Electronic Journal of Differential Equations, Vol. 2017 (2017), No. 07, pp. 1–7.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

UNIQUENESS FOR COEFFICIENT IDENTIFICATION IN ONE-DIMENSIONAL PARABOLIC EQUATIONS

XUEWEN WU, GUANGSHENG WEI

Communicated by Goong Chen

Abstract. This article concerns the question of uniqueness in the identification of coefficients in a one-dimensional parabolic partial differential equation. The solution of an initial-boundary value problem is observed at one internal point over a finite time interval. It is shown that, under the condition that the coefficient is known a priori on a subinterval, the coefficient on the entire interval is uniquely determined by such observation.

1. Introduction

We are concerned with the parabolic equation

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + q(x)u = 0 \tag{1.1}$$

for $0 < x < 1$ and $0 < t \le t_0$ with the initial and boundary conditions

$$u(x,0) = 0 \quad (0 < x < 1), \tag{1.2}$$

$$u(0,t) = 0, \quad u(1,t) = g(t) \quad (0 < t \le t_0). \tag{1.3}$$

Here the functions $q \in L^1[0,1]$ and $g \in AC[0,t_0]$ are real. We want to consider the problem of the unique identification of the coefficient q in (1.1) from the values $u(1,t)$ and $u(x_0,t)$ for all $t \in [0,t_0]$ and for some $x_0 \in (0,1)$, in terms of the inverse spectral theory of Sturm-Liouville differential operators.

It is obvious that uniqueness does not remain true without any assumption on (q,g). In fact, if we put $u(\zeta,t) \equiv 0$ in (1.1)-(1.3) for $\zeta = x_0, 1$, then $u \equiv 0$ follows for any coefficients q. However, it has been shown by Pierce [10] that this uniqueness problem is related closely to the spectrum of the Sturm-Liouville differential operator L which is defined as the realization in $L^2(0,1)$ of the differential expression

$$Ly := -y'' + q(x)y \tag{1.4}$$

with the Dirichlet boundary conditions

$$y(0) = y(1) = 0. \tag{1.5}$$

2010 *Mathematics Subject Classification.* 35R30, 34A55, 34B24.
Key words and phrases. Inverse problem; spectral data; parabolic equation.
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Submitted May 12, 2016. Published January 10, 2017.

As is well known [7], the operator L is self-adjoint in the space $L^2[0,1]$ and has a discrete spectrum consisting of simple real eigenvalues $\{\lambda_n\}$ for $n \in \mathbb{N}_0$. Pierce has shown that, under the assumption that x_0 is not in the set

$$\Omega := \{x \in (0,1) : \varphi_n(x) = 0 \text{ for some } n \in \mathbb{N}\}, \tag{1.6}$$

then the spectrum $\sigma(L)$ is uniquely determined by $u(\zeta, t)$ in (1.1)-(1.3) for $\zeta = x_0, 1$, where $\varphi_n(x)$ are the nontrivial eigenfunctions corresponding to eigenvalues λ_n. This shows that, for $x_0 \in \Omega$, there exists partial knowledge of the spectrum $\sigma(L)$ which is missing; that is, this partial knowledge can not be used to determine the coefficient q.

In the literature, there are many results (see [1, 4, 5, 6, 8, 9, 10, 11, 12, 13, 14] and the references therein) related to the problems of identification of the coefficient of (1.1). In particular, Ramm [11, 12, 13, 14, 15] considered initial-boundary value problem (1.1)-(1.3) and proved that either knowledge of $g(t)$, $u_x(1,t)$ or of $g(t)$, $u_x(0,t)$ and q on $[1/2, 1]$ (or $q(1/2 - x) = q(x)$) determines q uniquely. Moreover, Kimura and Suzuki [6] considered the uniqueness problem for (1.1)-(1.3) in the case when $x \in [0, \infty)$ and $q = 0$ on $[a, \infty)$ with $a > 0$.

In this note it is shown that, under condition that q in (1.1) is known a priori on a right subinterval of $[0,1]$, the coefficient q of initial and boundary problem (1.1)-(1.3) is uniquely determined by an observation of the solution at one interior point in space. The technique which we use to obtain this result is based on the uniqueness theorems of Gesztesy-Simon [3], Ramm [12, 13] and Wei-Xu [17] for inverse spectral theory of the Sturm-Liouville differential operators defined by (1.4) with partial information given on the potentials. Our results and their proofs of the paper will be stated in the next section.

2. Statement of results

Theorem 2.1. *Consider the parabolic equations defined as* (1.1)-(1.3)*, where* $q \in L^1[0,1]$ *and* $g \in AC(0, t_0]$ *for some* $t_0 > 0$ *are real functions with* $g \not\equiv 0$*. Let* $u(x,t)$ *be its a solution. Then* q *on* $[0,1]$ *is determined uniquely by* q *on* $[a,1]$*,* $u(x_0,t)$ *and* $g(t)$ *for* $t \in (0, t_0]$*, where* x_0 *and* a *satisfy one of the following conditions:*

(i) $x_0 \geq a$*, or*
(ii) $x_0 < a$ *and* $a \leq (1 - x_0)/2$.

Remark 2.2. For condition (i) above, the above uniqueness result can also be obtained by solving (1.1) on $[x_0, 1] \times [0, t_0]$ and using [11, Proposition 2.1], and this result for the case $x_0 = a$ has been given by Ramm in [15, p. 177]. However, we provide here another approach to prove the uniqueness result for the problem.

Remark 2.3. When $x_0 < a$, it is easy to ensure that $x_0 < 1/3$ and $a < 1/2$. As a typical example, knowing $x_0 = 1/4$ and knowing q on $[3/8, 1]$, then q on $[0,1]$ is determines uniquely in terms of $u(x_0, t)$ and $g(t)$ for $t \in (0, t_0]$.

Before proving the theorem, we shall first mention some preliminaries which will be needed subsequently. Consider the initial-value problems

$$-y'' + q(x)y = zy \tag{2.1}$$

on $[0,1]$ with initial conditions

$$y_-(0) = 0, \quad y'_-(0) = 1; \tag{2.2}$$

$$y_+(1) = 0, \quad y'_+(1) = 1. \tag{2.3}$$

Let $\varphi_- := \varphi_-(x, z)$ and $\varphi_+ := \varphi_+(x, z)$ be the solutions of problems (2.1)-(2.2) and problems (2.1) (2.3), respectively. If $z = \lambda_n$, then both $\varphi_-(x, \lambda_n) =: \varphi_{n,-}$ and $\varphi_+(x, \lambda_n) =: \varphi_{n,+}$ are eigenfunctions, corresponding to the eigenvalue λ_n, of the operator L defined by (1.4) and there holds the relation

$$\varphi_{n,+} = \kappa_n \varphi_{n,-}, \tag{2.4}$$

where $\kappa_n = 1/\varphi'_{n,-}(1)$ is called the norming constant corresponding to λ_n; κ_n is neither zero nor infinity. Let

$$\omega(\lambda) := \varphi_-(1, z), \tag{2.5}$$

$$\alpha_n = \int_0^1 |\varphi_-(x, \lambda_n)|^2 dx. \tag{2.6}$$

Then from [2], the eigenvalues of L are the zeros of the transcendental function $\omega(\lambda)$ and

$$\dot{\omega}(\lambda_n) = -\kappa_n \alpha_n \tag{2.7}$$

where $\dot{\omega}(z) = d\omega/dz$. It is well known [3] that the Weyl m-function is defined by

$$m_-(a, z) = -\frac{\varphi'_-(a, z)}{\varphi_-(a, z)} \tag{2.8}$$

for any $a \in [0, 1)$ and the potential q on $[0, a]$ is uniquely determined by $m_-(a, z)$ from Marchenko's fundamental uniqueness theorem [2] of inverse spectral theory.

Proof of Theorem 2.1. Let q be given on $[a, 1]$ with some $a \in (0, 1)$. Let q_1 and q_2 be two candidates for q extended to all of $[0, 1]$. Let $u_1(x, t)$, $u_2(x, t)$ be the solutions of the initial and boundary problem (1.1)-(1.3) corresponding to q_1 and q_2, respectively. By Theorem 2.1, our purpose here is to prove $q_1 = q_2$ on $[0, 1]$ under the assumptions that $u_1(\zeta, t) = u_2(\zeta, t)$ for $\zeta = 1, x_0$ and $q_1 = q_2$ on $[a, 1]$.

Let $\varphi_{1,\pm}(x, z)$ and $\varphi_{2,\pm}(x, z)$ be the solutions of (2.1) corresponding to q_1 and q_2, respectively, where $\varphi_{j,\pm}(x, z)$ satisfies the initial conditions (2.2) and (2.3), respectively. For $j = 1, 2$, we solve the equations (1.1) and we have from [16, pp. 215-216] that

$$u_j(x_0, t) = -\int_0^t g_j(\tau) K_j(t - \tau) d\tau, \tag{2.9}$$

where $t \in (0, t_0]$ and

$$K_j(t) = \sum_{n=1}^{\infty} \varphi'_{j,n}(1) \varphi_{j,n}(x_0) e^{-\lambda_{j,n} t}, \tag{2.10}$$

where $\varphi_{j,n}(x) = \varphi_{1,-}(x, \lambda_{j,n})/\|\varphi_{j,-}(\cdot, \lambda_{j,n})\|$. Since $u_1(x_0, t) = u_2(x_0, t)$, $u_1(1, t) = u_2(1, t)$; that is, $g_1(t) = g_2(t)$ for $t \in (0, t_0]$, from (2.9) it follows that

$$\int_0^t K_1(t - \tau) g_1(\tau) d\tau = \int_0^t K_2(t - \tau) g_1(\tau) d\tau. \tag{2.11}$$

This yields $K_1(t) = K_2(t)$ by using the property of Volterra-integral equation; that is,

$$\sum_{n=1}^{\infty} \varphi'_{1,n}(1) \varphi_{1,n}(x_0) e^{-\lambda_{1,n} t} = \sum_{n=1}^{\infty} \varphi'_{2,n}(1) \varphi_{2,n}(x_0) e^{-\lambda_{2,n} t}. \tag{2.12}$$

Whenever the eigenfunctions are uniformly bounded, the derivatives of the eigenfunctions at 1 are asymptotic to n and the eigenvalues are asymptotic to a multiple of n^2. Thus the series of K_1 and K_2 converges for all $t>0$ and, in fact, K_1, K_2 can be extended to an analytic function on the right half plane. We take the Laplace transforms termwise in both side of the equation (2.12); that is,

$$\int_0^\infty e^{-\lambda t}K_1(t)dt = \int_0^\infty e^{-\lambda t}K_2(t)dt$$

for $\operatorname{Re}\lambda > 0$, to obtain

$$\sum_{n=1}^{\infty}\frac{\varphi'_{1,n}(1)\varphi_{1,n}(x_0)}{\lambda+\lambda_{1,n}} = \sum_{n=1}^{\infty}\frac{\varphi'_{2,n}(1)\varphi_{2,n}(x_0)}{\lambda+\lambda_{2,n}} \tag{2.13}$$

for $\operatorname{Re}\lambda > 0$. By the asymptotic expression of the eigenfunctions and the eigenvalues, we can analytically continue both sides of (2.13) in λ when series above are convergent, so that (2.13) holds for $\lambda \in \mathbb{C}\backslash\{-\lambda_{1,n}\}_{n=1}^\infty \cup \{-\lambda_{2,n}\}_{n=1}^\infty$. Note that $\varphi'_{1,n}(1) \neq 0 \neq \varphi'_{2,n}(1)$. Then we obtain that if $\varphi_{1,n}(x_0) \neq 0$ then there exists $m(n) \in \mathbb{N}_0$ such that

$$\lambda_{1,n} = \lambda_{2,m(n)}, \tag{2.14}$$
$$\varphi'_{1,n}(1)\varphi_{1,n}(x_0) = \varphi'_{2,m(n)}(1)\varphi_{2,m(n)}(x_0). \tag{2.15}$$

Otherwise, let us assume $\lambda_{1,n} \neq \lambda_{2,m}$ for all $m \in \mathbb{N}_0$. Then we take a suitable disk which includes $-\lambda_{1,n}$ and does not include $\{-\lambda_{1,k}\}_{k\neq n} \cup \{-\lambda_{2,m}\}_{m=1}^\infty$. Integrating (2.13) in a disk, we have that $2\pi i \varphi'_{1,n}(1)\varphi_{1,n}(x_0) = 0$. This is a contradiction because of $\varphi'_{1,n}(1) \neq 0$.

Let us define the set

$$S := \{\lambda_{1,n} : \varphi_{1,n}(x_0) = 0\}. \tag{2.16}$$

Then $S^c := \sigma(L_1)\backslash S = \{\lambda_{1,n} | \varphi_{1,n}(x_0) \neq 0\}$. This implies that $S \subset \{\mu_n^-\}_{n=1}^\infty \cap \{\mu_n^+\}_{n=1}^\infty$, where $\{\mu_n^+\}_{n=1}^\infty$ and $\{\mu_n^-\}_{n=1}^\infty$ are the eigenvalues of operators $L_+y := -y'' + q_1 y$ on $[x_0, 1]$ with the Dirichlet boundary conditions $y(x_0) = 0 = y(1)$ and $L_-y := -y'' + q_1 y$ on $[0, x_0]$ with $y(0) = 0 = y(x_0)$, respectively. It should be noted that [2, p. 11]

$$\begin{gathered}
\mu_n^- = \frac{n^2\pi^2}{x_0^2} + \int_0^{x_0} q_1(\tau)d\tau + \varepsilon_n^-, \\
\mu_n^+ = \frac{n^2\pi^2}{(1-x_0)^2} + \int_{x_0}^{1} q_1(\tau)d\tau + \varepsilon_n^+,
\end{gathered}$$

where $\varepsilon_n^\pm \to 0$ as $n \to \infty$. Therefore, we have

$$\begin{aligned}
N_S(t) &= N_{\sigma(L_-)\cap\sigma(L_+)}(t) \\
&\leq \min\{1-x_0, x_0\}[\sqrt{t}/\pi] \\
&= \min\{1-x_0, x_0\}N_{\sigma(L_1)}(t)
\end{aligned} \tag{2.17}$$

for t sufficiently large, where $N_S(t) := \#\{\lambda_{1,n} \in S : \lambda_{1,n} \leq t\}$. This implies

$$\begin{aligned}
N_{S^c}(t) &\geq (1 - \min\{1-x_0, x_0\})N_{\sigma(L_1)}(t) \\
&= \max\{x_0, 1-x_0\}N_{\sigma(L_1)}(t).
\end{aligned}$$

We shall prove the theorem through the following two cases.

Case 1: $x_0 \geq a$. By the definitions of the norming constants $\kappa_{j,n}$ and $\alpha_{j,n}$ (see (2.4) and (2.6)), for $j = 1, 2$ and $n \in \mathbb{N}_0$ we infer that

$$\varphi_{j,n}(x) = \frac{\varphi_{j,-}(x, \lambda_{j,n})}{\|\varphi_{j,-}(x, \lambda_{j,n})\|} = \frac{\varphi_{j,+}(x, \lambda_{j,n})}{\kappa_{j,n}(\alpha_{j,n})^{1/2}}.$$

Taking into account (2.14) and (2.15), for $\lambda_{1,n} \in S^c$, the above equation yields

$$\frac{\varphi_{1,+}(x_0, \lambda_{1,n})}{\kappa_{1,n}^2 \alpha_{1,n}} = \frac{\varphi_{2,+}(x_0, \lambda_{1,n})}{\kappa_{2,m(n)}^2 \alpha_{2,m(n)}}. \tag{2.18}$$

It follows that $\varphi_{1,+}(x_0, \lambda_{1,n}) = \varphi_{2,+}(x_0, \lambda_{1,n})$ since $q_1 = q_2$ on $[a, 1]$ and $a \leq x_0$. This combined with (2.7) yield

$$\kappa_{1,n}\dot{\varphi}_{1,-}(1, \lambda_{1,n}) = \kappa_{2,m(n)}\dot{\varphi}_{2,-}(1, \lambda_{1,n}). \tag{2.19}$$

Let

$$U(a, z) := [\varphi_{1,-}(a, z), \varphi_{2,-}(a, z)] = \begin{vmatrix} \varphi_{1,-}(a, z) & \varphi_{2,-}(a, z) \\ \varphi'_{1,-}(a, z) & \varphi'_{2,-}(a, z) \end{vmatrix}. \tag{2.20}$$

By the assumption that $q_1 = q_2$ on $[a, 1]$, it is easy to see that

$$\begin{aligned} U(a, z) &= U(1, z) - \int_a^1 \frac{d}{dt}[\varphi_{1,-}(t, z), \varphi_{2,-}(t, z)]dt \\ &= U(1, z) + \int_a^1 (q_1 - q_2)(x)(\varphi_{1,-}\varphi_{2,-})(t, z)dt \\ &= U(1, z) \\ &= \begin{vmatrix} \varphi_{1,-}(1, z)\varphi_{2,-}(1, z) \\ \varphi'_{1,-}(1, z)\varphi'_{2,-}(1, z) \end{vmatrix}. \end{aligned} \tag{2.21}$$

From (2.14) and the definition of $\varphi_{j,-}(1, z)$, it follows that if $\lambda_{1,n} \in S^c$ then $\varphi_{j,-}(1, \lambda_{1,n}) = 0$ for $j = 1, 2$ and therefore $U(a, \lambda_{1,n}) = 0$. Furthermore, since

$$\dot{U}(a, z) = \begin{vmatrix} \dot{\varphi}_{1,-}(1, z) & \dot{\varphi}_{2,-}(1, z) \\ \varphi'_{1,-}(1, z) & \varphi'_{2,-}(1, z) \end{vmatrix} + \begin{vmatrix} \varphi_{1,-}(1, z) & \varphi_{2,-}(1, z) \\ \dot{\varphi}'_{1,-}(1, z) & \dot{\varphi}'_{2,-}(1, z) \end{vmatrix},$$

substituting $z = \lambda_{1,n} \in S^c$ into the above formula we have

$$\begin{aligned} \dot{U}(a, \lambda_{1,n}) &= \dot{\varphi}_{1,-}(1, \lambda_{1,n})\varphi'_{2,-}(1, \lambda_{1,n}) - \dot{\varphi}_{2,-}(1, \lambda_{1,n})\varphi'_{1,-}(1, \lambda_{1,n}) \\ &= \frac{\dot{\varphi}_{1,-}(1, \lambda_{1,n})}{\kappa_{2,m(n)}} - \frac{\dot{\varphi}_{2,-}(1, \lambda_{1,n})}{\kappa_{1,n}} = 0 \end{aligned}$$

by (2.4) and (2.19). This shows that $\lambda_{1,n}$ is the zero of $U(a, z)$ with multiplicity to at least 2.

Let us consider the function

$$F(z) = \frac{U(a, z)}{g^2(z)} \tag{2.22}$$

where

$$g(z) = \prod_{\lambda_{1,n} \in S^c} \Big(1 - \frac{z}{\lambda_{1,n}}\Big).$$

From (2.8) and [2, pp.12] we infer that

$$F(z) = \frac{\varphi'_{1,-}(a, z)\varphi'_{2,-}(a, z)}{g^2(z)}(-m_{1,-}(a, z)^{-1} + m_{2,-}(a, z)^{-1}) \tag{2.23}$$

and

$$g(z) = \frac{\varphi_{1,-}(1,z)}{C_0 \prod_{\lambda_{1,n}\in S}(1 - \frac{z}{\lambda_{1,n}})}.$$

It is easy to see that $F(z)$ is an entire function. In addition, from [3, pp. 2779,2782], we conclude that $F(z)$ satisfies $|F(z)| \le C_1 e^{c_2|z|^{1/2}}$ for all $z \in \mathbb{C}$. By (2.17) and [3, pp. 2770,2784] we have

$$\begin{gathered}
|g(iy)| \ge C|\varphi_{1,-}(1,iy)|^{x_0} \ge C|\varphi_{1,-}(1,iy)|^{a}, \\
|\varphi'_{j,-}(a,iy)| = \frac{1}{2}|e^{a\,\mathrm{Im}(\sqrt{i})|y|^{1/2}}|(1+o(1)), \\
|-m_{1,-}(\frac{1}{2},iy)^{-1} + m_{2,-}(\frac{1}{2},iy)^{-1}| = o(\frac{1}{|y|}),
\end{gathered}$$

as y (real) $\to \infty$. Here C is a positive constant. Consequently, we have

$$|F(iy)| \le \frac{\frac{1}{4}|e^{2a\,\mathrm{Im}(\sqrt{i})|y|^{1/2}}|(1+o(1))}{(\frac{1}{2})^{1-x_0}|e^{2a\,\mathrm{Im}(\sqrt{i})|y|^{1/2}}|(1+o(1))} \times o(\frac{1}{|y|}) \to 0 \tag{2.24}$$

as y (real) $\to \infty$. By [3, Proposition B.6], we obtain $F(z) \equiv 0$. This yields that $m_{1,-}(a,z) = m_{2,-}(a,z)$ for all $z \in \mathbb{C}$. That is, $q_1 = q_2$ on $[0,1]$ by Marchenko's uniqueness theorem [2].

Case 2: $x_0 < a$ and $a \le (1-x_0)/2$. In this case, we have $x_0 < (1-x_0)/2 < 1-x_0$ and from (2.17) we infer that

$$N_S(t) \le N_{\sigma(L_-)}(t) = x_0 N_{\sigma(L_1)}(t), \tag{2.25}$$

which implies

$$N_{S^c}(t) \ge (1-x_0)N_{\sigma(L_1)}(t) \tag{2.26}$$

for t sufficiently large. By [3, Theorem A.3], we obtain $q_1 = q_2$ on $[0,1]$. The proof is complete. $\square$

Theorem 2.4. *Consider the parabolic equations defined as* (1.1)-(1.3)*, where* $q \in L^1[0,1]$ *and* $g \in AC(0,t_0]$ *for some* $t_0 > 0$ *are real functions* ($g \not\equiv 0$)*. Let* $u(x,t)$ *be its a solution. Let* S *be defined as* (2.16) *for the operator* L *which is defined by* (1.4)*. Assume that*

$$N_S(t) \le AN_{\sigma(L)}(t) + B \tag{2.27}$$

for t *sufficiently large, where* $B \le 0$*. Then* q *on* $[0,1]$ *is uniquely determined by* q *on* $[a,1]$*,* $u(x_0,t)$ *and* $g(t)$ *for* $(0,t_0]$ *when* $x_0 < a$ *and* $a \le (1-A)/2$*.*

Remark 2.5. It is easy to see that $A \in [0, \min\{x_0, 1-x_0\}]$ in (2.17). Moreover, if $A = B = 0$; that is, $x_0 \notin \Omega$ defined by (1.6). Then $a = 1/2$ and therefore q on $[0,1]$ is uniquely determined by q on $[1/2,1]$, $u(x_0,t)$ and $g(t)$ for $(0,t_0]$.

Proof of Theorem 2.4. When $x_0 < a$ and $a \le (1-A)/2$, we know that

$$N_S(t) \le AN_{\sigma(L_1)}(t) + B < (1-2a)N_{\sigma(L_1)}(t). \tag{2.28}$$

Then we obtain

$$N_{S^c}(t) \ge 2aN_{\sigma(L_1)}(t) \tag{2.29}$$

for t sufficiently large. By [3, Theorem A.3], we obtain $q_1 = q_2$ on $[0,1]$. The proof is complete. $\square$

Acknowledgements. This research was supported in part by the NNSF under the grant number 11571212.

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Xuewen Wu

College of Mathematics and Information Science, Shaanxi Normal University, Xi'an 710062, China

E-mail address: xuewenwu1991@163.com

Guangsheng Wei

College of Mathematics and Information Science, Shaanxi Normal University, Xi'an 710062, China

E-mail address: weimath@vip.sina.com