemma 3.3 (i), we have

$$\begin{aligned}\|[\mathcal{R}_\alpha, v\cdot\nabla]\theta(t)\|_{L^a} &\le C\|[\mathcal{R}_\alpha, v\cdot\nabla]\theta(t)\|_{B^0_{a,1}}\\ &\le C\|\nabla v(t)\|_{L^a}\big(\|\theta(t)\|_{B^{1-\alpha}_{\infty,1}} + \|\theta(t)\|_{L^a}\big).\end{aligned} \tag{3.18}$$

From $\|\theta(t)\|_{L^a} \le \|\theta^0\|_{L^a}, \forall t \ge 0$, it follows that

$$\begin{aligned}\|\omega(t)\|_{L^a} &\le C\left(\|\omega^0\|_{L^p\cap L^\infty} + \|\theta^0\|_{B^{s-\alpha}_{p,1}}\right)\\ &\quad + C\int_0^t \|\omega(\tau)\|_{L^a}\left(\|\theta(\tau)\|_{B^{1-\alpha}_{\infty,1}} + \|\theta^0\|_{L^a}\right) d\tau.\end{aligned} \tag{3.19}$$

According to Gronwall's inequality we obtain

$$\|\omega\|_{L^a} \le Ce^{Ct}e^{C\|\theta\|_{L^1_t B^{1-\alpha}_{\infty,1}}}. \tag{3.20}$$

Next we estimate $\|\theta\|_{L^1_t B^{1-\alpha}_{\infty,1}}$, let $N \in \mathbb{N}$, by the Littlewood-Paley decomposition, by condition (1.4) we see that

$$\begin{aligned}&\|\theta\|_{L^1_t B^{1-\alpha}_{\infty,1}}\\ &\le \|\mathcal{S}_N\theta\|_{L^1_t B^{1-\alpha}_{\infty,1}} + \sum_{q\ge N} 2^{q(1-\alpha)}\|\Delta_q\theta\|_{L^1_t L^\infty}\end{aligned}$$

$$
\begin{aligned}
&\le Ct\|\theta^0\|_{L^\infty} + C\int_0^t \sum_{0\le q<N} \|\Delta_q\Lambda^{1-\alpha}\theta(\tau)\|_{L^\infty} d\tau + \sum_{q\ge N} 2^{q(1-\alpha)}\|\Delta_q\theta\|_{L^1_tL^\infty}\\
&\le Ct\|\theta^0\|_{L^\infty} + C\int_0^t \sum_{0\le q<N} 2^{q\frac{2}{b}}\|\Delta_q\Lambda^{1-\alpha}\theta(\tau)\|_{L^b} d\tau + \sum_{q\ge N} 2^{q(1-\alpha)}\|\Delta_q\theta\|_{L^1_tL^\infty}\\
&\le C\Big(\sum_{0\le q<N} 2^{q\frac{2}{b}}\Big)\sqrt{b\log b}\int_0^t \sup_{r\ge 2}\frac{\|\Lambda^{1-\alpha}\theta(\tau)\|_{L^r}}{\sqrt{r\log r}} d\tau\\
&\quad + Ct\|\theta^0\|_{L^\infty} + C\sum_{q\ge N} 2^{q(1-\alpha+\frac{2}{a})}\|\Delta_q\theta\|_{L^1_tL^a}\\
&\le Ct\|\theta^0\|_{L^\infty} + C2^{N\frac{2}{b}}\sqrt{b\log b} + C\sum_{q\ge N} 2^{q(1-\alpha+\frac{2}{a})}\|\Delta_q\theta\|_{L^1_tL^a}.
\end{aligned}
$$

Since $\alpha > 1/2$, we choose a large enough such that $1 - 2\alpha + 4/a < 0$, and using Lemma 3.4 (iv), we have

$$
\begin{aligned}
&\sum_{q\ge N} 2^{q(1-\alpha+\frac{2}{a})}\|\Delta_q\theta\|_{L^1_tL^a}\\
&\le \sum_{q\ge N} 2^{q(1-2\alpha+\frac{2}{a})}\Big(\|\theta^0\|_{L^a} + \|\theta^0\|_{L^\infty}\int_0^t \|\omega(\tau)\|_{L^a} d\tau\Big)\\
&\le C\|\theta^0\|_{L^a} + 2^{N(1-2\alpha+\frac{2}{a})}\|\theta^0\|_{L^\infty}\int_0^t \|\omega(\tau)\|_{L^a} d\tau.
\end{aligned}
\tag{3.21}
$$

Therefore,

$$
\begin{aligned}
\|\theta\|_{L^1_tB^{1-\alpha}_{\infty,1}} &\le C(t+1)\|\theta^0\|_{L^\infty\cap L^a} + C2^{N\frac{2}{b}}b^{\frac{1}{2}+\epsilon}\\
&\quad + 2^{N(1-2\alpha+\frac{2}{a})}\|\theta^0\|_{L^\infty}\int_0^t \|\omega(\tau)\|_{L^a} d\tau,
\end{aligned}
$$

for any $0 < \epsilon < 1/2$. Setting $b = N$ and selecting N as follows

$$
N = \Big[\frac{\log\big(e + \int_0^t \|\omega(\tau)\|_{L^a} d\tau\big)}{(2\alpha - 1 - 2/a)\log 2}\Big] + 2.
$$

Then

$$
\|\theta\|_{L^1_tB^{1-\alpha}_{\infty,1}} \le C(t+1)\|\theta^0\|_{L^\infty\cap L^a} + C\Big[\log\Big(e + \int_0^t \|\omega(\tau)\|_{L^a} d\tau\Big)\Big]^{\frac{1}{2}+\epsilon}. \tag{3.22}
$$

Combining (3.20) with (3.22), we have

$$
\begin{aligned}
\|\theta\|_{L^1_tB^{1-\alpha}_{\infty,1}} &\le C(t+1)\|\theta^0\|_{L^\infty\cap L^a} + C\Big[\log\Big(e + \int_0^t \|\omega(\tau)\|_{L^a} d\tau\Big)\Big]^{\frac{1}{2}+\epsilon}\\
&\le C(t+1)\|\theta^0\|_{L^\infty\cap L^a} + C\Big[\log\Big(e + tCe^{Ct}e^{C\|\theta\|_{L^1_tB^{1-\alpha}_{\infty,1}}}\Big)\Big]^{\frac{1}{2}+\epsilon}\\
&\le C(t^2+t+1)\big(\|\theta^0\|_{L^\infty\cap L^a} + 1\big) + C\|\theta\|^{\frac{1}{2}+\epsilon}_{L^1_tB^{1-\alpha}_{\infty,1}},
\end{aligned}
\tag{3.23}
$$

it follows that

$$
\|\theta\|_{L^1_tB^{1-\alpha}_{\infty,1}} \le C(1 + t + t^2).
$$

By (3.20), we obtain

$$
\|\omega(t)\|_{L^a} \le \Phi_1(t), \tag{3.24}
$$

where $\Phi_k(t), k = 1, 2, 3, \dots$ is same the as as in subsection 3.3.

Step 2: Estimation of $\|\omega(t)\|_{L^\infty}$. By using the maximum principle for the transport equation (3.3) we have

$$\|F(t)\|_{L^\infty} \le \|F(0)\|_{L^\infty} + \int_0^t \|[\mathcal{R}_\alpha, v\cdot\nabla]\theta(\tau)\|_{L^\infty} d\tau, \tag{3.25}$$

By using Lemma 3.4 (ii) with $p = \infty$, we see that (3.2) follows that

$$\|\mathcal{R}_\alpha\theta(t)\|_{L^\infty} \le \|\mathcal{R}_\alpha\theta^0\|_{L^\infty} + \int_0^t \|[\mathcal{R}_\alpha, v\cdot\nabla]\theta(\tau)\|_{L^\infty} d\tau. \tag{3.26}$$

For $\theta^0 \in B^{s-\alpha}_{p,1}$ with $0 < \alpha \le 1$, $s \ge 1 + 2/p$, we have $B^{s-\alpha}_{p,1} \hookrightarrow B^{s-\alpha-\frac{2}{p}}_{\infty,1} \hookrightarrow B^{1-\alpha}_{\infty,1}$, we have $\mathcal{R}_\alpha\theta^0 \in L^\infty$. From (3.25) and (3.26) we obtain

$$\begin{aligned}\|\omega\|_{L^\infty} &\le \|F(t)\|_{L^\infty} + \|\mathcal{R}_\alpha\theta(t)\|_{L^\infty}\\ &\le \|\omega^0\|_{L^\infty} + \|\theta^0\|_{B^{s-\alpha}_{p,1}} + 2\int_0^t \|[\mathcal{R}_\alpha, v\cdot\nabla]\theta(\tau)\|_{L^\infty} d\tau.\end{aligned} \tag{3.27}$$

Using the classical embedding $B^0_{\infty,1} \hookrightarrow L^\infty$, and Lemma 3.3 (i), we have

$$\begin{aligned}\|[\mathcal{R}_\alpha, v\cdot\nabla]\theta\|_{L^\infty} &\le C\|[\mathcal{R}_\alpha, v\cdot\nabla]\theta\|_{B^0_{\infty,1}}\\ &\le C(\|\omega\|_{L^\infty} + \|\omega\|_{L^a})\Big(\|\theta\|_{B^{\epsilon+1-\alpha}_{\infty,1}} + \|\theta\|_{L^a}\Big).\end{aligned} \tag{3.28}$$

Combining (3.27) and (3.28) we have

$$\begin{aligned}&\|\omega(t)\|_{L^\infty}\\ &\le C + C\int_0^t (\|\omega(\tau)\|_{L^\infty} + \|\omega(\tau)\|_{L^a})\Big(\|\theta(\tau)\|_{B^{\epsilon+1-\alpha}_{\infty,1}} + \|\theta(\tau)\|_{L^a}\Big) d\tau\\ &\le C + \|\omega\|_{L^\infty_t L^a}\Big(\|\theta\|_{L^1_t B^{\epsilon+1-\alpha}_{\infty,1}} + t\|\theta^0\|_{L^a}\Big)\\ &\quad + C\int_0^t \|\omega(\tau)\|_{L^\infty}\Big(\|\theta(\tau)\|_{B^{\epsilon+1-\alpha}_{\infty,1}} + \|\theta^0\|_{L^a}\Big) d\tau.\end{aligned} \tag{3.29}$$

We claim that

$$\|\theta\|_{L^1_t B^{\epsilon+1-\alpha}_{\infty,1}} \le \Phi_1(t), \tag{3.30}$$

for some suitable $\epsilon > 0$. In fact that, from step 1 and the Lemma 3.4 (iv) we have

$$\|\theta\|_{\widetilde{L}^1_t B^\alpha_{a,\infty}} \le \Phi_1(t),$$

and then for every $\sigma < \alpha$, we have

$$\|\theta\|_{L^1_t B^\sigma_{a,1}} \le \|\theta\|_{\widetilde{L}^1_t B^\alpha_{a,\infty}} \le \Phi_1(t).$$

We choose that $\sigma = \alpha - 1/a$, and we keep in mind that a satisfies $1 - 2\alpha + 4/a < 0$, therefore $2\alpha - 1 - 4/a > 0$, we choose ϵ satisfies $0 < \epsilon < 2\alpha - 1 - 3/a$, then we have $\epsilon + 1 - \alpha < \alpha - 3/a = \sigma - 2/a$ (here we note that the selected parameter a is make sure $\alpha > 3/a$, and we have $\sigma - 2/a > 0$). Therefore, we have the embedding $B^{\alpha-1/a}_{a,1} = B^\sigma_{a,1} \hookrightarrow B^{\sigma-2/a}_{\infty,1} \hookrightarrow B^{\epsilon+1-\alpha}_{\infty,1}$, and we finally obtain (3.30). Thus, by (3.30) and step 1, one has

$$\|\omega(t)\|_{L^\infty} \le \Phi_1(t) + C\int_0^t \|\omega(\tau)\|_{L^\infty}\Big(\|\theta(\tau)\|_{B^{\epsilon+1-\alpha}_{\infty,1}} + \|\theta^0\|_{L^a}\Big) d\tau. \tag{3.31}$$

Again using (3.30) and Gronwall's inequality, we obtain

$$\|\omega(t)\|_{L^\infty} \le \Phi_2(t).$$

Step 3: Estimation of $\|\nabla v(t)\|_{L^\infty}$. By using Lemma 3.1 and Lemma 3.4(iii), from (3.2) and (3.3) we have

$$\|F(t)\|_{B^0_{\infty,1}}+\|\mathcal{R}_\alpha\theta(t)\|_{B^0_{\infty,1}} \le \Big(C+\|[\mathcal{R}_\alpha, v\cdot\nabla]\theta\|_{L^1_tB^0_{\infty,1}}\Big)(1+\|\nabla v\|_{L^1_tL^\infty}), \quad (3.32)$$

Thanks to Lemma 3.3, and by step 1, step 2, we have

$$\begin{aligned} &\|[\mathcal{R}_\alpha, v\cdot\nabla]\theta\|_{L^1_tB^0_{\infty,1}} \\ &\le C\int_0^t (\|\omega(\tau)\|_{L^\infty}+\|\omega(\tau)\|_{L^a})\Big(\|\theta(\tau)\|_{B^{\epsilon+1-\alpha}_{\infty,1}}+\|\theta(\tau)\|_{L^a}\Big)d\tau \le \Phi_2(t). \end{aligned} \quad (3.33)$$

Therefore, we have

$$\|\omega(t)\|_{B^0_{\infty,1}} \le \|F(t)\|_{B^0_{\infty,1}}+\|\mathcal{R}_\alpha\theta(t)\|_{B^0_{\infty,1}} \le \Phi_2(t)\Big(1+\|\nabla v\|_{L^1_tL^\infty}\Big).$$

On the other hand, we have

$$\begin{aligned} \|\nabla v(t)\|_{L^\infty} &\le \|\nabla\Delta_{-1}v(t)\|_{L^\infty}+\sum_{q\in\mathbb{N}}\|\Delta_q\nabla v(t)\|_{L^\infty} \\ &\le \|\omega(t)\|_{L^a}+\|\omega(t)\|_{B^0_{\infty,1}} \\ &\le \Phi_1(t)+\|\omega(t)\|_{B^0_{\infty,1}}. \end{aligned} \quad (3.34)$$

Using (3.34), we have

$$\|\omega(t)\|_{B^0_{\infty,1}} \le \Phi_2(t)\Big(1+\int_0^t\|\omega(\tau)\|_{B^0_{\infty,1}}d\tau\Big).$$

From Gronwall's inequality we obtain $\|\omega(t)\|_{B^0_{\infty,1}} \le \Phi_3(t)$. Going back to (3.34), we obtain

$$\|\nabla v\|_{L^\infty} \le \Phi_3(t).$$

Next, we give the estimate of $\|v(t)\|_{B^s_{p,1}}, \|\theta(t)\|_{B^{s-\alpha}_{p,1}}$. Since $s \ge 1+2/p$, we have $B^{s-\alpha}_{p,1} \hookrightarrow B^{1-\alpha}_{\infty,1}$. By Lemma 3.4(i), we have

$$\begin{aligned} \|\theta(t)\|_{\widetilde{L}^\infty_t(B^{1-\alpha}_{\infty,1})}+\int_0^t\|\theta(\tau)\|_{B^1_{\infty,1}}d\tau &\le C\mathrm{e}^{CV(t)}(1+t)\|\theta^0\|_{B^{1-\alpha}_{\infty,1}} \\ &\le C\mathrm{e}^{CV(t)}(1+t)\|\theta^0\|_{B^{s-\alpha}_{p,1}}. \end{aligned} \quad (3.35)$$

On the other hand, by Lemma 3.2 we have

$$\|v(t)\|_{\widetilde{L}^\infty_t(B^{s-\alpha}_{p,1})} \le Ce^{CV(t)}\Big(\|v^0\|_{B^s_{p,1}}+\int_0^t\|\theta(\tau)\|_{B^{s-\alpha}_{p,1}}d\tau\Big).$$

Therefore, by the embedding $B^0_{\infty,1} \hookrightarrow L^\infty$ we have

$$\begin{aligned} &\int_0^t\|\nabla\theta(\tau)\|_{L^\infty}\|v(\tau)\|_{B^{s-\alpha}_{p,1}}d\tau \\ &\le \sup_{0\le\tau\le t}\|v(\tau)\|_{B^{s-\alpha}_{p,1}}\int_0^t\|\theta(\tau)\|_{B^1_{\infty,1}}d\tau \\ &\le Ce^{CV(t)}(1+t)\|\theta^0\|_{B^{s-\alpha}_{p,1}}\Big(\|v^0\|_{B^s_{p,1}}+\int_0^t\|\theta(\tau)\|_{B^{s-\alpha}_{p,1}}d\tau\Big). \end{aligned} \quad (3.36)$$

Applying (3.36) and (3.11), the estimate of θ reads as follows

$$\begin{aligned} &\|\theta(t)\|_{\widetilde{L}_t^\infty(B_{p,1}^{s-\alpha})} + \int_0^t \|\theta(\tau)\|_{B_{p,1}^s} d\tau \\ &\le C\mathrm{e}^{CV(t)}\Big(\|\theta^0\|_{B_{p,1}^{s-\alpha}}(1+t) + \int_0^t \|\nabla\theta(\tau)\|_{L^\infty}\|v(\tau)\|_{B_{p,1}^{s-\alpha}} d\tau\Big) \\ &\le C\mathrm{e}^{CV(t)}(1+t)\|\theta^0\|_{B_{p,1}^{s-\alpha}}\Big(1 + \|v^0\|_{B_{p,1}^s} + \int_0^t \|\theta(\tau)\|_{B_{p,1}^{s-\alpha}} d\tau\Big). \end{aligned} \tag{3.37}$$

Combining the estimates of v (applying Lemma 3.2 again) and (3.37), we have

$$\begin{aligned} &\|v(t)\|_{B_{p,1}^s} + \|\theta(t)\|_{B_{p,1}^{s-\alpha}} + \int_0^t \|\theta(\tau)\|_{B_{p,1}^s} d\tau \\ &\le C\mathrm{e}^{CV(t)}(1+t)\|\theta^0\|_{B_{p,1}^{s-\alpha}}\Big(1 + \|v^0\|_{B_{p,1}^s} + \int_0^t \|\theta(\tau)\|_{B_{p,1}^{s-\alpha}} d\tau\Big) \\ &\le \Phi_4(t)\Big(1 + \int_0^t \|\theta(\tau)\|_{B_{p,1}^{s-\alpha}} d\tau\Big). \end{aligned} \tag{3.38}$$

Therefore, by Gronwall's inequality, we finally obtain

$$\|v(t)\|_{B_{p,1}^s} + \|\theta(t)\|_{B_{p,1}^{s-\alpha}} + \int_0^t \|\theta(\tau)\|_{B_{p,1}^s} d\tau \le \Phi_5(t). \tag{3.39}$$

This completes the first part (when assumption (1.4) holds) of this theorem.

The proof of the second part (when assumption (1.5) holds) of this theorem is quite similar to the one in the first part. The main difference is the estimates of $\|\omega\|_{L^a}$ in step 1. We begin with (3.18), and by the embedding $B_{a,2}^0 \hookrightarrow L^a$ and Lemma 3.3 (i), we have

$$\begin{aligned} \|[\mathcal{R}_\alpha, v\cdot\nabla]\theta(t)\|_{L^a} &\le C\|[\mathcal{R}_\alpha, v\cdot\nabla]\theta(t)\|_{B_{a,2}^0} \\ &\le C\|\nabla v(t)\|_{L^a}\left(\|\theta(t)\|_{B_{\infty,2}^{1-\alpha}} + \|\theta(t)\|_{L^a}\right). \end{aligned} \tag{3.40}$$

It follows that

$$\begin{aligned} \|\omega(t)\|_{L^a} \le C &\left(\|\omega^0\|_{L^p\cap L^\infty} + \|\theta^0\|_{B_{p,1}^{s-\alpha}}\right) \\ &+ C\int_0^t \|\omega(\tau)\|_{L^a}\left(\|\theta(\tau)\|_{B_{\infty,2}^{1-\alpha}} + \|\theta^0\|_{L^a}\right) d\tau. \end{aligned} \tag{3.41}$$

According to Gronwall's inequality we obtain

$$\|\omega\|_{L^a} \le Ce^{Ct}e^{C\|\theta\|_{L_t^1 B_{\infty,2}^{1-\alpha}}}. \tag{3.42}$$

The estimate of $\|\theta\|_{L_t^1 B_{\infty,2}^{1-\alpha}}$ is as follows, let $N\in\mathbb{N}$, by the Littlewood-Paley decomposition, by condition (1.5) we see that

$$\begin{aligned} &\|\theta\|_{L_t^1 B_{\infty,2}^{1-\alpha}} \\ &\le \|\mathcal{S}_N\theta\|_{L_t^1 B_{\infty,2}^{1-\alpha}} + \|(\mathrm{Id}-\mathcal{S}_N)\theta\|_{L_t^1 B_{\infty,1}^{1-\alpha}} \\ &\le \|\mathcal{S}_N\theta\|_{L_t^1 B_{\infty,2}^{1-\alpha}} + \sum_{q\ge N} 2^{q(1-\alpha)}\|\Delta_q\theta\|_{L_t^1 L^\infty} \\ &\le Ct\|\theta^0\|_{L^\infty} + C\int_0^t \Big(\sum_{0\le q<N} \|\Delta_q\Lambda^{1-\alpha}\theta(\tau)\|_{L^\infty}^2\Big)^{1/2} d\tau + \sum_{q\ge N} 2^{q(1-\alpha)}\|\Delta_q\theta\|_{L_t^1 L^\infty} \end{aligned}$$

$$
\begin{aligned}
&\le C\sqrt{N}\int_0^t \|\Lambda^{1-\alpha}\theta(\tau)\|_{\dot B^0_{\infty,\infty}}d\tau + Ct\|\theta^0\|_{L^\infty} + C\sum_{q\ge N}2^{q(1-\alpha+\frac{2}{a})}\|\Delta_q\theta\|_{L^1_tL^a}\\
&\le C\sqrt{N} + Ct\|\theta^0\|_{L^\infty} + C\sum_{q\ge N}2^{q(1-\alpha+\frac{2}{a})}\|\Delta_q\theta\|_{L^1_tL^a}.
\end{aligned}
$$

Then, as for (3.21), we can choose suitable N such that

$$
\|\theta\|_{L^1_tB^{1-\alpha}_{\infty,2}} \le C(t+1)\|\theta^0\|_{L^\infty\cap L^a} + C\Big[\log\Big(e+\int_0^t\|\omega(\tau)\|_{L^a}d\tau\Big)\Big]^{1/2}. \tag{3.43}
$$

Combining (3.42) with (3.43), we have

$$
\|\theta\|_{L^1_tB^{1-\alpha}_{\infty,2}} \le \Phi_1(t).
$$

For the rest, we can follow the same process as above, and complete the proof. □

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