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ERGODICITY OF THE TWO-DIMENSIONAL MAGNETIC BENARD PROBLEM

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ABSTRACT. We study the two-dimensional magnetic Bénard problem with noise, white in time. We prove the well-posedness including the path-wise uniqueness of the generalized solution, and the existence of the unique invariant, and consequently ergodic, measure under random perturbation.

1. INTRODUCTION

The Bénard problem is concerned with the motion of a horizontal layer of viscous fluid heated from below and the magnetic Bénard problem similarly with the electrically conducting viscous fluid. Both problems have attracted much attention in the past (cf. [18, 38] for the Bénard problem and [19, 20, 33, 39, 40] for the magnetic Bénard problem). In particular, systems of stochastic partial differential equations for these problems have been studied intensively (cf. [2, 9, 10] for the Bénard problem and [3] for the magnetic Bénard problem). For such systems, the existence of a unique invariant measure, if proven, describes the statistical equilibrium to which the system approaches. In this manuscript, we show the existence and uniqueness of the invariant measure, which is consequently ergodic, for the magnetic Bénard problem.

2. PRELIMINARIES AND STATEMENT OF RESULTS

We consider a spatial domain $D = (0, L) \times (0, 1)$ with $L > 0$. We denote $\partial_t \triangleq \frac{\partial}{\partial t}$, $\partial_i \triangleq \frac{\partial}{\partial x_i}$, $i = 1, 2$ and let (e_1, e_2) represent the standard basis in $\mathbb{R}^2$. We let $u(x,t) = (u_1, u_2)(x,t), b(x,t) = (b_1, b_2)(x,t), p(x,t), \theta(x,t)$ be the velocity and magnetic vector fields, pressure and temperature scalar fields respectively. We furthermore denote by ν_1, ν_2, κ the kinematic viscosity, magnetic diffusivity, thermal diffusivity respectively and $s = H^2\nu_1\nu_2$ where H is the Hartman number. Finally we let $n^j(x,t), j = 1, 2$, be the Gaussian random fields, white noise in time, to be elaborated more below. With these notations, we consider

$$\partial_t u + (u \cdot \nabla)u - \nu_1 \Delta u + \nabla(p + \frac{s}{2}|b|^2) - s(b \cdot \nabla)b = \theta e_2 + n^1, \tag{2.1}$$

$$\partial_t b + (u \cdot \nabla)b - (b \cdot \nabla)u - \nu_2 \Delta b = n^2, \tag{2.2}$$

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$$\partial_t\theta + (u\cdot\nabla)\theta - \kappa\Delta\theta - u_2 = 0, \tag{2.3}$$

equipped with the boundary and initial conditions

$$\begin{gathered} u, p, b, \theta, \partial_1 u, \partial_1 b, \partial_1\theta, \text{ periodic in the } x_1\text{-direction with period } L,\\ u, b_2, \partial_2 b_1, \theta = 0 \quad \text{on } \{x_2 = 0\}\cup\{x_2=1\}, \end{gathered} \tag{2.4}$$

$$(u,b,\theta)(x,0) = (u_0, b_0, \theta_0)(x), \tag{2.5}$$

where x_i is the i-th coordinate of x (cf. [3, section 2.1.4 pg. 386]). For simplicity, hereafter we assume $s = 1$, write $\int f$ for $\int_D f(x)dx$ and when an equality or an inequality consists of a constant c that is of no significance, we write $A \approx B, A \lesssim B$ instead of $A = cB, A \leq cB$ respectively. For convenience, hereafter we frequently denote the solution by $y = (u, b, \theta)$. We now set up the standard functional setting and subsequently state our main results. We let

$$\begin{aligned} H_1 = H_2 &\triangleq \big\{u \in (L^2(D))^2 : \nabla\cdot u = 0,\ u_1|_{\{x_1=0\}} = u_1|_{\{x_1=L\}},\\ &\quad u_2|_{\{x_2=0\}} = u_2|_{\{x_2=1\}} = 0\big\},\\ H_3 &= (L^2(D))^2, \end{aligned}$$

and set $\mathbb{H} \triangleq H_1\times H_2\times H_3$, endowed with its norm for $y^j = (u^j, b^j, \theta^j)$,

$$(y^1, y^2) = \int y^1\cdot y^2, \quad |y^j|^2 = (y^j, y^j), \quad j = 1, 2.$$

We define

$$\begin{gathered} V_1 = \{u \in H_1\cap(H^1(D))^2 : u|_{\{x_2=0\}} = u|_{\{x_2=1\}} = 0, u \text{ is L-periodic in } x_1\},\\ V_2 = H_2\cap(H^1(D))^2,\\ V_3 = \{\theta\in H^1(D) : \theta|_{\{x_2=0\}} = \theta|_{\{x_2=1\}} = 0, \theta \text{ is L-periodic in } x_1\}. \end{gathered}$$

We let $\mathbb{V} \triangleq V_1\times V_2\times V_3$, and A_1 be the Stokes operator defined by its bilinear form

$$a_1(u^1, u^2) = \nu_1\sum_{i=1}^2\int\nabla u_i^1\cdot\nabla u_i^2, \quad a_1(u^j, u^j) = \|u^j\|_{V_1}^2, \quad u^1, u^2\in V_1\,.$$

Similarly A_2 is defined by its bilinear form

$$a_2(b^1, b^2) = \nu_2\sum_{i=1}^2\int\nabla b_i^1\cdot\nabla b_i^2, \quad a_2(b^j, b^j) = \|b^j\|_{V_2}^2, \quad b^1, b^2\in V_2\,.$$

A_3 is generated by the Dirichlet form

$$a_3(\theta^1, \theta^2) = \kappa\int\nabla\theta^1\cdot\nabla\theta^2, \quad a_3(\theta^j, \theta^j) = \|\theta^j\|_{V_3}^2, \quad \theta^1, \theta^2\in V_3\,.$$

Finally A be the operator defined by

$$a(y^1, y^2) = \sum_{i=1}^2\int\nu_1\nabla u_i^1\cdot\nabla u_i^2 + \nu_2\nabla b_i^1\cdot\nabla b_i^2 + \kappa\nabla\theta^1\cdot\nabla\theta^2, \quad a(y^j, y^j) = \|y^j\|_V^2,$$

$j = 1, 2$. We denote by $V_1^l = D(A_1^{1/2}), V_2^l = D(A_2^{1/2}), V_3^l = D(A_3^{1/2})$ and $\mathbb{V}^l \triangleq D(A^{1/2}), l\in\mathbb{R}$ (cf. [38, Chapter II Section 2.1]). We also define bilinear continuous mappings $B_i, i = 1,\dots,5$ to satisfy

$$\langle B_1(u^1, u^2), u^3\rangle = \int(u^1\cdot\nabla)u^2\cdot u^3, \quad \langle B_2(b^1, b^2), u^3\rangle = \int(b^1\cdot\nabla)b^2\cdot u^3,$$

$$\langle B_3(u^1,b^2)\cdot b^3\rangle = \int (u^1\cdot\nabla)b^2\cdot b^3, \quad \langle B_4(b^1,u^2),b^3\rangle = \int (b^1\cdot\nabla)u^2\cdot b^3,$$
$$\langle B_5(u^1,\theta^2),\theta^3\rangle = \int (u^1\cdot\nabla)\theta^2\theta^3.$$

For $y=(u,b,\theta)$, denoting

$$B(y,y) \triangleq \begin{pmatrix} B_1(u,u)-B_2(b,b) \\ B_3(u,b)-B_4(b,u) \\ B_5(u,\theta)\end{pmatrix}, \quad Ry \triangleq \begin{pmatrix}\theta e_2 \\ 0 \\ u_2\end{pmatrix}, \quad n \triangleq \begin{pmatrix} n^1 \\ n^2 \\ 0\end{pmatrix},$$

we rewrite (2.1)–(2.3) as

$$\partial_t y + Ay + B(y,y) = Ry + n. \tag{2.6}$$

We remark on a distinctive feature of the Bénard and magnetic Bénard problems in comparison to the Navier-Stokes equations (NSE) and the magnetohydrodynamics (MHD) system. Considering the case e.g. $n\equiv 0$ for the MHD system which is (2.6) at $R\equiv 0, \theta\equiv 0$, we see that there exists $\alpha'>0$, for example $\alpha'=\min\{\nu_1,\nu_2\}>0$, such that

$$(Ay,y)+(Ry,y)\geq \alpha'\|y\|^2_{\dot{H}^1};$$

i.e. $A+R$ is coercive on $\dot{H}^1(D)$, while in the case of the Bénard and the magnetic Bénard problems, such a property is valid only if $\min\{\nu_1,\nu_2,\kappa\}>0$ is sufficiently large (see [38, Chapter III Section 3.5]).

Concerning $n^j(t), j=1,2$, we let $\{g_i^j,\lambda_i^j\}_{i=1}^\infty$, $j=1,2,3$ be the eigenvectors and their corresponding eigenvalues of A_j respectively. Then we define $w^j(t)\triangleq \sum_{i=1}^\infty \beta_i^j(t)g_i^j$,

$$G^j w^j(t) \triangleq \sum_{i=1}^\infty \sigma_i^j\beta_i^j(t)g_i^j, \quad j=1,2 \tag{2.7}$$

where $\{\sigma_i^j\}_{i=1}^\infty, j=1,2$ are noise intensities, $\{\beta_i^j\}_{i=1}^\infty$, $j=1,2$ are families of independent standard one-dimensional Brownian motions defined for all t, on $(\Omega,\mathcal{F},\mathbb{P})$ with expectation with respect to $\mathbb{P}$ denoted by E and filtration of $\mathcal{F}_t=\sigma\{w(s)-w(\tau):\tau\leq s\leq t\}$ where $w\triangleq (w^1,w^2,0)$. We note that $\{g_i^j\}_{i=1}^\infty$, $j=1,2$ forms an orthonormal basis for H_j, $j=1,2,3$ respectively.

Now letting $G\triangleq (G^1,G^2,0), G\in L(\mathbb{H})$, we define our solution.

Definition 2.1. The stochastic process $y=(u,b,\theta)$ is a generalized solution over time interval $[t_0,T]$ of (2.6) if $\mathbb{P}$ a.e. $\omega\in\Omega$,

(1)
$$y(\cdot,\omega)\in C([t_0,T];\mathbb{H})\cap L^2(t_0,T;D(A_1^{1/4})\times D(A_2^{1/4})\times V_3), \tag{2.8}$$

(2) $y=(u,b,\theta)$ satisfies for all $\phi=(\phi^1,\phi^2,\phi^3)\in D(A)$,
$$\begin{aligned}&\langle y(t),\phi\rangle - \langle y(t_0),\phi\rangle + \int_{t_0}^t \langle Ay,\phi\rangle ds \\ &= -\int_{t_0}^t \langle B(y,y)+Ry,\phi\rangle ds + \langle Gw(t)-Gw(t_0),\phi\rangle,\end{aligned} \tag{2.9}$$

(3) $y=(u,b,\theta)$ is progressively measurable.

Let us now recall that irreducibility in $\mathbb{V}^l$ implies that the transition function $P(t,x,\Gamma) \triangleq \mathbb{P}(\{y(t;x) \in \Gamma\})$ satisfies

$$P(t,x,\Gamma) > 0 \quad \forall t > t_0,\ x \in \mathbb{V}^l$$

for all non-empty open set $\Gamma \subset \mathbb{V}^l$. Moreover, their associated Markovian semigroup $\{P_t\}$ is defined by

$$(P_t\psi)(x) \triangleq E[\psi(y(t;x))] \tag{2.10}$$

is Feller if $P_t(C_b(\mathbb{V}^l)) \subset C_b(\mathbb{V}^l)$ and strong Feller if $P_t(B_b(\mathbb{V}^l)) \subset C_b(\mathbb{V}^l)$ for all $t \in (t_0, T]$ where $B_b(\mathbb{V}^l), C_b(\mathbb{V}^l)$ are the sets of all real Borel functions that are bounded, continuous and bounded respectively. We define a dual semigroup P_t^* in $Pr(\mathbb{H})$, the space of probability measures on $\mathbb{H}$, by

$$\int_{\mathbb{H}} \psi d(P_t^*\mu) = \int_{\mathbb{H}} P_t\psi d\mu.$$

If $P_t^*\mu = \mu$, then $\mu \in Pr(\mathbb{H})$ is called an invariant measure for the dynamical system $(Pr(\mathbb{H}), P_t^*)$. We now state our first result.

Theorem 2.2. *Suppose that for both $j = 1, 2$, we have*

$$\sigma_i^j \neq 0 \quad \forall\, i = 1, 2, \dots; \tag{2.11}$$

$$\sigma_i^j \leq \frac{C}{i^{\frac{1}{4}+2\gamma_0}} \text{ for some } \gamma_0 > 0 \tag{2.12}$$

for i large enough. Then for all $y_{t_0} \triangleq (u,b,\theta)(t_0) \in \mathbb{H}$, there exists a unique process y that solves (2.9) *as defined in Definition 2.1 such that in addition to the regularity of* (2.8)*,*

$$u(\cdot,\omega) \in L^2(t_0,T; D(A_1^{\min\{\frac{1}{4}+\gamma,\frac{1}{2}\}})) \cap L^4(t_0,T; D(A_1^{1/4})),$$
$$b(\cdot,\omega) \in L^2(t_0,T; D(A_2^{\min\{\frac{1}{4}+\gamma,\frac{1}{2}\}})) \cap L^4(t_0,T; D(A_2^{1/4}))$$

for all $\gamma < \gamma_0$, $\mathbb{P}$-a.e. $\omega \in \Omega$. Moreover, it is a Markov process. Finally, there exists an invariant measure for (2.6)*.*

To prove the uniqueness of the invariant measure, it suffices to prove the irreducibility and strong Feller property ([8] and e.g. [6, Theorem 4.2.1]). Under a slightly more restrictive condition, this is possible.

Theorem 2.3. *Let $l \in \mathbb{Z}, l \geq 2$. Suppose that for both $j = 1, 2$, $\{\sigma_i^j\}_{i=1}^\infty$ satisfy* (2.11) *and*

$$\frac{c}{i^{\frac{l+1}{2}}} \leq \sigma_i^j < \frac{C}{i^{1/2}} \tag{2.13}$$

for i's large enough. Then for all $y_0 \triangleq (u_0,b_0,\theta_0) \in \mathbb{V}^l$, there exists a unique process y that solves (2.9) *as defined in Definition 2.1 at $t_0 = 0$ such that for $\mathbb{P}$-a.e. $\omega \in \Omega$,*

$$y \in C([0,T]; \mathbb{V}^l);$$

moreover, it is a Markov process. Finally, there exists a unique invariant measure μ of (2.9) *at $t_0 = 0$, supported in $\mathbb{V}^l$ such that $\lim_{t\to\infty} P(t,x,\Gamma) = \mu(\Gamma)$ for all $x \in \mathbb{V}^l, \Gamma \in \mathcal{B}(\mathbb{V}^l)$.*

Remark 2.4. (1) As a consequence of uniqueness, the invariant measure claimed in Theorem 2.3 is ergodic (cf. [6, Theorem 3.2.6]). To the best of the author's knowledge, both Theorems 2.2 and 2.3 represent new results on the magnetic Bénard problem. Moreover, although the existence of an invariant measure for the Bénard problem is shown in [10], not its uniqueness. As the magnetic Bénard problem when $b \equiv 0$ becomes the Bénard problem, Theorem 2.3 proves the uniqueness of the invariant measure obtained in [10] for the Bénard problem as well.

(2) The proofs of Theorems 2.2 and 2.3 follow the approach of [10, 12] closely. The proof is also inspired by the previous work on ergodicity of the NSE, Burgers' equation (see [5, 11, 14, 16, 17]); we also refer to [1, 22, 24, 25, 26, 31, 32, 34] as closely related important work. We remark however that in [10, 14], the path-wise uniqueness of the generalized solution was missing for a technical reason; in fact, in [14, Remark after Theorem 3.1], this uniqueness was stated as an open problem. We follow the approach of [13] to obtain such a path-wise uniqueness.

(3) For the generalization to l non-integer and $l = 1$ cases, we refer to the [12, Section 6]; we chose not to pursue this direction of research for simplicity of presentation.

3. Proof of Theorem 2.2: Existence

We consider two Ornstein-Uhlenbeck processes: for $\alpha \geq 0$, $j = 1, 2$, defined also for $t < 0$,

$$dz_\alpha^j + A_j z_\alpha^1(t)dt = -\alpha z_\alpha^j(t)dt + dG^j w^j, \tag{3.1}$$

for which their solutions are of the form

$$z_\alpha^j(t) = \int_{-\infty}^t e^{(-A_j - \alpha)(t-s)} G^j dw^j(s) \tag{3.2}$$

(cf. [7]) and we collect their properties in the Appendix. By (2.12) we see that there exists $\gamma_0 > 0$ such that

$$\sum_{i=1}^\infty \frac{(\sigma_i^j)^2}{(\lambda_i^j)^{\frac{1}{2} - 2\gamma_0}} < \infty$$

because $\lambda_i^j \sim i$ as $i \to \infty$. Thus, by Lemma 5.2 (1) we see that $\mathbb{P}$-a.s.

$$z_\alpha^j \in C([t_0, T]; D(A_j^{\gamma_0 + \frac{1}{4}})), \tag{3.3}$$

and by Lemma 5.1 (2)

$$G^j w^j \in C([t_0, T]; D(A_j^{\gamma_0 - \frac{1}{4}})). \tag{3.4}$$

Now we define

$$\overline{u} \triangleq u - z_\alpha^1, \quad \overline{b} \triangleq b - z_\alpha^2 \tag{3.5}$$

so that by (2.6) and (3.1),

$$\begin{aligned} &\partial_t \overline{u} + A_1 \overline{u} + B_1(\overline{u}, \overline{u}) + B_1(\overline{u}, z_\alpha^1) + B_1(z_\alpha^1, \overline{u}) - B_2(\overline{b}, \overline{b}) - B_2(\overline{b}, z_\alpha^2) - B_2(z_\alpha^2, \overline{b}) \\ &= -B_1(z_\alpha^1, z_\alpha^1) + B_2(z_\alpha^2, z_\alpha^2) + \theta e_2 + \alpha z_\alpha^1, \end{aligned} \tag{3.6}$$

$$\begin{aligned} &\partial_t \overline{b} + A_2 \overline{b} + B_3(\overline{u}, \overline{b}) + B_3(\overline{u}, z_\alpha^2) + B_3(z_\alpha^1, \overline{b}) - B_4(\overline{b}, \overline{u}) - B_4(\overline{b}, z_\alpha^1) - B_4(z_\alpha^2, \overline{u}) \\ &= -B_3(z_\alpha^1, z_\alpha^2) + B_4(z_\alpha^2, z_\alpha^1) + \alpha z_\alpha^2, \end{aligned} \tag{3.7}$$

$$\partial_t \theta + A_3\theta + B_5(\overline{u}, \theta) = -B_5(z_\alpha^1, \theta) + \overline{u}_2 + z_{\alpha,2}^1, \tag{3.8}$$

where $z_{\alpha,2}^1$ is the second component of z_α^1.

We now fix $\omega \in \Omega$ and take L^2-inner products of (3.6)–(3.8) with $(\overline{u}, \overline{b}, \theta)$. Their addition yields

$$\begin{aligned}
&\frac{1}{2}\partial_t |(\overline{u}, \overline{b}, \theta)|^2 + \|(\overline{u}, \overline{b}, \theta)\|_{\mathbb{V}}^2 \\
&= \langle B_1(\overline{u}, \overline{u}), z_\alpha^1\rangle - \langle B_2(\overline{b}, \overline{u}), z_\alpha^2\rangle + \langle B_1(z_\alpha^1, \overline{u}), z_\alpha^1\rangle - \langle B_2(z_\alpha^2, \overline{u}), z_\alpha^2\rangle \\
&\quad + \langle \theta e_2, \overline{u}\rangle + \alpha\langle z_\alpha^1, \overline{u}\rangle + \langle B_3(\overline{u}, \overline{b}), z_\alpha^2\rangle - \langle B_4(\overline{b}, \overline{b}), z_\alpha^1\rangle \\
&\quad + \langle B_3(z_\alpha^1, \overline{b}), z_\alpha^2\rangle - \langle B_4(z_\alpha^2, \overline{b}), z_\alpha^1\rangle + \alpha\langle z_\alpha^2, \overline{b}\rangle + \langle \overline{u}, e_2\theta\rangle + \langle z_\alpha^1, e_2\theta\rangle \\
&\le \|\overline{u}\|_{L^4}\|\overline{u}\|_{V_1}\|z_\alpha^1\|_{L^4} + \|\overline{b}\|_{L^4}\|\overline{u}\|_{V_1}\|z_\alpha^2\|_{L^4} + \|z_\alpha^1\|_{L^4}^2\|\overline{u}\|_{V_1} + \|z_\alpha^2\|_{L^4}^2\|\overline{u}\|_{V_1} \\
&\quad + |\theta||\overline{u}| + \alpha|z_\alpha^1||\overline{u}| + \|\overline{u}\|_{L^4}\|\overline{b}\|_{V_2}\|z_\alpha^2\|_{L^4} + \|\overline{b}\|_{L^4}\|\overline{b}\|_{V_2}\|z_\alpha^1\|_{L^4} \\
&\quad + \|z_\alpha^1\|_{L^4}\|\overline{b}\|_{V_2}\|z_\alpha^2\|_{L^4} + \|z_\alpha^2\|_{L^4}\|\overline{b}\|_{V_2}\|z_\alpha^1\|_{L^4} + \alpha|z_\alpha^2||\overline{b}| + |\overline{u}||\theta| + |z_\alpha^1||\theta| \\
&\lesssim |\overline{u}|^{1/2}\|\overline{u}\|_{V_1}^{3/2}\|z_\alpha^1\|_{L^4} + |\overline{b}|^{1/2}\|\overline{b}\|_{V_2}^{1/2}\|\overline{u}\|_{V_1}\|z_\alpha^2\|_{L^4} + \|z_\alpha^1\|_{L^4}^2\|\overline{u}\|_{V_1} \\
&\quad + \|z_\alpha^2\|_{L^4}^2\|\overline{u}\|_{V_1} + |\theta||\overline{u}| + |z_\alpha^1||\overline{u}| + |\overline{u}|^{1/2}\|\overline{u}\|_{V_1}^{1/2}\|\overline{b}\|_{V_2}\|z_\alpha^2\|_{L^4} \\
&\quad + |\overline{b}|^{1/2}\|\overline{b}\|_{V_2}^{3/2}\|z_\alpha^1\|_{L^4} + \|z_\alpha^1\|_{L^4}\|\overline{b}\|_{V_2}\|z_\alpha^2\|_{L^4} + |z_\alpha^2||\overline{b}| + |\overline{u}||\theta| + |z_\alpha^1||\theta| \\
&\le \frac{1}{2}\|(\overline{u}, \overline{b}, \theta)\|_{\mathbb{V}}^2 + c|(\overline{u}, \overline{b}, \theta)|^2(1 + \|(z_\alpha^1, z_\alpha^2)\|_{L^4}^4) + c\|(z_\alpha^1, z_\alpha^2)\|_{L^4}^4
\end{aligned} \tag{3.9}$$

where we used integration by parts, Hölder's inequalities, Gagliardo-Nirenberg inequalities of e.g. $\|u\|_{L^4(D)} \lesssim |u|^{1/2}\|u\|_{V_1}^{1/2}$ (cf. [28, Lemma 6.2], and more generally [30]), and Young's inequalities. We also relied on important cancelations such as

$$-\langle B_2(\overline{b}, \overline{b}), \overline{u}\rangle - \langle B_4(\overline{b}, \overline{u}), \overline{b}\rangle = 0,$$
$$-\langle B_2(z_\alpha^2, \overline{b}), \overline{u}\rangle - \langle B_4(z_\alpha^2, \overline{u}), \overline{b}\rangle = 0.$$

After subtracting $\frac{1}{2}\|(\overline{u}, \overline{b}, \theta)\|_{\mathbb{V}}^2$ from both sides of (3.9), we obtain by the Sobolev embedding of $D(A_j^{1/4}) \hookrightarrow L^4(D)$ and (3.3) that for all $t \in [t_0, T]$

$$|(\overline{u}, \overline{b}, \theta)(t)|^2 + \int_{t_0}^T \|(\overline{u}, \overline{b}, \theta)\|_{\mathbb{V}}^2 ds \lesssim |(\overline{u}, \overline{b}, \theta)(t_0)|^2 + 1 \lesssim 1. \tag{3.10}$$

Proposition 3.1. *Under the hypothesis of Theorem 2.2, $\alpha \ge 0$, on $[t_0, T]$ such that $(u, b, \theta)(t_0) \in \mathbb{H}$, there exists a unique solution $(\overline{u}, \overline{b}, \theta)$ to (3.6)–(3.8) such that $(\overline{u}(t_0), \overline{b}(t_0), \theta(t_0)) = (u(t_0) - z_\alpha^1(t_0), b(t_0) - z_\alpha^2(t_0), \theta(t_0))$ and*

$$(\overline{u}, \overline{b}, \theta) \in C([t_0, T]; \mathbb{H}) \cap L^2(t_0, T; \mathbb{V}) \quad \mathbb{P} - a.e.\ \omega \in \Omega. \tag{3.11}$$

Moreover, the solution is independent of α.

Proof. With the *a priori* estimates of (3.10), a standard Galerkin approximation scheme shows the existence of the solution (see [4, 28, 38]). We note that due to (3.5) the regularity of u and b are obtained from $\overline{u}, z_\alpha^1$ and $\overline{b}, z_\alpha^2$ respectively. This is the reason why in Definition 2.1, we require only $u \in L^2(t_0, T; D(A_1^{1/4})), b \in L^2(t_0, T; D(A_2^{1/4}))$ instead of $L^2(t_0, T; V_1), L^2(t_0, T; V_2)$.

We now prove the uniqueness. Suppose there exists $(\overline{u}^1, \overline{b}^1, \theta^1), (\overline{u}^2, \overline{b}^2, \theta^2)$ that are both solutions. Then defining

$$\delta\overline{u} \triangleq \overline{u}^1 - \overline{u}^2, \quad \delta\overline{b} \triangleq \overline{b}^1 - \overline{b}^2, \quad \delta\theta \triangleq \theta^1 - \theta^2, \tag{3.12}$$

we see that from (3.6)–(3.8),

$$\begin{aligned} \partial_t\delta\overline{u} + A_1\delta\overline{u} &= -B_1(\delta\overline{u}, \overline{u}^1) - B_1(\overline{u}^2, \delta\overline{u}) - B_1(\delta\overline{u}, z_\alpha^1) - B_1(z_\alpha^1, \delta\overline{u}) \\ &\quad + B_2(\delta\overline{b}, \overline{b}^1) + B_2(\overline{b}^2, \delta\overline{b}) + B_2(\delta\overline{b}, z_\alpha^2) + B_2(z_\alpha^2, \delta\overline{b}) + \delta\theta e_2, \end{aligned} \tag{3.13}$$

$$\begin{aligned} \partial_t\delta\overline{b} + A_2\delta\overline{b} &= -B_3(\delta\overline{u}, \overline{b}^1) - B_3(\overline{u}^2, \delta\overline{b}) - B_3(\delta\overline{u}, z_\alpha^2) - B_3(z_\alpha^1, \delta\overline{b}) \\ &\quad + B_4(\delta\overline{b}, \overline{u}^1) + B_4(\overline{b}^2, \delta\overline{u}) + B_4(\delta\overline{b}, z_\alpha^1) + B_4(z_\alpha^2, \delta\overline{u}), \end{aligned} \tag{3.14}$$

$$\partial_t\delta\theta + A_3\delta\theta = -B_5(\delta\overline{u}, \theta^1) - B_5(\overline{u}^2, \delta\theta) - B_5(z_\alpha^1, \delta\theta) + \delta\overline{u}_2, \tag{3.15}$$

with $\delta\overline{u}_2 = \overline{u}_2^1 - \overline{u}_2^2$, $\overline{u}_2^j$ being the second component of $\overline{u}^j$, $j = 1, 2$.

Taking L^2-inner products on (3.13)–(3.15) with $(\delta\overline{u}, \delta\overline{b}, \delta\theta)$ respectively we obtain similarly to (3.9),

$$\begin{aligned} &\frac{1}{2}\partial_t|(\delta\overline{u}, \delta\overline{b}, \delta\theta)|^2 + \|(\delta\overline{u}, \delta\overline{b}, \delta\theta)\|_{\mathbb{V}}^2 \\ &= -\int(\delta\overline{u}\cdot\nabla)\overline{u}^1\cdot\delta\overline{u} + \int(\delta\overline{u}\cdot\nabla)\delta\overline{u}\cdot z_\alpha^1 + \int(\delta\overline{b}\cdot\nabla)\overline{b}^1\cdot\delta\overline{u} - \int(\delta\overline{b}\cdot\nabla)\delta\overline{u}\cdot z_\alpha^2 \\ &\quad + \int\delta\theta e_2\cdot\delta\overline{u} - \int(\delta\overline{u}\cdot\nabla)\overline{b}^1\cdot\delta\overline{b} + \int(\delta\overline{u}\cdot\nabla)\delta\overline{b}\cdot z_\alpha^2 + \int(\delta\overline{b}\cdot\nabla)\overline{u}^1\cdot\delta\overline{b} \\ &\quad - \int(\delta\overline{b}\cdot\nabla)\delta\overline{b}\cdot z_\alpha^1 - \int(\delta\overline{u}\cdot\nabla)\theta^1\delta\theta + \int\delta\overline{u}\cdot\delta\theta e_2 \\ &\le \|\delta\overline{u}\|_{L^4}^2\|\overline{u}^1\|_{V_1} + \|\delta\overline{u}\|_{L^4}\|\delta\overline{u}\|_{V_1}\|z_\alpha^1\|_{L^4} + \|\delta\overline{b}\|_{L^4}\|\overline{b}^1\|_{V_2}\|\delta\overline{u}\|_{L^4} \\ &\quad + \|\delta\overline{b}\|_{L^4}\|\delta\overline{u}\|_{V_1}\|z_\alpha^2\|_{L^4} + |\delta\theta||\delta\overline{u}| + \|\delta\overline{u}\|_{L^4}\|\overline{b}^1\|_{V_2}\|\delta\overline{b}\|_{L^4} + \|\delta\overline{u}\|_{L^4}\|\delta\overline{b}\|_{V_2}\|z_\alpha^2\|_{L^4} \\ &\quad + \|\delta\overline{b}\|_{L^4}^2\|\overline{u}^1\|_{V_1} + \|\delta\overline{b}\|_{L^4}\|\delta\overline{b}\|_{V_2}\|z_\alpha^1\|_{L^4} + \|\delta\overline{u}\|_{L^4}\|\theta^1\|_{V_3}\|\delta\theta\|_{L^4} + |\delta\overline{u}||\delta\theta| \\ &\le \frac{1}{2}\|(\delta\overline{u}, \delta\overline{b}, \delta\theta)\|_{\mathbb{V}}^2 + c|(\delta\overline{u}, \delta\overline{b}, \delta\theta)|^2(\|(\overline{u}^1, \overline{b}^1, \theta^1)\|_{\mathbb{V}}^2 + \|(z_\alpha^1, z_\alpha^2)\|_{L^4}^4 + 1) \end{aligned}$$

where we use the crucial cancelations

$$\int(\overline{b}^2\cdot\nabla)\delta\overline{b}\cdot\delta\overline{u} + \int(\overline{b}^2\cdot\nabla)\delta\overline{u}\cdot\delta\overline{b} = 0,$$
$$\int(z_\alpha^2\cdot\nabla)\delta\overline{b}\cdot\delta\overline{u} + \int(z_\alpha^2\cdot\nabla)\delta\overline{u}\cdot\delta\overline{b} = 0.$$

Subtracting $\frac{1}{2}\|(\delta\overline{u}, \delta\overline{b}, \delta\theta)\|_{\mathbb{V}}^2$ from both sides, we obtain

$$\begin{aligned} &\partial_t|(\delta\overline{u}, \delta\overline{b}, \delta\theta)|^2 + \|(\delta\overline{u}, \delta\overline{b}, \delta\theta)\|_{\mathbb{V}}^2 \\ &\lesssim |(\delta\overline{u}, \delta\overline{b}, \delta\theta)|^2(\|(\overline{u}^1, \overline{b}^1, \theta^1)\|_{\mathbb{V}}^2 + \|(z_\alpha^1, z_\alpha^2)\|_{L^4}^4 + 1). \end{aligned} \tag{3.16}$$

Gronwall's inequality with (3.3) and (3.10) implies uniqueness.

Concerning the Markov property, it suffices to show that for $y = (u, b, \theta)$,

$$E[\psi(y(t; t_0, y(t_0)))|\mathcal{F}_s] = P_{t-s}(\psi)(y(s; t_0, y(t_0)))$$

for all $t_0 < s < t, y(t_0) \in \mathbb{H}, \psi \in C_b(\mathbb{H})$. This result is classical and can be found in [7, Theorem 9.14].

We have already shown that the solution $(\overline{u}, \overline{b}, \theta)$ to (3.6)–(3.8) is unique for all $\alpha \geq 0$. For a fixed $\alpha \geq 0$, we let $(\overline{u}_\alpha, \overline{b}_\alpha, \theta)$ be the unique solution to (3.6)–(3.8). Then by (3.5), $u_\alpha = \overline{u}_\alpha + z_\alpha^1, b_\alpha = \overline{b}_\alpha + z_\alpha^2$ and for $\alpha \geq 0$, we have $(\overline{u}_\alpha, \overline{b}_\alpha, \theta) \in L^2(0,T;\mathbb{V})$. Moreover, for $j = 1,2$, denoting by z^j, the solution to (3.1) at $\alpha = 0$, we see that they satisfy

$$\partial_t(z_\alpha^j - z^j) = -A_j(z_\alpha^j - z^j) - \alpha z_\alpha^j$$

so that

$$(z_\alpha^j - z^j)(t) = e^{-(t-t_0)A_j}(z_\alpha^j - z^j)(t_0) - \int_{t_0}^t e^{-(t-s)A_j}\alpha z_\alpha^j(s)ds$$

for all $\alpha \geq 0$ and it is well-known (cf. [35]) that $(z_\alpha^j - z^j)(t) \in L^2(t_0,T;V_j)$. Thus, $u_\alpha - z^1 \in L^2(t_0,T;V_1), b_\alpha - z^2 \in L^2(t_0,T;V_2)$ and therefore for all $\alpha \geq 0$, we obtain a unique solution (u_α, b_α) independent of α; it is also clear that θ is independent of α. □

Proposition 3.2. *Under the hypothesis of Theorem 2.2, suppose that for some* $\gamma \in (0,\frac{1}{2}) \cap (0, 2\gamma_0]$, *where* $\gamma_0 \in (0,\frac{3}{4}], y_{t_0} \triangleq y(t_0) \in D(A^\gamma)$. *Then* $\mathbb{P}$*-a.e.* $\omega \in \Omega$,

$$y \in C([t_0,T];D(A^{\min\{\gamma,\frac{1}{4}+\gamma_0\}})) \cap L^2([t_0,T];D(A^{\min\{\frac{1}{2}+\gamma,\frac{1}{4}+\gamma_0\}})).$$

Proof. For both $j = 1,2$, $z_\alpha^j \in C([t_0,T];D(A_j^{\gamma_0+\frac{1}{4}}))$ by (3.3) and hence due to (3.5), it suffices to show that

$$(\overline{u}, \overline{b}, \theta) \in C([t_0,T];D(A^{\min\{\gamma,\frac{1}{4}+\gamma_0\}})) \cap L^2([t_0,T];D(A^{\min\{\frac{1}{2}+\gamma,\frac{1}{4}+\gamma_0\}})).$$

We first remark that $|A_1^k \cdot| \approx |A_2^k \cdot|$. We take L^2-inner products on (3.6)–(3.8) with $(A_1^{2\gamma}\overline{u}, A_2^{2\gamma}\overline{b}, A_3^{2\gamma}\theta)$ to obtain

$$\begin{aligned}
&\frac{1}{2}\partial_t|(A_1^\gamma\overline{u}, A_2^\gamma\overline{b}, A_3^\gamma\theta)|^2 + |(A_1^{\frac{1}{2}+\gamma}\overline{u}, A_2^{\frac{1}{2}+\gamma}\overline{b}, A_3^{\frac{1}{2}+\gamma}\theta)|^2 \\
&= -\int [(\overline{u}\cdot\nabla)\overline{u} + (\overline{u}\cdot\nabla)z_\alpha^1 + (z_\alpha^1\cdot\nabla)\overline{u} - (\overline{b}\cdot\nabla)\overline{b} - (\overline{b}\cdot\nabla)z_\alpha^2 - (z_\alpha^2\cdot\nabla)\overline{b}]\cdot A_1^{2\gamma}\overline{u} \\
&\quad - \int [(z_\alpha^1\cdot\nabla)z_\alpha^1 - (z_\alpha^2\cdot\nabla)z_\alpha^2 - \theta e_2 - \alpha z_\alpha^1]\cdot A_1^{2\gamma}\overline{u} \\
&\quad - \int [(\overline{u}\cdot\nabla)\overline{b} + (\overline{u}\cdot\nabla)z_\alpha^2 + (z_\alpha^1\cdot\nabla)\overline{b} - (\overline{b}\cdot\nabla)\overline{u} - (\overline{b}\cdot\nabla)z_\alpha^1 - (z_\alpha^2\cdot\nabla)\overline{u}]\cdot A_2^{2\gamma}\overline{b} \\
&\quad - \int [(z_\alpha^1\cdot\nabla)z_\alpha^2 - (z_\alpha^2\cdot\nabla)z_\alpha^1 - \alpha z_\alpha^2]\cdot A_2^{2\gamma}\overline{b} \\
&\quad - \int [(\overline{u}\cdot\nabla)\theta + (z_\alpha^1\cdot\nabla)\theta + \overline{u}_2 + z_{\alpha,2}^1]A_3^{2\gamma}\theta.
\end{aligned}$$

We bound e.g.

$$\int (\overline{u}\cdot\nabla)\overline{u}\cdot A_1^{2\gamma}\overline{u} \leq |A_1^{-(\frac{1}{2}-\gamma)}[(\overline{u}\cdot\nabla)\overline{u}]||A_1^{\frac{1}{2}+\gamma}\overline{u}| \lesssim |A_1^{\frac{1}{4}+\frac{\gamma}{2}}\overline{u}|^2|A_1^{\frac{1}{2}+\gamma}\overline{u}|$$

by Hölder's inequality and Lemma 5.3. We compute similarly on other nonlinear terms to obtain

$$
\begin{aligned}
&\frac{1}{2}\partial_t|(A_1^{\gamma}\overline{u}, A_2^{\gamma}\overline{b}, A_3^{\gamma}\theta)|^2 + |(A_1^{\frac{1}{2}+\gamma}\overline{u}, A_2^{\frac{1}{2}+\gamma}\overline{b}, A_3^{\frac{1}{2}+\gamma}\theta)|^2 \\
&\lesssim (|A_1^{\frac{1}{4}+\frac{\gamma}{2}}\overline{u}|^2 + |A_1^{\frac{1}{4}+\frac{\gamma}{2}}\overline{u}||A_1^{\frac{1}{4}+\frac{\gamma}{2}}z_\alpha^1| + |A_1^{\frac{1}{4}+\frac{\gamma}{2}}\overline{b}|^2 + |A_1^{\frac{1}{4}+\frac{\gamma}{2}}\overline{b}||A_1^{\frac{1}{4}+\frac{\gamma}{2}}z_\alpha^2|)|A_1^{\frac{1}{2}+\gamma}\overline{u}| \\
&\quad + (|A_1^{\frac{1}{4}+\frac{\gamma}{2}}z_\alpha^1|^2 + |A_1^{\frac{1}{4}+\frac{\gamma}{2}}z_\alpha^2|^2 + |A_1^{\gamma-\frac{1}{2}}\theta| + |A_1^{\gamma-\frac{1}{2}}z_\alpha^1|)|A_1^{\frac{1}{2}+\gamma}\overline{u}| \\
&\quad + (|A_2^{\frac{1}{4}+\frac{\gamma}{2}}\overline{u}||A_2^{\frac{1}{4}+\frac{\gamma}{2}}\overline{b}| + |A_2^{\frac{1}{4}+\frac{\gamma}{2}}\overline{u}||A_2^{\frac{1}{4}+\frac{\gamma}{2}}z_\alpha^2| + |A_2^{\frac{1}{4}+\frac{\gamma}{2}}z_\alpha^1||A_2^{\frac{1}{4}+\frac{\gamma}{2}}\overline{b}|)|A_2^{\frac{1}{2}+\gamma}\overline{b}| \\
&\quad + (|A_2^{\frac{1}{4}+\frac{\gamma}{2}}z_\alpha^1||A_2^{\frac{1}{4}+\frac{\gamma}{2}}z_\alpha^2| + |A_2^{\gamma-\frac{1}{2}}z_\alpha^2|)|A_2^{\frac{1}{2}+\gamma}\overline{b}| \\
&\quad + (|A_3^{\frac{1}{4}+\frac{\gamma}{2}}\overline{u}||A_3^{\frac{1}{4}+\frac{\gamma}{2}}\theta| + |A_3^{\frac{1}{4}+\frac{\gamma}{2}}z_\alpha^1||A_3^{\frac{1}{4}+\frac{\gamma}{2}}\theta| + |A_3^{\gamma-\frac{1}{2}}\overline{u}| + |A_3^{\gamma-\frac{1}{2}}z_\alpha^1|)|A_3^{\frac{1}{2}+\gamma}\theta|.
\end{aligned}
\tag{3.17}
$$

Now $z_\alpha^j \in C([t_0,T]; D(A_j^{\gamma_0+\frac{1}{4}}))$ by (3.3) and $\gamma \in (0, 2\gamma_0]$ by hypothesis so that $\frac{1}{4}+\frac{\gamma}{2} \le \gamma_0 + \frac{1}{4}$. Thus,

$$
\begin{aligned}
&\frac{1}{2}\partial_t|(A_1^{\gamma}\overline{u}, A_2^{\gamma}\overline{b}, A_3^{\gamma}\theta)|^2 + |(A_1^{\frac{1}{2}+\gamma}\overline{u}, A_2^{\frac{1}{2}+\gamma}\overline{b}, A_3^{\frac{1}{2}+\gamma}\theta)|^2 \\
&\lesssim (1 + |A_1^{1/2}\overline{u}||A_1^{\gamma}\overline{u}| + |A_1^{1/2}\overline{b}||A_1^{\gamma}\overline{b}| + |A_1^{\frac{1}{2}}\theta||A_1^{\gamma}\theta|)|A_1^{\frac{1}{2}+\gamma}\overline{u}| \\
&\quad + (1 + |A_2^{1/2}\overline{u}||A_2^{\gamma}\overline{u}| + |A_2^{1/2}\overline{b}||A_2^{\gamma}\overline{b}|)|A_2^{\frac{1}{2}+\gamma}\overline{b}| \\
&\quad + (1 + |A_3^{1/2}\overline{u}||A_3^{\gamma}\overline{u}| + |A_3^{1/2}\theta||A_3^{\gamma}\theta|)|A_3^{\frac{1}{2}+\gamma}\theta| \\
&\le \frac{1}{2}|(A_1^{\frac{1}{2}+\gamma}\overline{u}, A_2^{\frac{1}{2}+\gamma}\overline{b}, A_3^{\frac{1}{2}+\gamma}\theta)|^2 + c(1 + |(A_1^{\gamma}\overline{u}, A_2^{\gamma}\overline{b}, A_3^{\gamma}\theta)|^2)(1 + \|(\overline{u}, \overline{b}, \theta)\|_V^2)
\end{aligned}
$$

where we used that $\gamma - \frac{1}{2} \le \frac{1}{4} + \frac{\gamma}{2}$, Gagliardo-Nirenberg and and Young's inequalities. Hence, by Gronwall's inequality we obtain

$$
\begin{aligned}
&|(A_1^{\gamma}\overline{u}, A_2^{\gamma}\overline{b}, A_3^{\gamma}\theta)(t)|^2 + \int_{t_0}^t |(A_1^{\frac{1}{2}+\gamma}\overline{u}, A_2^{\frac{1}{2}+\gamma}\overline{b}, A_3^{\frac{1}{2}+\gamma}\theta)|^2 ds \\
&\lesssim |(A_1^{\gamma}\overline{u}, A_2^{\gamma}\overline{b}, A_3^{\gamma}\theta)(t_0)|^2 e^{\int_{t_0}^t 1 + \|(\overline{u}, \overline{b}, \theta)\|_{\mathbb{V}}^2 ds}.
\end{aligned}
$$

The computation above may be repeated for γ replaced by $\min\{\gamma, \frac{1}{4}+\gamma_0\}$; therefore,

$$
\begin{aligned}
&|A^{\min\{\gamma, \frac{1}{4}+\gamma_0\}}(\overline{u}, \overline{b}, \theta)(t)|^2 + \int_{t_0}^t |(A^{\frac{1}{2}+\min\{\gamma, \frac{1}{4}+\gamma_0\}}(\overline{u}, \overline{b}, \theta)|^2 ds \\
&\lesssim |A^{\min\{\gamma, \frac{1}{4}+\gamma_0\}}(\overline{u}, \overline{b}, \theta)(t_0)|^2 e^{\int_{t_0}^t 1 + \|(\overline{u}, \overline{b}, \theta)\|_{\mathbb{V}}^2 ds}.
\end{aligned}
$$

Finally, as $(\overline{u}, \overline{b}, \theta)(t_0) = (u - z_\alpha^1, b - z_\alpha^2, \theta)(t_0)$ due to (3.5) where $(u, b, \theta)(t_0) \in D(A^{\gamma})$ by hypothesis and $z_\alpha^j \in C([t_0, T]; D(A_j^{\gamma_0+\frac{1}{4}}))$ by (3.3), we see that

$$
|A^{\min\{\gamma, \frac{1}{4}+\gamma_0\}}(\overline{u}, \overline{b}, \theta)(t_0)|^2 < \infty.
$$

Thus, by (3.11), the proof is complete. □

Proposition 3.3 (Dissipativity). *Let $y(t, t_0, (0, 0, \theta(t_0)), \omega)$ be the solution $y = (u, b, \theta)(\cdot, \omega)$ at time t that has values $(u, b, \theta)(t_0) = (0, 0, \theta(t_0)) \in \mathbb{H}$. Suppose that*

$-1 \le \theta(t_0) \le 1$ *a.e.* $x \in D$. *Then under the hypothesis of Theorem 2.2, there exists a random variable* $r(\omega)$ $\mathbb{P}$*-a.s. finite such that for* $\gamma \in (0, \frac{1}{2}) \cap (0, 2\gamma_0]$

$$\sup_{-\infty < t_0 \le 0} |A^\gamma y(0, t_0, y(t_0), \omega)| \le r(\omega). \tag{3.18}$$

Proof. We take L^2-inner products on (3.6), (3.7) with $(\overline{u}, \overline{b})$ to obtain similarly to (3.9), applying Hölder's, Gagliardo-Nirenberg and Young's inequalities,

$$\begin{aligned} &\frac{1}{2}\partial_t |(\overline{u}, \overline{b})|^2 + \|\overline{u}\|_{V_1}^2 + \|\overline{b}\|_{V_2}^2 \\ &\le \frac{1}{2}(\|\overline{u}\|_{V_1}^2 + \|\overline{b}\|_{V_2}^2) + \frac{\epsilon}{2}|(\overline{u}, \overline{b})|^2 + c(|\overline{u}|^2 \|(z_\alpha^1, z_\alpha^2)\|_{L^4}^4 \\ &\quad + |\overline{b}|^2 \|(z_\alpha^1, z_\alpha^2)\|_{L^4}^4 + \|(z_\alpha^1, z_\alpha^2)\|_{L^4}^4 + |(\theta, z_\alpha^1, z_\alpha^2)|^2) \end{aligned} \tag{3.19}$$

for $\epsilon > 0$ arbitrary small. We may choose $\epsilon > 0$ sufficiently small so that by Poincare's inequality,

$$\epsilon |(\overline{u}, \overline{b})|^2 + \frac{1}{2}(\|\overline{u}\|_{V_1}^2 + \|\overline{b}\|_{V_2}^2) \le \|\overline{u}\|_{V_1}^2 + \|\overline{b}\|_{V_2}^2.$$

Thus, subtracting $\frac{1}{2}(\|\overline{u}\|_{V_1}^2 + \|\overline{b}\|_{V_2}^2)$ from both sides of (3.19), we may obtain for some constant $c_0 > 0$,

$$\begin{aligned} &|(\overline{u}, \overline{b})(t)|^2 - |(\overline{u}, \overline{b})(t_0)|^2 e^{\int_{t_0}^t -\epsilon + c_0 \|(z_\alpha^1, z_\alpha^2)\|_{L^4}^4 ds} \\ &\le c \int_{t_0}^t (\|(z_\alpha^1, z_\alpha^2)\|_{L^4}^4 + |(\theta, z_\alpha^1, z_\alpha^2)|^2) e^{\int_s^t -\epsilon + c_0 \|(z_\alpha^1, z_\alpha^2)\|_{L^4}^4 dr} ds. \end{aligned} \tag{3.20}$$

Now $z_\alpha^j(t)$ is an ergodic process with values in $(L^4(D))^2$; thus,

$$\lim_{t_0 \to -\infty} \frac{1}{-t_0} \int_{t_0}^0 \|z_\alpha^j\|_{L^4}^4 ds = E[\|z_\alpha^j(0)\|_{L^4}^4]$$

while by Sobolev embedding of $D(A_j^{1/4}) \hookrightarrow L^4(D)$ and Lemma 5.2 (2), for all s, we know

$$\lim_{\alpha \to \infty} E[\|z_\alpha^j(s)\|_{L^4}^4] = 0$$

so that for $c_0 > 0$ fixed, we have

$$-\epsilon + c_0 E[\|(z_\alpha^1, z_\alpha^2)(0)\|_{L^4}^4] \le -\frac{\epsilon}{2}$$

for $\alpha > 0$ sufficiently large. Thus,

$$\lim_{t_0 \to -\infty} \frac{1}{-t_0} \int_{t_0}^0 -\epsilon + c_0 \|(z_\alpha^1, z_\alpha^2)\|_{L^4}^4 ds \le -\frac{\epsilon}{2}.$$

This implies that for given $\omega \in \Omega$ for some $\tau(\omega) < 0$ sufficiently small, we have

$$\sup_{t_0 < \tau(\omega)} \int_{t_0}^0 -\epsilon + c_0 \|(z_\alpha^1, z_\alpha^2)\|_{L^4}^4 ds \le (-\frac{\epsilon}{4})(-t_0) = (\frac{\epsilon}{4})(t_0). \tag{3.21}$$

Moreover, by (3.3) and the Sobolev embedding of $D(A_j^{\gamma_0 + \frac{1}{4}}) \hookrightarrow (L^4(D))^2$, there exists a constant $c(\omega)$ such that

$$|\sup_{\tau(\omega) \le t_0 \le 0} \int_{t_0}^0 -\epsilon + c_0 \|(z_\alpha^1, z_\alpha^2)\|_{L^4}^4 ds| \le c(\omega). \tag{3.22}$$

Thus, by (3.21) and (3.22) we have the bound

$$\int_{t_0}^{0} -\epsilon + c_0\|(z_\alpha^1, z_\alpha^2)\|_{L^4}^4 ds \le \begin{cases} (\frac{\epsilon}{4})(t_0) & \text{if } t_0 < \tau(\omega), \\ c(\omega) & \text{if } \tau(\omega) \le t_0, \end{cases}$$
$$\triangleq \overline{\epsilon}(t_0, \omega).$$

We observe that for a fixed $\tau(\omega)$, $\lim_{t_0 \to -\infty} \overline{\epsilon}(t_0, \omega) = -\infty$. Besides, for $\tau \le -1, \forall\, t \in [-1, 0]$, we also have

$$\int_{t_0}^{t} -\epsilon + c_0\|(z_\alpha^1, z_\alpha^2)\|_{L^4}^4 ds \le \overline{\epsilon}(t_0, \omega) + c(\omega).$$

Under our hypothesis, z_α^j is a stationary process with values in $(L^4(D))^2$; thus, for all $t < 0$,

$$\|z_\alpha^j(t, \omega)\|_{L^4} \le c(\omega)(1 + |t|), \quad |z_\alpha^j(t, \omega)| \le c(\omega)(1 + |t|) \tag{3.23}$$

for some $c(\omega) > 0$ taken larger than before if necessary. Thus, for all $t \in [-1, 0], t_0 \le t$, from (3.20),

$$|(\overline{u}, \overline{b})(t)|^2 \le |(\overline{u}, \overline{b})(t_0)|^2 e^{\overline{\epsilon}(t_0,\omega)+c(\omega)} + \int_{t_0}^{t} e^{\overline{\epsilon}(t_0,\omega)+c(\omega)} c(\omega)(1 + |s|) ds \le r_1(\omega) \tag{3.24}$$

for some $r_1(\omega)$ because of Proposition 5.4.

We go back to (3.19); subtracting $\frac{1}{2}(\|\overline{u}\|_{V_1}^2 + \|\overline{b}\|_{V_2}^2)$ from both sides of (3.19) leads to

$$\partial_t|(\overline{u}, \overline{b})|^2 + \|\overline{u}\|_{V_1}^2 + \|\overline{b}\|_{V_2}^2 \lesssim |(\overline{u}, \overline{b})|^2(1 + \|(z_\alpha^1, z_\alpha^2)\|_{L^4}^4) + \|(z_\alpha^1, z_\alpha^2)\|_{L^4}^4 + |\theta|^2.$$

Integrating over $[-1, 0]$ leads to

$$\int_{-1}^{0} \|\overline{u}\|_{V_1}^2 + \|\overline{b}\|_{V_2}^2 ds \le r_2(\omega) \tag{3.25}$$

for some $r_2(\omega)$ due to (3.23), (3.24) and Proposition 5.4. Next, taking L^2-inner products of (3.8) with θ, applying Young's inequality and integrating over $[-1, 0]$ give

$$\int_{-1}^{0} \|\theta\|_{V_3}^2 ds \le \frac{|\theta(-1)|^2}{2} + \int_{-1}^{0} |\theta|^2 + \frac{|\overline{u}|^2}{2} + \frac{|z_\alpha^1|^2}{2} ds \le r_3(\omega) \tag{3.26}$$

for some $r_3(\omega)$ due to Proposition 5.4, (3.23) and (3.24).

Next, from (3.17), we compute

$$\begin{aligned} &\frac{1}{2}\partial_t|(A_1^\gamma \overline{u}, A_2^\gamma \overline{b})|^2 + |(A_1^{\frac{1}{2}+\gamma} \overline{u}, A_2^{\frac{1}{2}+\gamma} \overline{b})|^2 \\ &\le |(A_1^{\frac{2}{2}+\gamma} \overline{u}, A_2^{\frac{1}{2}+\gamma} \overline{b})|^2 + c\Big(1 + |(A_1^{\frac{1}{4}+\frac{\gamma}{2}} \overline{u}, A_2^{\frac{1}{4}+\frac{\gamma}{2}} \overline{b})|^4 \\ &\quad + |(A_1^{\frac{1}{4}+\frac{\gamma}{2}} z_\alpha^1, A_2^{\frac{1}{4}+\frac{\gamma}{2}} z_\alpha^2)|^4 + \|\theta\|_{V_3}^2\Big). \end{aligned} \tag{3.27}$$

Subtracting $|(A_1^{\frac{2}{2}+\gamma}\overline{u}, A_2^{\frac{1}{2}+\gamma}\overline{b})|^2$ from both sides of (36) , using that $\frac{1}{4}+\frac{\gamma}{2} \leq \gamma_0 + \frac{1}{4}$ and Gagliardo-Nirenberg and Gronwall's inequalities over $[-1,t], t \in [-1,0]$ lead to

$$\begin{aligned}
&\sup_{t\in[-1,0]} |(A_1^{\gamma}\overline{u}, A_2^{\gamma}\overline{b})(t)|^2 \\
&\leq |(A_1^{\gamma}\overline{u}, A_2^{\gamma}\overline{b})(-1)|^2 e^{\int_{-1}^0 \|(\overline{u},\overline{b})\|_{V_1\times V_2}^2 ds} \\
&\quad + e^{\int_{-1}^0 \|(\overline{u},\overline{b})\|_{V_1\times V_2}^2 ds} \int_{-1}^0 1 + |(A_1^{\gamma_0+\frac{1}{4}} z_\alpha^1, A_2^{\gamma_0+\frac{1}{4}} z_\alpha^2)|^4 + \|\theta\|_{V_3}^2 ds \lesssim r_4(\omega)
\end{aligned} \tag{3.28}$$

for some $r_4(\omega)$ due to (3.25), (3.26) and (3.3). Finally, we go back to computations of (3.17) again to similarly obtain

$$\begin{aligned}
&\frac{1}{2}\partial_t |A_3^{\gamma}\theta|^2 + |A_3^{\frac{1}{2}+\gamma}\theta|^2 \\
&\leq |A_3^{\frac{1}{2}+\gamma}\theta|^2 + c(1 + |(A_1^{\frac{1}{4}+\frac{\gamma}{2}}\overline{u}, A_3^{\frac{1}{4}+\frac{\gamma}{2}}\theta)|^4 + |A_1^{\frac{1}{4}+\frac{\gamma}{2}} z_\alpha^1|^4) \\
&\leq |A_3^{\frac{1}{2}+\gamma}\theta|^2 + c(1 + \|\overline{u}\|_{V_1}^2 |A_1^{\gamma}\overline{u}|^2 + \|\theta\|_{V_3}^2 |A_3^{\gamma}\theta|^2 + |A_1^{\frac{1}{4}+\frac{\gamma}{2}} z_\alpha^1|^4).
\end{aligned}$$

Subtracting $|A_3^{\frac{1}{2}+\gamma}\theta|^2$ from both sides and applying Gronwall's inequality give

$$\begin{aligned}
|A_3^{\gamma}\theta(0)|^2 &\lesssim |A_3^{\gamma}\theta(-1)|^2 e^{\int_{-1}^0 \|\theta\|_{V_3}^2 ds} \\
&\quad + e^{\int_{-1}^0 \|\theta\|_{V_3}^2 ds} \int_{-1}^0 (1 + \|\overline{u}\|_{V_1}^2 |A_1^{\gamma}\overline{u}|^2 + |A_1^{\frac{1}{4}+\frac{\gamma}{2}} z_\alpha^1|^4) ds.
\end{aligned}$$

Thus, by (3.25), (3.26), (3.28) and (3.3), for all $t_0 \leq 0$, we obtain

$$|A_3^{\gamma}\theta(0, t_0, y(t_0), \omega)|^2 \leq r_5(\omega) \tag{3.29}$$

for some $r_5(\omega)$. With (3.28) and (3.29) we now conclude the proof by defining $r(\omega) \triangleq r_4(\omega) + r_5(\omega)$. □

Proof of Theorem 2.2. We now prove the path-wise uniqueness of the generalized solution y to (2.6). The issue here is that we do not know if $(u,b)(\cdot,\omega)$ belongs to $L^2(t_0,T; D(A_1^{1/2}) \times D(A_2^{1/2}))$, in contrast to the deterministic case, because $z_\alpha^j \in C([t_0,T]; D(A_j^{\gamma_0+\frac{1}{4}})), \gamma_0 > 0$ in (3.3), although from Proposition 3.1, we have $(\overline{u},\overline{b})(\cdot,\omega) \in L^2(t_0,T; V_1 \times V_2)$ for $\mathbb{P}$-a.e. $\omega \in \Omega$. The key observation from [13] is that along with the same type of classical results in [37], we only need $(u,b)(\cdot,\omega) \in L^4(t_0,T; D(A_1^{1/4}) \times D(A_2^{1/4}))$ to prove the path-wise uniqueness and this follows from the interpolation inequality of, e.g. in the case of u,

$$\|\overline{u}\|_{L^4(t_0,T;D(A_1^{1/4}))}^4 \lesssim \|\overline{u}\|_{L^\infty([t_0,T];H_1)}^2 \|\overline{u}\|_{L^2(t_0,T;V_1)}^2 \lesssim 1$$

and similarly for b. Therefore, by (3.3) and (3.5), we obtain $(u,b)(\cdot,\omega)$ belongs to $L^4(t_0,T; D(A_1^{1/4}) \times D(A_2^{1/4}))$ $\mathbb{P}$-a.e. $\omega \in \Omega$.

As in the proof of Proposition 3.1, we may consider $(u^1, b^1, \theta^1), (u^2, b^2, \theta^2)$ that both solve (2.6) so that defining $\delta u \triangleq u^1 - u^2, \delta b \triangleq b^1 - b^2$, $\delta\theta \triangleq \theta^1 - \theta^2$, we obtain

$$\begin{gathered}
\partial_t \delta u + A_1 \delta u + (\delta u \cdot \nabla) u^1 - (u^2 \cdot \nabla)\delta u - (\delta b \cdot \nabla) b^1 - (b^2 \cdot \nabla)\delta b = \delta\theta e_2, \\
\partial_t \delta b + A_2 \delta b + (\delta u \cdot \nabla) b^1 - (u^2 \cdot \nabla)\delta b - (\delta b \cdot \nabla) u^1 - (b^2 \cdot \nabla)\delta u = 0, \\
\partial_t \delta\theta + A_3 \delta\theta + (\delta u \cdot \nabla)\theta^1 + (u^2 \cdot \nabla)\delta\theta = \delta u_2.
\end{gathered}$$

Therefore, taking L^2-inner products with $(\delta u, \delta b, \delta\theta)$, a computation very similar to the proof of Proposition 3.1 shows that

$$\begin{aligned}
&\frac{1}{2}\partial_t|(\delta u, \delta b, \delta\theta)|^2 + \|(\delta u, \delta b, \delta\theta)\|_{\mathbb{V}}^2 \\
&\leq \frac{1}{2}\|(\delta u, \delta b, \delta\theta)\|_{\mathbb{V}}^2 + c|(\delta u, \delta b, \delta\theta)|^2(1 + \|(u^1, b^1)\|^4_{D(A_1^{1/4})\times D(A_2^{1/4})} + |\theta^1|^2\|\theta^1\|^2_{V_3}).
\end{aligned}$$

Thus, Gronwall's inequality implies $(\delta u, \delta b, \delta\theta) = 0$.

Next, we prove the existence of an invariant measure. With $r(\omega)$ from (3.18) we let $\Omega_N \triangleq \{\omega \in \Omega : r(\omega) \leq N\}$ so that $\Omega_N \nearrow \Omega$ $(N \to \infty)$, $\cup_N \Omega_N = \Omega, \lim_{N\to\infty} \mathbb{P}(\Omega_N) = 1$ and hence for all $\epsilon > 0$ fixed, there exists $N_\epsilon > 0$ sufficiently large so that $1 - \epsilon < \mathbb{P}(\Omega_{N_\epsilon})$. On the other hand, by Proposition 3.3

$$\mathbb{P}(\Omega_{N_\epsilon}) < \mathbb{P}(\{|y(0, t_0, y(t_0), \omega)|_{D(A^\gamma)} \leq N_\epsilon\})$$

where $y(0, t_0, y(t_0), \omega)$ is a solution to (2.6) at time 0 with value $y(t_0)$ at time t_0. Therefore,

$$1 - \epsilon < \mathbb{P}(\{|y(0, t_0, y(t_0), \omega)|_{D(A^\gamma)} \leq N_\epsilon\}).$$

By the compact embedding of $D(A^\gamma) \hookrightarrow \mathbb{H}$ for all $\gamma > 0$ (cf. [29, Theorem 16.1]), this implies that for all $\epsilon > 0$, there exists a compact set $K_\epsilon \subset \mathbb{H}$ such that $1 - \epsilon < \mathbb{P}(y(0, t_0, y(t_0), \omega) \in K_\epsilon)$.

Next, we let $\nu_\tau \triangleq \mathcal{L}(y(0, -\tau, y(-\tau), \omega))$ so that $\nu_\tau = P_\tau^* \delta_{y(0)} = \mathcal{L}(y_0(\tau, \omega)) = \mathcal{L}(y_{-\tau}(0, \omega)), \tau \geq 0$ and define

$$\mu_T \triangleq \frac{1}{T}\int_0^T \nu_\tau d\tau \tag{3.30}$$

so that we know $\{\mu_T\}_{T\geq 0}$ is tight. By Prokhorov's theorem [36], it is weakly convergent. Thus, the existence of an invariant measure now follows from the Krylov-Bogoliubov theorem [27], specifically [7, Corollary 11.8]. $\square$

4. Proof of Theorem 2.3: Uniqueness

We return to (2.6), denote by $z^j, j = 1, 2$, the solution to (3.1) at $\alpha = 0$, but for $t \geq 0$ with the additional condition that $z^j(0) = 0$, and let $\overline{u} \triangleq u - z^1, \overline{b} \triangleq b - z^2$ and hence similarly to (3.6)-(3.8) we obtain

$$\begin{aligned}
\partial_t\overline{u} + A_1\overline{u} &= -B_1(\overline{u}, \overline{u}) - B_1(\overline{u}, z^1) - B_1(z^1, \overline{u}) - B_1(z^1, z^1) \\
&\quad + B_2(\overline{b}, \overline{b}) + B_2(\overline{b}, z^2) + B_2(z^2, \overline{b}) + B_2(z^2, z^2) + \theta e_2, \\
\partial_t\overline{b} + A_2\overline{b} &= -B_3(\overline{u}, \overline{b}) - B_3(\overline{u}, z^2) - B_3(z^1, \overline{b}) - B_3(z^1, z^2) \\
&\quad + B_4(\overline{b}, \overline{u}) + B_4(\overline{b}, z^1) + B_4(z^2, \overline{u}) + B_4(z^2, z^1), \\
\partial_t\theta + A_3\theta &= -B_5(\overline{u}, \theta) - B_5(z^1, \theta) + \overline{u}_2 + z_2^1,
\end{aligned} \tag{4.1}$$

where z_2^1 is the second component of z^1

Proposition 4.1. *For $l \in \mathbb{Z}$, $l \geq 2$, suppose that for both $j = 1, 2$, $\sigma_i^j < \frac{C}{i^{1/2}}$. Then for all $y_0 \triangleq y(0) \in \mathbb{V}^l$, there exists a unique solution $y = (u, b, \theta)$ to*

$$y(t) + \int_0^t Ay + B(y, y)ds = y_0 + \int_0^t Ryds + Gw(t) \tag{4.2}$$

such that $y \in C([0, T]; \mathbb{V}^l)$ $\mathbb{P}$-a.s., and it is a Markov process, satisfying the Feller property in $\mathbb{V}^l$.

Proof. From our hypothesis that $\sigma_i^j < \frac{C}{i^{1/2}}$, we obtain for both $j = 1, 2$,

$$E[|A_j^{1/2} z^j(t)|^2] < \infty \tag{4.3}$$

(see [12]). Moreover, from the higher regularity of the initial data, taking L^2-inner products of (4.1) with $(A_1^l \overline{u}, A_2^l \overline{b}, A_3^l \theta)$ respectively and using the Banach algebra property of $H^s, s > 1$ in two-dimensional space, we compute

$$\begin{aligned}
&\frac{1}{2}\partial_t |(A_1^{1/2}\overline{u}, A_2^{1/2}\overline{b}, A_3^{1/2}\theta)|^2 + |(A_1^{\frac{l+1}{2}}\overline{u}, A_2^{\frac{l+1}{2}}\overline{b}, A_3^{\frac{l+1}{2}}\theta)|^2 \\
&= -\int A_1^{\frac{l-1}{2}}[B_1(\overline{u},\overline{u}) + B_1(\overline{u}, z^1) + B_1(z^1,\overline{u}) + B_1(z^1, z^2) \\
&\quad - B_2(\overline{b},\overline{b}) - B_2(\overline{b}, z^2) - B_2(z^2,\overline{b}) - B_2(z^2, z^2)] \cdot A_1^{\frac{l+1}{2}}\overline{u} + \int A_1^{1/2}\theta e_2 A_1^{1/2}\overline{u} \\
&\quad - \int A_2^{\frac{l-1}{2}}[B_3(\overline{u},\overline{b}) + B_3(\overline{u}, z^2) + B_3(z^1,\overline{b}) + B_3(z^1, z^2) \\
&\quad - B_4(\overline{b},\overline{u}) - B_4(\overline{b}, z^1) - B_4(z^2,\overline{u}) - B_4(z^2, z^1)] \cdot A_2^{\frac{l+1}{2}}\overline{b} \\
&\quad - \int A_3^{\frac{l-1}{2}}[B_5(\overline{u},\theta) + B_5(z^1,\theta)]A_3^{\frac{l+1}{2}}\theta + \int [A_3^{1/2}\overline{u}_2 + A_3^{1/2} z_2^1]A_3^{1/2}\theta \\
&\le \frac{1}{2}|(A_1^{\frac{l+1}{2}}\overline{u}, A_2^{\frac{l+1}{2}}\overline{b}, A_3^{\frac{l+1}{2}}\theta)|^2 \\
&\quad + c(\|\overline{u}\|_{V_1^l}^4 + \|(\overline{u}, z^1)\|_{V_1^l}^4 + \|(z^1, z^2)\|_{V_1^l}^4 + \|\overline{b}\|_{V_1^l}^4 + \|(\overline{b}, z^2)\|_{V_1^l}^4 + \|z^2\|_{V_1^l}^4 \\
&\quad + \|(\theta,\overline{u})\|_{V_1^l}^2 + \|(\overline{u},\overline{b})\|_{V_2^l}^4 + \|(\overline{u}, z^2)\|_{V_2^l}^4 + \|(z^1,\overline{b})\|_{V_2^l}^4 + \|(z^1, z^2)\|_{V_2^l}^4 \\
&\quad + \|(\overline{u},\theta)\|_{V_3^l}^4 + \|(z^1,\theta)\|_{V_3^l}^4 + \|(\overline{u}, z^1)\|_{V_3^l}^2).
\end{aligned}$$

Subtracting $\frac{1}{2}|(A_1^{\frac{l+1}{2}}\overline{u}, A_2^{\frac{l+1}{2}}\overline{b}, A_3^{\frac{l+1}{2}}\theta)|^2$ from both sides, and using the fact that we know from the basic estimate $\mathbb{P}$-almost surely,

$$\sup_{t\in[0,T]} |(\overline{u},\overline{b},\theta)|^2 + \int_0^T \|(\overline{u},\overline{b},\theta)\|_{\mathbb{V}}^2 dt \lesssim 1,$$

inductively we obtain

$$\sup_{t\in[0,T]} \|(\overline{u},\overline{b},\theta)\|_{\mathbb{V}^l}^2 + \int_0^T \|(\overline{u},\overline{b},\theta)\|_{\mathbb{V}^{l+1}}^2 dt \lesssim 1. \tag{4.4}$$

With such *a priori* estimates, a standard Galerkin approximation scheme proves the existence of the unique solution

$$\overline{u},\overline{b},\theta \in C([0,T];\mathbb{V}^l) \cap L^2(0,T;\mathbb{V}^{l+1})$$

and hence

$$u \in C([0,T];V_1^l), \quad b \in C([0,T];V_2^l)$$

as $z_j \in C([0,T];V_j^l), j = 1, 2$ by (4.3). □

4.1. **Irreducibility.** Let $B(v,\rho)$ be a ball of radius $\rho > 0$ centered at $v \in \mathbb{V}^l$. By denseness of $\mathbb{V}^{l+2}$ in $\mathbb{V}^l$, we find $y_\tau \in \mathbb{V}^{l+2}$ such that $\|v - y_\tau\|_{\mathbb{V}^l} < \rho/2$. Thus,

$$P(\tau, x, B(v,\rho)) \ge P(\tau, x, B(y_\tau, \frac{\rho}{2})) \tag{4.5}$$

where $P(t,x,\Gamma) = \mathbb{P}(\{y(t,x) \in \Gamma\})$.

Proposition 4.2. *For $l \in \mathbb{Z}, l \geq 2$, suppose that for both $j = 1, 2$, $\sigma_i^j < \frac{C}{i^{1/2}}$. Then, given any $\tau \in (0, T]$, $x = (x_1, x_2, x_3) = (u_0, b_0, \theta_0) \in \mathbb{V}^l, y_\tau \in \mathbb{V}^{l+2}$, there exists*

$$\hat{y} \in C([0, \tau]; \mathbb{V}^l) \cap C((0, \tau]; \mathbb{V}^{l+2}),$$
$$\hat{w}_G \in \mathrm{Lip}([0, \tau]; \mathbb{V}^l),$$

such that $\hat{y} = (\hat{u}, \hat{b}, \hat{\theta})$ is a solution to (4.2) with $\hat{w}_G = (G^1\hat{w}^1, G^2\hat{w}^2, 0)$, $\hat{y}(0) = x$, $\hat{y}(\tau) = y_\tau$.

Proof. Given $x \in \mathbb{V}^l$, we consider (4.2) with $G^1 w^1 = G^2 w^2 = 0$. Then we know there exists a unique $\hat{y} \in C([0, T]; \mathbb{V}^l) \cap L^2(0, T; \mathbb{V}^{l+1})$. Since $\hat{y} \in L^2(0, T; \mathbb{V}^{l+1})$, in particular, there exists $\tau_1 \in (0, \tau)$ arbitrary close to 0 such that $\hat{y}(\tau_1) \in \mathbb{V}^{l+1}$. Restarting from τ_1, we find

$$\hat{y} \in C([\tau_1, T]; \mathbb{V}^{l+1}) \cap L^2(\tau_1, T; \mathbb{V}^{l+2}).$$

Now we take $\tau_2 \in (\tau_1, \tau)$ arbitrary close to τ_1 so that $\hat{y}(\tau_2) \in \mathbb{V}^{l+2}$. We let

$$\hat{y}(t) = \hat{y}(\tau_2) + \frac{t - \tau_2}{\tau - \tau_2}[y_\tau - \hat{y}(\tau_2)] \tag{4.6}$$

for $t \in [\tau_2, \tau]$; we observe that $\hat{y}(t)|_{t=\tau} = y_\tau$. We set

$$\hat{\xi} \triangleq \partial_t \hat{y} + A\hat{y} + B(\hat{y}, \hat{y}) - R\hat{y};$$

it follows that

$$\hat{\xi} \in \mathbb{V}^{l+2} + C([\tau_2, \tau]; \mathbb{V}^l) + C([\tau_2, \tau]; \mathbb{V}^{l+1}) + C([\tau_2, \tau]; \mathbb{V}^{l+2}).$$

Now integrating $\hat{\xi}$ over $[\tau_2, t]$, we see that

$$\hat{w}_G(t) \triangleq \int_{\tau_2}^t \hat{\xi}(s)ds \in \mathrm{Lip}([\tau_2, \tau]; \mathbb{V}^l). \tag{4.7}$$

□

Next, with $\omega \in \Omega$ fixed we prove the following proposition:

Proposition 4.3. *For $l \in \mathbb{Z}, l \geq 2$, suppose $w_G, \hat{w}_G \in C^\gamma([0, T]; \mathbb{V}^{l+1-\frac{\epsilon}{2}}), \gamma > \frac{1}{2} - \frac{\epsilon}{4}, x \in \mathbb{V}^l$, for both $j = 1, 2$, $\sigma_i^j < \frac{c}{i^{1/2}}$ and $y, \hat{y}$ are the corresponding solutions to (4.2) with $w_G, \hat{w}_G$ respectively. Then*

$$\|y - \hat{y}\|_{C([0,T];\mathbb{V}^l)} \lesssim \|w_G - \hat{w}_G\|_{C^\gamma([0,T];V_1^{l+1-\frac{\epsilon}{2}} \times V_2^{l+1-\frac{\epsilon}{2}})}.$$

Proof. By [15, Theorem 5.2], we obtain for $j = 1, 2$,

$$z^j(t) = e^{-tA_j} G^j w^j(t) + \int_0^t A_j e^{-(t-s)A_j}(G^j w^j(t) - G^j w^j(s))ds.$$

Thus, with $z = (z^1, z^2), \hat{z} = (\hat{z}^1, \hat{z}^2)$ where for $j = 1, 2$,

$$d\hat{z}_j + A_j \hat{z}^j dt = G^j d\hat{w}^j, \quad \hat{z}^j(0) = 0,$$

so that similarly applying [15, Theorem 5.2], for $j = 1, 2$, we obtain

$$\hat{z}_j(t) = e^{-tA_j} G^j \hat{w}^j(t) + \int_0^t A_j e^{-(t-s)A_j}(G^j \hat{w}^j(t) - G^j \hat{w}^j(s))ds$$

and hence it follows that

$$\|z - \hat{z}\|_{C([0,T];V_1^l \times V_2^l)} \lesssim \|w_G - \hat{w}_G\|_{C^\gamma([0,T];V_1^{l+1-\frac{\epsilon}{2}} \times V_2^{l+1-\frac{\epsilon}{2}})}. \tag{4.8}$$

Now by hypothesis we have y and $\hat{y}$ as the corresponding solutions with $w_G, \hat{w}_G$ respectively so that defining

$$\overline{u} \triangleq u - z^1, \quad \overline{b} \triangleq b - z^2, \quad \overline{\hat{u}} \triangleq \hat{u} - \hat{z}^1, \quad \overline{\hat{b}} \triangleq \hat{b} - \hat{z}^2$$

and furthermore

$$\overline{U} \triangleq \overline{u} - \overline{\hat{u}}, \quad \overline{B} \triangleq \overline{b} - \overline{\hat{b}}, \quad \Theta \triangleq \theta - \hat{\theta}, \quad \overline{z}^1 \triangleq z^1 - \hat{z}^1, \quad \overline{z}^2 \triangleq z^2 - \hat{z}^2,$$

we obtain

$$\begin{aligned}
&\partial_t \overline{U} + A_1 \overline{U} + B_1(\overline{u} + z^1, \overline{U} + \overline{z}^1) + B_1(\overline{U} + \overline{z}_1, \overline{\hat{u}} + \hat{z}^1) \\
&- B_2(\overline{b} + z^2, \overline{B} + \overline{z}^2) - B_2(\overline{B} + \overline{z}^2, \overline{\hat{b}} + \hat{z}^2) = \Theta e_2, \\
&\partial_t \overline{B} + A_2 \overline{B} + B_3(\overline{u} + z^1, \overline{B} + \overline{z}^2) + B_3(\overline{U} + \overline{z}^1, \overline{\hat{b}} + \hat{z}^2) \\
&- B_4(\overline{b} + z^2, \overline{U} + \overline{z}^1) - B_4(\overline{B} + \overline{z}^2, \overline{\hat{u}} + \hat{z}^1), \\
&\partial_t \Theta + A_3 \Theta + B_5(\overline{u} + z^1, \Theta) + B_5(\overline{U} + \overline{z}^1, \hat{\theta}) - (\overline{U} + \overline{z}^1)_2,
\end{aligned}$$

where $(\overline{U} + \overline{z}^1)_2$ is the second component of $\overline{U} + \overline{z}^1$. Taking L^2-inner products with $(A_1^l \overline{U}, A_2^l \overline{B}, A_3^l \Theta)$ respectively gives

$$\begin{aligned}
&\frac{1}{2} \partial_t |(A_1^{1/2} \overline{U}, A_2^{1/2} \overline{B}, A_3^{1/2} \Theta)|^2 + |(A_1^{\frac{l+1}{2}} \overline{U}, A_2^{\frac{l+1}{2}} \overline{B}, A_3^{\frac{l+1}{2}} \Theta)|^2 \\
&= - \int A_1^{\frac{l-1}{2}} \operatorname{div}[(\overline{u} + z^1) \otimes (\overline{U} + \overline{z}_1) + (\overline{U} + \overline{z}_1) \otimes (\overline{\hat{u}} + \hat{z}^1) \\
&\quad - (\overline{b} + z^2) \otimes (\overline{B} + \overline{z}^2) - (\overline{B} + \overline{z}^2) \otimes (\overline{\hat{b}} + \hat{z}^2)] \cdot A_1^{\frac{l+1}{2}} \overline{U} + \int A_1^{1/2} \Theta e_2 \cdot A_1^{1/2} \overline{U} \\
&\quad - \int A_2^{\frac{l-1}{2}} \operatorname{div}[(\overline{u} + z^1) \otimes (\overline{B} + \overline{z}^2) + (\overline{U} + \overline{z}^1) \otimes (\overline{\hat{b}} + \hat{z}^2) \\
&\quad - (\overline{b} + z^2) \otimes (\overline{U} + \overline{z}^1) - (\overline{B} + \overline{z}^2) \otimes (\overline{\hat{u}} + \hat{z}^1)] \cdot A_2^{\frac{l+1}{2}} \overline{B} \\
&\quad - \int A_3^{\frac{l-1}{2}} \operatorname{div}[(\overline{u} + z^1) \Theta + (\overline{U} + \overline{z}^1) \hat{\theta}] A_3^{\frac{l+1}{2}} \Theta \\
&\quad - \int A_3^{1/2} \overline{U} \cdot A_3^{1/2} \Theta e_2 - \int A_3^{1/2} \overline{z}^1 \cdot A_3^{1/2} \Theta e_2 \\
&\leq \frac{1}{2} |(A_1^{\frac{l+1}{2}} \overline{U}, A_2^{\frac{l+1}{2}} \overline{B}, A_3^{\frac{l+1}{2}} \Theta)|^2 \\
&\quad + c(1 + \|(\overline{u}, z^1, \overline{\hat{u}}, \hat{z}^1, \overline{b}, z^2, \overline{\hat{b}}, \hat{z}^2, \hat{\theta})\|_{\mathbb{V}^l}^2) \|(\overline{U}, \overline{B}, \Theta)\|_{\mathbb{V}^l}^2 \\
&\quad + c(1 + \|(\overline{u}, z^1, \overline{\hat{u}}, \hat{z}^1, \overline{b}, z^2, \overline{\hat{b}}, \hat{z}^2, \hat{\theta})\|_{\mathbb{V}^l}^2) \|(\overline{z}_1, \overline{z}_2)\|_{V_1^l \times V_2^l}^2
\end{aligned}$$

by Young's inequalities and Banach algebra property. Therefore, after subtracting $\frac{1}{2} |(A_1^{\frac{l+1}{2}} \overline{U}, A_2^{\frac{l+1}{2}} \overline{B}, A_3^{\frac{l+1}{2}} \Theta)|^2$ from both sides, by Proposition 4.1, (4.3) and that $\overline{U}(0) = \overline{B}(0) = \Theta(0) = 0$, we obtain

$$\|(\overline{U}, \overline{B}, \Theta)\|_{C([0,T]; \mathbb{V}^l)} \lesssim \|(\overline{z}_1, \overline{z}_2)\|_{C([0,T]; V_1^l \times V_2^l)}. \tag{4.9}$$

Thus,

$$\begin{aligned}
\|y - \hat{y}\|_{C([0,T]; \mathbb{V}^l)} &\lesssim \|(\overline{U}, \overline{B}, \Theta)\|_{C([0,T]; \mathbb{V}^l)} + \|(\overline{z}_1, \overline{z}_2)\|_{C([0,T]; V_1^l \times V_2^l)} \\
&\lesssim \|(\overline{z}_1, \overline{z}_2)\|_{C([0,T]; V_1^l \times V_2^l)} \\
&\lesssim \|w_G - \hat{w}_G\|_{C^\gamma([0,T]; V_1^{l+1-\frac{\epsilon}{2}} \times V_2^{l+1-\frac{\epsilon}{2}})}
\end{aligned}$$

by (4.8) and (4.9). □

We now complete the proof that the Markovian semigroup P_t is irreducible in $\mathbb{V}^l$:

Proposition 4.4. *For $l \in \mathbb{Z}, l \geq 2$, suppose that for both $j = 1, 2$, $\{\sigma_i^j\}_{i=1}^{\infty}$ satisfies (2.11), $\sigma_i^j < \frac{C}{i^{1/2}}$, $x, v \in \mathbb{V}^l$, $\tau \in (0, T]$, $\rho > 0$. Then*
$$P(\tau, x, B(v, \rho)) > 0.$$

Proof. We denote by $y(\cdot, w_G; x)$ the solution to (4.2) emanating from x as initial data with $w_G = Gw = (G^1 w^1, G^2 w^2, 0)^T$ and obtain
$$\|y(\cdot, w_G; x) - \hat{y}(\cdot, \hat{w}_G; x)\|_{C([0,\tau];\mathbb{V}^l)} \leq c_0 \|w_G - \hat{w}_G\|_{W^{s,p}(0,\tau; V_1^{l-1+\frac{\epsilon}{2}} \times V_2^{l-1+\frac{\epsilon}{2}})}$$
for some $c_0 \geq 0$ by Proposition 4.3 if $\gamma > \frac{1}{2} - \frac{\epsilon}{4}$ and the one-dimensional Sobolev embedding of $W^{s,p}(0, \tau) \hookrightarrow C^{\gamma}([0, \tau])$ for all $\gamma \leq s - \frac{1}{p}$. Thus,
$$\begin{aligned} &\mathbb{P}(\{\|w_G - \hat{w}_G\|_{W^{s,p}(0,\tau; V_1^{l-1+\frac{\epsilon}{2}} \times V_2^{l-1+\frac{\epsilon}{2}})} < \frac{\rho}{2c_0}\}) \\ &\leq \mathbb{P}(\{\|y(\cdot, w_G; x) - \hat{y}(\cdot, \hat{w}_G; x)\|_{C([0,\tau];\mathbb{V}^l)} < \frac{\rho}{2}\}). \end{aligned} \tag{4.10}$$
It is shown in [12] that
$$\mathbb{P}(\{\|w_G - \hat{w}_G\|_{W^{s,p}(0,\tau; V_1^{l-1+\frac{\epsilon}{2}} \times V_2^{l-1+\frac{\epsilon}{2}})} < \frac{\rho}{2c_0}\} > 0 \tag{4.11}$$
because the law of Gw is a full measure on $W^{s,p}(0, T; V_1^{l-1+\frac{\epsilon}{2}} \times V_2^{l-1+\frac{\epsilon}{2}})$. Thus, applying (4.11) to (4.10) allows us to conclude that
$$0 < \mathbb{P}(\{\|y(\tau, w_G; x) - \hat{y}(\tau, \hat{w}_G; x)\|_{\mathbb{V}^l} < \frac{\rho}{2}\}).$$
By Proposition 4.2, we see that $0 < P(\tau, x, B(y_\tau, \frac{\rho}{2}))$. By (4.5), the proof is complete. □

4.2. **Strong Feller property.** We let $H_n^j = \text{span}\{g_1^j, \dots, g_n^j\}$ where we recall that $\{g_i^j\}_{i=1}^{\infty}$ are the complete orthonormal systems of eigenfunctions of $A_j, j = 1, 2, 3$ and $\pi_n^j : H_j \to H_n^j, \pi_n : \mathbb{H} \to \prod_{j=1}^3 H_n^j \triangleq H_n$ be the projection operator defined by $\pi_n^j x = \sum_{i=1}^n \langle x, g_i^j \rangle g_i^j$. For a fixed $R > 0$, we define a cutoff function $\Psi_R \in C^\infty(\mathbb{R})$ that satisfies
$$\Psi_R(t) = \begin{cases} 1 & t \in [-R, R], \\ 0 & t \in \mathbb{R} \setminus (-R-1, R+1). \end{cases}$$
We consider $y_n^{(R)} \triangleq (u_n^{(R)}, b_n^{(R)}, \theta_n^{(R)})$, for fixed $n \in \mathbb{N}$, that solves
$$\begin{aligned} &du_n^{(R)} + [A_1 u_n^{(R)} + \Psi_R(\|y_n^R\|_{\mathbb{V}^l}^2) \pi_n^1 B_1(u_n^{(R)}, u_n^{(R)}) \\ &- \Psi_R(\|y_n^{(R)}\|_{\mathbb{V}^l}^2) \pi_n^1 B_2(b_n^{(R)}, b_n^{(R)})] dt \\ &= \pi_n^1 \theta_n^{(R)} e_2 dt + \pi_n^1 G^1 dw^1, \\ &db_n^{(R)} + [A_2 b_n^{(R)} + \Psi_R(\|y_n^{(R)}\|_{\mathbb{V}^l}^2) \pi_n^2 B_3(u_n^{(R)}, b_n^{(R)}) \\ &- \Psi_R(\|y_n^{(R)}\|_{\mathbb{V}^l}^2) \pi_n^2 B_4(b_n^{(R)}, u_n^{(R)})] dt \\ &= \pi_n^2 G^2 dw^2, \end{aligned}$$
$$d\theta_n^{(R)} + [A_3 \theta_n^{(R)} + \Psi_R(\|y_n^{(R)}\|_{\mathbb{V}^l}^2) \pi_n^3 B_5(u_n^{(R)}, \theta_n^{(R)})] dt = \pi_n^3 u_{n,2}^{(R)} dt,$$

$$(u_n^{(R)}, b_n^{(R)}, \theta_n^{(R)})(0) = (\pi_n^1 x_1, \pi_n^2 x_2, \pi_n^3 x_3) = \pi_n x, \tag{4.12}$$

where $u_{n,2}^{(R)}$ is the second component of $u_n^{(R)}$, and its associated Markovian semigroup $P_{t,n}^{(R)}$. We prove the following proposition:

Proposition 4.5. *Under the hypothesis of Theorem 2.3, for all $t > 0$ and all $R > 0$, there exists a constant $L = L(R, t) \geq 0$ such that*

$$|P_{t,n}^{(R)}\psi(x) - P_{t,n}^{(R)}\psi(v)| \leq L\|x - v\|_{\mathbb{V}^l}$$

for all $n \in \mathbb{N}$, $x, v \in H_n$, $\psi \in C_b(H_n)$ such that $\|\psi\|_{C_b} \triangleq \sup_{x \in H_n} |\psi(x)| \leq 1$.

Proof. Firstly, $P_{t,n}^{(R)}$ is Feller and hence $P_{t,n}^{(R)}(C_b(\mathbb{V}^l)) \subset C_b(\mathbb{V}^l)$; moreover, by mean value theorem,

$$|P_{t,n}^{(R)}\psi(x) - P_{t,n}^{(R)}\psi(v)| \leq \sup_{k,h \in H_n, \|h\|_{\mathbb{V}^l} \leq 1} |DP_{t,n}^{(R)}\psi(k) \cdot h|\|x - v\|_{\mathbb{V}^l}$$

where $DP_{t,n}^{(R)}\psi(k) \cdot h$ denotes the derivative of the mapping $v \mapsto P_{t,n}^{(R)}\psi(v)$ at the point k in the direction h. We rewrite (4.12) similarly to (2.6) as

$$dy_n^{(R)} + [Ay_n^{(R)} + \Psi_R(\|y_n^{(R)}\|_{\mathbb{V}^l}^2)\pi_n B(y_n^{(R)}, y_n^{(R)})]dt = \pi_n R y_n^{(R)} dt + \pi_n G dw.$$

Thus, by Elworthy's formula (see e.g. [7, pg. 267]), for all $h \in H_n$ with an n-dimensional standard Wiener process β_n,

$$[DP_{t,n}^{(R)}\psi(x)] \cdot h = \frac{1}{t} E[\psi(y_n^{(R)}(t; x)) \int_0^t \langle (\pi_n G G^* \pi_n)^{-\frac{1}{2}} [Dy_n^{(R)}(s; x)] \cdot h, d\beta_n(s)\rangle].$$

By Burkholder-Davis-Gundy inequality (e.g. [23]),

$$|[DP_{t,n}^{(R)}\psi(x)] \cdot h| \leq \frac{c}{t}\|\psi\|_{C_b} E\Big[\Big(\int_0^t |(\pi_n G G^* \pi_n)^{-\frac{1}{2}} [Dy_n^{(R)}(s; x)] \cdot h|^2 ds\Big)^{1/2}\Big] \tag{4.13}$$

where we used that $\psi \in C_b(H_n)$. Due to (2.11) and (2.13) we have for a constant c independent of n,

$$|(\pi_n G G^* \pi_n)^{-\frac{1}{2}} v| \leq c\|v\|_{\mathbb{V}^{l+1}} \quad \forall\, y \in H_n. \tag{4.14}$$

We let

$$\overline{u}_n^{(R)} \triangleq u_n^{(R)} - \pi_n^1 z^1, \quad \overline{b}_n^{(R)} \triangleq b_n^{(R)} - \pi_n^2 z^2, \quad \overline{y}_n^{(R)} \triangleq (\overline{u}_n^{(R)}, \overline{b}_n^{(R)}, \theta_n^{(R)}),$$

where z^j, $j = 1, 2$ are solutions to (3.1) with $\alpha = 0$, $t \geq 0$, $z^j(0) = 0$ and also denote

$$\begin{aligned} Y_n^{(R)}(t) &\triangleq [D\overline{y}_n^{(R)}(t; x)] \cdot h \\ &= \begin{pmatrix} D_{x_1}\overline{u}_n^{(R)} \cdot h & D_{x_1}\overline{b}_n^{(R)} \cdot h & D_{x_1}\theta_n^{(R)} \cdot h \\ D_{x_2}\overline{u}_n^{(R)} \cdot h & D_{x_2}\overline{b}_n^{(R)} \cdot h & D_{x_2}\theta_n^{(R)} \cdot h \\ D_{x_3}\overline{u}_n^{(R)} \cdot h & D_{x_3}\overline{b}_n^{(R)} \cdot h & D_{x_3}\theta_n^{(R)} \cdot h \end{pmatrix} \triangleq \begin{pmatrix} \eta_1 & \eta_2 & \eta_3 \\ \eta_4 & \eta_5 & \eta_6 \\ \eta_7 & \eta_8 & \eta_9 \end{pmatrix} \end{aligned}$$

where $D \triangleq (D_{x_1}, D_{x_2}, D_{x_3})$ which is a derivative of the mapping $x \mapsto y_n^{(R)}(t; x)$. Thus, $\partial_t Y_n^{(R)} = \partial_t [D\overline{y}_n^{(R)}(t; x)] \cdot h$ and it can be checked that

$$\begin{aligned} &\partial_t \eta_1 + A_1 \eta_1 - \pi_n^1 \eta_3 e_2 \\ &= 2\Psi'_R(\|y_n^{(R)}\|_{\mathbb{V}^l}^2)\langle A^{1/2} y_n^{(R)}, A^{1/2} D_{x_1} y_n^{(R)} \cdot h\rangle \pi_n^1 [-B_1(u_n^{(R)}, u_n^{(R)}) + B_2(b_n^{(R)}, b_n^{(R)})] \\ &\quad + \Psi_R(\|y_n^{(R)}\|_{\mathbb{V}^l}^2)\pi_n^1 [-B_1(\eta_1, u_n^{(R)}) - B_1(u_n^{(R)}, \eta_1) + B_2(\eta_2, b_n^{(R)}) + B_2(b_n^{(R)}, \eta_2)], \end{aligned}$$

$$
\begin{aligned}
&\partial_t\eta_2 + A_2\eta_2\\
&= 2\Psi_R'(\|y_n^{(R)}\|_{\mathbb{V}^l}^2)\langle A^{1/2}y_n^{(R)}, A^{1/2}D_{x_1}y_n^{(R)}\cdot h\rangle\pi_n^2[-B_3(u_n^{(R)}, b_n^{(R)}) + B_4(b_n^{(R)}, u_n^{(R)})]\\
&\quad + \Psi_R(\|y_n^{(R)}\|_{\mathbb{V}^l}^2)\pi_n^2[-B_3(\eta_1, b_n^{(R)}) - B_3(u_n^{(R)}, \eta_2) + B_4(\eta_2, u_n^{(R)}) + B_4(b_n^{(R)}, \eta_1)],\\
&\quad \partial_t\eta_3 + A_3\eta_3 = -2\Psi_R'(\|y_n^{(R)}\|_{\mathbb{V}^l}^2)\langle A^{1/2}y_n^{(R)}, A^{1/2}D_{x_1}y_n^{(R)}\cdot h\rangle\pi_n^3B_5(u_n^{(R)}, \theta_n^{(R)})\\
&\qquad - \Psi_R(\|y_n^{(R)}\|_{\mathbb{V}^l}^2)\pi_n^3[B_5(\eta_1, \theta_n^{(R)}) + B_5(u_n^{(R)}, \eta_3)] + \pi_n^3\eta_{1,2},
\end{aligned}
$$

that

$$
\begin{aligned}
&\partial_t\eta_4 + A_1\eta_4 - \pi_n^1\eta_6 e_2\\
&= 2\Psi_R'(\|y_n^{(R)}\|_{\mathbb{V}^l}^2)\langle A^{1/2}y_n^{(R)}, A^{1/2}D_{x_2}y_n^{(R)}\cdot h\rangle\pi_n^1[-B_1(u_n^{(R)}, u_n^{(R)}) + B_2(b_n^{(R)}, b_n^{(R)})]\\
&\quad + \Psi_R(\|y_n^{(R)}\|_{\mathbb{V}^l}^2)\pi_n^1[-B_1(\eta_4, u_n^{(R)}) - B_1(u_n^{(R)}, \eta_4) + B_2(\eta_5, b_n^{(R)}) + B_2(b_n^{(R)}, \eta_5)],\\
&\partial_t\eta_5 + A_2\eta_5\\
&= 2\Psi_R'(\|y_n^{(R)}\|_{\mathbb{V}^l}^2)\langle A^{1/2}y_n^{(R)}, A^{1/2}D_{x_2}y_n^{(R)}\cdot h\rangle\pi_n^2[-B_3(u_n^{(R)}, b_n^{(R)}) + B_4(b_n^{(R)}, u_n^{(R)})]\\
&\quad + \Psi_R(\|y_n^{(R)}\|_{\mathbb{V}^l}^2)\pi_n^2[-B_3(\eta_4, b_n^{(R)}) - B_3(u_n^{(R)}, \eta_5) + B_4(\eta_5, u_n^{(R)}) + B_4(b_n^{(R)}, \eta_4)],\\
&\quad \partial_t\eta_6 + A_3\eta_6 = -2\Psi_R'(\|y_n^{(R)}\|_{\mathbb{V}^l}^2)\langle A^{1/2}y_n^{(R)}, A^{1/2}D_{x_2}y_n^{(R)}\cdot h\rangle\pi_n^3B_5(u_n^{(R)}, \theta_n^{(R)})\\
&\qquad - \Psi_R(\|y_n^{(R)}\|_{\mathbb{V}^l}^2)\pi_n^3[B_5(\eta_4, \theta_n^{(R)}) + B_5(u_n^{(R)}, \eta_6)] + \pi_n^3\eta_{4,2},
\end{aligned}
$$

and that

$$
\begin{aligned}
&\partial_t\eta_7 + A_1\eta_7 - \pi_n^1\eta_9 e_2\\
&= 2\Psi_R'(\|y_n^{(R)}\|_{\mathbb{V}^l}^2)\langle A^{1/2}y_n^{(R)}, A^{1/2}D_{x_3}y_n^{(R)}\cdot h\rangle\pi_n^1[-B_1(u_n^{(R)}, u_n^{(R)}) + B_2(b_n^{(R)}, b_n^{(R)})]\\
&\quad + \Psi_R(\|y_n^{(R)}\|_{\mathbb{V}^l}^2)\pi_n^1[-B_1(\eta_7, u_n^{(R)}) - B_1(u_n^{(R)}, \eta_7) + B_2(\eta_8, b_n^{(R)}) + B_2(b_n^{(R)}, \eta_8)],\\
&\partial_t\eta_8 + A_2\eta_8\\
&= 2\Psi_R'(\|y_n^{(R)}\|_{\mathbb{V}^l}^2)\langle A^{1/2}y_n^{(R)}, A^{1/2}D_{x_3}y_n^{(R)}\cdot h\rangle\pi_n^2[-B_3(u_n^{(R)}, b_n^{(R)}) + B_4(b_n^{(R)}, u_n^{(R)})]\\
&\quad + \Psi_R(\|y_n^{(R)}\|_{\mathbb{V}^l}^2)\pi_n^2[-B_3(\eta_7, b_n^{(R)}) - B_3(u_n^{(R)}, \eta_8) + B_4(\eta_8, u_n^{(R)}) + B_4(b_n^{(R)}, \eta_7)],\\
&\quad \partial_t\eta_9 + A_3\eta_9 = -2\Psi_R'(\|y_n^{(R)}\|_{\mathbb{V}^l}^2)\langle A^{1/2}y_n^{(R)}, A^{1/2}D_{x_3}y_n^{(R)}\cdot h\rangle\pi_n^3B_5(u_n^{(R)}, \theta_n^{(R)})\\
&\qquad - \Psi_R(\|y_n^{(R)}\|_{\mathbb{V}^l}^2)\pi_n^3[B_5(\eta_7, \theta_n^{(R)}) + B_5(u_n^{(R)}, \eta_9)] + \pi_n^3\eta_{7,2},
\end{aligned}
$$

where $\eta_{1,2}, \eta_{4,2}, \eta_{7,2}$ are the second components of η_1, η_4, η_7 respectively.

We take L^2-inner products with

$$
(A_1^l\eta_1, A_2^l\eta_2, A_3^l\eta_3, A_1^l\eta_4, A_2^l\eta_5, A_3^l\eta_6, A_1^l\eta_7, A_2^l\eta_8, A_3^l\eta_9)
$$

respectively to estimate for example,

$$
\begin{aligned}
&\frac{1}{2}\partial_t|A_1^{1/2}\eta_1|^2 + |A_1^{\frac{l+1}{2}}\eta_1|^2\\
&\lesssim \Psi_R'(\|y_n^{(R)}\|_{\mathbb{V}^l}^2)|A^{1/2}y_n^{(R)}||A^{1/2}Y_n^{(R)}|(|A_1^{1/2}u_n^{(R)}|^2 + |A_1^{1/2}b_n^{(R)}|^2)|A_1^{\frac{l+1}{2}}\eta_1|\\
&\quad + \Psi_R(\|y_n^{(R)}\|_{\mathbb{V}^l}^2)(|A_1^{1/2}\eta_1||A_1^{1/2}u_n^{(R)}| + |A_1^{1/2}\eta_2||A_1^{1/2}b_n^{(R)}|)|A_1^{\frac{l+1}{2}}\eta_1|\\
&\quad + |A_1^{1/2}\eta_3||A_1^{1/2}\eta_1|\\
&\lesssim c(R)(|A^{1/2}Y_n^{(R)}| + |A_1^{1/2}\eta_1| + |A_1^{1/2}\eta_2|)|A_1^{\frac{l+1}{2}}\eta_1| + |A_1^{1/2}\eta_3||A_1^{1/2}\eta_1|\\
&\le \frac{1}{2}|A_1^{\frac{l+1}{2}}\eta_1|^2 + c(R)|A^{1/2}Y_n^{(R)}|^2
\end{aligned}
$$

where we used Hölder's and Young's inequalities. Subtracting $\frac{1}{2}|A_1^{\frac{l+1}{2}}\eta_1|^2$ from both sides gives

$$\partial_t|A_1^{1/2}\eta_1|^2 + |A_1^{\frac{l+1}{2}}\eta_1|^2 \le c(R)|A^{1/2}Y_n^{(R)}|^2.$$

Similar computations on the other eight equations show that in sum

$$\partial_t|A^{1/2}Y_n^{(R)}|^2 + |A^{\frac{l+1}{2}}Y_n^{(R)}|^2 \le c(R)|A^{1/2}Y_n^{(R)}|^2. \tag{4.15}$$

Thus,

$$\|Y_n^{(R)}(t)\|_{\mathbb{V}^l}^2 \le \|h\|_{\mathbb{V}^l}^2 e^{c(R)t}$$

and hence integrating (4.15) gives

$$\int_0^t \|Y_n^{(R)}(s)\|_{\mathbb{V}^{l+1}}^2 ds \le c(R,t)e^{c(R)t}\|h\|_{\mathbb{V}^l}^2.$$

This implies

$$E[\int_0^t \|Y_n^{(R)}(s)\|_{\mathbb{V}^{l+1}}^2 ds] \le c(R,t)\|h\|_{\mathbb{V}^l}^2$$

so that

$$E[\int_0^t \|[Dy_n^{(R)}(s;x)]\cdot h\|_{\mathbb{V}^{l+1}}^2 ds] = E[\int_0^t \|Y_n^{(R)}(s)\|_{\mathbb{V}^{l+1}}^2 ds] \le c(R,t)\|h\|_{\mathbb{V}^l}^2. \tag{4.16}$$

Therefore,

$$\begin{aligned}
&\sup_{k,h\in H_n,\|h\|_{\mathbb{V}^l}\le 1} |[DP_{t,n}^{(R)}\psi(k)]\cdot h| \\
&\le \sup_{k,h\in H_n,\|h\|_{\mathbb{V}^l}\le 1} \frac{c}{t}\|\psi\|_{C_b} E[(\int_0^t |(\pi_n GG^*\pi_n)^{-\frac{1}{2}}[Dy_n^{(R)}(s;k)]\cdot h|^2 ds)^{1/2}] \\
&\le \sup_{k,h\in H_n,\|h\|_{\mathbb{V}^l}\le 1} \frac{c}{t}\|\psi\|_{C_b} E[(\int_0^t \|[Dy_n^{(R)}(s;k)]\cdot h\|_{\mathbb{V}^{l+1}}^2 ds)^{1/2}] \\
&\le \sup_{k,h\in H_n,\|h\|_{\mathbb{V}^l}\le 1} (\frac{\|\psi\|_{C_b}}{t})c(R,t)\|h\|_{\mathbb{V}^l} \le \frac{C(R,t)}{t}
\end{aligned}$$

by (4.13), (4.14), Hölder's inequality and (4.16). $\square$

We now consider

$$\begin{gathered}
du^{(R)} + [A_1u^{(R)} + \Psi_R(\|y^R\|_{\mathbb{V}^l}^2)B_1(u^{(R)},u^{(R)}) - \Psi_R(\|y^{(R)}\|_{\mathbb{V}^l}^2)B_2(b^{(R)},b^{(R)})]dt \\
= \theta^{(R)}e_2dt + G^1dw^1, \\
db^{(R)} + [A_2b^{(R)} + \Psi_R(\|y^{(R)}\|_{\mathbb{V}^l}^2)B_3(u^{(R)},b^{(R)}) - \Psi_R(\|y^{(R)}\|_{\mathbb{V}^l}^2)B_4(b^{(R)},u^{(R)})]dt \\
= G^2dw^2, \\
d\theta^{(R)} + [A_3\theta^{(R)} + \Psi_R(\|y^{(R)}\|_{\mathbb{V}^l}^2)B_5(u^{(R)},\theta^{(R)})]dt = u_2^{(R)}dt, \\
(u^{(R)},b^{(R)},\theta^{(R)})(0) = (x_1,x_2,x_3) = x,
\end{gathered}$$

with $P_t^{(R)}$ as its associated Markovian semigroup, pass the limit and obtain the following proposition:

Proposition 4.6. *Under the hypothesis of Theorem 2.3, for all $t > 0$ and all $R > 0$, there exists a constant $L = L(R, t) > 0$ such that*

$$|P_t^{(R)}\psi(x) - P_t^{(R)}\psi(v)| \le L\|x - v\|_{\mathbb{V}^l}$$

for all $x, v \in \mathbb{V}^l$, $\psi \in C_b(\mathbb{V}^l)$ such that $\|\psi\|_{C_b} \le 1$. Moreover, $P_t^{(R)}\psi$ is Lipschitz continuous for all $\psi \in B_b(\mathbb{V}^l)$.

Proof. The *a priori* estimates in Proposition 4.1 are available for the Galerkin approximation system (4.12); hence, by the well-known compact embedding results (see [4, 28, 38]), and passing to a subsequence if necessary and relabeling it, we obtain

$$y_n^{(R)} \to y^{(R)} \quad \text{a.e. } t,\ \mathbb{P}\text{-a.s. } (n \to \infty).$$

We let $\psi \in C_b(\mathbb{V}^l)$ so that $\psi \in C_b(H_n)$ and because $y_n^{(R)} \in H_n$, we obtain

$$\psi(y_n^{(R)}(\cdot; x)) \to \psi(y^{(R)}(\cdot; x)) \quad \text{in } L^1(0, T) \quad (n \to \infty)$$

by dominated convergence theorem. Taking expectation, using again that ψ is continuous and bounded, due to dominated convergence theorem, we obtain

$$E\Big[\int_0^T |\psi(y_n^{(R)}(\cdot; x)) - \psi(y^{(R)}(\cdot; x))|dt\Big] \to 0 \quad (n \to \infty)$$

and hence by Fubini's theorem

$$\int_0^T E[|\psi(y_n^{(R)}(\cdot, x)) - \psi(y^{(R)}(\cdot; x))|]dt \to 0 \quad (n \to \infty).$$

Taking a subsequence again, and relabeling it we obtain for a.e. $t \in [0, T]$,

$$E[|\psi(y_n^{(R)}(\cdot; x)) - \psi(y^{(R)}(\cdot; x))|] \to 0 \quad (n \to \infty). \tag{4.17}$$

By (2.10), we now see that for all $x, v \in \mathbb{V}^l$, $\psi \in C_b(\mathbb{V}^l)$ such that $\|\psi\|_{C_b} \le 1$, there exists $L = L(R, t) \ge 0$ such that

$$\begin{aligned}
&|P_t^{(R)}\psi(x) - P_t^{(R)}\psi(v)| \\
&\le |P_t^{(R)}\psi(x) - P_{t,n}^{(R)}\psi(x)| + |P_{t,n}^{(R)}\psi(x) - P_{t,n}^{(R)}\psi(v)| + |P_{t,n}^{(R)}\psi(v) - P_t^{(R)}\psi(v)| \\
&\le E[|\psi(y^{(R)}(\cdot; x)) - \psi(y_n^{(R)}(\cdot; x))|] + L\|x - v\|_{\mathbb{V}^l} \\
&\quad + E[|\psi(y^{(R)}(\cdot; v)) - \psi(y_n^{(R)}(\cdot; v))|] \\
&\to L\|x - v\|_{\mathbb{V}^l}
\end{aligned}$$

by Proposition 4.5 and (4.17). Since the trajectories of $y^{(R)}(t; x)$ in $\mathbb{V}^l$ are continuous, this holds for all $t \in [0, T]$. Considering the total variation of $\mathbb{V}^l$, this holds for all $\psi \in B_b(\mathbb{V}^l)$. □

The following proposition is now an immediate consequence of the above result.

Proposition 4.7. *Under the hypothesis of Theorem 2.3, for all $t > 0$,*

$$\lim_{R\to\infty} \|P(t, x, \cdot) - P^{(R)}(t, x, \cdot)\|_{\text{Var}} = 0$$

uniformly with respect to x in bounded sets of $\mathbb{V}^l$.

Proof of Theorem 2.3. We now complete the proof of the strong Feller property. By Proposition 4.7 with x, v in some ball of $\in \mathbb{V}^l$, for all $\epsilon > 0$ fixed, $t > 0$, there exists $R_\epsilon > 0$ such that $R \geq R_\epsilon$ implies

$$\|P(t,x,\cdot) - P^{(R)}(t,x,\cdot)\|_{\rm Var} + \|P(t,v,\cdot) - P^{(R)}(t,v,\cdot)\|_{\rm Var} < 2\epsilon.$$

Thus, for x, v such that $\|x - v\|_{\mathbb{V}^l} < \frac{\epsilon}{L}$ where L is that of Proposition 4.6,

$$\begin{aligned}
&\|P(t,x,\cdot) - P(t,v,\cdot)\|_{\rm Var}\\
&\leq \|P(t,x,\cdot) - P^{(R)}(t,x,\cdot)\|_{\rm Var} + \|P^{(R)}(t,x,\cdot) - P^{(R)}(t,v,\cdot)\|_{\rm Var}\\
&\quad + \|P^{(R)}(t,v,\cdot) - P(t,v,\cdot)\|_{\rm Var}\\
&\leq 2\epsilon + L\|x - v\|_{\mathbb{V}^l} \leq 3\epsilon
\end{aligned}$$

by Proposition 4.6. Thus, $P_t : B_b(\mathbb{V}^l) \to C_b(\mathbb{V}^l)$ for all $t \in (0,T]$; i.e. $\{P_t\}$ is strong Feller. □

5. Appendix

Lemma 5.1 ([10]). *Let $\{g_i^j, \lambda_i^j\}_{i=1}^\infty$ be the eigenvectors and their corresponding eigenvalues of A_j, $j = 1, 2$. Then*

(1) *for $f = \sum_{i=1}^\infty f_i(t) g_i^j(x)$, $j = 1, 2$, $\mathbb{P}$-a.s.,*

$$f \in D(A_j^k) \text{ if and only if } \sum_{i=1}^\infty (f_i(t))^2 (\lambda_i^j)^{2k} < \infty;$$

(2) *in particular, for $G^j w^j(t) = \sum_{i=1}^\infty \sigma_i^j \beta_i^j(t) g_i^j$, $j = 1, 2$, from (2.7), we have $\mathbb{P}$-a.s.,*

$$G^j w^j(t) \in D(A_j^k) \text{ if and only if } \sum_{i=1}^\infty (\sigma_i^j)^2 (\lambda_i^j)^{2k} < \infty.$$

Lemma 5.2 ([10]). *For $z_\alpha^j(t) = \int_{-\infty}^t e^{(-A_j - \alpha)(t-s)} dG^j w^j(s)$ in (3.2), where $G^j w^j(s) = \sum_{i=1}^\infty \sigma_i^j \beta_i^j(s) g_i^j$, $j = 1, 2$ from (2.7): we have $\mathbb{P}$-a.s.,*

$$z_\alpha^j \in D(A_j^k) \text{ if and only if } \sum_{i=1}^\infty \frac{(\sigma_i^j)^2 (\lambda_i^j)^{2k}}{2(\lambda_i^j + \alpha)} < \infty;$$

$$\lim_{\alpha \to \infty} E[|A_j^{1/4} z_\alpha^j|^4] = 0.$$

Lemma 5.3 ([21, Lemma 2.2]). *In an n-dimensional bounded domain D, let P be a continuous projector from $(L^r(D))^n$ to the closure of $\{u \in (C_0^\infty(D))^n : \nabla \cdot u = 0\}$ in $(L^r(D))^n$ and $A = -P\Delta$. Then for $0 \leq \delta < \frac{1}{2} + \frac{n(1 - \frac{1}{r})}{2}$*

$$\|A^{-\delta} P(u \cdot \nabla) v\|_{L^r} \lesssim_{\delta,\theta,\rho,r} \|A^\theta u\|_{L^r} \|A^\rho v\|_{L^r}$$

if $\delta + \theta + \rho \geq \frac{n}{2r} + \frac{1}{2}$, $\theta > 0$, $\rho > 0$, $\rho + \delta > 1/2$.

For the next result, see [10, Proposition 6.1] and [38, Chapter III, Section 3.5.2].

Proposition 5.4. *Suppose $-1 \leq \theta(x, t_0) \leq 1$ a.e. $x \in D$. Then $-1 \leq \theta(x,t) \leq 1$ a.e. $x \in D$, for all $t \geq t_0$, $\mathbb{P}$-almost surely.*

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