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EXISTENCE OF GLOBAL SOLUTIONS FOR REACTION DIFFUSION SYSTEMS MODELING THE ELECTRODEPOSITION OF ALLOYS WITH INITIAL DATA MEASURES

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ABSTRACT. In this work, we are interested in the mathematical model of reaction diffusion systems. The originality of our study is to work with concentrations appearing in reactors together with measure initial data. To validate this model, we prove the existence of global weak solutions. The "j" technique introduced by Pierre and Martin [18] is suitable for this type of solutions. However, its adaptation has some new technical difficulties that we have to overcome.

1. INTRODUCTION

A reaction diffusion system is a set of partial differential equations that can be understood to represent molecules reacting and diffusing over some space. They arise quite naturally in systems consisting of many chemical reactions or interacting components and they are widely used to describe some chemical, physical and biological systems where the principal ingredients of all these models are the second Fick's law and rate equation. In this paper we discuss the existence of global solutions for a reaction diffusion system modeling electrodeposition process with initial data measures.

We start by a simple history about this model problem. Electrodeposition is an attractive method for the fabrication of thin metal films and layered structures [14]. Structures with a wide range of compositions, morphologies, and functionalities can be deposited by varying the large number of experimental parameters available in electrochemical methods. In addition, electrochemistry offers a low-cost alternative to more involved deposition techniques. Also, electrodeposition is an alternative method for fabricating nanoengineered materials.

In general, electropolating is both an art and a science. Although based on several technologies and sciences, including chemistry, physics, chemical and electrical engineering. So the purposes of electropolating for which articles are electropolated are the appearence, protection, special surface properties and engineering or mechanical properties.

To understand this process, we will begin by a brief definition of this model, where we include some previous researches obtained by [3, 10, 12, 13, 17, 19, 20]

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where the discussion in the following will be limited to one group of alloy codeposition.

The classification of alloy codeposition systems developed by Brenner [10], including normal, anomalous, and induced codeposition. In our work, we will be interested in the second group. The next paragraph is devoted to explain why most of researchers have classified the Nickel-Iron electrodeposition as anomalous codeposition.

Fe-Ni deposition is classified as anomalous codeposition [10] because the discharge rate of the more noble compenent Ni inhibited, causing of the less noble metal Fe at a higher deposition rate than nickel. According to Dahms [12] and Dahms and Croll [13], Fe-Ni anomalous codeposition is due to the local pH rise at the interface due to the parallel parasitic hydrogen evolution reaction.

There are also some recent mathematical models that have been proposed for expalining this phenomena, include those by Hessami and Tobias[15], Grande and Talbot [14] and Matlosz [19], the mechanism of single-metal deposition of iron suggested by Bockris and Al [8] and of nickel suggested by Matulis and Slizys [20].

Hessami and Tobias [15] assumed that the electrodeposition of Fe-Ni occurs as a result of the reduction of both the bivalent metal ions, Ni^{2+} and Fe^{2+}, the monohydroxide ions $Fe(OH)^+$ and $\mathrm{Ni(OH)}^+$. According to this model, the dominant mechanism in the electrodeposition process was the reduction of divalent or bivalent ions rather than monohydroxide species. In this work, the ionic species of interest are: H^+, OH^-, Fe^{2+}, Ni^{2+}, FeOH^+ and NiOH^+ where the following homogeneous reactions are considered

$$
\begin{gathered}
\mathrm{H_2O} \rightleftharpoons \mathrm{H}^+ + \mathrm{OH}^- \\
\mathrm{FeOH}^+ \rightleftharpoons \mathrm{Fe}^{2+} + \mathrm{OH}^- \\
\mathrm{NiOH}^+ \rightleftharpoons \mathrm{Ni}^{2+} + \mathrm{OH}^-
\end{gathered}
$$

Then, we have Grande and Talbot [14] who proposed a one dimensionnel diffusion model, where they determined the effect of buffuring and the hydrolysis reactions on predicted surface PH and deposit composition. Their model includes the assumption that anomalous deposition of nickel and iron occurs due to the electrodeposition of their respective monohydroxide species.

On the other hand, we have the study of Alaa et al. [2] who studied the existence of global solutions for a Model of Nickel Iron alloy electrodeposition on rotating disk with quadratic nonlinearities in one dimension space. There Model addresses dissociation, diffusion, electomigration, convection and deposition of multiple ion species, where they have presented a generalization of [3] to ensure the global existence of classical solutions and their positivity, where in [3], the same researchers proved the existence and the positivity of weak solutions for their model problem without no restriction of growth on the nonlinear terms.

In this work, we study the existence of global solutions in more general case. So, instead of studying the problem of electrodeposition of Nickel-Iron alloy, we will consider that our model is composed of NS different species. We are interested in particular to the study of the following reaction diffusion systems of the type

$$\begin{gathered}
\frac{\partial \omega_i}{\partial t} - d_i \Delta \omega_i - m_i \operatorname{div}(\omega_i \nabla \phi) = S_i(\omega) \quad \text{in } Q_T, \\
-\Delta \phi = F(\omega) \quad \text{in } Q_T, \\
-d_i \frac{\partial \omega_i}{\partial \upsilon} - m_i \omega_i \frac{\partial \phi}{\partial \upsilon} = 0 \quad \text{on } \Sigma_T, \\
\phi(t,x) = 0 \quad \text{on } \Sigma_T, \\
\phi(0,x) = \phi_0(x) \quad \text{in } \Omega, \\
\omega_i(0,.) = \mu_i \quad \text{in } \Omega,
\end{gathered} \tag{1.1}$$

Where Ω denotes an open and bounded subset of $\mathbb{R}^N$, with smooth boundary $\partial\Omega$. The normal exterior derivative on $\partial\Omega$ is denoted by ∂_υ and we have $Q_T =]0,T[\times\Omega$, $\Sigma_T =]0,T[\times\partial\Omega$ with T is a nonnegative constant.

The components ω_i's represent the concentrations of NS species considered during the electrodeposition, z_i are the charges and m_i is the electrical mobility, d_i is the diffusion coefficients associated to each one of our species and ϕ designates the electric potential, S_i are the reaction terms or production rate and F is a bounded function in $L^\infty(Q_T)$ which depends on the concentrations $\omega = (\omega_1, \omega_2, \dots, \omega_{NS})$ and also on the fixed charge concentration. We suppose that S_i depends continuously on the ω_i's. We also assume that d_i are nonnegative constants for each $i = 1, \dots, NS$.

The layout of this work is as follows. We begin in the second section by defining notation and essential concepts, after that we consider our problem and we expose the principal result. The third paragraph is devoted to the proof of the principal result by passing through an approximate problem then obtaining the appropriate estimations of ω_i, ϕ and S_i to pass to the limit and prove that the solution of truncated system converges to the solution of our model problem (1.1).

1.1. **Notion of weak solution.** Throughout this paper we make the following assumptions: for all $i = 1, \dots, NS$,

$$\mu_i \in M_b^+(\Omega), \tag{1.2}$$

and

$$\phi_0 \in L^\infty(\Omega). \tag{1.3}$$

There exists $\Theta \in L^\infty(Q_T)$, such that

$$\begin{gathered}
|F(t,x,r)| \le \Theta(t,x) \quad \text{a.e. } (t,x) \in Q_T, \\
\forall r \in [0,+\infty)^{NS}.
\end{gathered} \tag{1.4}$$

The total mass control is preserved on time if for all $r \in [0,+\infty)^{NS}$

$$\sum_{i=1}^{NS} S_i(r) \le K\Big(\sum_{i=1}^{NS} r_i\Big) + N \quad \text{where } K, N \ge 0\,. \tag{1.5}$$

The nonnegativity of solutions is preserved if and only if the quasi-positivity condition is satisfied

$$S_i(r_1, \dots, r_{i-1}, 0, r_{i+1}, \dots, r_{NS}) \ge 0. \tag{1.6}$$

After all, to expect the existence of global solutions in time, more structure must be required on S_i. Additional assumptions usually come from the underlying model.

We assume the existence of a lower triangular invertible matrix $Q = (q_{ij})_{1\le i,j\le NS}$ with nonnegative coefficients, such that

$$\begin{gathered}\exists L, M \in (0,+\infty)^{NS},\ \forall (t,x,r) \in (0,T)\times\Omega\times[0,+\infty)^{NS},\\ QS(t,x,r) \le L\Big(\sum_{1\le i\le NS} r_i\Big) + M\,,\end{gathered} \tag{1.7}$$

where $S(r) = (S_1(r), S_2(r), \dots, S_{NS}(r))$. To finish this paragraph, we recall the following notation and definitions:

$$\begin{gathered}C_0^\infty(Q_T) = \{\varphi : Q_T \to \mathbb{R} \text{ indefinitely derivable with compact support in } Q_T\},\\ C_b(\Omega) = \{\varphi : \Omega \to \mathbb{R} \text{ a continuous and bounded function in } \Omega\},\\ M_b(\Omega) = \{\mu_i \text{ bounded Radon measure in } \Omega\},\\ M_b^+(\Omega) = \{\mu_i \text{ bounded nonnegative Radon measure in } \Omega\}.\end{gathered}$$

Definition 1.1. Let $\omega_i \in C(]0,T[; L^1(\Omega))$ and $\mu_i \in M_b(\Omega)$. We say that $\omega_i(0,.) = \mu_i$ in $M_b(\Omega)$ if for every $\varphi \in C_b(\Omega)$,

$$\lim_{t\to 0}\int_\Omega \omega_i(t,x)\varphi dx = \langle \mu_i, \varphi\rangle,$$

Now, we introduce the notion of weak solution that we will use in this work.

Definition 1.2. A weak solution of problem (1.1), is a couple of functions $(\omega,\phi) = (\omega_1, \omega_2, \dots, \omega_{NS}, \phi)$ such that $\omega \in C(]0,T[; L^1(\Omega)^{NS}) \cap L^1(0,T; W^{1,1}(\Omega)^{NS})$, $\phi \in L^\infty(0,T; W_0^{1,\infty}(\Omega))$ and $S_i(\omega) \in L^1(Q_T)$. For all $1 \le i \le NS$, the couple (ω,ϕ) satisfies

$$\begin{gathered}\frac{\partial \omega_i}{\partial t} - d_i\Delta\omega_i - m_i \operatorname{div}(\omega_i\nabla\phi) = S_i(\omega) \quad \text{in } D'(Q_T), -\Delta\phi = F(\omega) \quad \text{in } D'(Q_T)\\ \phi(0,x) = \phi_0(x) \quad \text{in } \Omega\\ \omega_i(0,.) = \mu_i \quad \text{in } M_b(\Omega)\end{gathered} \tag{1.8}$$

We mention here that if $\mu = (\mu_1, \mu_2, \dots, \mu_{NS})$ belongs to $L^2(\Omega)^{NS}$. Then, we could talk about a strong solution which is defined in the following sense.

Definition 1.3. A strong solution of problem (1.1) is a couple of functions $(\omega,\phi) = (\omega_1, \omega_2, \dots, \omega_{NS}, \phi)$ such that $\mu \in L^2(\Omega)^{NS}$, $\mu_i \ge 0$, $\omega \in C([0,T]; L^2(\Omega)^{NS}) \cap L^2(0,T; H^1(\Omega)^{NS})$, $\phi \in L^\infty(0,T; W_0^{1,\infty}(\Omega))$ and $S_i(\omega) \in L^1(Q_T)$ for all $1 \le i \le NS$ and satisfying:

$$\begin{gathered}\text{for all } v \in C^1(Q_T) \text{ such that } v(T,.) = 0,\\ -\int_{Q_T}\omega_i\frac{\partial v}{\partial t} + d_i\int_{Q_T}\nabla\omega_i\nabla v + m_i\int_{Q_T}\omega_i\nabla\phi\nabla v - \langle\mu_i, v(0,x)\rangle\\ = \int_{Q_T} S_i(\omega)v \quad \text{for all } \theta \in D(\Omega) \text{ and } t \in]0,T[\\ \int_\Omega \nabla\phi\nabla\theta = \int_\Omega F(\omega)\theta\\ \phi(0,x) = \phi_0(x) \quad \text{in } \Omega,\\ \omega_i(0,x) = \mu_i(x) \quad \text{in } \Omega\,.\end{gathered} \tag{1.9}$$

2. Main results

Theorem 2.1. *We assume that* (1.2), (1.3), (1.4), (1.5), (1.6) *and* (1.7) *hold. Then the problem* (1.1) *has a weak solution* (ω, ϕ) *satisfying* $\omega_i \geq 0$ *in* Q_T*, for all* $i = 1, \dots, NS$.

2.1. **Proof of main result.**

Approximating scheme. To approximate our problem, we truncate the initial data $(\mu_i)_{1 \leq i \leq NS}$ as follows

$$\mu_i^n \in C_0^\infty(\Omega) \quad \text{such that } \mu_i^n \geq 0, \quad \|\mu_i^n\|_{L^1(\Omega)} \leq \|\mu_i\|_{M_b(\Omega)},\ \mu_i^n \to \mu_i \tag{2.1}$$

in $M_b(\Omega)$. To each nonlinearity S_i we associate the function $\hat{S}_i$ such that

$$\begin{aligned} \hat{S}_i(r) &= \hat{S}_i(r_1, r_2, \dots, r_{NS}) \\ &= \begin{cases} S_i(r_1, r_2, \dots, r_{NS}) & \text{if } (r_1, r_2, \dots, r_{NS}) \in [0, +\infty)^{NS} \\ S_i(r_1, \dots, r_{j-1}, 0, r_{j+1}, \dots, r_m) & \text{if } r_j \leq 0\,. \end{cases} \end{aligned} \tag{2.2}$$

Also we consider the truncated function $\eta_n \in C_0^\infty(\mathbb{R}^{NS})$, satisfying

$$\begin{gathered} 0 \leq \eta_n \leq 1, \\ \eta_n(r) = \begin{cases} 1 & \text{if } |r| \leq n \\ 0 & \text{if } |r| \geq n+1. \end{cases} \end{gathered}$$

Then for $r \in \mathbb{R}^{NS}$ we define

$$S_i^n(r) = \eta_n(r)\hat{S}_i(r) \quad \text{for all } n \geq 1 \tag{2.3}$$

Proposition 2.2 ([4])**.** *We assume that* (1.1), (1.2), (1.3), (1.4), (1.5) *and* (1.6) *are satisfied. Then for each* n *there exists a strong solution* (ω_n, ϕ_n) *of*

$$\begin{gathered} \frac{\partial \omega_{i,n}}{\partial t} - d_i \Delta\omega_{i,n} - m_i \operatorname{div}(\omega_{i,n} \nabla\phi_n) = S_i^n(\omega_n) \quad \text{in } Q_T, \\ -\Delta\phi_n = F(\omega_n) \quad \text{in } Q_T, \\ -d_i \frac{\partial \omega_{i,n}}{\partial \upsilon} - m_i \omega_{i,n} \frac{\partial \phi_n}{\partial \upsilon} = 0 \quad \text{in } \Sigma_T, \\ \phi_n(t,x) = 0 \quad \text{in } \Sigma_T, \\ \phi_n(0,x) = \phi_0(x) \quad \text{in } \Omega, \\ \omega_{i,n}(0,.) = \mu_i^n \quad \text{in } \Omega. \end{gathered} \tag{2.4}$$

Here the solutions $\omega_{i,n}$ *are nonnegative.*

3. A priori estimate

In the following section, we give some a priori estimates for proving that under suitable additional assumptions, the solution of (2.4) converges to a solution of (1.1) as n tends to ∞. To ensure the existence of solution, we use the main structural assumptions on the nonlinearities. First we need to prove the following lemma.

Lemma 3.1. *Under the assumptions of Proposition 2.2 we have*

$$\int_\Omega \sum_{i=1}^{NS} |\omega_{i,n}(t)| \leq e^{tK}\Big[\sum_{i=1}^{NS} \|\mu_i\|_{M_b(\Omega)} + NK^{-1}(1 - e^{-tK})\Big] \tag{3.1}$$

Proof. We have

$$\frac{\partial}{\partial t}\Big(\sum_{i=1}^{NS}\omega_{i,n}\Big)-\sum_{i=1}^{NS}d_i\Delta\omega_{i,n}-\operatorname{div}\Big(\sum_{i=1}^{NS}m_i\omega_{i,n}\nabla\phi_n\Big)=\sum_{i=1}^{NS}S_i^n(\omega_n)$$

from (1.5), we there exist $K,N\geq 0$ such that

$$\frac{\partial}{\partial t}\Big(\sum_{i=1}^{NS}\omega_{i,n}\Big)-\sum_{i=1}^{NS}d_i\Delta\omega_{i,n}-\operatorname{div}\Big(\sum_{i=1}^{NS}m_i\omega_{i,n}\nabla\phi_n\Big)\leq K\Big(\sum_{i=1}^{NS}\omega_{i,n}\Big)+N. \tag{3.2}$$

Integrating over Ω and using the boundary conditions, we obtain

$$\partial_t\int_\Omega W_n(t)\leq\int_\Omega[K(W_n)+N],$$

where $W_n=\sum_{i=1}^{NS}\omega_{i,n}$. Hence

$$\int_\Omega\frac{\partial(W_ne^{-sK})}{\partial t}\leq\int_\Omega Ne^{-sK}.$$

Integrating the previous inequality on $[0,t]$, for all $0<t<T$, we obtain

$$\int_\Omega W_n(t)e^{tL}-\int_\Omega W_n(0,x)\leq\int_\Omega NK^{-1}(1-e^{-tK}),$$

hence

$$\int_\Omega W_n(t)\leq e^{tK}[\sum_{i=1}^{NS}\int_\Omega\mu_i^n+NK^{-1}(1-e^{-tK})]$$

By using the fact that for every $i=1,\dots,NS$ $\|\mu_i^n\|_{L^1(\Omega)}\leq\|\mu_i\|_{M_b(\Omega)}$. Then the proof is complete. □

Here, we note that $(S_i^n)_{1\leq i\leq NS}$ satisfies the same assumptions as $(S_i)_{1\leq i\leq NS}$, and especially the structure (1.7) that we use in the following lemma.

Lemma 3.2. *Let the assumptions of Proposition 2.2 are satisfied and* (1.7) *be satisfied. Then, for each $T>0$, there exists a constant C depending on T, L_i, M_i, q_{ij} and $\|\mu_i\|_{M_b(\Omega)}$ for all $1\leq j\leq i\leq NS$ such that*

$$\int_{Q_T}\sum_{1\leq i\leq NS}|S_i^n(\omega_n)|\,dt\,dx\leq C.$$

Proof. We denote by C_0 any constant depending only on the initial data and T. Then for all $t\in[0,T]$, we have $\int_\Omega(\omega_{i,n})(t)\leq C_0$ for all $1\leq i\leq NS$. Now, we take the equation satisfied by $\omega_{i,n}$ and we sum the NS equations to obtain that, for $1\leq i\leq NS$ and for $1\leq j\leq i$, we have

$$\sum_{j=1}^{i}q_{ij}\frac{\partial(\omega_{j,n})}{\partial t}-\sum_{j=1}^{i}q_{ij}d_j\Delta\omega_{j,n}-\operatorname{div}\Big(\sum_{j=1}^{i}q_{ij}m_j\omega_{j,n}\nabla\phi_n\Big)=\sum_{j=1}^{i}q_{ij}S_j^n(\omega_n)$$

Integrating on Q_T, we have

$$\sum_{j=1}^{i}q_{i,j}\int_{Q_T}\frac{\partial(\omega_{j,n})}{\partial t}-\sum_{j=1}^{i}q_{ij}\int_{\Sigma_T}(d_j\frac{\partial\omega_{j,n}}{\partial\upsilon}+m_j\omega_{j,n}\frac{\partial\phi_n}{\partial\upsilon})=\sum_{j=1}^{i}q_{ij}\int_{Q_T}S_j^n(\omega_n)\,.$$

Then using the boundary conditions, we obtain

$$\int_{Q_T} \sum_{j=1}^{i} q_{ij} \frac{\partial(\omega_{j,n})}{\partial t} = \int_{Q_T} \sum_{j=1}^{i} q_{ij} S_j^n(\omega_n);$$

therefore,

$$\sum_{j=1}^{i} q_{ij} \int_{\Omega} \omega_{j,n}(T) = \sum_{j=1}^{i} q_{ij} \int_{Q_T} S_j^n(\omega_n) + \sum_{j=1}^{i} q_{ij} \int_{\Omega} \omega_{j,n}(0,x)\,.$$

The nonnegativity of solutions yields

$$-\sum_{j=1}^{i} q_{ij} \int_{Q_T} S_j^n(\omega_n) \le \sum_{j=1}^{i} q_{ij} \int_{\Omega} \mu_j^n \le \sum_{j=1}^{i} q_{ij} \|\mu_j\|_{M_b(\Omega)} \tag{3.3}$$

This leads us to the estimate

$$\int_{Q_T} h_i(\omega_n) \le \sum_{j=1}^{i} q_{ij} \|\mu_j\|_{M_b(\Omega)} + \int_{Q_T} M_i + L_i\Big(\sum_{l=1}^{NS} \omega_{l,n}\Big), \tag{3.4}$$

where

$$h_i(\omega_n) = -\sum_{j=1}^{i} q_{ij} S_j^n(\omega_n) + M_i + L_i\Big(\sum_{l=1}^{NS} \omega_{l,n}\Big)$$

From the above, we have

$$\sum_{j=1}^{i} q_{ij} S_j^n(\omega_n) = -h_i(\omega_n) + M_i + L_i(\sum_{l=1}^{NS} \omega_{l,n}).$$

By using (3.3), (3.4) and the previous equality, we obtain

$$\|\sum_{j=1}^{i} q_{ij} S_j^n(\omega_n)\|_{L^1(Q_T)} \le C;$$

therefore, for $1 \le i \le NS$, we have $\|S_i^n(\omega_n)\|_{L^1(Q_T)} \le C$. □

Let ϕ_n be the unique solution of the elliptic problem

$$\begin{gathered} -\Delta\phi_n = F(\omega_n) \quad \text{on } Q_T = (0,T) \times \Omega, \\ \phi_n(t,x) = 0 \quad \text{on } \Sigma_T = (0,T) \times \partial\Omega, \\ \phi_n(0,x) = \phi_0(x) \quad \text{on } \Omega, \end{gathered} \tag{3.5}$$

where $F(\omega)$ is a bounded function in $L^\infty(Q_T)$, ϕ_0 is also bounded in $L^\infty(\Omega)$ and ϕ_n is the solution of equation (3.5).

Lemma 3.3. *There exists a constant C depending on T and on the L^∞ norm of ϕ_0, such that*

$$\|\phi_n\|_{L^\infty(0,T;W_0^{1,\infty}(\Omega))} \le C.$$

Proof. We have that for all $t \in]0,T[$, the function ϕ_n is the unique solution of the elliptic problem (3.5), where

$$\begin{gathered} \phi_n(t,x) = \int_{\Omega} G(s,x)\theta^n(t,s)ds, \\ \theta^n(t,s) = F(t,s,\omega_n), \quad s \in \Omega \end{gathered}$$

where G denotes the Green's function associated to the Poisson equation. Then we have

$$\|F(t,s,\omega_n)\|_{L^\infty(Q_T)} \le C,$$

hence

$$\|\phi_n\|_{L^\infty(0,T;W_0^{1,\infty}(\Omega))} \le C.$$

$\square$

Lemma 3.4. *Let $\omega_{i,n}$ be a solution of (2.4). Then, for every $T>0$, the mapping*

$$(\mu_i^n, S_i^n) \in\ L^1(\Omega)\times L^1(Q_T) \mapsto \omega_{i,n} \in L^1(Q_T) \tag{3.6}$$

is compact. Moreover, it is continuous from $L^1(\Omega)\times L^1(Q_T)$ to $C(]0,T[;L^1(\Omega))$. Moreover, $(\omega_{i,n})_{1\le i\le NS}$ is compact in $L^1(0,T;W^{1,1}(\Omega))$ and for the trace compactness, we use the continuity of the trace operator from $W^{1,1}(\Omega)$ to $L^1(\partial\Omega)$, then the trace mapping is also compact in $L^1(\Sigma_T)$.

For more details, we refer the readers to [23, Lemma 5.6].

Proposition 3.5. *Under the hypothesis of Lemme 3.2, there exists $(\omega_i)_{1\le i\le NS}$ in $L^1(Q_T)$ with $\nabla\omega_i \in [L^1(Q_T)^N]^{NS}$ such that, up to a subsequence, we have the following convergence*

$$\omega_{i,n} \to \omega_i \quad \textit{in } L^1(Q_T) \textit{ and a.e. in } Q_T,$$
$$\nabla\omega_{i,n} \to \nabla\omega_i \quad \textit{in } [L^1(Q_T)^N]^{NS} \textit{ and a.e. in } Q_T.$$

4. Convergence

Now, we show that (ω,ϕ) is a solution of (1.1). From Proposition 3.5, for $i = 1,\dots,N$, we have

$$\begin{gathered} \omega_{i,n} \to \omega_i \quad \text{in } L^1(Q_T), \\ \nabla\omega_{i,n} \to \nabla\omega_i \quad \text{in } [L^1(Q_T)^N]^{NS} \end{gathered} \tag{4.1}$$

and if we extract a new subsequence, then we can assume that it converges

$$\omega_{i,n} \to \omega_i \quad \text{almost everywhere in } Q_T, \tag{4.2}$$

and

$$\mu_i^n \to \mu_i \quad \text{in } M_b(\Omega) \tag{4.3}$$

Since S_i is continuous with respect to ω and by construction of S_i^n, we have

$$S_i^n(\omega_n) \to S_i(\omega) \quad \text{almost everywhere in } Q_T \tag{4.4}$$

For the proof of Theorem 2.1, we pass to the limit in the approximate problem when n tends to infinity. In fact, we need to prove that the convergence in (4.4) holds in $L^1(Q_T)$. To fill up the gap between those two kinds of convergence, we state the following lemma.

Lemma 4.1. *Let σ_n be a sequence in $L^1(Q_T)$ and $\sigma \in L^1(Q_T)$ such that*

$$\sigma_n \to \sigma \textit{ almost everywhere in } Q_T, \tag{4.5}$$
$$\sigma_n \textit{ is uniformly integrable in } Q_T\,. \tag{4.6}$$

Then $\sigma_n \to \sigma$ in $L^1(Q_T)$.

Proof. Condition (4.6) is implied by: for each $\varepsilon > 0$, there exists $\theta > 0$ such that

$$(K \subset Q_T \text{ measurable, } \operatorname{meas}(K) < \theta) \Longrightarrow \int_K |\sigma_n| < \varepsilon \quad \forall n$$

In this proof, we have to show that $S_i^n(\omega_n)$ are not only bounded in $L^1(Q_T)$ but uniformly integrable. We can not realize this without an extra assumption on (S_i) which is (1.7), which we assume in the first section. □

Theorem 4.2. *Assume the conditions in Proposition 2.2 and* (1.7) *are satisfied. Then, for all* $(\mu_i)_{1\le i\le NS}$ *in* $M_b(\Omega)$ *and* $\phi_0 \in L^\infty(\Omega)$*, there exists a weak solution* (ω,ϕ) *to problem* (1.1).

Lemma 4.3 ([16])**.** *Let* σ_n *be a sequence in* $L^1(Q_T)$*. Then the following instructions are equivalent*

- (i) σ_n *is uniformly bounded in* $L^1(Q_T)$*;*
- (ii) *there exists a function* $J : (0,\infty) \to (0,\infty)$ *with* $J(0^+) = 0$ *and*
  - (a) J *is convex,* J' *is concave,* $J' \ge 0$*;*
  - (b) $\lim_{r\to+\infty} \frac{J(r)}{r} = +\infty$*;*
  - (c) $\sup_n \int_{Q_T} J(|\sigma_n|) < \infty$.

Proof of Theorem 4.2

By Proposition 2.2, we have the existence of the solution (ω_n, ϕ_n) to the approximate problem (2.4).

Since ϕ_n is uniformly bounded in $L^\infty(0,T;W^{1,\infty}(\Omega))$, we deduce the existence of $\phi \in L^\infty(0,T;W^{1,\infty}(\Omega))$, such that

$$\nabla\phi_n \to \nabla\phi \quad \text{in the topology of } \sigma(L^\infty(Q_T), L^1(Q_T)) \tag{4.7}$$

Next we show that

$$\omega_{i,n}\nabla\phi_n \to \omega_i\nabla\phi \quad \text{in } D'(Q_T)$$

To this end, we will prove that

$$\omega_{i,n}\nabla\phi_n \to \omega_i\nabla\phi \quad \text{in the topology of } \sigma(L^1(Q_T), L^\infty(Q_T)) \tag{4.8}$$

For $v \in L^\infty(Q_T)$, we have

$$\begin{aligned}
&\int_{Q_T} v(\omega_{i,n}\nabla\phi_n - \omega_i\nabla\phi)\,dx\,dt \\
&= \int_{Q_T} v\nabla\phi_n(\omega_{i,n} - \omega_i)\,dx\,dt + \int_{Q_T} v\omega_i(\nabla\phi_n - \nabla\phi)\,dx\,dt
\end{aligned}$$

Concerning the first term in this equality, we have

$$\Big|\int_{Q_T} v\nabla\phi_n(\omega_{i,n} - \omega_i)\,dx\,dt\Big| \le \|v\|_{L^\infty(Q_T)}\|\nabla\phi_n\|_{L^\infty(Q_T)}\|\omega_{i,n} - \omega_i\|_{L^1(Q_T)}$$

and by (4.1), we obtain

$$\int_{Q_T} v\nabla\phi_n(\omega_{i,n} - \omega_i)\,dx\,dt \to 0.$$

The second term approcahes zero because $\nabla\phi_n$ converges to $\nabla\phi$ in the topology of $\sigma(L^\infty(Q_T), L^1(Q_T))$ and $v\omega_i \in L^1(Q_T)$. So from the convergence result, we obtain

$$\frac{\partial\omega_{i,n}}{\partial t} - d_i\Delta\omega_{i,n} - m_i\operatorname{div}(\omega_{i,n}\nabla\phi_n) \to \frac{\partial\omega_i}{\partial t} - d_i\Delta\omega_i - m_i\operatorname{div}(\omega_i\nabla\phi) \quad \text{in } D'(Q_T)$$

From (4.2) and (4.7), we have

$$-\Delta\phi_n \to -\Delta\phi \quad \text{in } D'(Q_T)$$

Furthermore,

$$F(t,x,\omega_n) \to F(t,x,\omega) \quad \text{a.e. in } Q_T.$$

According to (1.4) and by applying the Lebesgue theorem, we obtain

$$-\Delta\phi_n(t,.) \to -\Delta\phi(t,.) = F(t,.,\omega) \quad \text{strongly in } L^1(\Omega)\,.$$

Now, we define J as in the lemma 4.3 where (ii)(c) is replaced by

$$\sup_n \int_{Q_T} J\Big(\sum_{i=1}^{NS} \omega_{i,n}\Big) < \infty, \quad \sup_n \int_\Omega J\Big(\sum_{i=1}^{NS}\sum_{j=1}^{i} q_{ij}\mu_j^n\Big) < \infty\,. \tag{4.9}$$

Which is possible by lemma 4.3 since

$$\sum_{i=1}^{NS} \omega_{i,n} \to \sum_{i=1}^{NS} \omega_i \quad \text{in } L^1(Q_T)$$

and

$$\sum_{i=1}^{NS}\sum_{j=1}^{i} q_{ij}\mu_j^n \to \sum_{i=1}^{NS}\sum_{j=1}^{i} q_{ij}\mu_j \quad \text{in } M_b(\Omega)\,.$$

Putting

$$j(r) = \int_0^r \min(J'(s),(J^*)^{-1}(s))ds, \tag{4.10}$$

where J^* is the conjugate function of J, j satisfies (ii)(a) and (ii)(b) and we have

$$\forall r>0 \quad j(r) \le J(r) \quad J^*(j'(r)) \le r\,. \tag{4.11}$$

Our goal is to show that

$$\sup_n \int_{Q_T} j'\Big(\sum_{i=1}^{NS}\sum_{j=1}^{i} q_{ij}\omega_{j,n}\Big)\Big(\sum_{i=1}^{NS} |S_i^n(\omega_n)|\Big) < \infty\,. \tag{4.12}$$

First of all, we indicate how the proof of the theorem can be completed. We see that estimation (4.12) implies the uniform integrability of $(S_i^n(\omega_n))_{1\le i\le NS}$ in Q_T. Indeed, let $\varepsilon>0$ and K be a measurable set of Q_T. Then

$$\begin{aligned}\int_K |S_i^n(\omega_n)| &\le \int_{K\cap[\sum_{i=1}^{NS}\sum_{j=1}^{i} q_{ij}\omega_{j,n}<k]} \sup_{0\le\omega_{1,n},\dots,\omega_{NS,n}\le k} |S_i^n(\omega_n)| \\ &\quad + \int_{K\cap[\sum_{i=1}^{NS}\sum_{j=1}^{i} q_{ij}\omega_{j,n}>k]} |S_i^n(\omega_n)| \\ &\le I_1 + I_2\,.\end{aligned}$$

For I_1, we find that

$$\sup_{0\le\omega_{1,n},\dots,\omega_{NS,n}\le k} |S_i^n(t,x,\omega_n)| \le C(k),$$

where the constant C depends on L_i, M_i for all $i=1,\dots,NS$ and $k(\varepsilon)$, consequently

$$I_1 \le C(k(\varepsilon))\,\text{meas}(K) < \frac{\varepsilon}{2} \quad \text{if } \text{meas}(K) < \frac{\varepsilon}{2[C(k(\varepsilon))]}\,.$$

We estimated the second integral as follows

$$I_2 \le \frac{1}{j'(k)} \int_{Q_T} j'\Big(\sum_{i=1}^{NS}\sum_{j=1}^{i} q_{ij}\omega_{j,n}\Big)|S_i^n(\omega_n)|$$

Where we have used the convexity of the function J and (4.10) to obtain the desired result. Thanks to (4.12), this may be less than $\frac{\varepsilon}{2}$, by choosing $k = k(\varepsilon)$ sufficiently large and depend only on ε.

This proves the uniform integrability of $S_i^n(\omega_n)$. We use now the almost everywhere convergence of $S_i^n(\omega_n)$ in Q_T, then, we obtain that $S_i^n(\omega_n)$ converges in $L^1(Q_T)$, which completes the proof of the theorem.

Returning now to the proof of the estimate (4.12). We set

$$\begin{gathered}
T_{1,n} = M_1 + L_1\Big(\sum_{i=1}^{NS}\omega_{i,n}\Big) - q_{1,1}S_1^n(\omega_n) \ge 0,\\
T_{2,n} = M_2 + L_2\Big(\sum_{i=1}^{NS}\omega_{i,n}\Big) - q_{2,1}S_1^n(\omega_n) - q_{2,2}S_2^n(\omega_n) \ge 0,\\
\dots\\
T_{NS,n} = M_{NS} + L_{NS}\Big(\sum_{i=1}^{NS}\omega_{i,n}\Big) - q_{NS,1}S_1^n(\omega_n) - \dots - q_{NS,NS}S_{NS}^n(\omega_n) \ge 0
\end{gathered} \tag{4.13}$$

which implies

$$\sum_{i=1}^{NS} T_{i,n} = \sum_{i=1}^{NS} M_i + \Big(\sum_{l=1}^{NS}\omega_{l,n}\Big)\sum_{i=1}^{NS} L_i - \sum_{i=1}^{NS}\sum_{j=1}^{i} q_{ij}S_j^n(\omega_n) \ge 0\,.$$

For every $j = 1, \dots, i$ and $i = 1, \dots, NS$ we have

$$\frac{\partial\omega_{j,n}}{\partial t} = d_j\Delta\omega_{j,n} + m_j\operatorname{div}(\omega_{j,n}\nabla\phi_n) + S_j^n(\omega_n)\,.$$

We multiply the first equation by $(q_{ij})_{1\le i,j\le NS}$ and we sum the three equations, so we obtain

$$\begin{aligned}
\frac{\partial}{\partial t}\Big(\sum_{j=1}^{i} q_{ij}\omega_{j,n}\Big) &= \Big(\sum_{j=1}^{i} q_{ij}d_j\Delta\omega_{j,n}\Big) + \operatorname{div}\Big(\Big(\sum_{j=1}^{i} q_{ij}m_j\omega_{j,n}\Big)\nabla\phi_n\Big)\\
&\quad + \sum_{j=1}^{i} q_{ij}S_j^n(\omega_n)\,.
\end{aligned}$$

Then, we multiply by $j'(\sum_{i=1}^{NS}\sum_{j=1}^{i} q_{ij}\omega_{j,n})$ and integrate over Q_T to obtain

$$\begin{aligned}
&\int_\Omega j\Big(\sum_{i=1}^{NS}\sum_{j=1}^{i} q_{ij}\omega_{j,n})(T) + \int_{Q_T} j'\Big(\sum_{i=1}^{NS}\sum_{j=1}^{i} q_{ij}\omega_{j,n}\Big)\sum_{i=1}^{NS} T_{i,n}\\
&= J_1 + J_2 + \int_\Omega j\Big(\sum_{i=1}^{NS}\sum_{j=1}^{i} q_{ij}\mu_j^n\Big)
\end{aligned} \tag{4.14}$$

where

$$J_1=\int_{Q_T} j'\Big(\sum_{i=1}^{NS}\sum_{j=1}^{i} q_{ij}\omega_{j,n}\Big)\Big[\Big(\sum_{i=1}^{NS}\sum_{j=1}^{i} q_{ij}d_j\Delta\omega_{j,n}\Big)+\operatorname{div}\big((\sum_{i=1}^{NS}\sum_{j=1}^{i} q_{ij}m_j\omega_{j,n})\nabla\phi_n\big)\Big]$$

Concerning the term J_2, we have

$$J_2=\int_{Q_T} j'\Big(\sum_{i=1}^{NS}\sum_{j=1}^{i} q_{ij}\omega_{j,n}\Big)\Big[\sum_{i=1}^{NS} M_i+\sum_{i=1}^{NS} L_i\Big(\sum_{l=1}^{NS}\omega_{l,n}\Big)\Big]$$

To estimate this term, we use the inequality

$$j'(r).s\le J(s)+J^*(j'(r))\le J(s)+r, \tag{4.15}$$

so that

$$\begin{aligned} J_2\le \int_{Q_T}&\Big[J\Big(\sum_{i=1}^{NS} M_i\Big)+\Big(\sum_{i=1}^{NS}\sum_{j=1}^{i} q_{ij}\omega_{j,n}\Big)\Big]\\ &+\int_{Q_T}\sum_{i=1}^{NS} L_i\Big[J\Big(\sum_{l=1}^{NS}\omega_{l,n}\Big)+\Big(\sum_{i=1}^{NS}\sum_{j=1}^{i} q_{ij}\omega_{j,n}\Big)\Big] \end{aligned}$$

By using the Lemma 3.1, the choice of J_2 and (4.9), we see that J_2 is bounded independently of n.

On the other hand, if we can control the term J_1. Then, from (4.9), (4.14) and the estimate on J_2, we obtain the desired result. By definition of $T_{i,n}$ we have

$$T_n=M+L\Big(\sum_{l=1}^{NS}\omega_{l,n}\Big)-QS^n(\omega_n),$$

where $T_n=(T_{1,n},\dots,T_{NS,n})$, $L=(L_1,\dots,L_{NS})$, $M=(M_1,\dots,M_{NS})$ and $S^n(\omega_n)=(S_1^n(\omega_n),\dots,S_{NS}^n(\omega_n))$.

Since Q is an invertible matrix, we obtain

$$S^n(\omega_n)=Q^{-1}M+\Big(\sum_{l=1}^{NS}\omega_{l,n}\Big)Q^{-1}L-Q^{-1}T_n \tag{4.16}$$

which gives an estimation on the sum of nonlinearities S_i for all $i=1,\dots,NS$.

For J_1 we have the estimate

$$\begin{aligned} J_1&=\int_{Q_T} j'\Big(\sum_{i=1}^{NS}\sum_{j=1}^{i} q_{ij}\omega_{j,n}\Big)\Big(\sum_{i=1}^{NS}\sum_{j=1}^{i} d_jq_{ij}\Delta\omega_{j,n}\Big)\\ &\quad+\int_{Q_T} j'\Big(\sum_{i=1}^{NS}\sum_{j=1}^{i} q_{ij}\omega_{j,n}\Big)\operatorname{div}\Big(\sum_{i=1}^{NS}\sum_{j=1}^{i} m_jq_{ij}\omega_{j,n}\nabla\phi_n\Big)\\ &=-\int_{Q_T} j''\Big(\sum_{i=1}^{NS}\sum_{j=1}^{i} q_{ij}\omega_{j,n})\nabla\Big(\sum_{i=1}^{NS}\sum_{j=1}^{i} q_{ij}\omega_{j,n}\Big)[\sum_{i=1}^{NS}\sum_{j=1}^{i} d_jq_{ij}\nabla\omega_{j,n}\\ &\quad+\sum_{i=1}^{NS}\sum_{j=1}^{i} m_jq_{ij}\omega_{j,n}\nabla\phi_n]. \end{aligned}$$

We set $W_n = \sum_{i=1}^{NS}\sum_{j=1}^{i} q_{ij}\omega_{j,n}$ and integrate by parts to have

$$J_1 = -\int_{Q_T} j''(W_n)\nabla W_n\Big[\sum_{i=1}^{NS}\sum_{j=1}^{i} d_j q_{ij}\nabla\omega_{j,n} + \sum_{i=1}^{NS}\sum_{j=1}^{i} m_j q_{ij}\omega_{j,n}\nabla\phi_n\Big]$$

Next, we use Hölder's and Young's inequalities to obtain

$$J_1 \le \tilde{C}\int_{Q_T} j''(W_n)\Big(\sum_{i=1}^{NS}|\omega_{i,n}|^2 + \sum_{i=1}^{NS}|\nabla\omega_{i,n}|^2\Big), \tag{4.17}$$

where the constant $\tilde{C}$ depends on $\max_{1\le j\le i}(d_j)$, $\max_{1\le j\le i}(m_j)$, q_{ij}, $\|\nabla\phi_n\|_{L^\infty(Q_T)}$, and $1\le j\le i\le NS$.

Since j' is concave, $j''(r)\le \frac{j'(r)}{r}$ and we have

$$\int_{Q_T} j''(W_n)\Big(\sum_{i=1}^{NS}\omega_{i,n}^2\Big) \le \int_{Q_T} j'(W_n)\Big(\sum_{i=1}^{NS}\omega_{i,n}\Big)$$

hence

$$\int_{Q_T} j''(W_n)\Big(\sum_{i=1}^{NS}\omega_{i,n}^2\Big) \le \int_{Q_T} J\Big(\sum_{i=1}^{NS}\omega_{i,n}Big) + W_n$$

which allows to say that this term is uniformly bounded. It remains to show that the first and the second term in (4.17) are also uniformly bounded. First, we have

$$\frac{\partial\omega_{1,n}}{\partial t} - d_1\Delta\omega_{1,n} - m_1\operatorname{div}(\omega_{1,n}\nabla\phi_n) = S_1^n(\omega_n)\,.$$

Then we multiply by q_{11} to obtain

$$\frac{\partial(q_{11}\omega_{1,n})}{\partial t} - d_1\Delta(q_{11}\omega_{1,n}) - m_1\operatorname{div}(q_{11}\omega_{1,n}\nabla\phi_n)] = q_{11}S_1^n(\omega_n). \tag{4.18}$$

Now, we multiply by $j'(q_{11}\omega_{1,n})$ and integrate over Q_T to obtain

$$\begin{aligned}
&\int_{Q_T}\frac{\partial(j(q_{11}\omega_{1,n}))}{\partial t} + d_1\int_{Q_T} j''(q_{11}\omega_{1,n})|\nabla(q_{11}\omega_{1,n})|^2\\
&+ m_1\int_{Q_T} j''(q_{11}\omega_{1,n})(q_{11}\omega_{1,n})\nabla\phi_n\nabla(q_{11}\omega_{1,n})\\
&= \int_{Q_T} j'(q_{11}\omega_{1,n})q_{11}S_1^n(\omega_n),
\end{aligned}$$

which implies

$$\begin{aligned}
&\int_\Omega j(q_{11}\omega_{1,n})(T) + d_1\int_{Q_T} j''(q_{11}\omega_{1,n})|\nabla(q_{11}\omega_{1,n})|^2\\
&\le \int_\Omega j(q_{11}\mu_1^n) - m_1\int_{Q_T} j''(q_{11}\omega_{1,n})(q_{11}\omega_{1,n})\nabla\phi_n\nabla(q_{11}\omega_{1,n})\\
&\quad + \int_{Q_T} j'(q_{11}\omega_{1,n})\Big[M_1 + L_1\Big(\sum_{l=1}^{NS}\omega_{l,n}\Big)\Big]\,.
\end{aligned}$$

Using again Hölder's and Young's inequalities, we obtain

$$\int_\Omega j(q_{11}\omega_{1,n})(T) + d_1\int_{Q_T} j''(q_{11}\omega_{1,n})|\nabla(q_{11}\omega_{1,n})|^2$$

$$\begin{aligned}&\le \int_\Omega j(q_{11}\mu_1^n) + C_{\varepsilon_1}\int_{Q_T} j''(q_{11}\omega_{1,n})(q_{11}\omega_{1,n})^2\\&\quad + \varepsilon_1 \int_{Q_T} j''(q_{11}\omega_{1,n})|\nabla(q_{11}\omega_{1,n})|^2 + \int_{Q_T} j'(q_{11}\omega_{1,n})\Big[M_1 + L_1\Big(\sum_{l=1}^{NS}\omega_{l,n}\Big)\Big];\end{aligned}$$

therefore,

$$\begin{aligned}&\int_\Omega j(q_{11}\omega_{1,n})(T) + (d_1-\varepsilon_1)\int_{Q_T} j''(q_{11}\omega_{1,n})|\nabla(q_{11}\omega_{1,n})|^2\\&\le \int_\Omega j(q_{11}\mu_1^n) + C_{\varepsilon_1}\int_{Q_T} j''(q_{11}\omega_{1,n})(q_{11}\omega_{1,n})^2\\&\quad + \int_{Q_T} j'(q_{11}\omega_{1,n})\Big[M_1 + L_1\Big(\sum_{l=1}^{NS}\omega_{l,n}\Big)\Big].\end{aligned}$$

Then we have

$$\begin{aligned}\int_{Q_T} j''(q_{11}\omega_{1,n})(q_{11}\omega_{1,n})^2 &\le \int_{Q_T} j'(q_{11}\omega_{1,n})(q_{11}\omega_{1,n})\\&\le \int_{Q_T} J(q_{11}\omega_{1,n}) + q_{11}\omega_{1,n}\,.\end{aligned}$$

Finally, we add the condition $j''(q_{11}\omega_{1,n}) \ge j''(W_n)$. Then, we deduce easily that $\int_{Q_T} j''(W_n)|\nabla\omega_{1,n}|^2$ is uniformly bounded. Similarly, we show that, for all $i = 2,\dots,NS$, the terms $\int_{Q_T} j''(W_n)|\nabla\omega_{i,n}|^2$ are uniformly bounded. Then we conclude that $\int_{Q_T} j'\Big(\sum_{i=1}^{NS}\sum_{j=1}^{i} q_{ij}\omega_{j,n}\Big)\sum_{i=1}^{NS} T_{i,n}$ is also uniformly bounded. Then, we use (4.16) and the definition of each term $T_{i,n}$ for all $1 \le i \le NS$, to obtain the uniformly bound of the second term I_2 and also to deduce the equi-integrability of S_i^n which completes the proof.

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