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POSITIVE ALMOST PERIODIC SOLUTIONS TO INTEGRAL EQUATIONS WITH SUPERLINEAR PERTURBATIONS VIA A NEW FIXED POINT THEOREM IN CONES

JING-YUN ZHAO, HUI-SHENG DING, GASTON M. N'GUÉRÉKATA

Abstract. In this article, we establish a new fixed point theorem for nonlinear operators with superlinear perturbations in partially ordered Banach spaces, Then we use the fixed point theorem to prove the existence of positive almost periodic solutions to some integral equations with superlinear perturbations. Also, a concrete example is given to illustrate our results.

1. Introduction

Cooke and Kaplan [6] initiated the study on the nonlinear delay integral equation

$$x(t) = \int_{t-\tau}^{t} f(s, x(s))ds, \tag{1.1}$$

which is a model for the spread of some infectious diseases. Afterwards, Fink and Gatica [12] firstly studied the existence of positive almost periodic solution to equation (1.1). Since the work of Fink and Gatica, there has been of great interest for many mathematicians to study the existence of positive almost periodic type solutions to (1.1). There is a large body of literature on this topic (see, e.g., [1, 2, 3, 4, 5, 8, 10, 13, 14, 16] and references therein).

Among the above references on almost periodic type solutions to (1.1), there are several interesting works on generalized variants of equation (1.1). For example, Ait Dads and Ezzinbi [1] considered the neutral integral equation

$$x(t) = \gamma x(t-\tau) + (1-\gamma)\int_{t-\tau}^{t} f(s, x(s))ds, \tag{1.2}$$

Ait Dads and Ezzinbi [2] studied the infinite delay integral equation

$$x(t) = \int_{-\infty}^{t} a(t-s)f(s, x(s))ds, \tag{1.3}$$

and Ait Dads et al [4] generalized equation (1.3), i.e., they discussed the more general infinite delay integral equation

$$x(t) = \int_{-\infty}^{t} a(t, t-s)f(s, x(s))ds. \tag{1.4}$$

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In fact, (1.1) is also a special case of equation (1.4). This is so because if

$$a(t,s) = \begin{cases} 1 & s \in [0,\tau],\ t \in \mathbb{R}, \\ 0 & s > \tau,\ t \in \mathbb{R}, \end{cases}$$

then (1.4) becomes (1.1).

Motivated by the ideas in [1] and [4], Ding, Chen, and N'Guérékata [8] studied the integral equation

$$x(t) = \alpha(t)x(t-\beta) + \int_{-\infty}^{t} a(t,t-s)f(s,x(s))ds + h(t,x(t)), \tag{1.5}$$

which unifies (1.1)-(1.4). Recently, Bellour and Ait Dads [5] studied the nonlinear integro-differential equation with neutral delay

$$x(t) = \gamma x(t-\sigma(t)) + (1-\gamma)\int_{t-\sigma(t)}^{t} f(s,x(s),x'(s))ds. \tag{1.6}$$

As noted in [4] and [5], the above variants of (1.1) include many important integral and functional equations that arise in biomathematics.

Very recently, the authors of this paper [16] studied the existence of S-asymptotically periodic solutions for the delay integral equation with superlinear perturbations

$$x(t) = \alpha(t)x^n(t-\beta) + \int_{t-\tau(t)}^{t} f(s,x(s))ds, \tag{1.7}$$

where $n \geq 1$. We aim is to make further study on this direction, i.e., we aim to investigate the existence of positive almost periodic solution to the integral equations with superlinear perturbations,

$$x(t) = \alpha(t)x^n(t-\beta) + \int_{-\infty}^{t} a(t,t-s)f(s,x(s))ds + h(t,x(t)), \tag{1.8}$$

where $n \geq 1$. We will use a different method from [16]. In fact, we will first establish a new fixed point theorem for nonlinear operators with superlinear perturbations in partially ordered Banach spaces, and then apply the obtained fixed point theorem to equation (1.8).

2. Preliminaries

Throughout the rest of this paper, we denote by $\mathbb{N}$ the set of positive integers, by $\mathbb{R}$ the set of all real numbers, by $\mathbb{R}^+$ the set of nonnegative real numbers, by X a real Banach space with the norm $\|\cdot\|$, by Ω a subset of X, by $L^1(\mathbb{R}^+)$ the set of all Lebesgue measurable functions $f:\mathbb{R}\to\mathbb{R}^+$ with $\int_{\mathbb{R}}|f(t)|dt < +\infty$ and denote

$$\|f\|_{L^1(\mathbb{R}^+)} = \int_{\mathbb{R}} |f(t)|dt.$$

Next, let us recall some definitions, notation and basic results about almost periodic functions. For more details, we refer the reader to [11, 15].

Definition 2.1. A continuous function $f:\mathbb{R}\to X$ is called almost periodic if for every $\varepsilon > 0$ there exists $l(\varepsilon) > 0$ such that every interval I of length $l(\varepsilon)$ contains a number τ with the property that

$$\sup_{t\in\mathbb{R}} \|f(t+\tau) - f(t)\| < \varepsilon.$$

We denote by $AP(X)$ the set of all such functions.

Definition 2.2. A continuous function $f : \mathbb{R} \times \Omega \to X$ is called almost periodic in t uniformly for $x \in \Omega$ if for every $\varepsilon > 0$ and for every compact subset $K \subset \Omega$ there exists $l(\varepsilon, K) > 0$ such that every interval I of length $l(\varepsilon, K)$ contains a number τ with the property that
$$\sup_{t\in\mathbb{R}, x\in K} \|f(t+\tau, x) - f(t, x)\| < \varepsilon.$$
We denote by $AP(\mathbb{R} \times \Omega, X)$ the set of all such functions.

Lemma 2.3 ([15])**.** *The following assertions hold:*

(a) *$AP(X)$ is a Banach space equipped with the supremum norm.*
(b) *$f, g \in AP(\mathbb{R})$ implies that $f \cdot g \in AP(\mathbb{R})$.*
(c) *$f \in AP(X)$ implies that $f(\cdot - c) \in AP(X)$ for every $c \in \mathbb{R}$.*

Lemma 2.4 ([15])**.** *Let $f \in AP(\mathbb{R} \times \Omega, X)$, $g \in AP(X)$ and $\overline{g(\mathbb{R})} \subset \Omega$. Then $f(\cdot, g(\cdot)) \in AP(X)$.*

Lemma 2.5 ([4])**.** *Let $f \in AP(\mathbb{R})$ and $a : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}^+$ satisfying $t \mapsto a(t, \cdot)$ being in $AP(L^1(\mathbb{R}^+))$. Then, $F \in AP(\mathbb{R})$, where $F(t) = \int_{-\infty}^t a(t, t-s) f(s) ds$ for all $t \in \mathbb{R}$.*

We also need to recall some basic notation about cones (for more details, we refer the reader to [7]). Let X be a real Banach space, and θ be the zero element in X. A closed and convex set K in X is called a cone if the following two conditions are satisfied:

(i) if $x \in K$, then $\lambda x \in K$ for every $\lambda \geq 0$;
(ii) if $x \in K$ and $-x \in K$, then $x = \theta$.

A cone K induces a partial ordering $\leq$ in X by
$$x \leq y \quad \text{if and only if} \quad y - x \in K.$$
For any given $u, v \in K$ with $u \leq v$,
$$[u, v] := \{x \in X | u \leq x \leq v\}.$$
A cone K is called normal if there exists a constant $k > 0$ such that
$$\theta \leq x \leq y \quad \text{implies} \quad \|x\| \leq k\|y\|.$$
We denote by K^o the interior of K. A cone K is called a solid cone if $K^o \neq \emptyset$.

An operator $T : K \to K$ is called increasing if $\theta \leq x \leq y$ implies $Tx \leq Ty$, and is called decreasing if $\theta \leq x \leq y$ implies $Tx \geq Ty$.

3. Main results

The following theorem is a generalization of [8, Theorem 3.1] and [9, Theorem 2.1]. As one will see, although the organization of the proof is more or less similar to that of [8, Theorem 3.1], the proof is more tricky and more delicate.

Theorem 3.1. *Let $n \geq 1$ be a constant, K be a normal solid cone in a real Banach space X, and A be an operator defined on $K \times K \times K$ by*
$$A(x, y, z) = B(x, y, z) + D(x), \quad x, y, z \in K,$$
where $B : K \times K \times K \to K$ and $D : K \to K$. Assume that the following conditions hold:

(H1) *For every* $x, y, z \in K^o$, $B(\cdot, y, z)$ *is increasing in* K^o, $B(x, \cdot, z)$ *is decreasing in* K^o, *and* $B(x, y, \cdot)$ *is decreasing in* K^o. *Moreover,* $D : K^o \to K^o$ *is an increasing operator and* $D(tx) = t^n D(x)$ *for every* $x \in K^o$ *and* $t > 0$.

(H2) *There exist* $x_0, y_0 \in K^o$ *with* $x_0 \le y_0$, $A(x_0, y_0, x_0) \ge x_0$ *and* $A(y_0, x_0, y_0) \le y_0$.

(H3) *For every* $x, y, z \in [t_0 x_0, t_0^{-1} y_0]$, $B(x, y, z) \in K^o$, *where* $t_0 = \sup\{t > 0 : x_0 \ge t y_0\}$.

(H4) *There exists a function* $\varphi : (0, 1) \times K^o \times K^o \to (0, +\infty)$ *such that for every* $t \in (0, 1)$ *and* $x, y, z \in [t_0 x_0, t_0^{-1} y_0]$,
$$B(tx, t^{-1}y, z) \ge \varphi(t, x, y) B(x, y, z) \quad \text{and} \quad \varphi(t, x, y) > \varepsilon_{t_0}(t - t^n) + t,$$
where $\varepsilon_\lambda = \inf\{t > 0 : D(\lambda^{-1} y_0) \le t B(\lambda x_0, \lambda^{-1} y_0, \lambda^{-1} y_0)\}$ *for every* $\lambda \in [t_0, 1]$. *Moreover, it holds*
$$\inf_{x,y \in [x_0, y_0]} \varphi(t, x, y) > \varepsilon_{t_0}(t - t^n) + t, \quad t \in (0, 1). \tag{3.1}$$

(H5) *There exists a constant* $L > 0$ *such that for all* $x, y, z_1, z_2 \in K^o$ *with* $z_1 \ge z_2$,
$$B(x, y, z_1) - B(x, y, z_2) \ge -L(z_1 - z_2). \tag{3.2}$$

Then A *has a unique fixed point* x^* *in* $[x_0, y_0]$, *i.e.,* $A(x^*, x^*, x^*) = x^*$.

Proof. It is easy to see that for every $x, y, z \in K^o$, $A(\cdot, y, z)$ is increasing in K^o, $A(x, \cdot, z)$ is decreasing in K^o, and $A(x, y, \cdot)$ is decreasing in K^o. Note that
$$\varepsilon_\lambda = \inf\{t > 0 : D(\lambda^{-1} y_0) \le t B(\lambda x_0, \lambda^{-1} y_0, \lambda^{-1} y_0)\}, \quad \lambda \in [t_0, 1],$$
it is easy to see that
$$D(\lambda^{-1} y_0) \le \varepsilon_\lambda B(\lambda x_0, \lambda^{-1} y_0, \lambda^{-1} y_0).$$
Then, for every $x, y, z \in [\lambda x_0, \lambda^{-1} y_0]$, we have
$$D(x) \le D(\lambda^{-1} y_0) \le \varepsilon_\lambda B(\lambda x_0, \lambda^{-1} y_0, \lambda^{-1} y_0) \le \varepsilon_\lambda B(x, y, z),$$
Thus, it holds
$$A(x, y, z) \le (1 + \varepsilon_\lambda) B(x, y, z)$$
i.e.,
$$B(x, y, z) \ge \frac{1}{1 + \varepsilon_\lambda} A(x, y, z), \quad \lambda \in [t_0, 1],\ x, y, z \in [\lambda x_0, \lambda^{-1} y_0].$$
In addition, it follows from the definition of ε_λ that ε_λ is decreasing in λ. We divide the remaining proof by three steps.

Step 1. In view of the above observations and (H4), for every $\lambda \in [t_0, 1]$, $x, y, z \in [\lambda x_0, \lambda^{-1} y_0]$ and $t \in (0, 1)$, we have
$$\begin{aligned}
A(tx, t^{-1}y, z) &= B(tx, t^{-1}y, z) + D(tx) \\
&\ge \varphi(t, x, y) B(x, y, z) + t^n D(x) \\
&= tA(x, y, z) + [\varphi(t, x, y) - t] B(x, y, z) + (t^n - t) D(x) \\
&\ge tA(x, y, z) + [\varphi(t, x, y) - t] B(x, y, z) - \varepsilon_\lambda (t - t^n) B(x, y, z) \\
&= tA(x, y, z) + [\varphi(t, x, y) - t - \varepsilon_\lambda (t - t^n)] B(x, y, z) \\
&\ge tA(x, y, z) + \frac{\varphi(t, x, y) - t - \varepsilon_\lambda (t - t^n)}{1 + \varepsilon_\lambda} A(x, y, z) \\
&= \phi_\lambda(t, x, y) A(x, y, z),
\end{aligned} \tag{3.3}$$

where for every $\lambda \in [t_0, 1]$, ϕ_λ is defined by

$$\phi_\lambda(t,x,y) = t + \frac{\varphi(t,x,y) - t - \varepsilon_\lambda(t - t^n)}{1 + \varepsilon_\lambda}, \quad t \in (0,1),\ x,y \in [\lambda x_0, \lambda^{-1} y_0].$$

By (H4), for every $\lambda \in [t_0, 1]$, $x, y \in [\lambda x_0, \lambda^{-1} y_0]$ and $t \in (0,1)$, it holds

$$\varphi(t,x,y) > \varepsilon_{t_0}(t - t^n) + t \ge \varepsilon_\lambda(t - t^n) + t,$$

which means that

$$\phi_\lambda(t,x,y) > t. \tag{3.4}$$

Moreover, by (3.1),

$$\inf_{x,y \in [x_0, y_0]} \phi_1(t,x,y) > t, \quad t \in (0,1). \tag{3.5}$$

Step 2. By using (H5) and a similar proof to [9, Theorem 2.1], one can show that for every $x, y \in [t_0 x_0, t_0^{-1} y_0]$, there exists a unique point in $[t_0 x_0, t_0^{-1} y_0]$, which we denote by $\Psi(x,y)$, such that

$$A(x, y, \Psi(x,y)) = \Psi(x,y).$$

Also, $\Psi(\cdot, y)$ is increasing, and $\Psi(x, \cdot)$ is decreasing. Moreover, for every $\lambda \in [t_0, 1]$ and $x, y \in [\lambda x_0, \lambda^{-1} y_0]$, it holds

$$\Psi(x,y) \in [\lambda x_0, \lambda^{-1} y_0].$$

Then, combining (3.3) with the fact that A is decreasing for the third argument, for every $\lambda \in (t_0, 1]$, it holds

$$\begin{aligned} \Psi(tx, t^{-1}y) &= A(tx, t^{-1}y, \Psi(tx, t^{-1}y)) \\ &\ge A(tx, t^{-1}y, \Psi(x,y)) \\ &\ge \phi_\lambda(t,x,y) A(x, y, \Psi(x,y)) \\ &= \phi_\lambda(t,x,y) \Psi(x,y), \end{aligned} \tag{3.6}$$

for all $t \in [\frac{t_0}{\lambda}, 1)$ and $x, y \in [\lambda x_0, \lambda^{-1} y_0]$. Moreover, denoting $\phi_{t_0}(1,x,y) = 1$, (3.6) holds for $\lambda \in [t_0, 1]$, $t \in [\frac{t_0}{\lambda}, 1]$ and $x, y \in [\lambda x_0, \lambda^{-1} y_0]$.

Step 3. Let $u_0 = x_0$, $v_0 = y_0$ and

$$u_n = \Psi(u_{n-1}, v_{n-1}),\ v_n = \Psi(v_{n-1}, u_{n-1}),\ n \in \mathbb{N}.$$

It follows from Step 2 that

$$u_0 \le u_1 \le \cdots \le u_n \le \cdots \le v_n \le \cdots \le v_1 \le v_0.$$

Let $t_n = \sup\{t > 0 : u_n \ge t v_n\}$, $n \in \mathbb{N}$. Then, $u_n \ge t_n v_n$, $n \in \mathbb{N}$, and

$$0 < t_0 \le t_1 \le \cdots \le t_n \le \cdots \le 1.$$

Let $\xi = \lim_{n\to\infty} t_n$. Next, we prove $\xi = 1$ by contradiction, i.e., we assume that $\xi \in (0,1)$. Noting that $\xi \ge t_0$, $\xi v_n, \frac{u_n}{\xi} \in [\xi x_0, \xi^{-1} y_0]$ and $\frac{t_n}{\xi} \in [\frac{t_0}{\xi}, 1]$, by using (3.6), we obtain

$$\begin{aligned} u_{n+1} &= \Psi(u_n, v_n) \\ &\ge \Psi(t_n v_n, t_n^{-1} u_n) \\ &\ge \Psi\Big(\frac{t_n}{\xi} \cdot \xi v_n, \frac{\xi}{t_n} \cdot \frac{u_n}{\xi}\Big) \end{aligned}$$

$$\begin{aligned}&\geq \phi_\xi\big(\frac{t_n}{\xi}, \xi v_n, \frac{u_n}{\xi}\big)\Psi\big(\xi v_n, \frac{u_n}{\xi}\big)\\ &\geq \frac{t_n}{\xi}\Psi\big(\xi v_n, \frac{u_n}{\xi}\big).\end{aligned}$$

Again by using (3.6), we have

$$\Psi(\xi v_n, \xi^{-1} u_n) \geq \phi_1(\xi, v_n, u_n)\Psi(v_n, u_n) = \phi_1(\xi, v_n, u_n)v_{n+1}.$$

Then, we have $u_{n+1} \geq \frac{t_n}{\xi}\phi_1(\xi, v_n, u_n)v_{n+1}$, and thus

$$t_{n+1} \geq \frac{\phi_1(\xi, v_n, u_n)}{\xi}t_n \geq \frac{\inf_{x,y\in[x_0,y_0]}\phi_1(\xi, x, y)}{\xi}t_n,$$

which contradicts with $\lim_{n\to\infty} t_n = \xi$ since $\frac{\inf_{x,y\in[x_0,y_0]}\phi_1(\xi,x,y)}{\xi} > 1$ by (3.5).

In view of

$$0 \leq u_{n+p} - u_n \leq v_n - u_n \leq v_n - t_n v_n \leq (1 - t_n)v_0, \quad n, p \in \mathbb{N},$$

we conclude that u_n is convergent in X, and we denote $\lim_{n\to\infty} u_n = x^*$. In addition, noting that $u_n \leq x^*$ for all $n \in \mathbb{N}$, we have

$$0 \leq v_n - x^* \leq v_n - u_n \leq (1 - t_n)v_0, \quad n \in \mathbb{N},$$

which means that $\lim_{n\to\infty} v_n = x^*$. Moreover, by the monotonicity of Ψ, it is not difficult to show that $\Psi(x^*, x^*) = x^*$, and x^* is a unique fixed point of Ψ in $[x_0, y_0]$.

Combining Step 2 and Step 3, one obtains

$$x^* = \Psi(x^*, x^*) = A(x^*, x^*, \Psi(x^*, x^*)) = A(x^*, x^*, x^*).$$

Also, x^* is the unique fixed point of A in $[x_0, y_0]$. □

Now, we are ready to establish our existence result for equation (1.8), i.e.,

$$x(t) = \alpha(t)x^n(t-\beta)) + \int_{-\infty}^{t} a(t, t-s)f(s, x(s))ds + h(t, x(t)), \quad t \in \mathbb{R}.$$

For convenience, we denote by $AP(\mathbb{R}\times\mathbb{R}^+, \mathbb{R}^+)$ the set of all nonnegative functions in $AP(\mathbb{R} \times \mathbb{R}^+, \mathbb{R})$.

Theorem 3.2. *Assume that the function f in (1.8) admits the decomposition:*

$$f(t, x) = \sum_{i=1}^{p} f_i(t, x)g_i(t, x), \quad t \in \mathbb{R},\ x \in \mathbb{R}^+$$

for some $p \in \mathbb{N}$. Moreover, the following conditions hold:

(H6) *$\alpha \in AP(\mathbb{R})$ with positive infimum. f_i, g_i, $h \in AP(\mathbb{R} \times \mathbb{R}^+, \mathbb{R}^+)$ ($i = 1, 2, \dots, p$) satisfy that for every $t \in \mathbb{R}$ and $i \in \{1, 2, \dots, p\}$, $f_i(t, \cdot)$ is increasing in $\mathbb{R}^+$, $g_i(t, \cdot)$ is decreasing in $\mathbb{R}^+$, and $h(t, \cdot)$ is decreasing in $\mathbb{R}^+$. In addition, there exists a constant $L > 0$ such that*

$$h(t, z_1) - h(t, z_2) \geq -L(z_1 - z_2), \quad \forall t \in \mathbb{R},\ \forall z_1 \geq z_2 \geq 0.$$

(H7) *a is a function from $\mathbb{R} \times \mathbb{R}^+$ to $\mathbb{R}^+$ and the function $t \mapsto a(t, \cdot)$ is in $AP(L^1(\mathbb{R}^+))$.*

(H8) *There exist two constants $d_0 > c_0 > 0$ such that*

$$\inf_{t\in\mathbb{R}} \int_{-\infty}^{t} a(t,t-s) \sum_{i=1}^{p} f_i(s,c_0) g_i(s,d_0) ds \geq c_0,$$

$$\|\alpha\| d_0^n + \sup_{t\in\mathbb{R}} \Big[\int_{-\infty}^{t} a(t,t-s) \sum_{i=1}^{p} f_i(s,d_0) g_i(s,c_0) ds + h(t,d_0) \Big] \leq d_0.$$

(H9) *There exist $\varphi_i,\ \psi_i : (0,1) \times (0,+\infty) \to (0,1]$ such that*

$$f_i(t,\lambda x) \geq \varphi_i(\lambda,x) f_i(t,x), \quad \text{and} \quad g_i(t,\lambda^{-1} y) \geq \psi_i(\lambda,y) g_i(t,y),$$

for all $x,y > 0$, $\lambda \in (0,1)$, $t \in \mathbb{R}$ and $i \in \{1,2,\dots,p\}$. Moreover,

$$\inf_{x,y\in[\frac{c_0^2}{d_0},\frac{d_0^2}{c_0}]} \varphi_i(\lambda,x)\psi_i(\lambda,y) > \gamma(\lambda - \lambda^n) + \lambda, \quad \lambda \in (0,1),\ i = 1,2,\dots,p,$$

where

$$\gamma = \frac{\|\alpha\| (\frac{d_0^2}{c_0})^n}{\inf_{t\in\mathbb{R}} \big[\int_{-\infty}^{t} a(t,t-s) \sum_{i=1}^{p} f_i\big(s,\frac{c_0^2}{d_0}\big) g_i\big(s,\frac{d_0^2}{c_0}\big) ds + h\big(t,\frac{d_0^2}{c_0}\big) \big]}.$$

Then (1.8) *has an almost periodic solution with positive infinimum.*

Proof. Let $K = \{x \in AP(\mathbb{R}) : \inf_{t\in\mathbb{R}} x(t) \geq 0\}$. It is easy to verify that K is a normal and solid cone in $AP(\mathbb{R})$, and $K^o = \{x \in AP(\mathbb{R}) : \inf_{t\in\mathbb{R}} x(t) > 0\}$. For $x,y,z \in K^o$ and $t \in \mathbb{R}$, define $D(x)(t) = \alpha(t) x^n(t-\beta)$,

$$B(x,y,z)(t) = \int_{-\infty}^{t} a(t,t-s) \sum_{i=1}^{p} f_i(s,x(s)) g_i(s,y(s)) ds + h(t,z(t)),$$

and $A(x,y,z) = B(x,y,z) + D(x)$.

Next, we verify all the assumptions of Theorem 3.1. By (H6), (H7), Lemmas 2.3–2.5, it is not difficult to show that B is an operator from $K \times K \times K$ to K and D is an operator from K to K. Also, it follows directly from (H6) that assumptions (H1) and (H5) hold. In addition, taking $x_0(t) \equiv c_0$ and $y_0(t) \equiv d_0$, assumption (H2) follows from (H8).

Let us verify (H3). It is easy to see $t_0 = \frac{c_0}{d_0}$. For every $x,y,z \in [t_0 x_0, t_0^{-1} y_0]$, we have

$$\begin{aligned}
B(x,y,z)(t) &\geq \int_{-\infty}^{t} a(t,t-s) \sum_{i=1}^{p} f_i(s,t_0 x_0) g_i(s,t_0^{-1} y_0) ds \\
&= \int_{-\infty}^{t} a(t,t-s) \sum_{i=1}^{p} f_i(s,t_0 c_0) g_i(s,t_0^{-1} d_0) ds \\
&\geq \int_{-\infty}^{t} a(t,t-s) \sum_{i=1}^{p} \varphi_i(t_0,c_0)\psi_i(t_0,d_0) f_i(s,c_0) g_i(s,d_0) ds \\
&\geq t_0 \int_{-\infty}^{t} a(t,t-s) \sum_{i=1}^{p} f_i(s,c_0) g_i(s,d_0) ds \\
&\geq t_0 c_0 > 0, \quad t \in \mathbb{R},
\end{aligned}$$

which means that $B(x,y,z) \in K^o$. Thus, assumption (H3) holds.

It remains to show that (H4) holds. For $x, y \in K^o$, we denote
$$(x,y)^- = \min\{\inf_{t\in\mathbb{R}} x(t), \inf_{t\in\mathbb{R}} y(t)\}, \quad (x,y)^+ = \max\{\sup_{t\in\mathbb{R}} x(t), \sup_{t\in\mathbb{R}} y(t)\}.$$
Then, by (H9), for all $x, y > 0$, $\lambda \in (0,1)$, $t \in \mathbb{R}$ and $i \in \{1, 2, \dots, p\}$,
$$f_i(t, \lambda x) \geq \varphi_i(\lambda, x) f_i(t, x), \quad \text{and} \quad g_i(t, \lambda^{-1} y) \geq \psi_i(\lambda, y) g_i(t, y),$$
which yields that for every $t \in \mathbb{R}$, $\lambda \in (0,1)$ and $x, y, z \in K^o$,
$$\begin{aligned} B(\lambda x, \lambda^{-1} y, z)(t) &= \int_{-\infty}^t a(t, t-s) \sum_{i=1}^p f_i(s, \lambda x(s)) g_i(s, \lambda^{-1} y(s)) ds + h(t, z(t)) \\ &\geq \phi(\lambda, x, y) B(x, y, z)(t), \end{aligned}$$
where
$$\phi(\lambda, x, y) := \min_{1\leq i\leq p} \Big(\inf_{u,v\in[(x,y)^-,(x,y)^+]} \varphi_i(\lambda, u)\psi_i(\lambda, v)\Big).$$
By the definition of γ, for every $t \in \mathbb{R}$, we have
$$\begin{aligned} D(t_0^{-1} y_0)(t) &= D\Big(\frac{d_0^2}{c_0}\Big)(t) \\ &\leq \|\alpha\| \Big(\frac{d_0^2}{c_0}\Big)^n \\ &\leq \gamma \inf_{t\in\mathbb{R}} \Big[\int_{-\infty}^t a(t, t-s) \sum_{i=1}^p f_i\Big(s, \frac{c_0^2}{d_0}\Big) g_i\Big(s, \frac{d_0^2}{c_0}\Big) ds + h\Big(t, \frac{d_0^2}{c_0}\Big)\Big] \\ &\leq \gamma B\Big(\frac{c_0^2}{d_0}, \frac{d_0^2}{c_0}, \frac{d_0^2}{c_0}\Big)(t) = \gamma B(t_0 x_0, t_0^{-1} y_0, t_0^{-1} y_0)(t), \end{aligned}$$
which means that $\gamma \geq \varepsilon_{t_0}$. Then, by (H9), for every $\lambda \in (0,1)$ and $x, y \in [t_0 x_0, t_0^{-1} y_0] = [\frac{c_0^2}{d_0}, \frac{d_0^2}{c_0}]$, it holds
$$\begin{aligned} \phi(\lambda, x, y) &= \min_{1\leq i\leq p} \Big(\inf_{u,v\in[(x,y)^-,(x,y)^+]} \varphi_i(\lambda, u)\psi_i(\lambda, v)\Big) \\ &\geq \min_{1\leq i\leq p} \Big(\inf_{u,v\in[\frac{c_0^2}{d_0}, \frac{d_0^2}{c_0}]} \varphi_i(\lambda, u)\psi_i(\lambda, v)\Big) \\ &> \gamma(\lambda - \lambda^n) + \lambda \\ &\geq \varepsilon_{t_0}(\lambda - \lambda^n) + \lambda. \end{aligned}$$
Similarly, one can show that
$$\inf_{x,y\in[x_0,y_0]} \phi(\lambda, x, y) > \varepsilon_{t_0}(\lambda - \lambda^n) + \lambda, \quad \lambda \in (0,1).$$
Thus, (H4) holds.

Now, Theorem 3.1 gives that A has a unique fixed point in $[c_0, d_0]$, and thus (1.8) has an almost periodic solution with positive infinimum. □

Next, we give a simple example to show that our assumptions on (1.8) can be satisfied.

Example 3.3. Let $\alpha(t) \equiv 1/20$, $n = 4/3$, $\beta = 1$, $a(t,s) = \exp(-s^2)$, $p = 1$,
$$f_1(t, x) = (1 + \frac{|\sin t + \sin \pi t|}{30}) \sqrt[3]{x^2 + x}, \quad g_1(t, x) \equiv 1, \quad h(t, x) = \frac{\sin^2 t + \sin^2 \sqrt{2} t}{20(1+x)}.$$

It is easy to see that (H6) and (H7) hold. Let $c_0 = 1$ and $d_0 = 2$. We have

$$\inf_{t\in\mathbb{R}} \int_{-\infty}^{t} a(t,t-s) \sum_{i=1}^{p} f_i(s,c_0) g_i(s,d_0) ds \geq \frac{\sqrt{\pi}}{2} \sqrt[3]{2} \geq 1 = c_0,$$

and

$$\begin{aligned} &\|\alpha\| d_0^n + \sup_{t\in\mathbb{R}} \Big[\int_{-\infty}^{t} a(t,t-s) \sum_{i=1}^{p} f_i(s,d_0) g_i(s,c_0) ds + h(t,d_0) \Big] \\ &\leq \frac{2^{4/3}}{20} + \frac{1 + 16\sqrt{\pi}\sqrt[3]{6}}{30} \leq 2 = d_0, \end{aligned}$$

which means that (H8) holds. Moreover, noting that

$$\begin{aligned} \gamma &= \frac{\|\alpha\| \big(\frac{d_0^2}{c_0}\big)^n}{\inf_{t\in\mathbb{R}} \big[\int_{-\infty}^{t} a(t,t-s) \sum_{i=1}^{p} f_i\big(s, \frac{c_0^2}{d_0}\big) g_i\big(s, \frac{d_0^2}{c_0}\big) ds + h\big(t, \frac{d_0^2}{c_0}\big) \big]} \\ &\leq \frac{\frac{4^{4/3}}{20}}{\frac{\sqrt{\pi}}{2} \sqrt[3]{\frac{1}{4} + \frac{1}{2}}} < \frac{1}{2}, \end{aligned}$$

and for every $\lambda \in (0,1)$, $x > 0$, and $t \in \mathbb{R}$,

$$\frac{f_1(t,\lambda x)}{f_1(t,x)} \geq \lambda^{2/3}, \quad \lambda^{2/3} > (\lambda - \lambda^{4/3}) + \lambda,$$

we conclude that (H9) holds with $\varphi_1(\lambda,x) \equiv \lambda^{2/3}$ and $\psi_1(\lambda,x) \equiv 1$.

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Jing-Yun Zhao
College of Mathematics and Information Science, Jiangxi Normal University, Nanchang, Jiangxi 330022, china
E-mail address: 1475551804@qq.com

Hui-Sheng Ding (corresponding author)
College of Mathematics and Information Science, Jiangxi Normal University, Nanchang, Jiangxi 330022, china
E-mail address: dinghs@mail.ustc.edu.cn

Gaston M. N'Guérékata
Department of Mathematics, Morgan State University, 1700 E. Cold Spring Lane, Baltimore, MD 21251, USA
E-mail address: Gaston.N'Guerekata@morgan.edu, nguerekata@aol.com