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ASYMPTOTICALLY PERIODIC SOLUTIONS OF VOLTERRA INTEGRAL EQUATIONS

MUHAMMAD N. ISLAM

Abstract. We study the existence of asymptotically periodic solutions of a nonlinear Volterra integral equation. In the process, we obtain the existence of periodic solutions of an associated nonlinear integral equation with infinite delay. Schauder's fixed point theorem is used in the analysis.

1. Introduction

Although many research have been done on periodic solutions of differential and integral equations, not much has been done on asymptotically periodic solutions of such equations. References [1, 5, 7, 9, 10] are among the few that we have found on asymptotically periodic solutions. Article [5], which motivated us to write the present article, is about asymptotically periodic and periodic solutions of Volterra integral equations. The results of our work in the present paper differ substantially from the work of [5] in terms of assumptions and methods of proof.

In [10, p. 631], a result on an asymptotically periodic solutions of a Volterra integral equation under certain growth, monotonicity, and sign conditions on the kernel and on its derivative is given. Articles [1, 7, 9] are on difference equations where the asymptotically periodic solutions are studied. On periodic solutions, we refer to the following partial list [4, 6, 8, 10, 11, 12, 13], and the references therein.

Let $\mathbb{R} = (-\infty, \infty)$ and $\mathbb{R}_+ = [0, \infty)$. We consider the nonlinear Volterra equation

$$x(t) = a(t) + \int_0^t C(t,s) f(s, x(s)) ds, \tag{1.1}$$

and the associated integral equation with infinite delay

$$x(t) = b(t) + \int_{-\infty}^t D(t,s) g(s, x(s)) ds. \tag{1.2}$$

Throughout this article, we assume that $a : \mathbb{R} \to \mathbb{R}$ and $b : \mathbb{R} \to \mathbb{R}$ are bounded continuous functions, $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous and bounded for bounded x, $C(t,s)$ is continuous on $0 \le s \le t < \infty$, and $D(t,s)$ is continuous on $-\infty < s \le t < \infty$. In addition to these continuity assumptions, we assume that there exists a positive constant T and a function $q : \mathbb{R} \to \mathbb{R}$ such that

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$a(t) = b(t) + q(t), b(t+T) = b(t), q(t) \to 0$ as $t \to \infty$, $g(t+T, x) = g(t, x)$, and $D(t+T, s+T) = D(t, s)$. We call these as our basic assumptions. We prove, under suitable conditions, that (1.1) has a continuous asymptotically T-periodic solution, and that (1.2) has a continuous T-periodic solution.

Definition 1.1. A function x is asymptotically T-periodic if there exists a T-periodic function y and a function z such that $x(t) = y(t) + z(t)$ with $z(t) \to 0$ as $t \to \infty$.

The function y in the above definition will be referred as the T-periodic part of x. In this article we show that (1.1) has an asymptotically T-periodic solution, and that the T-periodic part of that solution is indeed a T-periodic solution of (1.2).

We employ Schauder's fixed point theorem for the existence of asymptotically T-periodic solution. Like many fixed point theorems, Schauder's theorem requires a compact mapping. For problems on finite domains, this compactness is normally obtained by Arzela-Ascoli's theorem. Since the domain of an asymptotically periodic function is unbounded, Arzela-Ascoli's theorem does not apply in our work. We obtain the required compactness following a method found in [2, 3]. The researchers who study existence results for problems on unbounded domains employing fixed point theory, will find this method very useful for the required compactness property.

We assume

(H1) there exists real valued continuous functions $Q(t, s)$, $0 \le s \le t < \infty$ and $h : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$, with $C(t, s)f(t, x) = D(t, s)g(t, x) + Q(t, s)h(t, x)$, and
$$\lim_{t\to\infty} \int_0^t |Q(t, s)|ds = 0;$$
(H2) the function $t \mapsto \int_{-\infty}^t |D(t, s)|ds$ is continuous, and
$$\int_{-\infty}^t |D(t, s)|ds \le d^* < \infty,$$
for all $t \in \mathbb{R}$.

For any positive constant ρ, let
$$B_\rho := \{x \in \mathbb{R} : |x| \le \rho\}.$$
Assume

(H3) (i) $(mf)_\rho = \sup_{x\in B_\rho,\, t\in\mathbb{R}_+} |f(t, x)| < \infty$,
(ii) $(mg)_\rho = \sup_{x\in B_\rho,\, t\in\mathbb{R}_+} |g(t, x)| < \infty$,
(iii) $(mh)_\rho = \sup_{x\in B_\rho,\, t\in\mathbb{R}_+} |h(t, x)| < \infty$.

Remark 1.2. When Q satisfies condition (H1) then it is easy to see that Q satisfies the integrability condition
$$\sup_{t\ge 0} \int_0^t |Q(t, s)|ds \le q^* < \infty. \tag{1.3}$$

Remark 1.3. When D satisfies the integrability condition in (H2), then D satisfies
$$\lim_{\tau\to\infty} \int_{-\infty}^t |D(t+\tau, s)|ds = 0, \tag{1.4}$$
uniformly in t.

Here is a proof of Remark 1.3. Since D satisfies (H2), we can write

$$\int_{-\infty}^{t} |D(t,s)|ds = \int_{-\infty}^{t-nT} |D(t,s)|ds + \int_{t-nT}^{t} |D(t,s)|ds$$

Now taking limit on both sides as $n \to \infty$, we obtain

$$\int_{-\infty}^{t} |D(t,s)|ds = \lim_{n\to\infty} \int_{-\infty}^{t-nT} |D(t,s)|ds + \int_{-\infty}^{t} |D(t,s)|ds$$

This implies

$$\lim_{n\to\infty} \int_{-\infty}^{t-nT} |D(t,s)|ds = 0$$

Since $D(t,s) = D(t+T, s+T)$, we can write $D(t,s) = D(t+nT, s+nT)$. Now we conclude the proof by showing that

$$\begin{aligned}
\lim_{n\to\infty} \int_{-\infty}^{t-nT} |D(t,s)|ds &= \lim_{n\to\infty} \int_{-\infty}^{t-nT} |D(t+nT, s+nT)|ds \\
&= \lim_{n\to\infty} \int_{-\infty}^{t} |D(t+nT, s)|ds \\
&= \lim_{\tau\to\infty} \int_{-\infty}^{t} |D(t+\tau, s)|ds
\end{aligned}$$

Remark 1.4. When (H1)-(H3) hold then C satisfies the integrability condition

$$\sup_{t\geq 0} \int_{0}^{t} |C(t,s)|ds \leq c^* < \infty, \tag{1.5}$$

which follows easily when (H2), (1.3), and (H3) is applied on $C(t,s)f(t,x) = D(t,s)g(t,x) + Q(t,s)h(t,x)$ of (H1).

Lemma 1.5. *In addition to the basic assumptions, let assumptions* (H1)–(H3) *hold and let x be a continuous asymptotically T-periodic solution function of* (1.1) *with $|x(t)| \leq \rho,\ t \geq 0$. Then the function*

$$n(t) = \int_{0}^{t} C(t,s)f(s, x(s))ds$$

is continuous and asymptotically T-periodic. Moreover, the T-periodic part of $n(t)$ is

$$\varphi(t) = \int_{-\infty}^{t} D(t,s)g(s, \bar{\pi}(s))ds,$$

where $\bar{\pi}$ is the T-periodic extension of π, with $x = \pi + \sigma$, $\pi(t+T) = \pi(t)$, and $\sigma(t) \to 0$ as $t \to \infty$.

Proof. The continuity of $n(t)$ follows easily from the assumptions. Also, it is easy to verify that $\varphi(t+T) = \varphi(t)$. Now we show that $|n(t) - \varphi(t)| \to 0$, as $t \to \infty$. This will prove that $n(t)$ is asymptotically T-periodic, with $n(t) - \varphi(t) = \alpha(t)$.

$$\begin{aligned}
&|n(t) - \varphi(t)| \\
&= \Big| \int_{0}^{t} C(t,s)f(s,x(s))ds - \int_{-\infty}^{t} D(t,s)g(s,\bar{\pi}(s))ds \Big|
\end{aligned}$$

$$
\begin{aligned}
&= \Big| \int_0^t D(t,s)g(s,x(s))ds + \int_0^t Q(t,s)h(s,x(s))ds - \int_{-\infty}^t D(t,s)g(s,\bar{\pi}(s))ds \Big| \\
&= \Big| \int_0^{t-\tau} D(t,s)g(s,x(s))ds + \int_{t-\tau}^t D(t,s)g(s,x(s))ds + \int_0^t Q(t,s)h(s,x(s))ds \\
&\quad - \int_{-\infty}^{t-\tau} D(t,s)g(s,\bar{\pi}(s))ds - \int_{t-\tau}^t D(t,s)g(s,\bar{\pi}(s))ds \Big| \qquad (1.6) \\
&\le \int_{-\infty}^t |D(t+\tau,s)\|g(s,x(s))|ds + \int_{-\infty}^t |D(t+\tau,s)||g(s,\bar{\pi}(s))|ds \\
&\quad + \int_0^t |Q(t,s)\|h(s,x(s))|ds + \int_{t-\tau}^t |D(t,s)\|g(s,x(s)) - g(s,\bar{\pi}(s))|ds
\end{aligned}
$$

In the above calculations we have used assumption (H1), and replaced $\int_0^{t-\tau} D(t,s)ds$ by $\int_0^t D(t+\tau,s)ds$ where $0 < \tau < t$. Then we have used $\int_0^t |D(t+\tau,s)|ds \le \int_{-\infty}^t |D(t+\tau,s)|ds$.

Let $x, \pi \in B_\rho$. Using assumption (H3) in (1.6) yields,

$$
\begin{aligned}
|n(t) - \varphi(t)| \le 2(mg)_\rho &\int_{-\infty}^t |D(t+\tau,s)|ds \\
&+ \int_{t-\tau}^t |D(t,s)\|g(s,x(s)) - g(s,\bar{\pi}(s))|ds \qquad (1.7) \\
&+ (mh)_\rho \int_0^t |Q(t,s)|ds
\end{aligned}
$$

Let $\epsilon > 0$ be arbitrary. Each of the three terms on the right hand side of (1.7) can be made less than $\frac{\epsilon}{3}$ for sufficiently large t.

First term: By (1.4), there exists a $\tau > 0$ such that for $t > \tau$,

$$
\int_{-\infty}^t |D(t+\tau,s)|ds < \frac{\epsilon}{6(mg)_\rho}
$$

which makes the first term less than $\epsilon/3$.

Second term: The function g is continuous, and $|x(t) - \pi(t)| \to 0$ as $t \to \infty$. Therefore, $|g(t,x(t)) - g(t,\pi(t))| \to 0$ as $t \to \infty$. This means there exists a $T_1 > \tau$ such that for $t > T_1$, we can make

$$
\int_{t-\tau}^t |D(t,s)\|g(s,x(s)) - g(s,\bar{\pi}(s))|ds < \frac{\epsilon}{3}.
$$

Third term: From assumption (H1) we see that $\int_0^t Q(t,s)ds \to 0$ as $t \to \infty$. Therefore, there exists a T_2 such that for $t > T_2$,

$$
\int_0^t |Q(t,s)|ds < \frac{\epsilon}{3(mh)_\rho},
$$

which means the third term is less than $\epsilon/3$.

Let $T = \max\{T_1, T_2\}$. Then for $t > T$, it follows from (1.7) that

$$
|n(t) - \varphi(t)| < \epsilon.
$$

This concludes the proof. $\square$

Let

$$B = \{x : x \text{ is continuous and bounded on } \mathbb{R}_+\}.$$

Then B is a Banach space with the norm $\|x\| = \sup_{t\in\mathbb{R}+} |x(t)|$. Let

$$B_l = \{x \in B : \exists \lim_{t\to\infty} x(t) \in \mathbb{R}\}.$$

A convenient compactness criterion, given below in Lemma 1.6, holds on this space.

Lemma 1.6 ([2, 3]). *A family $A \subset B_l$ is relatively compact if and only if*

(a) *$\mathcal{A}$ is uniformly bounded,*
(b) *$\mathcal{A}$ is equicontinuous on compact subsets of $\mathbb{R}_+$,*
(a) *$\mathcal{A}$ is equiconvergent.*

Theorem 1.7 (Schauder's Fixed Point Theorem). *If S is a closed, bounded, convex subset of a Banach space X, and $H : S \to S$ is completely continuous, then H has a fixed point in S.*

An operator is completely continuous if it is continuous and it maps bounded sets into relatively compact sets.

2. Existence Theorems

Theorem 2.1. *Suppose* (H1)–(H3) *along with the basic assumptions hold. Then* (1.1) *has a continuous asymptotically T-periodic solution.*

Proof. Let

$$M = \|b\| + d^*(mg)_\rho, \quad N = \|q\| + c^*(mf)_\rho + d^*(mg)_\rho, \tag{2.1}$$

where c^* and d^* are the constants of (1.5) and (H2) respectively. And $(mf)_\rho$ and $(mg)_\rho$ are the constants of (H3) (i) and (ii) respectively.

Suppose there exists a $\rho > 0$ such that

$$M + N = \|b\| + \|q\| + c^*(mf)_\rho + 2d^*(mg)_\rho \le \rho. \tag{2.2}$$

Let S_ρ be the set of functions $x \in B$, $x = \pi + \sigma$, $\pi(t + T) = \pi(t)$, $\sigma(t) \to 0$ as $t \to \infty$, $\|\pi\| \le M$, and $\|\sigma\| \le N$. Clearly, $\|x\| \le \rho$, and the set S_ρ is a closed and convex subset of the Banach space B.

Define H on S_ρ as follows. For $x \in S_\rho$,

$$Hx(t) = a(t) + \int_0^t C(t,s)f(s,x(s))ds. \tag{2.3}$$

Since $x \in S_\rho$, $x = \pi + \sigma$, $\pi(t + T) = \pi(t)$, $\sigma(t) \to 0$ as $t \to \infty$ for some π and σ. From Lemma 1.5 we know that

$$n(t) = \int_0^t C(t,s)f(s,x(s))ds$$

is continuous and asymptotically T-periodic, and that the T-periodic part of $n(t)$ is

$$\varphi(t) = \int_{-\infty}^t D(t,s)g(s,\bar{\pi}(s))ds,$$

where $\bar{\pi}$ is the T-periodic extension of π on $\mathbb{R}$. Let $\alpha(t) = n(t) - \varphi(t)$. Then $\alpha(t) \to 0$ as $t \to 0$. By our basic assumptions, $a(t) = b(t) + q(t)$, $b(t + T) = b(t)$, $q(t) \to 0$ as $t \to \infty$. Therefore from (2.3), we can write

$$Hx(t) = b(t) + q(t) + \varphi(t) + \alpha(t) = (b(t) + \varphi(t)) + (q(t) + \alpha(t)) = u(t) + v(t), \tag{2.4}$$

where $u(t) = b(t) + \varphi(t)$ and $v(t) = q(t) + \alpha(t)$. Clearly, u is continuous and T-periodic since both b and φ are of these properties. Similarly, the function v is bounded, continuous, and $v(t) \to 0$ as $t \to \infty$, because both q and α have the same properties. Note that

$$|\varphi(t)| \le \int_{-\infty}^{t} |D(t,s)||g(s,\bar{\pi}(s))|ds \le d^*(mg)_\rho,$$

which implies $\|\varphi\| \le d^*(mg)_\rho$. Also,

$$|\alpha(t)| \le |n(t)| + |\varphi(t)| \le \int_0^t |C(t,s)||f(s,x(s))|ds + \|\varphi\| \le c^*(mf)_\rho + d^*(mg)_\rho,$$

from which we obtain $\|\alpha\| \le c^*(mf)_\rho + d^*(mg)_\rho$. Therefore,

$$\|u\| \le \|b\| + \|\varphi\| \le \|b\| + d^*(mg)_\rho = M,$$
$$\|v\| \le \|q\| + \|\alpha\| \le \|q\| + c^*(mf)_\rho + d^*(mg)_\rho = N.$$

So, from (2.4) and (2.2), we find

$$|Hx(t)| \le M + N \le \rho. \tag{2.5}$$

This shows that H maps from S_ρ into itself i.e., $HS_\rho \subseteq S_\rho$, and hence HS_ρ is uniformly bounded.

Now we show that H is a continuous operator, and that the set HS_ρ is relatively compact. For the continuity of the operator H, define operators U and V as follows. For each $x \in S_\rho$,

$$(Ux)(t) = \int_0^t C(t,s)x(s)ds,$$
$$(Vx)(t) = f(t,x(t)),$$

for all $t \in \mathbb{R}_+$. Clearly, V is continuous in x because f is. The operator U is a linear operator and hence is continuous. The continuity of the operator H is then follows from $Hx = a + (U \circ V)x$, for all $x \in S_\rho$.

We show the relative compactness of HS_ρ by showing that every sequence in HS_ρ has a subsequence that converges to an element in HS_ρ. Let $\{x_m\}$ be an arbitrary sequence in HS_ρ. Then by (2.4), each $x_m = u_m + v_m$, with u_m being T-periodic and $v_m(t) \to 0$ as $t \to \infty$. Here $\|x_m\| \le \rho$ with $\|u_m\| \le M$ and $\|v_m\| \le N$.

The sequence $\{u_m\}$ is a continuous bounded T-periodic functions. Therefore, there exists a subsequence $\{u_{m_k}\}$ that converges uniformly to a continuous bounded T-periodic function, say u. Now, consider the corresponding sequence $\{v_{m_k}\}$ of functions on $\mathbb{R}+$. Since all members of this sequence satisfy $\|v_{m_k}\| \le N$, the sequence is uniformly bounded. Also, $\lim_{t\to\infty} v_{m_k}(t) = 0$, for all members of this sequence, and hence, the sequence is equiconvergent. Now, we show that the sequence is equicontinuous on compact subsets of $\mathbb{R}+$. To show this, it is sufficient to show the equicontinuity on an arbitrary interval $[0,T]$, for a $T > 0$.

Let $t_1, t_2 \in [0,T]$. Without loss of generality, we assume $t_1 < t_2$. For notational simplicity, let us write $j = m_k$. Then

$$\begin{aligned} |v_j(t_1) - v_j(t_2)| &\le |q(t_1) - q(t_2)| + |\alpha_j(t_1) - \alpha_j(t_2)| \\ &\le |q(t_1) - q(t_2)| + |n_j(t_1) - n_j(t_2)| + |\varphi_j(t_1) - \varphi_j(t_2)|. \end{aligned} \tag{2.6}$$

Employing (H3)(i) on the expression for $n(t)$ we can write

$$|n_j(t_1) - n_j(t_2)| \le (mf)_\rho\{\int_0^{t_1} |C(t_1, s) - C(t_2, s)|ds + \int_{t_1}^{t_2} |C(t_2, s)|ds\}. \tag{2.7}$$

Employing (H3) (ii) on the expression for $\varphi(t)$ we can write

$$|\varphi_j(t_1) - \varphi_j(t_2)| \le (mg)_\rho|\int_{-\infty}^{t_1} D(t_1, s)ds - \int_{-\infty}^{t_2} D(t_2, s)ds| \tag{2.8}$$

Let $\epsilon > 0$ be arbitrary. Since q is continuous, there exists a $\delta_1 > 0$ such that $|q(t_1)-q(t_2)| < \frac{\epsilon}{3}$ when $|t_1-t_2| < \delta_1$. By the continuity of C, one can see from (2.6) that there exists a $\delta_2 > 0$ such that $|n_j(t_1) - n_j(t_2)| < \frac{\epsilon}{3}$ when $|t_1 - t_2| < \delta_2$. We know from Remark 1.4 that assumption (H2) implies the continuity of the function $\int_{-\infty}^{t} D(t, s)ds$ in t. Therefore, from (H2) and (2.8) there exists a $\delta_3 > 0$ such that $|\varphi_j(t_1) - \varphi_j(t_2)| < \frac{\epsilon}{3}$ when $|t_1 - t_2| < \delta_3$. Let $\delta = \min\{\delta_1, \delta_2, \delta_3\}$. Then from (2.6) we have $|v_j(t_1) - v_j(t_2)| < \epsilon$ when $|t_1 - t_2| < \delta$. This concludes that the sequence $\{v_j\}$ i.e., the sequence $\{v_{m_k}\}$ is equicontinuous on compact subsets of $\mathbb{R}+$.

Then by Lemma 1.6, there exists a subsequence $\{v_{m_{k_l}}\}$ that converges to a function, say v on $\mathbb{R}+$. As the limit function v has the properties that v is continuous, bounded and $v(t) \to 0$ as $t \to \infty$, and $\|v\| \le N$. Now, consider the corresponding sequence $\{u_{m_{k_l}}\}$, which is it self a subsequence of $\{u_{m_k}\}$. We already found that $\{u_{m_k}\}$ converges to u. Thus, the subsequence $\{u_{m_{k_l}}\}$ also converges to u. As the limit function, u is continuous and T-periodic. It is now clear from the very construction that for the sequence $\{x_m\}$, there exists a subsequence $\{x_{m_{k_l}}\} = \{u_{m_{k_l}}\} + \{v_{m_{k_l}}\}$ that converges to $x = u + v$, where u is T-periodic, $\|u\| \le M$, $v(t) \to 0$ as $t \to \infty$, $\|v\| \le N$. Then $\|x\| \le \|u\| + \|v\| \le M + N = \rho$. This means the limit function is in HS_ρ. This concludes the proof that the set HS_ρ is relatively compact.

By Schauder's fixed point theorem, there exists a function x in HS_ρ such that $x = Hx$; the function x is a solution of (1.1). This concludes the proof of Theorem 2.1. □

Theorem 2.2. *Suppose* (H1)–(H3) *along with the basic assumptions hold. Then* (1.2) *has a continuous T-periodic solution.*

Proof. Let $x = \pi + \sigma$, $\pi(t + T) = \pi(t)$, $\sigma(t) \to 0$ as $t \to \infty$ is an asymptotically T-periodic solution of (1.1). Then from (1.1) we obtain

$$\pi(t) + \sigma(t) = b(t) + q(t) + \int_0^t C(t, s)f(s, x(s))ds. \tag{2.9}$$

By Lemma 1.5,

$$\int_0^t C(t, s)f(s, x(s))ds = \int_{-\infty}^t D(t, s)g(s, \pi(s))ds + \alpha(t), \tag{2.10}$$

where $\alpha(t) \to 0$ as $t \to \infty$. Therefore, combining (2.9) and (2.10), we find

$$\pi(t) + \sigma(t) = b(t) + \int_{-\infty}^t D(t, s)g(s, \pi(s))ds + \alpha(t) + q(t) \tag{2.11}$$

Equating the T-periodic part from both sides of (2.11) we obtain

$$\pi(t) = b(t) + \int_{-\infty}^t D(t, s)g(s, \pi(s))ds.$$

It is easy to verify that $\pi(t+T) = \pi(t)$. Therefore, π is a T-periodic solution of (1.2). This concludes the proof of Theorem 2.2, showing that (1.2) has a continuous T-periodic solution. $\square$

As an example, consider the Volterra equation

$$x(t) = \sin t + e^{-|t|} + \int_0^t (\cos t + se^{-s})x(s)ds, \tag{2.12}$$

Here $a(t)$ of (1.1) is $\sin t + e^{-|t|}$, which is clearly asymptotically 2π periodic with $b(t)$ of (1.2) being $\sin t$ and $q(t) = e^{-|t|}$. Also, in this equation, we consider the functions $C(t,s) = e^s \cos t + s$, and $f(t, x(t)) = e^{-t}x(t)$. Note that we can write

$$C(t,s)f(t,x(t)) = e^{s-t}x(t)\cos t + se^{-t}x(t).$$

Let $D(t,s) = e^{s-t}$, $g(t,x(t)) = x(t)\cos t$, $Q(t,s) = se^{-t}$, and $h(t,x(t)) = x(t)$. Clearly, D and Q satisfy assumptions (H1) and (H2). For any fixed positive ρ, the conditions in (H3) hold with all three constants $(mf)_\rho$, $(mf)_\rho$, and $(mf)_\rho$ being ρ. Therefore, equation (2.12) has an asymptotically 2π periodic solution x with $\|x\| \le \rho$, and the 2π periodic part of this solution is indeed a periodic solution of the associated integral equation

$$x(t) = \sin t + \int_{-\infty}^t e^{s-t}x(s)\cos s\, ds.$$

Remark 2.3. It is important to understand that equations (1.1) and (1.2) are of different nature. An asymptotically periodic solution function of (1.1) is defined on $[0,\infty)$, where as a periodic solution function of (1.2) is defined on $(-\infty,\infty)$. We have shown in Theorem 2.1 that (1.1) has an asymptotically periodic solution x with $x(t) = \pi(t) + \sigma(t), \pi(t+T) = \pi(t), \sigma(t) \to 0$ as $t \to \infty$. In Theorem 2.2, we have shown that (1.2) has a T-periodic solution and that solution is the T-periodic extension of the periodic part π of the solution x of (1.1). More explicitly, let $\tilde{\pi}(t)$, defined on $(-\infty,\infty)$, be the T-periodic extension of $\pi(t)$, defined on $[0,\infty)$. Then $\tilde{\pi}$ is the periodic solution of (1.2).

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Muhammad N. Islam
Department of Mathematics, University of Dayton, Dayton, OH 45469-2316, USA
E-mail address: mislam1@udayton.edu