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EXISTENCE, BOUNDARY BEHAVIOR AND ASYMPTOTIC BEHAVIOR OF SOLUTIONS TO SINGULAR ELLIPTIC BOUNDARY-VALUE PROBLEMS

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Abstract. In this article, we consider the singular elliptic boundary-value problem

$$-\Delta u + f(u) - u^{-\gamma} = \lambda u \text{ in } \Omega, \quad u > 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega.$$

Using the upper-lower solution method, we show the existence and uniqueness of the solution. Also we study the boundary behavior and asymptotic behavior of the positive solutions.

1. Introduction

In this article, we consider the singular elliptic boundary-value problem

$$\begin{gathered} -\Delta u + f(u) - u^{-\gamma} = \lambda u \quad \text{in } \Omega, \\ u > 0 \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $N \geq 2$, with $C^{2,\beta}$ boundary $\partial\Omega$, $\gamma > 0$, $\beta \in (0,1)$ and $\lambda > 0$ is a real parameter. We use the following assumptions in this article.

- (A1) $f : \mathbb{R}^+ \to \mathbb{R}$ is a continuous function.
- (A2) $s^{-1}f(s)$ is strictly increasing for $s > 0$.
- (A3) $f : \mathbb{R}^+ \to \mathbb{R}$ is strictly increasing.

Existence, boundary behavior and asymptotic behavior of solutions for nonlinear elliptic boundary value problems have been intensively studied in the previous decades. Berestycki [1] considered the problem

$$\begin{gathered} Lu + f(x,u) = \lambda a u, \quad \text{in } \Omega, \\ u = 0, \quad \text{on } \partial\Omega, \end{gathered} \tag{1.2}$$

where L is a second order uniformly elliptic operator, $a \in C^0(\bar{\Omega})$, $a > 0$ in $\bar{\Omega}$ and under the conditions that $f(x,0) = 0$, $s \to (f(x,s)/s)$ is strictly increasing for $s > 0$

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and $\lim_{s\to\infty}(f(x,s)/s)=+\infty$, it is proved that (1.2) has one and only one positive solution $u_\lambda \in W^{2,p}(\Omega)$ for $\lambda > \lambda_1$, where λ_1 denotes the principle eigenvalue of

$$L\phi = \lambda a\phi \text{ in } \Omega, \quad \phi = 0 \text{ on } \partial\Omega.$$

As a special case for (1.2), Fraile, López-Gómez and Delis [6] studied

$$\begin{gathered} -\Delta u = \lambda u + f(u) - u^{p+1} \quad \text{in } \Omega, \\ u > 0 \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.3}$$

and described the structure of the positive solutions of (1.3) in detail. Some other asymptotic behavior studies have been shown in [16, 17, 18].

Notice that in the above f needs to be continuous or differential at $x=0$. On the other hand, singular elliptic boundary value problems in various forms have been studied extensively by many authors, see [2, 3, 4, 5, 7, 8, 10, 11, 12, 15, 19, 21, 22]. For instance, for the problem

$$\begin{gathered} -\Delta u - u^{-\alpha} = (\lambda u)^p \quad \text{in } \Omega, \\ u > 0 \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.4}$$

by results in [4, 7, 8, 19], it follows that if $\lambda = 0$, (1.4) possesses a unique classical solution and at least one classical solution for $\lambda \neq 0$ and $p < 1$. Coclite and Palmieri [3] showed that if $0 < p < 1$, (1.4) has at least one solution u_λ for all $\lambda \geq 0$; and if $p \geq 1$, there exist $\tilde\lambda \in (0,+\infty]$ such that (1.4) has at least one classical solution for all $\lambda \in [0,\tilde\lambda)$ and has no solution for $\tilde\lambda < \lambda$. See also results in [2, 12, 22]. A more generalized work in [15] is about for the singular problem

$$\begin{gathered} -\Delta u + K(x)u^{-\alpha} = \lambda u^p \quad \text{in } \Omega, \\ u > 0 \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega\,. \end{gathered} \tag{1.5}$$

If $K(x) < 0$ for all $x \in \bar\Omega$, then (1.5) has one and only one solution u_λ for any $\lambda \in \mathbb{R}$, $c_1 d(x) \leq u_\lambda(x) \leq c_2 d(x)$ for any $x \in \bar\Omega$ and some $c_1, c_2 > 0$ independent of x; if $K(x) > 0$ for all $x \in \bar\Omega$, there exist a $\tilde\lambda > 0$ such that (1.5) has at least one solution u_λ for $\lambda > \tilde\lambda$, and (1.5) has no solution for $\lambda < \tilde\lambda$. Other results can be found in [4, 5, 10, 12, 21].

Up to now, there are only a few results on existence, boundary behavior and asymptotic behavior of positive solutions for (1.1). Our goal in this paper is to show existence, boundary behavior and asymptotic behavior of the solutions for singular elliptic boundary-value problem (1.1). Using the upper-lower solution method, we obtain that (1.1) has at least one solution, and if $0 < \gamma < 1$, (1.1) has one and only one solution. In the meanwhile, the boundary behavior of the solution is established for $0 < \gamma < 1$. Finally, we obtain the asymptotic behavior of solutions under a special form of $f(u)$.

2. Existence and uniqueness of a solution for (1.1)

First, we introduce notation and present some lemmas. In the next lemma, $W^{k,q}(\Omega)$ denotes the usual Sobolev space.

Lemma 2.1 ([4]). *Let ϑ_0, ϑ be bounded open domains in $\mathbb{R}^n$ with $\bar{\vartheta}_0 \subset \vartheta$. Suppose L is a second order uniformly elliptic operator with coefficients continuous in $\bar{\vartheta}$ and $q > n$. Then there is a constant K such that*

$$\|w\|_{W^{2,q}(\vartheta_0)} \leq K(\|Lw\|_{L^q(\vartheta)} + \|w\|_{L^q(\vartheta)})$$

for all $w \in W^{2,q}(\vartheta)$. The constant K depends on n, q, the diameter of ϑ, the distance from ϑ_0 to $\partial\vartheta$, the ellipticity constant of L, and bounds for the coefficients of L (in $L^\infty(\vartheta)$) and the moduli of continuity of the coefficients.

We consider the nonlinear elliptic boundary-value problem

$$\begin{gathered} Lu + f(x,u) = 0 \quad \text{in } \Omega, \\ Bu = g \quad \text{on } \partial\Omega, \end{gathered} \tag{2.1}$$

where L is a second order uniformly elliptic operator

$$L = \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2 u}{\partial x^i \partial x^j} + \sum_{i=1}^{n} b_i \frac{\partial u}{\partial x^i}, \quad x = (x^1, \dots, x^n),$$

and B is one of the boundary operators

$$Bu = u,$$

or

$$Bu = \frac{\partial u}{\partial v} + \beta(x) u, \quad x \in \partial\Omega.$$

Here $\partial/\partial v$ denotes the outward conormal derivative, and we assume $\beta \geq 0$ everywhere on the boundary, $\partial\Omega$.

Lemma 2.2 ([14]). *Let there exist two smooth functions $u_0(x) \geq v_0(x)$ such that*

$$\begin{gathered} Lu_0 + f(x,u_0) \leq 0, \quad Bu_0 \geq g, \\ Lv_0 + f(x,v_0) \geq 0, \quad Bv_0 \leq g. \end{gathered}$$

Assume f is a smooth function on $\min v_0 \leq u \leq \max u_0$. Then there exists a regular solution w of

$$Lw + f(x,w) = 0, \quad Bw = g,$$

such that $v_0 \leq w \leq u_0$.

Let ϕ_1 denote an eigenfunction corresponding to the first eigenvalue λ_1 of

$$\begin{gathered} -\Delta u = \lambda u \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{gathered}$$

As is known, ϕ_1 belongs to $C^{2,\beta}(\bar{\Omega})$, $\phi_1 > 0$ in Ω and $\lambda_1 > 0$.

Lemma 2.3 ([12]). $\int_\Omega (\frac{1}{\phi_1})^s dx < \infty$ *if and only if* $s < 1$.

Assume that (1.1) has a positive solution u_λ and let $x_0 \in \Omega$ be the point where u_λ reaches its maximum. Thus, $-\Delta u_\lambda(x_0) \geq 0$, which concludes that

$$\lambda \geq -u_\lambda^{-(\gamma+1)}(x_0) + \frac{f(u_\lambda(x_0))}{u_\lambda(x_0)}\,.$$

Define

$$G(s) = -s^{-(\gamma+1)} + \frac{f(s)}{s}, s > 0.$$

Theorem 2.4. *If $f(u)$ satisfies* (A1) *and* (A2)*, then* (1.1) *has at least one solution $u_\lambda \in C(\bar{\Omega}) \cap C^{2,\beta}(\Omega)$ for $\lambda > \lambda_1$, and there exist a constant $C > 0$ such that $u_\lambda \geq C\phi_1$. Moreover, if $0 < \gamma < 1$,* (1.1) *has one and only one solution for $\lambda > \lambda_1$.*

Proof. (i) (Existence) First we consider the solution of the nonlinear elliptic boundary-value problem

$$\begin{gathered} -\Delta u + f(u) - u^{-\gamma} = \lambda u \quad \text{in } \Omega, \\ u > 0 \quad \text{in } \Omega, \\ u = 1/k \quad \text{on } \partial\Omega, \end{gathered} \tag{2.2}$$

where $1/k < G^{-1}(\lambda)$, $k \in N_+$. Set

$$\overline{u_\lambda^{(k)}}(x) = G^{-1}(\lambda),$$

then

$$\begin{aligned} -\Delta\overline{u_\lambda^{(k)}} + f(\overline{u_\lambda^{(k)}}) - \overline{u_\lambda^{(k)}}^{-\gamma} &= f(G^{-1}(\lambda)) - (G^{-1}(\lambda))^{-\gamma} \\ &= G^{-1}(\lambda)[\frac{f(G^{-1}(\lambda))}{G^{-1}(\lambda)} - (G^{-1}(\lambda))^{-(\gamma+1)}] \\ &= G^{-1}(\lambda)G(G^{-1}(\lambda)) \\ &= \lambda G^{-1}(\lambda) = \lambda\overline{u_\lambda^{(k)}}. \end{aligned}$$

So

$$\begin{gathered} -\Delta\overline{u_\lambda^{(k)}} + f(\overline{u_\lambda^{(k)}}) - \overline{u_\lambda^{(k)}}^{-\gamma} \geq \lambda\overline{u_\lambda^{(k)}} \quad \text{in } \Omega, \\ \overline{u_\lambda^{(k)}} = G^{-1}(\lambda) \quad \text{on } \partial\Omega. \end{gathered}$$

Therefore, $\overline{u_\lambda^{(k)}}(x)$ is a super-solution of (2.2). Put

$$\underline{u_\lambda^{(k)}}(x) = h_k G^{-1}(\lambda - \lambda_1)(\phi_1(x) + \delta_k),$$

where $\delta_k = 1/k$, $h_k = 1/(\delta_k + 1)$, $h_k G^{-1}(\lambda - \lambda_1)\delta_k \leq 1/k$. Since

$$\underline{u_\lambda^{(k)}} \leq 1/k \quad \text{on } \partial\Omega,$$

and

$$\begin{aligned} & -\Delta\underline{u_\lambda^{(k)}} + f(\underline{u_\lambda^{(k)}}) - \underline{u_\lambda^{(k)}}^{-\gamma} \\ & = \lambda_1 h_k G^{-1}(\lambda - \lambda_1)\phi_1(x) + f(h_k G^{-1}(\lambda - \lambda_1)(\phi_1(x) + \delta_k)) \\ & \quad - [h_k G^{-1}(\lambda - \lambda_1)(\phi_1(x) + \delta_k)]^{-\gamma} \\ & \leq h_k G^{-1}(\lambda - \lambda_1)(\phi_1(x) + \delta_k)\{\lambda_1 + \frac{f(h_k G^{-1}(\lambda - \lambda_1)(\phi_1(x) + \delta_k))}{h_k G^{-1}(\lambda - \lambda_1)(\phi_1(x) + \delta_k)} \\ & \quad - [h_k G^{-1}(\lambda - \lambda_1)(\phi_1(x) + \delta_k)]^{-(\gamma+1)}\} \\ & \leq h_k G^{-1}(\lambda - \lambda_1)(\phi_1(x) + \delta_k)\{\lambda_1 + \frac{f(G^{-1}(\lambda - \lambda_1))}{G^{-1}(\lambda - \lambda_1)} - [G^{-1}(\lambda - \lambda_1)]^{-(\gamma+1)}\} \\ & = h_k G^{-1}(\lambda - \lambda_1)(\phi_1(x) + \delta_k)(\lambda_1 + \lambda - \lambda_1) \\ & = \lambda h_k G^{-1}(\lambda - \lambda_1)(\phi_1(x) + \delta_k) \\ & = \lambda\underline{u_\lambda^{(k)}}, \end{aligned}$$

we obtain that $\underline{u_\lambda^{(k)}}(x)$ is a sub-solution of (2.2).

When $k=1$, standard elliptic arguments imply that there exist a solution $u_\lambda^{(1)} \in C(\bar{\Omega}) \cap C^{2,\beta}(\Omega)$ such that

$$\underline{u_\lambda^{(1)}} \le u_\lambda^{(1)} \le \overline{u_\lambda^{(1)}}, \quad \text{in } \bar{\Omega}.$$

Now taking $u_\lambda^{(1)}(x)$ and $\underline{u_\lambda^{(2)}}(x)$ as a pair of super and sub-solutions for (2.2), we obtain a solution $u_\lambda^{(2)} \in C(\bar{\Omega}) \cap C^{2,\beta}(\Omega)$ such that

$$\underline{u_\lambda^{(2)}} \le u_\lambda^{(2)} \le u_\lambda^{(1)} \quad \text{in } \bar{\Omega}.$$

In this manner we find a sequence $\{u_\lambda^{(k)}\}$, such that

$$\underline{u_\lambda^{(k)}} \le u_\lambda^{(k)} \le u_\lambda^{(k-1)} \le \overline{u_\lambda^{(1)}} \quad \text{in } \bar{\Omega}.$$

Therefore $\lim_{k\to\infty} u_\lambda^{(k)}(x) = u_\lambda(x)$ exists everywhere in $\bar{\Omega}$, where $u_\lambda > 0$ in Ω and $u_\lambda = 0$ on $\partial\Omega$. It remains to see that $u_\lambda \in C^2(\Omega)$ and

$$-\Delta u_\lambda + f(u_\lambda) - u_\lambda^{-\gamma} = \lambda u_\lambda, \quad \text{in } \Omega.$$

Choose open subsets ϑ_1, ϑ_2 of Ω so that $\bar{\vartheta}_2 \subset \vartheta_1 \subset \bar{\vartheta}_1 \subset \Omega$ and $q > N$. By Lemma 2.1 there is a constant $K = K(N, q, \vartheta_1, \vartheta_2, \Delta)$ such that

$$\begin{aligned}
\|u_\lambda^{(k)}\|_{W^{2,q}(\vartheta_2)} &\le K(\|\Delta u_\lambda^{(k)}\|_{L^q(\vartheta_1)} + \|u_\lambda^{(k)}\|_{L^q(\vartheta_1)}) \\
&= K(\|f(u_\lambda^{(k)}) - (u_\lambda^{(k)})^{-\gamma} - \lambda u_\lambda^{(k)}\|_{L^q(\vartheta_1)} + \|u_\lambda^{(k)}\|_{L^q(\vartheta_1)}),
\end{aligned}$$

which implies that $\{u_\lambda^{(k)} | \, k \in N_+\}$ is bounded in $W^{2,q}_{\rm loc}(\Omega)$. Therefore $u_\lambda^{(k)} \to u_\lambda$ weakly in $W^{2,q}_{\rm loc}(\Omega)$. Choose $\alpha \in (0,1)$ and $q > N(1-\alpha)^{-1}$. If follows from the Sobolev embedding theorems that $\{u_\lambda^{(k)}\}$ is compact in $C^{1,\alpha}_{\rm loc}(\Omega)$. Thus we have $u_\lambda \in W^{2,p}_{\rm loc}(\Omega)$. The L^q regularity theory for Δ now implies $u_\lambda \in W^{3,q}_{\rm loc}(\Omega)$ and hence $u_\lambda \in C^{2,\alpha}_{\rm loc}(\Omega) \subset C^2(\Omega)$. Notice that Lemma 2.2 ensures that $u_\lambda^{(k)}(x)$ is a solution of (2.2), so

$$-\Delta u_\lambda^{(k)} + f(u_\lambda^{(k)}) - (u_\lambda^{(k)})^{-\gamma} = \lambda u_\lambda^{(k)} \quad \text{in } \Omega.$$

In conjunction with the results above, let $k \to \infty$, therefore

$$-\Delta u_\lambda + f(u_\lambda) - u_\lambda^{-\gamma} = \lambda u_\lambda \quad \text{in } \Omega.$$

Consequently, (1.1) has at least one solution $u_\lambda \in C(\bar{\Omega}) \cap C^{2,\beta}(\Omega)$ for $\lambda > \lambda_1$.

Furthermore, since $\lim_{k\to\infty} u_\lambda^{(k)}(x) = u_\lambda(x)$ and

$$h_k G^{-1}(\lambda - \lambda_1)(\phi_1(x) + \delta_k) = \underline{u_\lambda^{(k)}}(x) \le u_\lambda^{(k)}(x), \quad \text{in } \Omega,$$

$G^{-1}(\lambda - \lambda_1) > 0$, $\delta_k = 1/k \to 0, h_k = 1/(\delta_k + 1) \to 1$ as $k \to \infty$, so there exist a constant $C > 0$ such that $C\phi_1 \le u_\lambda$ in Ω.

(ii) (Uniqueness) If $0 < \gamma < 1$, we put

$$h(s) = \lambda s - f(s) + s^{-\gamma}, \quad s > 0,$$

hence $s^{-1}h(s)$ is strictly decreasing for $s > 0$.

Assume that $v(x)$ is another solution of (1.1). Then we argue by contradiction. Notice that

$$\Delta u + h(u) = 0, \quad \Delta v + h(v) = 0,$$

and $u, v > 0$ in Ω, $u = v$ on $\partial\Omega$. Since $C\phi_1(x) \le u_\lambda(x)$, it is easy to see that $\Delta u \in L^1(\Omega)$ by Lemma 2.3. If $u(x) = v(x)$ is not true, we can assume that $u(x) > v(x)$, then there exist $\varepsilon_0, \delta_0 > 0$, and a ball $B \Subset \Omega$ such that

$$u(x) - v(x) > \varepsilon_0, \quad x \in B,$$
$$\int_B uv(\frac{h(v)}{v} - \frac{h(u)}{u})dx > \delta_0.$$

Let

$$M = \max\{1, \|\Delta u\|_{L^1(\Omega)}\}, \quad \varepsilon = \min\{1, \varepsilon_0, \frac{\delta_0}{4M}\},$$

and θ be a smooth function on R, such that $\theta(t) = 0$ if $t \le \frac{1}{2}$; $\theta(t) = 1$ if $t \ge 1$; $\theta(t) \in (0, 1)$ if $t \in (\frac{1}{2}, 1)$, $\theta'(t) \ge 0$ for $t \in \mathbb{R}$. Then for $\varepsilon > 0$, define the function $\theta_\varepsilon(t)$ by

$$\theta_\varepsilon(t) = \theta(\frac{t}{\varepsilon}), \quad t \in \mathbb{R}.$$

It follows from $\theta_\varepsilon(t) \ge 0$ for $t \in \mathbb{R}$ that

$$(v\Delta u - u\Delta v)\theta_\varepsilon(u - v) = uv(\frac{h(v)}{v} - \frac{h(u)}{u})\theta_\varepsilon(u - v) \quad \text{in } \Omega.$$

On the other hand, by the continuity of u, v and θ_ε, and the fact that $u = v$ on $\partial\Omega$. It is easy to see that there exist a subdomain $\hat{\Omega}$ such that $B \subset \hat{\Omega} \Subset \Omega$ satisfying that $u(x) - v(x) < \frac{\varepsilon}{2}$ for all $x \in \Omega \backslash \hat{\Omega}$. Then

$$\int_{\hat{\Omega}} (v\Delta u - u\Delta v)\theta_\varepsilon(u - v)dx = \int_{\hat{\Omega}} uv(\frac{h(v)}{v} - \frac{h(u)}{u})\theta_\varepsilon(u - v)dx.$$

Denote

$$\Theta_\varepsilon(t) = \int_0^t s\theta'_\varepsilon(s)ds, \quad t \in \mathbb{R}\,.$$

It is easy to verify that $0 \le \Theta_\varepsilon(t) \le 2\varepsilon$ for $t \in \mathbb{R}$, and $\Theta_\varepsilon(t) = 0$ for $t < \frac{\varepsilon}{2}$. Therefore

$$\begin{aligned}
&\int_{\hat{\Omega}} (v\Delta u - u\Delta v)\theta_\varepsilon(u - v)dx \\
&= \int_{\partial\hat{\Omega}} v\theta_\varepsilon(u - v)\frac{\partial u}{\partial n}ds - \int_{\hat{\Omega}} (\nabla u \cdot \nabla v)\theta_\varepsilon(u - v)dx \\
&\quad - \int_{\hat{\Omega}} v\nabla u\theta'_\varepsilon(u - v)(\nabla u - \nabla v)dx - \int_{\partial\hat{\Omega}} u\theta_\varepsilon(u - v)\frac{\partial v}{\partial n}ds \\
&\quad + \int_{\hat{\Omega}} (\nabla v \cdot \nabla u)\theta_\varepsilon(u - v)dx + \int_{\hat{\Omega}} u\nabla v\theta'_\varepsilon(u - v)(\nabla u - \nabla v)dx \\
&= \int_{\hat{\Omega}} u\theta'_\varepsilon(u - v)(\nabla v - \nabla u)(\nabla u - \nabla v)dx + \int_{\hat{\Omega}} (u - v)\theta'_\varepsilon(u - v)\nabla u(\nabla u - \nabla v)dx \\
&\le \int_{\hat{\Omega}} \nabla u\nabla(\Theta'_\varepsilon(u - v))dx \\
&= \int_{\partial\hat{\Omega}} \Theta_\varepsilon(u - v)\frac{\partial u}{\partial n}ds - \int_{\hat{\Omega}} \Delta u\Theta_\varepsilon(u - v)dx \\
&\le 2\varepsilon \int_{\hat{\Omega}} |\Delta u|dx \\
&\le 2\varepsilon M < \frac{\delta_0}{2}.
\end{aligned}$$

However,

$$\begin{aligned}\int_{\hat\Omega} uv(\frac{h(v)}{v}-\frac{h(u)}{u})\theta_\varepsilon(u-v)dx &\geq \int_B uv(\frac{h(v)}{v}-\frac{h(u)}{u})\theta_\varepsilon(u-v)dx\\ &= \int_B uv(\frac{h(v)}{v}-\frac{h(u)}{u})dx > \delta_0,\end{aligned}$$

which is a contradiction. Thus the uniqueness is proved. □

Our method to prove the uniqueness of the solution is similar to and motivated by the proof of Shi and Yao [15, Lemma 2.3].

3. The boundary behavior of the solution to (1.1)

Theorem 3.1. *If $f(u)$ satisfies* (A1)–(A3) *and $0<\gamma<1$, then the solution u_λ of* (1.1) *is strictly increasing with respect to λ. Furthermore, there exist two positive constants $c_1, c_2 > 0$ depending on λ such that $c_1 d(x) \leq u_\lambda(x) \leq c_2 d(x)$ in Ω.*

Proof. (i) (Dependence on λ) We assume $0<\lambda_1<\lambda_2$, and u_{λ_1}, u_{λ_2} are corresponding unique solution to (1.1). Since $C\phi_1(x) \leq u_\lambda(x)$, it is easy to see that $\Delta u_\lambda \in L^1(\Omega)$ by Lemma 2.3. Thus,

$$\begin{aligned}0 &= \Delta u_{\lambda_2} - f(u_{\lambda_2}) + u_{\lambda_2}^{-\gamma} + \lambda_2 u_{\lambda_2}\\ &= \Delta u_{\lambda_1} - f(u_{\lambda_1}) + u_{\lambda_1}^{-\gamma} + \lambda_1 u_{\lambda_1}\\ &< \Delta u_{\lambda_1} - f(u_{\lambda_1}) + u_{\lambda_1}^{-\gamma} + \lambda_2 u_{\lambda_1}\end{aligned}$$

for $x\in\Omega$ and $u_{\lambda_1}(x) = u_{\lambda_2}(x)$ on $\partial\Omega$. Therefore, similar to the proof of Theorem 2.4 (ii),

$$u_{\lambda_1}(x) \leq u_{\lambda_2}(x), \quad x\in\bar\Omega.$$

Moreover, by the maximum principle,

$$u_{\lambda_1}(x) < u_{\lambda_2}(x), \quad x\in\Omega.$$

So u_λ is increasing with respect to λ.

(ii) (Bounds for the solution) Fix $\lambda>0$, let u_λ be the unique solution of (1.1). There exists a unique nonnegative solution $\xi\in C^{2,\beta}(\bar\Omega)$ of

$$\begin{gathered}-\Delta\xi = 1 \quad \text{in } \Omega,\\ \xi = 0 \quad \text{on } \partial\Omega.\end{gathered}$$

By the weak maximum principle (see [9]), $\xi>0$ in Ω. Put $z(x)=c\xi(x)$. Consider that we can find $\check c>0$ small enough such that

$$\lambda z(x) + (z(x))^{-\gamma} \geq \lambda\check c\|\xi\|_\infty + (\check c\|\xi\|_\infty)^{-\gamma},$$

and $\hat c>0$ small enough such that

$$\hat c[\frac{-f(\hat c\|\xi\|_\infty) + (\hat c\|\xi\|_\infty)^{-\gamma}}{2\hat c} - 1] + \hat c\|\xi\|_\infty[\frac{-f(\hat c\|\xi\|_\infty) + (\hat c\|\xi\|_\infty)^{-\gamma}}{2\hat c\|\xi\|_\infty} + \lambda] \geq 0\,.$$

Select $0<c<\min\{\check c,\hat c\}$. Then

$$\begin{aligned}&\Delta z(x) + \lambda z(x) - f(z(x)) + (z(x))^{-\gamma}\\ &\geq -c + \lambda c\|\xi\|_\infty - f(c\|\xi\|_\infty) + (c\|\xi\|_\infty)^{-\gamma}\end{aligned}$$

$$
\begin{aligned}
&= c[\frac{-f(c\|\xi\|_\infty) + (c\|\xi\|_\infty)^{-\gamma}}{2c} - 1] \\
&\quad + c\|\xi\|_\infty[\frac{-f(c\|\xi\|_\infty) + (c\|\xi\|_\infty)^{-\gamma}}{2c\|\xi\|_\infty} + \lambda] \\
&\geq 0
\end{aligned}
$$

for each $x \in \Omega$. Therefore, $z(x)$ is a sub-solution of (1.1) for $c > 0$ small enough. Since $\xi \in C^{2,\beta}(\bar{\Omega})$, $\xi > 0$ in Ω, and $\xi = 0$ on $\partial\Omega$, by Gilbarg and Trudinger [9, Lemma 3.4],

$$
\frac{\partial \xi}{\partial \nu}(y) < 0, \quad \forall y \in \partial\Omega.
$$

Therefore, there exist a positive constant c_0 such that

$$
\frac{\partial \xi}{\partial \nu}(y) = \lim_{x\in\Omega, x\to y} \frac{\xi(y) - \xi(x)}{|x-y|} \leq -c_0, \quad \forall y \in \partial\Omega.
$$

So for each $y \in \Omega$, there exist $r_y > 0$, such that

$$
\frac{\xi(x)}{|x-y|} \geq \frac{c_0}{2}, \quad \forall x \in B_{r_y}(y) \cap \Omega. \tag{3.1}
$$

Using the compactness of $\partial\Omega$, we can find a finite number k of balls $B_{ry_i}(y_i)$ such that

$$
\partial\Omega \subset \cup_{i=1}^k B_{ry_i}(y_i).
$$

Moreover, assume that for small $d_0 > 0$,

$$
\{x \in \Omega : d(x) < d_0\} \subset \cup_{i=1}^k B_{ry_i}(y_i).
$$

By (3.1) we obtain

$$
\xi(x) \geq \frac{c_0}{2} d(x), \quad \forall x \in \Omega \text{ with } d(x) < d_0.
$$

This fact, combined with $\xi > 0$ in Ω, shows that for some constant $\tilde{c} > 0$,

$$
\xi(x) \geq \tilde{c} d(x), \quad \forall x \in \Omega.
$$

Thus, $z(x) \geq c_1 d(x)$ in Ω for some constant $c_1 > 0$ follows by the definition of z. Since

$$
\Delta u_\lambda - f(u_\lambda) + u_\lambda^{-\gamma} + \lambda u_\lambda = 0 \leq \Delta z - f(z) + z^{-\gamma} + \lambda z,
$$

and $u_\lambda, z > 0$ in Ω, $u_\lambda = z$ on $\partial\Omega$, $\Delta z \in L^1(\Omega)$. It follows that $u_\lambda \geq z$ in $\bar{\Omega}$. Therefore, from the above proof, $c_1 d(x) \leq u_\lambda(x)$ for all $x \in \Omega$, where c_1 is a positive constant.

Next, we prove that $u_\lambda(x) \leq c_2 d(x)$ for some constant $c_2 > 0$. Our method is similar to that by Gui and Lin [10]. Using the smoothness of $\partial\Omega$, we can find $\delta \in (0,1)$ such that for all

$$
x_0 \in \Omega_\delta := \{x \in \Omega; d(x) \leq \delta\},
$$

there exists a $y \in \mathbb{R}^N \setminus \bar{\Omega}$, with $d(y, \partial\Omega) = \delta$, and $d(x_0) = |x_0 - y| - \delta$. Let $K > 1$ be such that $\operatorname{diam}(\Omega) < (K-1)\delta$ and let ω be the unique solution of the Dirichlet problem

$$
\begin{gathered}
-\Delta\omega = \lambda\omega - f(\omega) + \omega^{-\gamma} \quad \text{in } B_K(0) \setminus \overline{B_1}(0), \\
\omega > 0 \quad \text{in } B_K(0) \setminus \overline{B_1}(0), \\
\omega = 0 \quad \text{on } \partial(B_K(0) \setminus \overline{B_1}(0)),
\end{gathered}
$$

where $B_r(0)$ is the open ball in $\mathbb{R}^N$ of radius r and centered at the origin. By uniqueness, ω is radially symmetric. Hence, $\omega(x) = \tilde{\omega}(|x|)$ and

$$\begin{gathered} \tilde{\omega}'' + \frac{N-1}{r}\tilde{\omega}' + \lambda\tilde{\omega} - f(\tilde{\omega}) + \tilde{\omega}^{-\gamma} = 0 \quad \text{for } r \in (1, K), \\ \tilde{\omega} > 0 \quad \text{in } (1, K), \\ \tilde{\omega}(1) = \tilde{\omega}(K) = 0. \end{gathered} \tag{3.2}$$

Integrating in (3.2) yields

$$\begin{aligned} \tilde{\omega}'(t) &= \tilde{\omega}'(a)a^{N-1}t^{1-N} - t^{1-N}\int_a^t r^{N-1}[\lambda\tilde{\omega}(r) - f(\tilde{\omega}(r)) + (\tilde{\omega}(r))^{-\gamma}]dr \\ &= \tilde{\omega}'(b)b^{N-1}t^{1-N} + t^{1-N}\int_t^b r^{N-1}[\lambda\tilde{\omega}(r) - f(\tilde{\omega}(r)) + (\tilde{\omega}(r))^{-\gamma}]dr, \end{aligned}$$

where $1 < a < t < b < K$. Since $-f(\tilde{\omega}) + \tilde{\omega}^{-\gamma} \in L^1(1, K)$, we deduce that both $\tilde{\omega}'(1)$ and $\tilde{\omega}'(K)$ are finite. Then $\tilde{\omega} \in C^2(1, K) \cap C^1[1, K]$ and

$$\omega(x) \le C\min\{K - |x|, |x| - 1\}, \quad \text{for any } x \in B_K(0) \setminus B_1(0). \tag{3.3}$$

Let us fix $x_0 \in \Omega_\delta$. Then we can find $y_0 \in \mathbb{R}^N \setminus \bar{\Omega}$ with

$$d(y_0, \partial\Omega) = \delta, \quad d(x_0) = |x_0 - y| - \delta.$$

Thus, $\Omega \subset B_{K\delta}(y_0) \setminus B_\delta(y_0)$. Define

$$v(x) = c\omega(\frac{x - y_0}{\delta}), x \in \bar{\Omega}.$$

We show that v is a super-solution of (1.1) provided that c is large enough. Indeed, if $c > \max\{1, \delta^2\}$, then

$$\begin{aligned} &\Delta v + \lambda v - f(v) + v^{-\gamma} \\ &\le \frac{c}{\delta^2}(\tilde{\omega}''(r) + \frac{N-1}{r}\tilde{\omega}'(r)) + \lambda(c\tilde{\omega}(r)) - f(c\tilde{\omega}(r)) + (c\tilde{\omega}(r))^{-\gamma}, \end{aligned}$$

where

$$r = \frac{|x - y_0|}{\delta} \in (1, K).$$

Using assumption (A2) we obtain

$$\lambda(c\tilde{\omega}) - f(c\tilde{\omega}) + (c\tilde{\omega})^{-\gamma} \le c(\lambda\tilde{\omega} - f(\tilde{\omega}) + (\tilde{\omega})^{-\gamma}) \quad \text{in } (1, K).$$

The above relation leads us to

$$\begin{aligned} &\Delta v + \lambda v - f(v) + v^{-\gamma} \\ &\le \frac{c}{\delta^2}(\tilde{\omega}'' + \frac{N-1}{r}\tilde{\omega}' + \lambda\tilde{\omega} - f(\tilde{\omega}) + \tilde{\omega}^{-\gamma}) = 0. \end{aligned}$$

Since $\Delta u_\lambda \in L^1(\Omega)$, then $u_\lambda \le v$ in Ω. This combined with (3.3) yields

$$u_\lambda(x_0) \le v(x_0) \le C\min\{K - \frac{|x_0 - y_0|}{\delta}, \frac{|x_0 - y_0|}{\delta} - 1\} \le \frac{C}{\delta}d(x_0).$$

Hence $u_\lambda \le (C \setminus \delta)d(x)$ in Ω_δ and the last inequality follows. $\square$

4. Asymptotic behavior of the solution

In this section, we consider the asymptotic behavior of the positive solution of (1.1) under the assumption that $f(u) = u^{p+1}, p > 0$, which satisfy (A1)–(A3). Thus,

$$\begin{gathered} -\Delta u + u^{p+1} - u^{-\gamma} = \lambda u \quad \text{in } \Omega, \\ u > 0 \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{4.1}$$

Notice that the function $g(u)$ defined by

$$g(u) = u^{-(\gamma+1)} - u^p, \quad u > 0, \tag{4.2}$$

is continuous. Thus $\lim_{u\to+\infty} g(u) = -\infty$. In terms of $g(u)$ problem (4.1) can be written as

$$-\Delta u = (\lambda + g(u))u.$$

In the next two theorems we collect some general features and estimate the positive solutions of (4.1) for λ large.

Theorem 4.1. *The following assertions hold*

(i) $u_\lambda \le c_\lambda$ *for any positive solution* u_λ *of* (4.1)*, where* $c_\lambda > 0$ *is the largest real number such that*

$$\lambda + g(c_\lambda) = 0, \tag{4.3}$$

and $g(u)$ *is the function defined by* (4.2)*. Moreover,*

$$\lim_{\lambda\to\infty} \frac{c_\lambda}{\lambda^{1/p}} = 1. \tag{4.4}$$

(ii) *Given* $\varepsilon > 0$ *arbitrary, there exists* $\lambda(\varepsilon) > \sigma_1^\Omega$ *such that*

$$|\frac{1}{\lambda}\frac{1}{u_\lambda^{\gamma+1}}| \le \varepsilon \tag{4.5}$$

for all $\lambda \ge \lambda(\varepsilon)$ *on any compact subsets of* Ω*.*

Proof. (i) Assume that (4.1) has a positive solution and let $x_0 \in \Omega$ be the point where u_λ reaches its maximum. Obviously, $-\Delta u_\lambda(x_0) \ge 0$, and

$$\lambda + g(u_\lambda(x_0)) \ge 0.$$

Consider that $\lambda + g(u)$ is strictly decreasing for $u > 0$ and $\lambda + g(c_\lambda) = 0$. Then, any positive solution u_λ of (4.1) satisfies $u_\lambda(x_0) \le c_\lambda$, where x_0 is the point at which u_λ reaches its maximum, hence $u_\lambda \le c_\lambda$. As $\lim_{u\to+\infty} g(u) = -\infty$, we have $\lim_{\lambda\to+\infty} c_\lambda = +\infty$. From (4.2) and (4.3) it follows easily that

$$\frac{\lambda^{1/p}}{c_\lambda} = [1 - \frac{1}{c_\lambda^{\gamma+p+1}}]^{1/p},$$

and since $c_\lambda \to \infty$ as $\lambda \to \infty$, one has

$$\lim_{\lambda\to\infty} \frac{1}{c_\lambda^{\gamma+p+1}} = 0 \quad \text{and} \quad \lim_{\lambda\to\infty} \frac{\lambda^{1/p}}{c_\lambda} = 1.$$

Hence (4.4) holds.

(ii) Since $G^{-1}(\lambda-\lambda_1)\phi_1(x) < u_\lambda(x)$ from the proof of Theorem 2.4 (i), it is easy to see that $u_\lambda \to \infty$ as $\lambda \to \infty$ on any compact subsets of Ω. Let $\varepsilon > 0$. There exists $\lambda_1(\varepsilon)$, if $\lambda \geq \lambda_1(\varepsilon)$ such that

$$|\frac{1}{u^{\gamma+p+1}}| \leq \frac{\varepsilon}{3}.$$

As $\lim_{\lambda\to\infty} \frac{c_\lambda^p}{\lambda} = 1$, there exists $\lambda_2(\varepsilon) > 0$, if $\lambda \geq \lambda_2(\varepsilon)$ such that

$$\frac{c_\lambda^p}{\lambda} \leq \frac{3}{2}.$$

Set

$$\lambda(\varepsilon) = \max\{\lambda_1(\varepsilon), \lambda_2(\varepsilon)\}.$$

Let $\lambda \geq \lambda(\varepsilon)$ and assume that (4.1) has a positive solution u_λ. Then

$$|\frac{1}{\lambda}\frac{1}{u_\lambda(x)^{\gamma+1}}| = \Big\{\frac{u_\lambda(x)}{c_\lambda}\Big\}^p \frac{c_\lambda^p}{\lambda}|\frac{1}{u_\lambda(x)^{\gamma+p+1}}| \leq \frac{3}{2}\frac{\varepsilon}{3} = \frac{\varepsilon}{2}$$

because $u_\lambda \leq c_\lambda$. Thus, (4.5) holds. □

Note that $\theta_{\lambda,m}$ is the unique positive solution of the problem

$$\begin{gathered} -\frac{1}{\lambda}\Delta u = mu - u^{p+1} \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{gathered}$$

Theorem 4.2. *For each $\varepsilon > 0$ arbitrary, there exists $\lambda(\varepsilon) > \sigma_1^\Omega$ such that*

$$\lambda^{1/p}\theta_{\lambda,1-\varepsilon} \leq u_\lambda \leq \lambda^{1/p}\theta_{\lambda,1+\varepsilon} \tag{4.6}$$

for all $\lambda \geq \lambda(\varepsilon)$ on any compact subsets of Ω. In particular,

$$\lim_{\lambda\to\infty} \frac{u_\lambda}{\lambda^{1/p}} = 1 \tag{4.7}$$

uniformly on any compact subsets of Ω.

Proof. The charge of variable $u = \lambda^{1/p}v$ transforms (4.1) into

$$\begin{gathered} -\frac{1}{\lambda}\Delta v = v + \frac{1}{\lambda^{(\gamma+p+1)/p}v^\gamma} - v^{p+1} \quad \text{in } \Omega, \\ v > 0 \quad \text{in } \Omega, \\ v = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{4.8}$$

where $p > 0$, $\gamma > 0$ and $\lambda > 0$ is a real parameter.

By Theorem 2.4 (see [6]), to prove this theorem it is sufficient to show that

$$\theta_{\lambda,1-\varepsilon} \leq v_\lambda \leq \theta_{\lambda,1+\varepsilon}$$

for all $\lambda \geq \lambda(\varepsilon)$ on any compact subsets of Ω.

Fixed $\varepsilon > 0$, we first show that there exists $\lambda_1(\varepsilon) > \sigma_1^\Omega$ such that $\theta_{\lambda,1-\varepsilon} \leq v_\lambda$ for all $\lambda \geq \lambda_1(\varepsilon)$ on any compact subsets of Ω. Let v_λ be a positive solution of (4.8). If $\theta_{\lambda,1-\varepsilon} = v_\lambda$ for λ large we have concluded. If $\theta_{\lambda,1-\varepsilon} \neq v_\lambda$, then

$$\frac{v_\lambda^{p+1} - \theta_{\lambda,1-\varepsilon}^{p+1}}{v_\lambda - \theta_{\lambda,1-\varepsilon}} = \theta_{\lambda,1-\varepsilon}^p + Q(x),$$

with $Q(x) > 0$ for some $x \in \Omega$ and hence

$$\sigma_1[-\frac{1}{\lambda}\Delta - 1 + \varepsilon + \frac{v_\lambda^{p+1} - \theta_{\lambda,1-\varepsilon}^{p+1}}{v_\lambda - \theta_{\lambda,1-\varepsilon}}] > \sigma_1[-\frac{1}{\lambda}\Delta - 1 + \varepsilon + \theta_{\lambda,1-\varepsilon}^p].$$

By the definition of $\theta_{\lambda,1-\varepsilon}$, we find that

$$\sigma_1[-\frac{1}{\lambda}\Delta - 1 + \varepsilon + \theta_{\lambda,1-\varepsilon}^p] = 0.$$

Thus,

$$\sigma_1[-\frac{1}{\lambda}\Delta - 1 + \varepsilon + \frac{v_\lambda^{p+1} - \theta_{\lambda,1-\varepsilon}^{p+1}}{v_\lambda - \theta_{\lambda,1-\varepsilon}}] > 0.$$

On the other hand, after some straightforward manipulations it follows from (4.8) that

$$\big[-\frac{1}{\lambda}\Delta - 1 + \varepsilon + \frac{v_\lambda^{p+1} - \theta_{\lambda,1-\varepsilon}^{p+1}}{v_\lambda - \theta_{\lambda,1-\varepsilon}}\big](v_\lambda - \theta_{\lambda,1-\varepsilon}) = \big[\varepsilon + \frac{1}{\lambda}\frac{1}{(\lambda^{1/p}v_\lambda)^{\gamma+1}}\big]v_\lambda. \tag{4.9}$$

Moreover, it follows from (4.5) that there exists $\lambda_1(\varepsilon) > \sigma_1^\Omega$ such that

$$\varepsilon + \frac{1}{\lambda}\frac{1}{(\lambda^{1/p}v_\lambda)^{\gamma+1}} > 0$$

for all $\lambda \ge \lambda_1(\varepsilon)$ on any compact subsets of Ω. Therefore, applying the maximum principle to (4.9) we find that $\theta_{\lambda,1-\varepsilon} \le v_\lambda$ for all $\lambda \ge \lambda_1(\varepsilon)$ and any positive solution v_λ of (4.8) on any compact subsets of Ω (see [13, 20]).

We now prove that there exists $\lambda_2(\varepsilon) > \sigma_1^\Omega$ such that $v_\lambda \le \theta_{\lambda,1+\varepsilon}$ for all $\lambda \ge \lambda_2(\varepsilon)$ on any compact subsets of Ω. Let v_λ be a positive solution of (4.8). If $\theta_{\lambda,1+\varepsilon} = v_\lambda$ for λ large we have concluded. If $\theta_{\lambda,1+\varepsilon} \ne v_\lambda$ for λ, then

$$\frac{\theta_{\lambda,1+\varepsilon}^{p+1} - v_\lambda^{p+1}}{\theta_{\lambda,1+\varepsilon} - v_\lambda} = \theta_{\lambda,1+\varepsilon}^p + \hat{Q}(x),$$

with $\hat{Q}(x) > 0$ for some $x \in \Omega$. Thus, arguing as above we find that

$$\sigma_1\big[-\frac{1}{\lambda}\Delta - 1 - \varepsilon + \frac{\theta_{\lambda,1+\varepsilon}^{p+1} - v_\lambda^{p+1}}{\theta_{\lambda,1+\varepsilon} - v_\lambda}\big] > \sigma_1\big[-\frac{1}{\lambda}\Delta - 1 - \varepsilon + \theta_{\lambda,1+\varepsilon}^p\big] = 0.$$

On the other hand, after some straightforward manipulations it follows from (4.8) that

$$\big[-\frac{1}{\lambda}\Delta - 1 - \varepsilon + \frac{\theta_{\lambda,1+\varepsilon}^{p+1} - v_\lambda^{p+1}}{\theta_{\lambda,1+\varepsilon} - v_\lambda}\big](\theta_{\lambda,1+\varepsilon} - v_\lambda) = \big[\varepsilon - \frac{1}{\lambda}\frac{1}{(\lambda^{1/p}v_\lambda)^{\gamma+1}}\big]v_\lambda. \tag{4.10}$$

Moreover, it follows from (4.5) that there exists $\lambda_2(\varepsilon) > \sigma_1^\Omega$ such that

$$\varepsilon - \frac{1}{\lambda}\frac{1}{(\lambda^{1/p}v_\lambda)^{\gamma+1}} > 0$$

for all $\lambda \ge \lambda_2(\varepsilon)$ on any compact subsets of Ω. Therefore, applying the maximum principle to (4.10) we find that $v_\lambda \le \theta_{\lambda,1+\varepsilon}$ for all $\lambda \ge \lambda_2(\varepsilon)$ and any positive solution v_λ of (4.8) on any compact subsets of Ω (see [13, 20]).

Taking $\lambda(\varepsilon) = \max\{\lambda_1(\varepsilon), \lambda_2(\varepsilon)\}$, the proof of (4.6) is completed. The rest of the proof follows from [6, Theorem 2.1]. $\square$

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References

[1] H. Berestycki; *Le nombre de solutions de certains problémes semi-linéaires elliptiques*, J. Funct. Anal. 40 (1981), 1-29.

[2] M. M. Coclite; *On a singular nonlinear Dirichlet problem, II*, Boll. Unione Mat. Ital. Sez B(7). 5 (1991), 955-975.

[3] M. M. Coclite, G. Palmieri; *On a singular nonlinear Dirichlet problem*, Comm. Partial Differential Equations. 14 (1989), 1315-1327.

[4] M. G. Crandall, P. H. Rabinowitz, L.Tartar; *On a Dirichlet problem with a singular nonlinearity*, Comm. Partial Differential Equations. 2 (1977), 193-222.

[5] M. Del Pino; *A global estimate for the gradient in a singular elliptic boundary value problem*, Proc. Roy. Soc. Edinburgh. Sect A. 122 (1992), 341-352.

[6] J. M. Fraile, J. López-Gómez, J. C. Delis; *On the global structure of the set of positive solutions of some semilinear elliptic boundary value problems*, J. Differential. Equations. 123 (1995), 180-212.

[7] W. Fulks, J. S. Maybee; *A singular non-linear equation*, Osaka Math. J. 12 (1960), 1-19.

[8] S. M. Gomes; *On a singular nonlinear elliptic problem*, SIAM J. Math. Anal. 17 (1986), 1359-1369.

[9] D. Gilbarg, N. S. Trudinger; *Elliptic partial differential equations of second order*, second ed. Springer-Verlag, Berlin(1983).

[10] C. Gui, F. H. Lin; *Regularity of an elliptic problem with a singular nonlinearity*, Proc. Roy. Soc. Edinburgh. Sect A. 123 (1993), 1021-1029.

[11] N. Hirano, C. Saccon, N. Shioji; *Brezis-Nirenberg type theorems and multiplicity of positive solutions for a singular elliptic problem*, J. Differential. Equations. 245 (2008), 1997-2037.

[12] A. C. Lazer, P. J. McKenna; *On a singular nonlinear elliptic boundary-value problem*, Proc. Amer. Math. Soc. 3 (1991), 720-730.

[13] M. H. Protter, H. F. Weinberger; *Maximum principles in differential equations*, Prentice-Hall, Englewood Cliffs, NJ, 1967.

[14] D. H. Sattinger; *Monotone methods in nonlinear elliptic and parabolic boundary value problems*, Indiana University Mathematics Journal, 21 (1972), 979-1000.

[15] J. Shi, M. Yao; *On a singular nonlinear semilinear elliptic problem*, Proc. Roy. Soc. Edinburgh. Sect A. 128 (1998), 1389-1401.

[16] T. Shibata; *Precise spectral asymptotics for nonlinear Sturm-Liouville problems*, J. Differential. Equations. 180 (2002), 374-394.

[17] T. Shibata; *Three-term spectral asymptotics for nonlinear Sturm-Liouville problems*, Nonlinear. Differ. Equ. Appl. 9 (2002), 239-254.

[18] T. Shibata; *Global behavior of the branch of positive solutions for nonlinear Sturm-Liouville problems*, Annali di Matematica Pura ed Applicata. 186 (2007), 525-537.

[19] C. A. Stuart; *Existence and approximation of solutions of non-linear elliptic equations*, Math. Z. 147 (1976), 53-63.

[20] W. Walter; *A theorem on elliptic inequalities with an application to gradient bounds*, Math. Z. 200 (1989), 293-299.

[21] Z. Zhang; *On a Dirichlet problem with a singular nonlinearity*, J. Math. Anal. Appl. 194 (1995), 103-113.

[22] Z. Zhang, J. Yu; *On a singular nonlinear Dirichlet problem with a convection term*, SIAM J. Math. Anal. 32 (2000), 916-927.

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