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GLOBAL REGULARITY CRITERIA FOR THE n-DIMENSIONAL BOUSSINESQ EQUATIONS WITH FRACTIONAL DISSIPATION

ZUJIN ZHANG

Abstract. We consider the n-dimensional Boussinesq equations with fractional dissipation, and establish a regularity criterion in terms of the velocity gradient in Besov spaces with negative order.

1. Introduction

In this article, we study the n-dimensional Boussinesq equations with fractional dissipation,

$$\begin{gathered}\partial_t\mathbf{u}+(\mathbf{u}\cdot\nabla)\mathbf{u}+\mathit{\Lambda}^{2\alpha}\mathbf{u}+\nabla\mathit{\Pi}=\vartheta\mathbf{e}_n,\\ \partial_t\vartheta+(\mathbf{u}\cdot\nabla)\vartheta=0,\\ \nabla\cdot\mathbf{u}=0,\\ \mathbf{u}(0)=\mathbf{u}_0,\quad \vartheta(0)=\vartheta_0,\end{gathered}\tag{1.1}$$

where $\mathbf{u}:\mathbb{R}^+\times\mathbb{R}^n\to\mathbb{R}^n$ is the velocity field; $\vartheta:\mathbb{R}^+\times\mathbb{R}^n\to\mathbb{R}$ is a scalar function representing the temperature in the content of thermal convection (see [8]) and the density in the modeling of geophysical fluids (see [9]); $\mathit{\Pi}$ is the the fluid pressure; $\mathbf{e}_n$ is the unit vector in the x_n direction; and $\mathit{\Lambda}:=(-\Delta)^{\frac{1}{2}}$, $\alpha\geq 0$ is a real number.

When $\alpha=1$, Equation (1.1) reduces to the classical Boussinesq equations, which are frequently used in the atmospheric sciences and oceanographic turbulence where rotation and stratification are important (see [8, 9]). If $\vartheta=0$, then (1.1) becomes the generalized Navier-Stokes equation, which was first considered by Lions [7], where he showed the global regularity once $\alpha\geq\frac{1}{2}+\frac{n}{4}$. One may refer the reader to [5, 10] for recent advances. Xiang-Yan [12], Yamazaki [13] and Ye [14] were able to extend Lions's result to system (1.1), where there is no diffusion in the ϑ equation. And it remains an open problem for the global-in-time smooth for (1.1) with $0<\alpha<\frac{1}{2}+\frac{n}{4}$. The purpose of the present paper is to establish a blow-up criterion as follows.

Theorem 1.1. *Let $0<\alpha<\frac{1}{2}+\frac{n}{4}$, $(\mathbf{u}_0,\vartheta_0)\in H^s(\mathbb{R}^n)$ with $s>1+\frac{n}{2}$ and $\nabla\cdot\mathbf{u}_0=0$. Assume that $(\mathbf{u},\vartheta)$ be the smooth local unique solution pair to (1.1) with initial data*

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$(\mathbf{u}_0, \vartheta_0)$. *If additionally,*

$$\nabla \mathbf{u} \in L^{\frac{2\alpha}{2\alpha-\gamma}}(0,T;\dot{B}^{-\gamma}_{\infty,\infty}(\mathbb{R}^n)) \tag{1.2}$$

for some $0<\gamma<2\alpha$, *then the solution* $(\mathbf{u},\vartheta)$ *can be extended smoothly beyond* T.

Here, $\dot{B}^{-\gamma}_{\infty,\infty}(\mathbb{R}^n)$ is the homogeneous Besov space with negative order, which contains classical Lebesgue space $L^{\frac{n}{\gamma}}(\mathbb{R}^n)$, see [1, Chapter 2]. In the proof of Theorem 1.1 in Section 2, we shall frequently use the following refined Gagliardo-Nirenberg inequality.

Lemma 1.2 ([1, Theorem 2.42]). *Let* $2<q<\infty$ *and* γ *be a positive real number. Then a constant* C *exists such that*

$$\|f\|_{L^q} \le C\|f\|^{1-\frac{2}{q}}_{\dot{B}^{-\gamma}_{\infty,\infty}} \|f\|^{2/q}_{\dot{H}^{\gamma(\frac{q}{2}-1)}}. \tag{1.3}$$

Remark 1.3. Our result extends that of Kozono-Shimada [6]. Indeed, the Navier-Stokes equations corresponds to (1.1) with $\vartheta=0$ and $\alpha=1$.

Remark 1.4. In [3] (see also the end-point smallness condition in [2]), Geng-Fan proved a regularity criterion

$$\mathbf{u} \in L^{\frac{2}{1-r}}(0,T;\dot{B}^{-r}_{\infty,\infty}(\mathbb{R}^3)) \quad (-1<r<1,\ r\neq 0) \tag{1.4}$$

for system (1.1) with $\alpha=1$ and $n=3$. Thus our result generalizes (1.4) also, in view of the fact that

$$C_1\|\nabla f\|_{\dot{B}^{-1-r}_{\infty,\infty}} \le \|f\|_{\dot{B}^{-r}_{\infty,\infty}} \le C_2\|\nabla f\|_{\dot{B}^{-1-r}_{\infty,\infty}}.$$

Moreover, our result (1.2) is valid for (1.1) with arbitrarily large n and arbitrarily small α.

Interested readers are referred to [11] for blow-up criterion for (1.1) without diffusion in the $\mathbf{u}$ equation.

2. Proof of Theorem 1.1

It is not difficult to prove that there exists a $T_0>0$ and a unique smooth solution $(\mathbf{u},\vartheta)$ to (1.1) on $[0,T_0]$. We only need to establish the a priori estimates. Therefore, in the following calculations, we assume that the solution $(\mathbf{u},\vartheta)$ is sufficiently smooth on $[0,T]$.

First, taking the inner product of $(1.1)_1$ and $(1.1)_2$ with $\mathbf{u},\vartheta$ in $L^2(\mathbb{R}^n)$ respectively, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|(\mathbf{u},\vartheta)\|^2_{L^2} + \|\Lambda^\alpha \mathbf{u}\|^2_{L^2} = \int_{\mathbb{R}^n} \vartheta \mathbf{e}_n \cdot \mathbf{u}\,\mathrm{d}x \le \frac{1}{2}\|(\mathbf{u},\vartheta)\|^2_{L^2}.$$

Applying Gronwall inequality, we deduce

$$\|(\mathbf{u},\vartheta)\|_{L^\infty(0,t;L^2(\mathbb{R}^n))} + \|\Lambda^\alpha \mathbf{u}\|_{L^2(0,t;L^2(\mathbb{R}^n))} \le C. \tag{2.1}$$

For $k>0$, applying Λ^k to $(1.1)_1$, and testing the resulting equations by $\Lambda^k\mathbf{u}$ respectively, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Lambda^k\mathbf{u}\|_{L^2}^2+\|\Lambda^{k+\alpha}\mathbf{u}\|_{L^2}^2\\
&=-\int_{\mathbb{R}^n}\Lambda^k[(\mathbf{u}\cdot\nabla)\mathbf{u}]\cdot\Lambda^k\mathbf{u}\,\mathrm{d}x+\int_{\mathbb{R}^n}\Lambda^k(\vartheta\mathbf{e}_n)\cdot\Lambda^k\mathbf{u}\,\mathrm{d}x\\
&=-\int_{\mathbb{R}^3}\big\{\Lambda^k[(\mathbf{u}\cdot\nabla)\mathbf{u}]-(\mathbf{u}\cdot\nabla)(\Lambda^k\mathbf{u})\big\}\cdot\Lambda^k\mathbf{u}\,\mathrm{d}x+\int_{\mathbb{R}^n}\Lambda^k(\vartheta\mathbf{e}_n)\cdot\Lambda^k\mathbf{u}\,\mathrm{d}x\\
&\equiv I_1^k+I_2^k.
\end{aligned}\tag{2.2}$$

We may use the following commutator estimates of Kato-Ponce [4]:

$$\|\Lambda^k(fg)-f\Lambda^kg\|_{L^p}\le C\big[\|\nabla f\|_{L^{p_1}}\|\Lambda^{k-1}g\|_{L^{p_2}}+\|\Lambda^kf\|_{L^{p_3}}\|g\|_{L^{p_4}}\big]\tag{2.3}$$

with

$$1<p,p_2,p_3<\infty,\quad 1\le p_1,p_4\le\infty,\quad \frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}=\frac{1}{p_3}+\frac{1}{p_4}$$

to bound I_1^k as

$$\begin{aligned}
I_1^k&\le C\|\Lambda^k[(\mathbf{u}\cdot\nabla)\mathbf{u}]-(\mathbf{u}\cdot\nabla)(\Lambda^k\mathbf{u})\|_{L^{\frac{4(k+\gamma+\alpha-1)}{2k+3\gamma+2\alpha-2}}}\|\Lambda^k\mathbf{u}\|_{L^{\frac{4(k+\gamma+\alpha-1)}{2k+\gamma+2\alpha-2}}}\\
&\le C\|\nabla\mathbf{u}\|_{L^{\frac{2(k+\gamma+\alpha-1)}{\gamma}}}\|\Lambda^k\mathbf{u}\|_{L^{\frac{4(k+\gamma+\alpha-1)}{2k+\gamma+2\alpha-2}}}\cdot\|\Lambda^k\mathbf{u}\|_{L^{\frac{4(k+\gamma+\alpha-1)}{2k+\gamma+2\alpha-2}}}\\
&\le C\|\nabla\mathbf{u}\|_{\dot B^{-\gamma}_{\infty,\infty}}^{\frac{k+\alpha-1}{k+\gamma+\alpha-1}}\|\nabla\mathbf{u}\|_{\dot H^{k+\alpha-1}}^{\frac{\gamma}{k+\gamma+\alpha-1}}\Big(\|\Lambda^k\mathbf{u}\|_{\dot B^{-(k-1+\gamma)}_{\infty,\infty}}^{\frac{\gamma}{2(k+\gamma+\alpha-1)}}\|\Lambda^k\mathbf{u}\|_{\dot H^{\frac{\gamma(k+\gamma-1)}{2k+\gamma+2\alpha-2}}}^{\frac{2k+\gamma+2\alpha-2}{2(k+\gamma+\alpha-1)}}\Big)^2\\
&\le C\|\nabla\mathbf{u}\|_{\dot B^{-\gamma}_{\infty,\infty}}^{\frac{k+\alpha-1}{k+\gamma+\alpha-1}}\|\Lambda^{k+\alpha}\mathbf{u}\|_{L^2}^{\frac{\gamma}{k+\gamma+\alpha-1}}\|\nabla\mathbf{u}\|_{\dot B^{-\gamma}_{\infty,\infty}}^{\frac{\gamma}{k+\gamma+\alpha-1}}\|\Lambda^k\mathbf{u}\|_{\dot H^{\frac{\gamma(k+\gamma-1)}{2k+\gamma+2\alpha-2}}}^{\frac{2k+\gamma+2\alpha-2}{k+\gamma+\alpha-1}}\\
&\le C\|\nabla\mathbf{u}\|_{\dot B^{-\gamma}_{\infty,\infty}}\|\Lambda^{k+\alpha}\mathbf{u}\|_{L^2}^{\frac{\gamma}{k+\gamma+\alpha-1}}\\
&\quad\times\Big(\|\Lambda^k\mathbf{u}\|_{L^2}^{1-\frac{\gamma(k+\gamma-1)}{\alpha(2k+\gamma+2\alpha-2)}}\|\Lambda^{k+\alpha}\mathbf{u}\|_{L^2}^{\frac{\gamma(k+\gamma-1)}{\alpha(2k+\gamma+2\alpha-2)}}\Big)^{\frac{2k+\gamma+2\alpha-2}{k+\gamma+\alpha-1}}\\
&\le C\|\nabla\mathbf{u}\|_{\dot B^{-\gamma}_{\infty,\infty}}\|\Lambda^k\mathbf{u}\|_{L^2}^{\frac{2\alpha-\gamma}{\alpha}}\|\Lambda^{k+\alpha}\mathbf{u}\|_{L^2}^{\frac{\gamma}{\alpha}}\\
&\le C\|\nabla\mathbf{u}\|_{\dot B^{-\gamma}_{\infty,\infty}}^{\frac{2\alpha}{2\alpha-\gamma}}\|\Lambda^k\mathbf{u}\|_{L^2}^2+\frac{1}{2}\|\Lambda^{k+\alpha}\mathbf{u}\|_{L^2}^2.
\end{aligned}\tag{2.4}$$

Substituting (2.4) in (2.2), we find

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\Lambda^k\mathbf{u}\|_{L^2}^2+\|\Lambda^{k+\alpha}\mathbf{u}\|_{L^2}^2\le C\|\nabla\mathbf{u}\|_{\dot B^{-\gamma}_{\infty,\infty}}^{\frac{2\alpha}{2\alpha-\gamma}}\|\Lambda^k\mathbf{u}\|_{L^2}^2+2I_2^k.\tag{2.5}$$

Now, we treat $2I_2^k$ step by step. If $0<k\le\alpha$, then

$$\begin{aligned}
2I_2^k&=2\int_{\mathbb{R}^n}\vartheta\mathbf{e}_n\cdot\Lambda^{2k}\mathbf{u}\,\mathrm{d}x\\
&\le 2\|\vartheta\|_{L^2}\|\Lambda^{2k}\mathbf{u}\|_{L^2}\\
&\le C\|\vartheta\|_{L^2}\big(\|\mathbf{u}\|_{L^2}+\|\Lambda^{k+\alpha}\mathbf{u}\|_{L^2}\big)\quad\big(H^{k+\alpha}(\mathbb{R}^n)\subset\dot H^{2k}(\mathbb{R}^n)\big)\\
&\le C+\frac{1}{2}\|\Lambda^{k+\alpha}\mathbf{u}\|_{L^2}^2\quad(\text{by }(2.1)).
\end{aligned}\tag{2.6}$$

Substituting (2.6) into (2.5), we apply Gronwall inequality to deduce

$$\|\Lambda^k(\mathbf{u},\vartheta)\|_{L^\infty(0,t;L^2(\mathbb{R}^n))}+\|\Lambda^{k+\alpha}\mathbf{u}\|_{L^2(0,t;L^2(\mathbb{R}^n))}\le C\quad(0<k\le\alpha).\tag{2.7}$$

Suppose we have already the statement for some $0 \le l \in \mathbb{N}$,

$$\|\Lambda^k(\mathbf{u},\vartheta)\|_{L^\infty(0,t;L^2(\mathbb{R}^n))}+\|\Lambda^{k+\alpha}\mathbf{u}\|_{L^2(0,t;L^2(\mathbb{R}^n))} \le C \quad (\forall\, l\alpha < k \le (l+1)\alpha), \tag{2.8}$$

we wish to deduce higher-order estimate

$$\|\Lambda^{k+\alpha}(\mathbf{u},\vartheta)\|_{L^\infty(0,t;L^2(\mathbb{R}^n))}+\|\Lambda^{k+2\alpha}\mathbf{u}\|_{L^2(0,t;L^2(\mathbb{R}^n))} \le C. \tag{2.9}$$

Indeed, as long as (2.8) holds, we may dominate $2I_2^{k+\alpha}$ as

$$\begin{aligned} 2I_2^{k+\alpha} &= 2\int_{\mathbb{R}^n} \Lambda^{k+\alpha}(\vartheta\mathbf{e}_n)\cdot\Lambda^{k+\alpha}\mathbf{u}\,\mathrm{d}x \\ &= 2\int_{\mathbb{R}^n} \Lambda^{k}(\vartheta\mathbf{e}_n)\cdot\Lambda^{k+2\alpha}\mathbf{u}\,\mathrm{d}x \\ &\le 2\|\Lambda^k\vartheta\|_{L^2}\|\Lambda^{k+2\alpha}\mathbf{u}\|_{L^2} \\ &\le 2\|\Lambda^k\vartheta\|_{L^2}^2+\frac{1}{2}\|\Lambda^{k+2\alpha}\mathbf{u}\|_{L^2}^2. \end{aligned} \tag{2.10}$$

Putting (2.10) into (2.5) with k replaced by $k+\alpha$, and using (2.8), we deduce (2.9) as desired.

Now prove that (2.7) and (2.8) imply (2.9), we see readily that

$$\|\Lambda^s\mathbf{u}\|_{L^\infty(0,t;L^2(\mathbb{R}^n))}+\|\Lambda^{s+\alpha}\mathbf{u}\|_{L^2(0,t;L^2(\mathbb{R}^n))} \le C. \tag{2.11}$$

With this good estimate of the velocity field, we are now in a position to treat that of ϑ. Applying Λ^s to $(1.1)_2$, and testing the resultant equation by $\Lambda^s\vartheta$, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Lambda^s\vartheta\|_{L^2}^2 \\ &= -\int_{\mathbb{R}^n}\Lambda^s[(\mathbf{u}\cdot\nabla)\vartheta]\cdot\Lambda^s\vartheta\,\mathrm{d}x \\ &= -\int_{\mathbb{R}^n}\{\Lambda^s[(\mathbf{u}\cdot\nabla)\vartheta]-(\mathbf{u}\cdot\nabla)\Lambda^s\vartheta\}\cdot\Lambda^s\vartheta\,\mathrm{d}x \\ &\le C\Big(\|\nabla\mathbf{u}\|_{L^\infty}\|\Lambda^s\vartheta\|_{L^2}+\|\nabla\vartheta\|_{L^\infty}\|\Lambda^s\mathbf{u}\|_{L^2}\Big)\|\Lambda^s\vartheta\|_{L^2} \quad \text{(by (2.3))} \\ &\le C\Big(\|\mathbf{u}\|_{L^2}+\|\Lambda^s\mathbf{u}\|_{L^2}\Big)\|\Lambda^s\vartheta\|_{L^2}^2+\Big(\|\vartheta\|_{L^2}+\|\Lambda^s\vartheta\|_{L^2}\Big)\|\Lambda^s\mathbf{u}\|_{L^2}\|\Lambda^s\vartheta\|_{L^2} \\ &\quad \text{(by } H^s(\mathbb{R}^n)\subset W^{1,\infty}(\mathbb{R}^n)) \\ &\le C+C\|\Lambda^s\vartheta\|_{L^2}^2 \quad \text{(by (2.1) and (2.11))}. \end{aligned} \tag{2.12}$$

Applying Gronwall inequality, we obtain

$$\|\Lambda^s\vartheta\|_{L^\infty(0,t;L^2(\mathbb{R}^n))} \le C.$$

With this estimate and (2.11), we complete the proof.

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Zujin Zhang

School of Mathematics and Computer Sciences, Gannan Normal University, Ganzhou 341000, Jiangxi, China

E-mail address: zhangzujin361@163.com, phone (86) 07978393663