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NONOSCILLATION CRITERIA AND ENERGY FUNCTIONAL FOR EVEN-ORDER HALF-LINEAR TWO-TERM DIFFERENTIAL EQUATIONS

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Abstract. We investigate oscillatory properties of even-order half-linear differential equations and conditions for negativity of the associated energy functional. First, using the relationship between positivity of the functional and nonoscillation of the investigated equation, we prove Hille-Nehari type nonoscillation criteria which extend criteria known in the linear case. In the second part of the paper, we present conditions which guarantee that the energy functional attains a negative value, i.e., it is unbounded below.

1. Introduction

We consider the even-order half-linear two-term differential equation

$$(-1)^n \big(t^\alpha \Phi(y^{(n)})\big)^{(n)} + c(t)\Phi(y) = 0, \tag{1.1}$$

where $\Phi(y) = |y|^{p-2}y$, $p > 1$, is the odd power function and $\alpha \in \mathbb{R}$. If $p = 2$, then (1.1) reduces to the *linear* even-order Sturm-Liouville differential equation

$$(-1)^n \big(t^\alpha y^{(n)}\big)^{(n)} + c(t)y = 0 \tag{1.2}$$

whose oscillation and spectral theory is relatively deeply developed. We refer to the books [16, 22], the papers [2, 4, 6, 13, 14, 17, 19], and the references given therein.

Equation (1.1) is a particular case of the general even-order half-linear differential equation

$$\sum_{k=0}^{n} (-1)^k \big(r_k(t)\Phi(y^{(k)})\big)^{(k)} = 0 \tag{1.3}$$

which, in the linear case $p = 2$, takes the form

$$\sum_{k=0}^{n} (-1)^k \big(r_k(t)y^{(k)}\big)^{(k)} = 0. \tag{1.4}$$

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The investigation of oscillatory properties of (1.4) is based on the relationship between this equation and its quadratic energy functional

$$\mathcal{F}(y;a,b) = \int_a^b \Big[\sum_{k=0}^{n} r_k(t)\big(y_k(t)\big)^2\Big]\,dt \tag{1.5}$$

and on the fact that using the substitution

$$x = \begin{pmatrix} y \\ y' \\ \vdots \\ y^{(n-1)} \end{pmatrix}, \quad u = \begin{pmatrix} \sum_{k=1}^{n}(-1)^{k-1}\big(r_k y^{(k)}\big)^{(k)} \\ \vdots \\ -\big(r_n y^{(n)}\big)' + r_{n-1}y^{(n-1)} \\ r_n y^{(n)} \end{pmatrix}$$

equation (1.4) can be written as the linear Hamiltonian system

$$x' = Ax + B(t)u, \quad u' = C(t)x - A^T u \tag{1.6}$$

with

$$B(t) = \operatorname{diag}\big\{0,\dots,0,\frac{1}{r_n(t)}\big\}, \quad C(t) = \operatorname{diag}\{r_0(t),\dots,r_{n-1}(t)\},$$
$$A = A_{i,j} = \begin{cases} 1 & j = i+1,\ i = 1\dots,n-1,\\ 0 & \text{elsewhere.}\end{cases}$$

In particular, using the so-called Reid Roundabout Theorem for (1.6) (see [21, Theorem 6.3, p. 284]), it is proved that $\mathcal{F}(y;T,\infty) > 0$ for every $0 \not\equiv y \in W_0^{n,2}[T,\infty)$ (the definition of this space is recalled later) if and only if no nontrivial solution of (1.4) has more than one zero point of multiplicity n in $[T,\infty)$, i.e., there exists no pair of distinct points $t_1, t_2 \in [T,\infty)$ such that

$$y^{(i)}(t_1) = 0 = y^{(i)}(t_2), \quad i = 0,\dots,n-1. \tag{1.7}$$

Following the linear case, equation (1.3) is said to be *nonoscillatory* if there exists $T \in \mathbb{R}$ such that for any nontrivial solution of this equation there is no pair of distinct points in $[T,\infty)$ such that (1.7) holds. Points t_1, t_2 with this property are said to be *conjugate points* relative to (1.3).

Equation (1.3) can be written as a Hamiltonian type system

$$x' = Ax + B(t)\Phi^{-1}(u), \quad u' = C(t)\Phi(x) - A^T u \tag{1.8}$$

with

$$x = \begin{pmatrix} y \\ y' \\ \vdots \\ y^{(n-1)} \end{pmatrix}, \quad u = \begin{pmatrix} \sum_{k=1}^{n}(-1)^{k-1}\big(r_k \Phi(y^{(k)})\big)^{(k)} \\ \vdots \\ -\big(r_n \Phi(y^{(n)})\big)' + r_{n-1}\Phi(y^{(n-1)}) \\ r_n \Phi(y^{(n)}) \end{pmatrix}. \tag{1.9}$$

The functions Φ, Φ^{-1} of a vector argument are defined in a natural way as

$$\Phi(x) = \begin{pmatrix} \Phi(x_1) \\ \Phi(x_2) \\ \vdots \\ \Phi(x_n) \end{pmatrix}, \quad \Phi^{-1}(u) = \begin{pmatrix} \Phi^{-1}(u_1) \\ \Phi^{-1}(u_2) \\ \vdots \\ \Phi^{-1}(u_n) \end{pmatrix}$$

for column vectors $x = (x_i)_{i=1}^n$ and $u = (u_i)_{i=1}^n$, where the scalar function $\Phi^{-1}(y) = |y|^{q-2}y$ is the inverse function of Φ, i.e., q can be expressed as $q = \frac{p}{p-1}$. The number q is called the *conjugate exponent* of p and satisfies the equality $\frac{1}{p} + \frac{1}{q} = 1$.

However, a Roundabout type theorem for (1.8) is missing, so the theory of (1.8) and (1.3) is much less developed than in the linear case. Concerning oscillatory properties of (1.1) and (1.3), as far as we know, only the papers [10, 20] and the book [9, Sec. 9.4] deal with this problem.

This article consists essentially of two parts. The first one can be regarded as a continuation of [10]. In our paper we prove Hille-Nehari nonoscillation criteria for (1.1) which extend previously proved (in [4, 8]) nonoscillation criteria for (1.2). The second one is devoted to the investigation of conditions which imply that the p-degree energy functional associated with (1.1) attains a negative value.

2. Preliminary results

In our investigation, an important role is played by the test functions from certain Sobolev spaces which are defined as follows. We denote

$$\begin{aligned} W_0^{n,p}[T,\infty) = \Big\{ y\colon [T,\infty) \to \mathbb{R} : y^{(n-1)} \in \mathcal{AC}[T,\infty);\ y^{(n)} \in \mathcal{L}^p(T,\infty);\\ \text{there exists } T_1 > T \text{ such that } y(t) = 0 \text{ for } t \geq T_1\\ \text{and } y^{(i)}(T) = 0 \text{ for } i = 0,\dots,n-1 \Big\} \end{aligned}$$

and

$$\begin{aligned} W_0^{n,p}(\mathbb{R}) = \Big\{ y\colon \mathbb{R} \to \mathbb{R} : y^{(n-1)} \in \mathcal{AC}(\mathbb{R});\ y^{(n)} \in \mathcal{L}^p(\mathbb{R});\\ \text{and there exists } T_1 \in \mathbb{R} \text{ such that } y(t) = 0 \text{ for } |t| \geq T_1 \Big\}. \end{aligned}$$

We use the following variational lemma which is proved e.g. in [9, Sec. 9.4].

Lemma 2.1. *Suppose that there exists $T \in \mathbb{R}$ such that*

$$\mathcal{F}(y;T,\infty) = \int_T^\infty \Big[\sum_{k=0}^n r_k(t) |y^{(k)}|^p \Big]\, dt > 0 \tag{2.1}$$

for every nontrivial $y \in W_0^{n,p}[T,\infty)$. Then equation (1.3) *is nonoscillatory, i.e., no solution of* (1.3) *has more than one zero point of multiplicity n in $[T,\infty)$.*

Another principal tool we use is the Wirtinger type inequality which we will apply in the following form, see [9, Lemma 2.1.1].

Lemma 2.2. *Let M be a positive continuously differentiable function for which $M'(t) \neq 0$ in $[T,\infty)$ and let $y \in W_0^{1,p}[T,\infty)$. Then*

$$\int_T^\infty |M'(t)||y|^p\, dt \leq p^p \int_T^\infty \frac{M^p(t)}{|M'(t)|^{p-1}} |y'|^p\, dt. \tag{2.2}$$

If we take $(-\infty,\infty)$ instead of $[T,\infty)$ and $W_0^{1,p}(\mathbb{R})$ instead of $W_0^{1,p}[T,\infty)$ in Lemma 2.2, then the corresponding statement also holds.

The previous inequality, with $M^p(t)/|M'(t)|^{p-1} = t^\alpha$ and $\alpha \neq p-1$, applied to $y \in W_0^{1,p}[T,\infty)$, reduces to the inequality

$$\int_T^\infty t^\alpha |y'|^p\, dt \geq \gamma_{p,\alpha} \int_T^\infty t^{\alpha-p} |y|^p\, dt, \quad \gamma_{p,\alpha} = \Big(\frac{|p-1-\alpha|}{p} \Big)^p. \tag{2.3}$$

If $\alpha = p-1$, then we have the inequality

$$\int_T^\infty t^{p-1}|y'|^p\,dt \ge \gamma_p \int_T^\infty \frac{|y|^p}{t\log^p t}\,dt, \quad \gamma_p = \gamma_{p,0} = \Big(\frac{p-1}{p}\Big)^p. \tag{2.4}$$

We will also use the following auxiliary inequality.

Lemma 2.3. *Let $\beta \in \mathbb{R}$ and $y \in W_0^{1,p}[T,\infty)$, then*

$$\int_T^\infty \frac{|y|^p}{t^{p\beta+1}\log^p t}\,dt \le \frac{1}{\gamma_p}\int_T^\infty t^{p-1}\big|\big(\frac{y}{t^\beta}\big)'\big|^p\,dt. \tag{2.5}$$

Proof. For $y \in W_0^{1,p}[T,\infty)$, we denote $z = y/t^\beta$ and by using integration by parts and the Hölder inequality we have

$$\begin{aligned}
\int_T^\infty \frac{|y|^p}{t^{p\beta+1}\log^p t}\,dt &= \int_T^\infty \frac{|z|^p}{t\log^p t}\,dt \\
&= \frac{1}{1-p}\cdot\frac{|z|^p}{\log^{p-1} t}\Big|_T^\infty - \frac{p}{1-p}\int_T^\infty \frac{\Phi(z)}{t^{1/q}\log^{p-1} t}\cdot\frac{z'}{t^{-\frac{1}{q}}}\,dt \\
&\le \frac{p}{p-1}\Big(\int_T^\infty \frac{|z|^p}{t\log^p t}\,dt\Big)^{1/q}\Big(\int_T^\infty t^{p-1}|z'|^p\,dt\Big)^{1/p} \\
&\le \gamma_p^{-\frac{1}{q}}\big(\frac{p}{p-1}\big)\Big(\int_T^\infty t^{p-1}|z'|^p\,dt\Big)^{1/q}\Big(\int_T^\infty t^{p-1}|z'|^p\,dt\Big)^{1/p} \\
&= \frac{1}{\gamma_p}\int_T^\infty t^{p-1}\big|\big(\frac{y}{t^\beta}\big)'\big|^p\,dt,
\end{aligned}$$

where between the third and the forth line of the previous computation inequality (2.4) has been used. $\square$

The proof of the next lemma can be found e.g. in [4].

Lemma 2.4. *Let $m \in \{0,\dots,n-1\}$, then*

$$y^{(n)} = \Big\{\frac{1}{t}\big[t^{m+1}\big(\frac{y}{t^m}\big)'\big]^{(m)}\Big\}^{(n-m-1)}.$$

3. Nonoscillation criteria

In this section we formulate and prove Hille-Nehari type nonoscillation criteria for (1.1). As we have pointed out in [10], an important role in the investigation of oscillatory properties of (1.1) plays the fact whether or not $\alpha \in \{p-1, 2p-1,\dots,np-1\} =: \mathcal{M}_p$, the case $\alpha \notin \mathcal{M}_p$ being easier than the other one. The next theorem deals with the case $\alpha \in \mathcal{M}_p$.

Theorem 3.1. *Suppose that $\alpha = jp-1$ for some $j \in \{1,\dots,n\}$ and*

$$\liminf_{t\to\infty} \log^{p-1} t \int_t^\infty c_-(s)s^{p(n-j)}\,ds > K \tag{3.1}$$

where $c_-(t) = \min\{0, c(t)\}$ and

$$K = -\frac{1}{p}\big(\frac{p-1}{p}\big)^{p-1}[(j-1)!(n-j)!]^p.$$

Then equation (1.1) is nonoscillatory.

Proof. Denote $k = \frac{np-1-\alpha}{p} = n-j \in \mathbb{N}$ and for $y \in W_0^{n,p}[T,\infty)$ denote $z = \frac{y}{t^k}$. Let T be so large that the limited expression in (3.1) is greater than K for $t \geq T$. Using Lemma 2.3 (to obtain the last line from the previous one), we have

$$\begin{aligned}
\int_T^\infty c(t)|y|^p\,dt &\geq \int_T^\infty c_-(t)|y|^p\,dt = \int_T^\infty c_-(t)t^{pk}\big|\big(\frac{y}{t^k}\big)\big|^p\,dt \\
&= p\int_T^\infty c_-(t)t^{pk}\Big(\int_T^t \Phi(z)z'\,ds\Big)dt \\
&= p\int_T^\infty \Phi(z)z'\frac{1}{\log^{p-1} t}\log^{p-1} t\int_t^\infty c_-(s)s^{pk}\,ds\,dt \\
&> pK\int_T^\infty \Phi(z)z'\frac{1}{\log^{p-1} t}\,dt \geq pK\int_T^\infty \frac{|\Phi(z)|}{t^{1/q}\log^{p-1} t}\cdot t^{1/q}|z'|\,dt \\
&\geq pK\Big(\int_T^\infty \frac{|y|^p}{t^{pk+1}\log^{q(p-1)} t}\,dt\Big)^{1/q}\Big(\int_T^\infty t^{\frac{p}{q}}|z'|^p\,dt\Big)^{1/p} \\
&= pK\Big(\int_T^\infty \frac{|y|^p}{t^{pk+1}\log^{p} t}\,dt\Big)^{1/q}\Big(\int_T^\infty t^{p-1}\big|\big(\frac{y}{t^k}\big)'\big|^p\,dt\Big)^{1/p} \\
&\geq \frac{pK}{\gamma_p^{1/q}}\int_T^\infty t^{p-1}\big|\big(\frac{y}{t^k}\big)'\big|^p\,dt
\end{aligned}$$

for nontrivial $y \in W_0^{n,p}[T,\infty)$. The second line of the previous computation comes from the equality $(|z|^p)' = p\Phi(z)z'$ by integrating over $[T,t]$ and using the definition of z. To obtain the fifth line the Hölder inequality is used together with the equality $|\Phi(z)|^q = |z|^p$.

Next we apply Lemma 2.4 to $\int_T^\infty t^\alpha|y^{(n)}|^p\,dt$. We put $m = k$ in Lemma 2.4, i.e., $n-m-1 = (n-k)-1 = j-1$. Further, denote

$$u(t) = t^{k+1}\Big(\frac{y(t)}{t^k}\Big)', \quad v(t) = \frac{1}{t}\Big[t^{k+1}\Big(\frac{y(t)}{t^k}\Big)'\Big]^{(k)} = \frac{1}{t}\,[u(t)]^{(n-j)}.$$

Then using repeated application of the Wirtinger inequality (2.3) we have

$$\begin{aligned}
\int_T^\infty t^\alpha|y^{(n)}|^p\,dt &= \int_T^\infty t^{jp-1}|v^{(j-1)}|^p\,dt \\
&\geq [(j-1)!]^p\int_T^\infty t^{p-1}|v|^p\,dt \\
&= [(j-1)!]^p\int_T^\infty t^{-1}|u^{(n-j)}|^p\,dt \\
&\geq [(j-1)!(n-j)!]^p\int_T^\infty t^{-1-(n-j)p}|u|^p\,dt \\
&= [(j-1)!(n-j)!]^p\int_T^\infty t^{-1-(n-j)p}t^{(n-j+1)p}\big|\big(\frac{y}{t^k}\big)'\big|^p\,dt \\
&= [(j-1)!(n-j)!]^p\int_T^\infty t^{p-1}\big|\big(\frac{y}{t^k}\big)'\big|^p\,dt
\end{aligned}$$

for $y \in W_0^{n,p}[T,\infty)$. Summarizing the previous computations,

$$\int_T^\infty \big\{t^\alpha|y^{(n)}|^p + c(t)|y|^p\big\}\,dt$$

$$> \Big\{[(j-1)!(n-j)!]^p + \frac{pK}{\gamma_p^{1/q}}\Big\} \int_T^\infty t^{p-1}\big|\big(\frac{y}{t^k}\big)'\big|^p\,dt = 0$$

for nontrivial $y \in W_0^{n,p}[T,\infty)$. This means, by Lemma 2.1, that (1.1) is nonoscillatory. □

The next example illustrates the nonoscillation criterion in Theorem 3.1 and shows that the constant K in (3.1) cannot be improved.

Example 3.2. Consider the equation

$$(-1)^n\big(t^{jp-1}\Phi(y^{(n)})\big)^{(n)} + \frac{\gamma}{t^{(n-j)p+1}\log^2 t}\Phi(y) = 0 \tag{3.2}$$

for some $j \in \{1,\dots,n\}$. Then

$$\log^{p-1} t \int_t^\infty \frac{\gamma s^{(n-j)p}}{s^{(n-j)p+1}\log^p s} = \frac{\gamma}{p-1}.$$

Hence, by Theorem 3.1, equation (3.2) is nonoscillatory if

$$\gamma > -\big(\frac{p-1}{p}\big)^p[(j-1)!(n-j)!]^p.$$

In particular, if $n = 1$ in (3.2), then $j = 1$ and the criterion from Theorem 3.1 complies with the known result that the second order equation

$$-\big(t^{p-1}\Phi(y')\big)' + \frac{\gamma}{t\log^p t}\Phi(y) = 0$$

is nonoscillatory if and only if $\gamma \ge -\big(\frac{p-1}{p}\big)^p$. Note also that we cannot apply Theorem 3.1 if the limit in (3.1) equals the constant K as shows the example of the second order Riemann-Weber type equation

$$-\big(t^{p-1}\Phi(y')\big)' + \Big[-\big(\frac{p-1}{p}\big)^p\frac{1}{t\log^p t} + \frac{\mu}{t\log^p t\log^2(\log t)}\Big]\Phi(y) = 0$$

which is nonoscillatory if $\mu \ge -\frac{1}{2}\big(\frac{p-1}{p}\big)^{p-1}$ and oscillatory in the opposite case, see [12].

The fundamental role in the proof of the next theorem is played by a nonoscillation criterion for the *second order* half-linear differential equations. To formulate it, consider the pair of second order differential equations

$$-\big(r(t)\Phi(x')\big)' + c(t)\Phi(x) = 0 \tag{3.3}$$

and its perturbation

$$-\big(r(t)\Phi(x')\big)' + [c(t)+d(t)]\Phi(x) = 0, \tag{3.4}$$

where r, c, d are continuous functions with $r(t) > 0$. The following nonoscillation criterion is proved in [7, Theorem 3].

Proposition 3.3. *Suppose that* (3.3) *is nonoscillatory and possesses a positive solution h satisfying*

(i) $h'(t) \neq 0$ *for large t;*

(ii)
$$\int^\infty \frac{dt}{r(t)h^2(t)|h'(t)|^{p-2}} = \infty;$$

(iii) *There exists a finite limit*

$$\lim_{t\to\infty} r(t)h(t)\Phi(h'(t)) =: L \neq 0.$$

Moreover, suppose that the integral $\int^\infty d(t)h^p(t)\,dt$ *is convergent. Then equation* (3.4) *is nonoscillatory provided*

$$\liminf_{t\to\infty} G(t)\int_t^\infty d(s)h^p(s)\,ds > -\frac{1}{2q}, \tag{3.5}$$

$$\limsup_{t\to\infty} G(t)\int_t^\infty d(s)h^p(s)\,ds < \frac{3}{2q}, \tag{3.6}$$

where $G(t) = \int^t r^{-1}(s)h^{-2}(s)|h'(s)|^{2-p}\,ds$ *and* q *is the conjugate exponent of* p.

Note that the previous proposition is proved in [7] under the assumption $h'(t) > 0$, but a straightforward modification of the proof shows that it extends also to the case when $h'(t) < 0$ for large t.

The energy functional on an interval $[T,\infty)$ associated with (3.4) is

$$\int_T^\infty \left[r(t)|y'|^p + (c(t)+d(t))|y|^p\right]\,dt$$

and this functional is positive for every $0 \not\equiv y \in W_0^{1,p}[T,\infty)$ if and only if (3.4) is nonoscillatory and T is sufficiently large, see [9].

In the next theorem we use the notation

$$\gamma_{n,p,\alpha} := \prod_{j=1}^n \Big(\frac{|jp-1-\alpha|}{p}\Big)^p$$

and we investigate (1.1) as a perturbation of the Euler type half-linear differential equation

$$(-1)^n\left(t^\alpha \Phi(y^{(n)})\right)^{(n)} - \frac{\gamma_{n,p,\alpha}}{t^{np-\alpha}}\Phi(y) = 0.$$

Theorem 3.4. *Suppose that* $\alpha \notin \{p-1, 2p-1, \dots, np-1\}$ *and the integral*

$$\int^\infty \Big(c(t) + \frac{\gamma_{n,p,\alpha}}{t^{\alpha-np}}\Big)\, t^{np-1-\alpha}\,dt$$

is convergent. Equation (1.1) *is nonoscillatory provided*

$$\liminf_{t\to\infty} \log t \int_t^\infty \Big[c(s) + \frac{\gamma_{n,p,\alpha}}{s^{np-\alpha}}\Big]\, s^{np-1-\alpha}\,ds > -\frac{p(p-1)}{2(np-1-\alpha)^2}\gamma_{n,p,\alpha}, \tag{3.7}$$

$$\limsup_{t\to\infty} \log t \int_t^\infty \Big[c(s) + \frac{\gamma_{n,p,\alpha}}{s^{np-\alpha}}\Big]\, s^{np-1-\alpha}\,ds < \frac{3p(p-1)}{2(np-1-\alpha)^2}\gamma_{n,p,\alpha}. \tag{3.8}$$

Proof. Denote $d_0(t) := (c(t) + \gamma_{n,p,\alpha}t^{\alpha-np})$. The energy functional on $[T,\infty)$ associated with (1.1) is

$$\begin{aligned}
\mathcal{F}(y) &= \int_T^\infty \left[t^\alpha |y^{(n)}|^p + c(t)|y|^p\right]dt \\
&= \int_T^\infty \left(t^\alpha |y^{(n)}|^p - \gamma_{n,p,\alpha}t^{\alpha-np}|y|^p\right)dt + \int_T^\infty d_0(t)|y|^p\,dt.
\end{aligned}$$

The first term in the first integral on the previous line can be estimated using the Wirtinger inequality as follows

$$\int_T^\infty t^\alpha |y^{(n)}|^p \, dt \ge \gamma_{n-1,p,\alpha} \int_T^\infty t^{\alpha-(n-1)p} |y'|^p \, dt.$$

Using this inequality,

$$\begin{aligned}
\mathcal{F}(y) &= \gamma_{n-1,p,\alpha} \Big\{ \int_T^\infty \Big[\frac{t^\alpha}{\gamma_{n-1,p,\alpha}} |y^{(n)}|^p + \Big(\frac{d_0(t)}{\gamma_{n-1,p,\alpha}} - \frac{\gamma_{n,p,\alpha} t^{\alpha-np}}{\gamma_{n-1,p,\alpha}} \Big) |y|^p \Big] \, dt \Big\} \\
&\ge \gamma_{n-1,p,\alpha} \Big\{ \int_T^\infty \Big[\frac{|y'|^p}{t^{(n-1)p-\alpha}} + \Big(\frac{d_0(t)}{\gamma_{n-1,p,\alpha}} - \Big(\frac{|np-1-\alpha|}{p} \Big)^p \frac{1}{t^{np-\alpha}} \Big) |y|^p \Big] \, dt \Big\}.
\end{aligned}$$

The last integral is the energy functional associated with the second order half-linear differential equation

$$-\Big(t^{\alpha-(n-1)p} \Phi(x') \Big)' + \Big[-\Big(\frac{|np-1-\alpha|}{p} \Big)^p t^{\alpha-np} + \frac{d_0(t)}{\gamma_{n-1,p,\alpha}} \Big] \Phi(x) = 0 \tag{3.9}$$

and this functional is positive for every $0 \not\equiv y \in W_0^{1,p}[T,\infty)$ if and only if (3.9) is nonoscillatory and T is sufficiently large.

Next, we apply Proposition 3.3 to (3.9) with

$$r(t) = t^{\alpha-(n-1)p}, \quad c(t) = -\Big(\frac{|np-1-\alpha|}{p} \Big)^p t^{\alpha-np} \quad \text{and} \quad d(t) = \frac{d_0(t)}{\gamma_{n-1,p,\alpha}}.$$

The equation

$$-\Big(t^{\alpha-(n-1)p} \Phi(x') \Big)' - \Big(\frac{|np-1-\alpha|}{p} \Big)^p t^{\alpha-np} \Phi(x) = 0$$

has a solution $h(t) = t^{(np-1-\alpha)/p}$ (i.e. nonoscillatory) for which $h'(t) \neq 0$ for $t > 0$. By a direct computation we have

$$\begin{gathered}
r(t) h(t) \Phi(h'(t)) = \Phi\Big(\frac{np-1-\alpha}{p} \Big) \neq 0, \\
r(t) h^2(t) |h'(t)|^{p-2} = \Big(\frac{|np-1-\alpha|}{p} \Big)^{p-2} t,
\end{gathered}$$

hence (ii) and (iii) of Proposition 3.3 are satisfied. Moreover,

$$G(t) = \int^t r^{-1}(s) h^{-2}(s) |h'(s)|^{2-p} \, ds = \Big(\frac{p}{|np-1-\alpha|} \Big)^{p-2} \log t.$$

Then (3.5) reads as follows (note that $q = p/(p-1)$)

$$\liminf_{t\to\infty} \Big(\frac{p}{|np-1-\alpha|} \Big)^{2-p} \log t \int_t^\infty \frac{d_0(s)}{\gamma_{n-1,p,\alpha}} s^{np-1-\alpha} \, ds > -\frac{1}{2} \Big(\frac{p-1}{p} \Big)$$

and substituting for $d_0(s)$ we have

$$\begin{aligned}
\liminf_{t\to\infty} \log t \int_t^\infty \Big(c(s) + \frac{\gamma_{n,p,\alpha}}{s^{np-\alpha}} \Big) s^{np-1-\alpha} \, ds &> -\frac{p-1}{2p} \Big(\frac{|np-1-\alpha|}{p} \Big)^{p-2} \gamma_{n-1,p,\alpha} \\
&= -\frac{p(p-1)\gamma_{n,p,\alpha}}{2(np-1-\alpha)^2}.
\end{aligned}$$

Similarly, (3.6) reduces to

$$\limsup_{t\to\infty} \log t \int_t^\infty \Big(c(s) + \frac{\gamma_{n,p,\alpha}}{s^{np-\alpha}} \Big) s^{np-1-\alpha} \, ds < \frac{3p(p-1)\gamma_{n,p,\alpha}}{2(np-1-\alpha)^2}.$$

Hence, if (3.7), (3.8) hold, equation (3.9) is nonoscillatory and the functional

$$\int_T^\infty \Big[\frac{|y'|^p}{t^{(n-1)p-\alpha}} - \Big(\frac{|np-1-\alpha|}{p}\Big)^p \frac{|y|^p}{t^{np-\alpha}} + \frac{d_0(t)}{\gamma_{n-1,p,\alpha}}|y|^p\Big]\,dt > 0$$

if T is sufficiently large what we needed to prove. ☐

Remark 3.5. If $d(t) \le 0$ in (3.4), then, of course, condition (3.6) is redundant. If $d(t) \ge 0$, then (3.4) is a minorant to (3.3) and its nonoscillation follows from the half-linear Sturmian theory, see [9].

Corollary 3.6. *Consider the higher order Riemann-Weber type half-linear differential equation*

$$(-1)^n\big(t^\alpha\Phi(y^{(n)})\big)^{(n)} - \Big[\frac{\gamma_{n,p,\alpha}}{t^{np-\alpha}} + \frac{\mu}{t^{np-\alpha}\log^2 t}\Big]\Phi(y) = 0 \tag{3.10}$$

with $\alpha \notin \mathcal{M}_p$. *Then* (3.10) *is nonoscillatory if*

$$\mu < \frac{p(p-1)\gamma_{n,p,\alpha}}{2(np-1-\alpha)^2}. \tag{3.11}$$

Proof. We denote $c(t) = -[\frac{\gamma_{n,p,\alpha}}{t^{np-\alpha}} + \frac{\mu}{t^{np-\alpha}\log^2 t}]$ and we show that assumptions of Theorem 3.4 are satisfied. We have

$$\int_t^\infty \Big(c(s) + \frac{\gamma_{n,p,\alpha}}{s^{\alpha-np}}\Big)\, s^{np-1-\alpha}\,ds = -\int_t^\infty \frac{\mu}{s\log^2 s}\,ds = -\frac{\mu}{\log t}.$$

Condition (3.8) is obvious (see proof of Theorem 3.4 and Remark 1) and condition (3.7) is reduced to the condition

$$\mu < \frac{p(p-1)\gamma_{n,p,\alpha}}{2(np-1-\alpha)^2}.$$

☐

Example 3.7. Consider the case $n = 1$ in the previous corollary. Then equation (3.10) reduces to the second order Riemann-Weber type equation

$$\big(t^\alpha\Phi(y')\big)' + \Big[\Big(\frac{|p-1-\alpha|}{p}\Big)^p t^{\alpha-p} + \frac{\mu}{t^{p-\alpha}\log^2 t}\Big]\Phi(y) = 0. \tag{3.12}$$

It is known, see [11], that this equation is nonoscillatory if

$$\mu \le \mu_{p,\alpha}, \quad \mu_{p,\alpha} := \frac{p-1}{2p}\Big(\frac{|p-1-\alpha|}{p}\Big)^{p-2}$$

and oscillatory in the opposite case. This result shows that inequality in (3.11) is exact since in the case $n = 1$

$$\frac{p(p-1)\gamma_{n,p,\alpha}}{2(np-1-\alpha)^2} = \frac{p(p-1)}{2(p-1-\alpha)^2}\Big(\frac{|p-1-\alpha|}{p}\Big)^p = \mu_{p,\alpha}.$$

This result also shows that the constant in the right-hand side of inequality (3.7) cannot be improved.

4. Negativity of the energy functional

As a motivation, let us consider the second order half-linear differential equation

$$-\big(r(t)\Phi(y')\big)' + c(t)\Phi(x) = 0 \tag{4.1}$$

with continuous functions c, r and $r(t) > 0$. It was proved in [5] that if

$$\int_{-\infty} r^{1-q}(t)\,dt = \infty = \int^{\infty} r^{1-q}(t)\,dt, \quad \frac{1}{p} + \frac{1}{q} = 1,$$

and

$$\int_{-\infty}^{\infty} c(t)\,dt \le 0, \quad c(t) \not\equiv 0,$$

then (4.1) is *conjugate* on $\mathbb{R}$, i.e., there exists a nontrivial solution with at least two different zeros on $\mathbb{R}$. Conjugacy of (4.1) is equivalent to the existence of a nontrivial function $y \in W_0^{1,p}(\mathbb{R})$ for which the energy functional associated with (4.1)

$$\mathcal{F}(y;\mathbb{R}) = \int_{-\infty}^{\infty} \left[r(t)|y'|^p + c(t)|y|^p\right]\,dt \tag{4.2}$$

attains a negative value. In the terminology of linear equations, see [15], a differential operator with the property that there exists a function from a suitable Sobolev space for which the associated energy functional is negative is called *supercritical.*

Concerning the $2n$-order linear differential equation

$$(-1)^n \left(r(t)y^{(n)}\right)^{(n)} + c(t)y = 0 \tag{4.3}$$

a similar statement was proved first in [18] for a fourth order linear equation and later it was extended to general $2n$-order equation (4.3) in [1]. This result says that if there exists an integer m, $0 \le m \le n-1$, such that

$$\int_{-\infty} t^{2m} r^{-1}(t)dt = \infty = \int^{\infty} t^{2m} r^{-1}(t)dt$$

and there exists a polynomial $Q(t) = a_k t^k + \cdots + a_1 t + a_0$ of the degree $0 \le k \le n-m-1$ such that

$$\int_{-\infty}^{\infty} Q^2(t)c(t)\,dt < 0,$$

then (4.3) is conjugate on $\mathbb{R}$, i.e., there exists a nontrivial solution of (4.3) having two different zeros of multiplicity n in $\mathbb{R}$. Again, this statement is equivalent to the fact that the associated energy functional

$$\int_{-\infty}^{\infty} \left[r(t)|y^{(n)}|^2 + c(t)|y|^2\right]\,dt$$

attains a negative value for some $y \in W_0^{n,p}(\mathbb{R})$. In the next theorem we present a partial extension of these results to (1.1) with $\alpha = 0$.

Theorem 4.1. *Suppose that*

$$\int_{-\infty}^{\infty} c(t)\,dt < 0 \tag{4.4}$$

and $c(t) \le 0$ for t close to $-\infty$ and ∞. Then the energy functional

$$\mathcal{F}_n(y;\mathbb{R}) = \int_{-\infty}^{\infty} \left[|y^{(n)}|^p + c(t)|y|^p\right]\,dt \tag{4.5}$$

associated with the equation

$$(-1)^n\big(\Phi(y^{(n)})\big)^{(n)} + c(t)\Phi(y) = 0 \tag{4.6}$$

attains a negative value over $W_0^{n,p}(\mathbb{R})$.

Proof. According to (4.4), there exist $t_1 < t_2$ such that

$$\int_{t_1}^{t_2} c(t)\,dt =: -\varepsilon < 0 \quad \text{and} \quad c(t) \le 0,\ t \in (-\infty, t_1] \cup [t_2, \infty).$$

Let $t_0 < t_1 < t_2 < t_3$ (the values t_0, t_3 will be specified later) and define the test function as follows

$$y(t) = \begin{cases} 0 & t \in (-\infty, t_0], \\ f(t) & t \in [t_0, t_1], \\ 1 & t \in [t_1, t_2], \\ g(t) & t \in [t_2, t_3], \\ 0 & t \in [t_3, \infty). \end{cases}$$

The function f is defined using the following construction (the construction of g will be specified later). To simplify the notation, we denote $\delta := q - 1$ (q is the conjugate exponent of p). Let

$$y_1(t) = (t-t_0)^n,\ y_2(t) = (t-t_0)^{\delta+n}, \dots, y_n(t) = (t-t_0)^{(n-1)\delta+n}.$$

These functions are solutions of $\Phi(y^{(n)}) = C_k(t-t_0)^k$, $k = 0, \dots, n-1$ for suitable constants C_k (i.e., of $\big(\Phi(y^{(n)})\big)^{(n)} = 0$) for $t \ge t_0$. We define f as a linear combination $f = c_1y_1 + \dots + c_ny_n$ where the constants $c_1, \dots, c_n$ we define in such a way that f satisfies the conditions

$$f(t_1) = 1,\ f^{(i)}(t_1) = 0,\ i = 1, \dots, n-1,$$

(because we need $y \in W_0^{n,p}(\mathbb{R})$). This means that the constants $c_1, \dots, c_n$ form a solution of the linear system (where we denote $T := t_1 - t_0$)

$$\begin{aligned}
1 &= T^n c_1 + T^{\delta+n} c_2 + \dots + T^{(n-1)\delta+n} c_n, \\
0 &= nT^{n-1} c_1 + (\delta+n)T^{\delta+n-1} c_2 + \dots + [(n-1)\delta+n]T^{(n-1)\delta+n-1} c_n, \\
&\dots \\
0 &= \Delta_{i,1}T^{n-i} c_1 + \Delta_{i,2}T^{\delta+n-i} c_2 + \dots + \Delta_{i,n}T^{(n-1)\delta+n-i} c_n, \\
&\dots \\
0 &= n!Tc_1 + \Delta_{n,2}T^{\delta+1} c_2 + \dots + \Delta_{n,n}T^{(n-1)\delta+1} c_n,
\end{aligned}$$

where we have used the notation $\Delta_{i,j} := [(j-1)\delta + n] \dots [(j-1)\delta + n - i + 2]$. The determinant of the matrix of this linear system can be expressed as follows. We factor out $T^{(j-1)\delta}$ from the j-th column and then T^{n-i+1} from the i-th row. Then it remains to calculate the determinant (where we explicitly write the quantities

$\Delta_{i,j}$) $\det \Delta_n$, where

$$\Delta_n := \begin{pmatrix} 1 & 1 & \dots & 1 \\ n & (\delta+n) & \dots & (n-1)\delta+n \\ n(n-1) & (\delta+n)(\delta+n-1) & \dots & [(n-1)\delta+n][(n-1)\delta+n-1] \\ & & \vdots & \\ \prod_{l=1}^{i}(n-l+1) & \prod_{l=1}^{i}(\delta+n-l+1) & \dots & \prod_{l=1}^{i}[(n-1)\delta+n-l+1] \\ & & \vdots & \\ n! & (\delta+n)\cdots(\delta+2) & \dots & [(n-1)\delta+n]\cdots[(n-1)\delta+2] \end{pmatrix}.$$

In the last section we show that this determinant is nonzero, so the determinant of the linear system for c_k is $D := T^{\frac{n(n+1)}{2}+\frac{n(n-1)\delta}{2}} \det \Delta_n \neq 0$. By the Cramer rule we find that the coefficients $c_k = c_k(T)$ can be expressed as

$$c_k(T) = h_k T^{-n-\frac{n(n-1)\delta}{2}} T^{\frac{n(n-1)\delta}{2}-(k-1)\delta} = h_k T^{-n-(k-1)\delta} \tag{4.7}$$

the constants h_k can be expressed explicitly, but their values are not important for our computations at this moment. Consequently,

$$\begin{aligned} f^{(n)}(t) &= c_1(T)y^{(n)}(t) + \dots + c_n(T)y_n^{(n)}(t) = c_1(T)n! + c_2(T)\tilde{h}_2(t-t_0)^{\delta} + \dots \\ &\quad \dots + c_k(T)\tilde{h}_k(t-t_0)^{(k-1)\delta} + \dots + c_n(T)\tilde{h}_n(t-t_0)^{(n-1)\delta}, \end{aligned}$$

where $\tilde{h}_k = [(k-1)\delta+n]\cdots[(k-1)\delta+1]$. Consequently, in view of (4.7) and using the Jensen inequality for the function $x \mapsto |x|^p$, we have

$$\begin{aligned} \int_{t_0}^{t_1} |f^{(n)}(t)|^p\,dt &= \int_{t_0}^{t_1} \Big|\sum_{k=1}^{n} h_k\tilde{h}_k T^{-n-(k-1)\delta}(t-t_0)^{(k-1)\delta}\Big|^p\,dt \\ &\le \int_{t_0}^{t_1} n^{p-1}\sum_{k=1}^{n} \big|h_k\tilde{h}_k T^{-n-(k-1)\delta}T^{(k-1)\delta}\big|^p\,dt \\ &= CT^{-pn+1} \to 0 \quad \text{as } T\to\infty, \text{ i.e., as } t_0 \to -\infty, \end{aligned}$$

where $C = n^{p-1}\sum_{k=1}^{n}\Big|h_k\tilde{h}_k\Big|^p$.

The construction of the function g is similar. It is a function satisfying the boundary condition $g(t_2) = 1$, $g^{(i)}(t_2) = 0$, $i = 1,\dots,n-1$, $g^{(i)}(t_3) = 0$, $i = 0,\dots,n-1$. This function we construct as a linear combination of the functions

$$\tilde{y}_k(t) = (t_3-t)^{(k-1)\delta+n}, \quad k = 1,\dots,n.$$

Similarly as for the function f, we have

$$\int_{t_2}^{t_3} |g^{(n)}(t)|^p\,dt \to 0 \quad \text{as } t_3 \to \infty. \tag{4.8}$$

Summarizing the previous computations, we see that t_0, t_3 can be chosen in such a way that

$$\int_{t_0}^{t_1} |f^{(n)}(t)|^p\,dt < \frac{\varepsilon}{4}, \quad \int_{t_2}^{t_3} |g^{(n)}(t)|^p\,dt < \frac{\varepsilon}{4}.$$

Then we have

$$\int_{t_0}^{t_3}\Big[|y^{(n)}(t)|^p + c(t)|y(t)|^p\Big]\,dt = \int_{t_0}^{t_1}|f^{(n)}(t)|^p\,dt + \int_{t_0}^{t_1} c(t)|f(t)|^p\,dt - \int_{t_1}^{t_2} c(t)\,dt$$

$$\begin{aligned}&+\int_{t_2}^{t_3}|g^{(n)}(t)|^p\,dt+\int_{t_2}^{t_3}c(t)|g(t)|^p\,dt\\&<\frac{\varepsilon}{4}-\varepsilon+\frac{\varepsilon}{4}<0,\end{aligned}$$

where we have used that $c(t)\le 0$ for $t\in(-\infty,t_1]\cup[t_2,\infty)$. □

The formulation of the statement and construction of the test function in the proof of the next theorem is a modification of Theorem 4.1. The meaning of the this theorem from the point of view of the oscillation theory of higher order half-linear differential equations is discussed at the end of this section.

Theorem 4.2. *Suppose that $c(t)\le 0$ for large t. If*

$$\int^{\infty}c(t)\,dt=-\infty, \tag{4.9}$$

then there exists $T\in\mathbb{R}$ such that the energy functional

$$\int_T^{\infty}\big[\big|y^{(n)}\big|^p+c(t)|y|^p\big]\,dt \tag{4.10}$$

associated with equation (4.6) *attains a negative value over $W_0^{n,p}[T,\infty)$.*

Proof. Let $T\in\mathbb{R}$ be arbitrarily large and $T<t_0<t_1<t_2<t_3$. Define the function y essentially in the same way as in the previous proof, only comparing with that proof, the function f may be arbitrary function satisfying at t_0 and t_1 the boundary condition $f^{(i)}(t_0)=0$, $i=0,\dots,n-1$, $f(t_1)=1$, $f^{(i)}(t_1)=0$, $i=1,\dots,n-1$. We denote $K=\int_{t_0}^{t_1}|f^{(n)}(t)|^p\,dt+\int_{t_0}^{t_1}c(t)|f^p(t)|\,dt$. Now, we take t_2 so large that $c(t)\le 0$ for $t\ge t_2$ and

$$\int_{t_1}^{t_2}c(t)\,dt<-3K.$$

The function g is then the same as in the previous proof with t_3 so large that $\int_{t_2}^{t_3}|g^{(n)}(t)|^p\,dt<K$. Then, for the function y constructed in this way, we have

$$\begin{aligned}\int_T^{\infty}\big[|y^{(n)}|^p\,dt+c(t)|y|^p\big]\,dt&=\int_{t_0}^{t_1}|f^{(n)}(t)|^p\,dt+\int_{t_0}^{t_1}c(t)|f(t)|^p\,dt\\&\quad+\int_{t_1}^{t_2}c(t)\,dt+\int_{t_2}^{t_3}|g^{(n)}(t)|^p\,dt+\int_{t_2}^{t_3}c(t)|g(t)|^p\,dt\\&\le K-3K+K<0,\end{aligned}$$

what we needed to prove. □

Remark 4.3. (a) If $p=2$ in Lemma 2.1, i.e., we consider linear equation (1.4) and the associated quadratic functional (1.5), we have *equivalence* in Lemma 2.1. This equivalence is based on the so-called Reid Roundabout theorem for associated linear Hamiltonian differential systems.

An analogue of the Roundabout theorem is missing for half-linear Hamiltonian type system (1.8), so we only have one implication in Lemma 2.1 at this moment. Nevertheless, we conjecture that the equivalence holds also in the half-linear case, this problem is a subject of the present investigation (note that this conjecture is true for second order equations (3.3), see [9, Chap. 2]). Having proved this conjecture, the construction of the test function in the proofs of Theorem 4.1 and

Theorem 4.2 can be used to establish various oscillation criteria for (1.1) similarly as in the linear case in [3, 4, 8, 17, 19].

(b) Since the energy functionals associated with half-linear equations are homogeneous (of degree p), the fact that these functionals attain a negative value also means that they are unbounded below.

5. A technical result

In this section we prove that the determinant of the matrix Δ_n from the previous section is really nonzero, so the constants $c_k(T)$, $k = 1, \dots, n$, can be computed using the Cramer rule. This result may be known for people working in the linear algebra, but we have not found it in the literature, so we present it here.

Recall that we consider the matrix (with $\delta > 0$)

$$\Delta_n := \begin{pmatrix} 1 & 1 & \dots & 1 \\ n & (\delta+n) & \dots & (n-1)\delta+n \\ n(n-1) & (\delta+n)(\delta+n-1) & \dots & [(n-1)\delta+n][(n-1)\delta+n-1] \\ & & \vdots & \\ \prod_{l=1}^{i-1}(n-l+1) & \prod_{l=1}^{i-1}(\delta+n-l+1) & \dots & \prod_{l=1}^{i-1}[(n-1)\delta+n-l+1] \\ & & \vdots & \\ n! & (\delta+n)\cdots(\delta+2) & \dots & [(n-1)\delta+n]\cdots[(n-1)\delta+2] \end{pmatrix}.$$

Lemma 5.1. *Let $\delta > 0$ and $n \in \mathbb{N}$. Then*

$$\det(\Delta_n) = \delta^{\frac{n(n-1)}{2}} \prod_{k=1}^{n} (k-1)!.$$

Proof. Let $n \in \mathbb{N}$ be arbitrary but fixed in the following considerations. Denote $A := \Delta_n$, where $A = (a_{i,j})_{i,j=1}^n$. Hence

$$a_{i,j} = \prod_{l=1}^{i-1} [(j-1)\delta + n - l + 1].$$

Using elementary row operations, we will find a triangular matrix with the determinant equal to that of the original matrix A. For this purpose, we will construct a finite sequence of square matrices $A^{[1]}, \dots, A^{[n]}$ such that $A^{[1]} = (a_{i,j}^{[1]})_{i,j=1}^n = A$ and the matrix $A^{[k]} = (a_{i,j}^{[k]})_{i,j=1}^n$ will be obtained from the matrix $A^{[k-1]}$ by applying $n-1-(k-2)$ elementary row operations for $k = 2, \dots, n$. More precisely, we obtain the matrix $A^{[k]}$ by subtracting a suitable multiple of $(i-1)$-th row of the matrix $A^{[k-1]}$ from the i-th row of the matrix $A^{[k-1]}$, and we repeat this for each $i \geq k$ (for $i < k$, the rows $(a_{i,j}^{[k]})_{j=1,\dots,n}$ will be the same as in matrix $A^{[k-1]}$). As a suitable multiple of $(i-1)$-th row, we consider such multiple, which after subtracting from i-th row gives the zero on the first nonzero position of this i-th row.

Before constructing such a sequence of matrices, note that the first row of the matrix $A^{[1]}$ will be the same as first rows of the matrices $A^{[1]}, \dots, A^{[n]}$, then, in particular, we have

$$a_{1,1}^{[m]} = a_{1,1}^{[1]} = 1$$

for $m = 1, \dots, n$.

Now let us construct the matrix $A^{[2]}$. Let $i \in \{2, \dots, n\}$ be arbitrary, but fixed and let us consider the rows $(a^{[1]}_{i-1,j})_{j=1,\dots,n}$ and $(a^{[1]}_{i,j})_{j=1,\dots,n}$, i.e., the submatrix written as

$$\begin{pmatrix} \prod_{l=1}^{i-2}(n-l+1) & \dots & \prod_{l=1}^{i-2}[(j-1)\delta+n-l+1] & \dots \\ \prod_{l=1}^{i-1}(n-l+1) & \dots & \prod_{l=1}^{i-1}[(j-1)\delta+n-l+1] & \dots \end{pmatrix}.$$

After subtracting the $(i-1)$-th row multiplied by $n-(i-1)+1$ from the i-th row of the matrix $A^{[1]}$, we obtain entries of the i-th row of the matrix $A^{[2]}$, i.e.,

$$\begin{aligned} a^{[2]}_{i,j} &= \prod_{l=1}^{i-1}[(j-1)\delta+n-l+1] - (n-i+2)\prod_{l=1}^{i-2}[(j-1)\delta+n-l+1] \\ &= \prod_{l=1}^{i-2}[(j-1)\delta+n-l+1]\cdot\{(j-1)\delta+n-(i-1)+1-(n-i-2)\} \\ &= (j-1)\delta\prod_{l=1}^{i-2}[(j-1)\delta+n-l+1] \end{aligned}$$

for $j = 1, \dots, n$.

Note that the second row of the matrix $A^{[2]}$ will be again the same as second rows of the matrices $A^{[3]}, \dots, A^{[n]}$, then, in particular, we have

$$a^{[m]}_{2,2} = a^{[2]}_{2,2} = (2-1)\delta\prod_{l=1}^{0}[(2-1)\delta+n-l+1] = \delta$$

for $m = 2, \dots, n$.

We obtain similar relations for the i-th row of the matrix $A^{[2]}$, where $i \in \{3, \dots, n\}$. Then we have

$$a^{[3]}_{i,j} = (j-2)(j-1)\delta^2\prod_{l=1}^{i-3}[(j-1)\delta+n-l+1]$$

for $j = 1, \dots, n$.

Now we describe the general procedure of constructing of the matrix $A^{[k]}$ from the matrix $A^{[k-1]}$. Again, let $i \in \{k, k+1, \dots, n\}$ be arbitrary, but fixed and let us consider rows $(a^{[k-1]}_{i-1,j})_{j=1,\dots,n}$, $(a^{[k-1]}_{i,j})_{j=1,\dots,n}$, which are of the form

$$\begin{pmatrix} 0 & \dots & 0 & (k-2)(k-3)\cdot\dots\cdot 2\delta^{k-2}\prod_{l=1}^{i-k}[(k-2)\delta+n-l+1] & \dots \\ 0 & \dots & 0 & (k-2)(k-3)\cdot\dots\cdot 2\delta^{k-2}\prod_{l=1}^{i-(k-1)}[(k-2)\delta+n-l+1] & \dots \end{pmatrix.$$
$$\begin{matrix} \dots & (j-1)(j-2)\cdot\dots\cdot(j-(k-2))\delta^{k-2}\prod_{l=1}^{i-k}[(j-1)\delta+n-l+1] \\ \dots & (j-1)(j-2)\cdot\dots\cdot(j-(k-2))\delta^{k-2}\prod_{l=1}^{i-(k-1)}[(j-1)\delta+n-l+1] \end{matrix}\Bigg),$$

with the $(k-1)$-th column containing first nonzero coefficients. Therefore, we subtract the $(i-1)$-th row multiplied by $(k-2)\delta+n-(i-(k-1))+1$ from the i-th row of the matrix $A^{[k-1]}$ to obtain entries of the i-th row of the matrix $A^{[k]}$, i.e.

$$\begin{aligned} a^{[k]}_{i,j} &= (j-1)(j-2)\dots(j-(k-2))\delta^{k-2}\prod_{l=1}^{i-(k-1)}[(j-1)\delta+n-l+1] \\ &\quad - \{(k-2)\delta+n-(i-(k-1))+1\}\cdot(j-1)(j-2)\cdot\dots\cdot(j-(k-2)) \end{aligned}$$

$$
\begin{aligned}
&\times \delta^{k-2} \prod_{l=1}^{i-k} [(j-1)\delta + n - l + 1] \\
&= (j-1)(j-2)\dots(j-(k-2))\delta^{k-2} \prod_{l=1}^{i-k} [(j-1)\delta + n - l + 1] \\
&\quad \times \{(j-1)\delta + n - (i-(k-1)) + 1 - ((k-2)\delta + n - (i-(k-1)) + 1)\} \\
&= (j-1)(j-2)\cdot \dots \cdot (j-(k-1))\delta^{k-1} \prod_{l=1}^{i-k} [(j-1)\delta + n - l + 1]
\end{aligned}
$$

for $j = 1, \dots, n$. Such $a_{i,j}^{[k]}$ correspond to the expected form, which had to be proved.

The k-th rows of the matrices $A^{[k]}, A^{[k+1]}, \dots, A^{[n]}$ are equal, and it follows

$$
\begin{aligned}
a_{k,k}^{[m]} = a_{k,k}^{[k]} &= (k-1)(k-2)\cdot \dots \cdot (k-(k-1))\delta^{k-1} \prod_{l=1}^{0} [(k-1)\delta + n - l + 1] \\
&= (k-1)!\delta^{k-1}
\end{aligned}
$$

for $m = k, k+1 \dots, n$.

From this construction, it is clear, that $A^{[n]}$ is an upper triangular matrix and that the diagonal elements of $A^{[n]}$ are

$$
\left(a_{1,1}^{[n]}, a_{2,2}^{[n]}, \dots, a_{n,n}^{[n]}\right) = \left(a_{1,1}^{[1]}, a_{2,2}^{[2]}, \dots, a_{n,n}^{[n]}\right).
$$

Obviously, the determinant of the matrix A has not changed by performed operations, therefore, it holds

$$
\det(A) = \det(A^{[n]}) = \prod_{k=1}^{n} a_{k,k}^{[n]} = \prod_{k=1}^{n} \left[(k-1)!\delta^{k-1}\right] = \delta^{\frac{n(n-1)}{2}} \prod_{k=1}^{n} (k-1)!,
$$

which proves the lemma. $\square$

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