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FRACTIONAL BOUNDARY VALUE PROBLEMS WITH MULTIPLE ORDERS OF FRACTIONAL DERIVATIVES AND INTEGRALS

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Abstract. In this article we study a new class of boundary value problems for fractional differential equations and inclusions with multiple orders of fractional derivatives and integrals, in both fractional differential equation and boundary conditions. The Sadovski's fixed point theorem is applied in the single-valued case while, in multi-valued case, the nonlinear alternative for contractive maps is used. Some illustrative examples are also included.

1. Introduction

Fractional differential equations have attracted more and more attention in recent years, which is partly because of their numerous applications in many branches of science and engineering including fluid flow, signal and image processing, fractals theory, control theory, electromagnetic theory, fitting of experimental data, optics, potential theory, biology, chemistry, diffusion, and viscoelasticity, etc. For a detailed account of applications and recent results on initial and boundary value problems of fractional differential equations, we refer the reader to a series of books and papers [1, 2, 4, 5, 6, 7, 8, 10, 13, 16, 17, 19, 20, 23, 24, 25] and references cited therein.

In this article, we investigate boundary value problems which contains multiple orders of fractional derivatives and integrals, in both fractional differential equation and boundary conditions. More precisely, we consider the following boundary value problems which consist from the differential equation

$$\left(\lambda D^{\alpha} + (1-\lambda) D^{\beta}\right) x(t) = f(t, x(t)), \quad t \in (0, T), \tag{1.1}$$

which includes two fractional derivatives, supplemented by boundary conditions with:

- two fractional derivatives

$$x(0) = 0, \quad \mu D^{\gamma_1} x(T) + (1-\mu) D^{\gamma_2} x(T) = \gamma_3, \tag{1.2}$$

or

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- two fractional integrals

$$x(0) = 0, \quad \mu I^{\delta_1} x(T) + (1-\mu) I^{\delta_2} x(T) = \delta_3, \tag{1.3}$$

or

- one fractional derivative and one fractional integral

$$x(0) = 0, \quad \mu D^{\gamma_1} x(T) + (1-\mu) I^{\delta_2} x(T) = \gamma_3, \tag{1.4}$$

where D^ϕ is the Riemann-Liouville or Caputo fractional derivative of order $\phi \in \{\alpha, \beta, \gamma_1, \gamma_2\}$ such that $1 < \alpha, \beta \le 2$ and $0 < \gamma_1, \gamma_2 < \alpha - \beta$, $\gamma_3, \delta_3 \in \mathbb{R}$, I^χ is the Riemann-Liouville fractional integral of order $\chi \in \{\delta_1, \delta_2\}$, $0 < \lambda \le 1$, $0 \le \mu \le 1$ are given constants and $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

Also we consider the multi-valued analogues of boundary value problems above by studying the differential inclusion

$$\left(\lambda D^\alpha + (1-\lambda) D^\beta\right) x(t) \in F(t, x(t)), \quad t \in (0, T), \tag{1.5}$$

supplemented by boundary conditions (1.2)-(1.4), where $F : [0, T] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued function ($\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$).

In fact, fractional calculus provide an excellent tool for the description of memory and hereditary properties of various materials and processes in mathematical modeling. The fractional differential equation (1.1) and inclusion (1.5) subject to boundary conditions (1.2), (1.3) and (1.4) describe models of physical problems in which often some parameters have been adjusted to suitable situations. The values of these parameters can be change the effects of fractional derivatives and integrals. Especially, in this paper, the linear adjusting or convex combination is used.

Recently in [15] we studied problem (1.1)-(1.2) with four Riemann-Liouville fractional derivatives. Existence and uniqueness results were proved by using Banach's fixed point theorem, Krasnoselskii's fixed point theorem and Leray-Schauder's nonlinear alternative. Similar results for the boundary value problems (1.1)-(1.2) to (1.1)-(1.4) can be established also for Caputo fractional derivatives, with obvious modifications.

In this article we prove an existence result for the boundary value problem (1.1)-(1.2), with four Caputo type fractional derivatives, via Sadovski's fixed point theorem and an existence result for the multi-valued analogue (1.5)-(1.2), by means of nonlinear alternative for contractive maps.

This article is organized as follows. In section 2, we present the framework in which the boundary value problems (1.1)-(1.2), (1.1)-(1.3), (1.1)-(1.4), are formulated in a fixed point equation. Section 3 is devoted to the problem (1.1)-(1.2) and Section 4 to the problem (1.5)-(1.2). Illustrative examples are also presented.

2. Preliminaries

In this section, we introduce some notation and definitions of fractional calculus [13, 19] and present preliminary results needed in our proofs later.

Definition 2.1. The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $g : (0, \infty) \to \mathbb{R}$ is defined by

$$J^\alpha g(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} g(s) ds,$$

provided the right-hand side is point-wise defined on $(0, \infty)$, where Γ is the Gamma function.

Definition 2.2. The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a continuous function $g : (0, \infty) \to \mathbb{R}$ is defined by

$$D^\alpha g(t) = \frac{1}{\Gamma(n-\alpha)} \Big(\frac{d}{dt}\Big)^n \int_0^t \frac{g(s)}{(t-s)^{\alpha-n+1}} ds, \quad n-1 < \alpha < n,$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of real number α, provided the right-hand side is point-wise defined on $(0, \infty)$.

Definition 2.3. The Caputo derivative of order q for a function $f : [0, \infty) \to \mathbb{R}$ can be written as

$${}^cD^q f(t) = D^q \Big(f(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} f^{(k)}(0) \Big), \quad t > 0,\ n-1 < q < n.$$

Remark 2.4. If $f(t) \in C^n[0, \infty)$, then

$${}^cD^q f(t) = \frac{1}{\Gamma(n-q)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{q+1-n}} ds = I^{n-q} f^{(n)}(t), \quad t > 0,\ n-1 < q < n.$$

Lemma 2.5. *For $q > 0$, the general solution of the fractional differential equation ${}^cD^q x(t) = 0$ is given by*

$$x(t) = c_0 + c_1 t + \dots + c_{n-1} t^{n-1},$$

where $c_i \in \mathbb{R}$, $i = 0, 1, 2, \dots, n-1$ ($n = [q] + 1$).

In view of Lemma 2.5, it follows that

$$I^q\ {}^cD^q x(t) = x(t) + c_0 + c_1 t + \dots + c_{n-1} t^{n-1}, \tag{2.1}$$

for some $c_i \in \mathbb{R}$, $i = 0, 1, 2, \dots, n-1$ ($n = [q] + 1$).

Lemma 2.6. *The boundary value problem*

$$\begin{gathered} \big(\lambda D^\alpha + (1-\lambda) D^\beta\big) x(t) = \omega(t), \quad t \in (0, T), \\ x(0) = 0, \quad \mu D^{\gamma_1} x(T) + (1-\mu) D^{\gamma_2} x(T) = \gamma_3, \end{gathered} \tag{2.2}$$

is equivalent to the integral equation

$$\begin{aligned} x(t) &= \frac{\lambda - 1}{\lambda\Gamma(\alpha-\beta)} \int_0^t (t-s)^{\alpha-\beta-1} x(s) ds + \frac{1}{\lambda\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \omega(s) ds \\ &\quad + \frac{t}{\Lambda_1} \Big(\gamma_3 - \frac{\mu(\lambda-1)}{\lambda\Gamma(\alpha-\beta-\gamma_1)} \int_0^T (T-s)^{\alpha-\beta-\gamma_1-1} x(s) ds \\ &\quad - \frac{\mu}{\lambda\Gamma(\alpha-\gamma_1)} \int_0^T (T-s)^{\alpha-\gamma_1-1} \omega(s) ds \\ &\quad - \frac{(1-\mu)(\lambda-1)}{\lambda\Gamma(\alpha-\beta-\gamma_2)} \int_0^T (T-s)^{\alpha-\beta-\gamma_2-1} x(s) ds \\ &\quad - \frac{1-\mu}{\lambda\Gamma(\alpha-\gamma_2)} \int_0^T (T-s)^{\alpha-\gamma_2-1} \omega(s) ds \Big), \quad t \in J := [0, T], \end{aligned} \tag{2.3}$$

where the non zero constant Λ_1 is defined by

$$\Lambda_1 = \frac{\mu T^{1-\gamma_1}}{\Gamma(2-\gamma_1)} + \frac{(1-\mu) T^{1-\gamma_2}}{\Gamma(2-\gamma_2)}. \tag{2.4}$$

Proof. From the first equation of (2.2), we have

$$D^{\alpha}x(t) = \frac{\lambda-1}{\lambda}D^{\beta}x(t) + \frac{1}{\lambda}\omega(t), \quad t \in J. \tag{2.5}$$

Taking the Riemann-Liouville fractional integral of order α to both sides of (2.5), we obtain

$$\begin{aligned} x(t) = &\frac{\lambda-1}{\lambda\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1}x(s)ds + \frac{1}{\lambda\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\omega(s)ds \\ &+ C_1 + C_2 t, \end{aligned}$$

for C_1, $C_2 \in \mathbb{R}$. The first boundary condition of (2.2) implies that $C_1 = 0$. Hence

$$x(t) = \frac{\lambda-1}{\lambda\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1}x(s)ds + \frac{1}{\lambda\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\omega(s)ds + C_2 t. \tag{2.6}$$

Applying the Caputo fractional derivative of order $\psi \in \{\gamma_1, \gamma_2\}$ such that $0 < \psi < \alpha - \beta$ to (2.6), we have

$$\begin{aligned} D^{\psi}x(t) = &\frac{\lambda-1}{\lambda\Gamma(\alpha-\beta-\psi)}\int_0^t (t-s)^{\alpha-\beta-\psi-1}x(s)ds \\ &+ \frac{1}{\lambda\Gamma(\alpha-\psi)}\int_0^t (t-s)^{\alpha-\psi-1}\omega(s)ds + C_2\frac{1}{\Gamma(2-\psi)}t^{1-\psi}. \end{aligned}$$

Substituting the values $\psi = \gamma_1$ and $\psi = \gamma_2$ to the above relation and using the second condition of (2.2), we obtain

$$\begin{aligned} \gamma_3 = &\frac{\mu(\lambda-1)}{\lambda\Gamma(\alpha-\beta-\gamma_1)}\int_0^T (T-s)^{\alpha-\beta-\gamma_1-1}x(s)ds \\ &+ \frac{\mu}{\lambda\Gamma(\alpha-\gamma_1)}\int_0^T (T-s)^{\alpha-\gamma_1-1}\omega(s)ds + \frac{\mu T^{1-\gamma_1}}{\Gamma(2-\gamma_1)}C_2 \\ &+ \frac{(1-\mu)(\lambda-1)}{\lambda\Gamma(\alpha-\beta-\gamma_2)}\int_0^T (T-s)^{\alpha-\beta-\gamma_2-1}x(s)ds \\ &+ \frac{1-\mu}{\lambda\Gamma(\alpha-\gamma_2)}\int_0^T (T-s)^{\alpha-\gamma_2-1}\omega(s)ds + \frac{(1-\mu)T^{1-\gamma_2}}{\Gamma(2-\gamma_2)}C_2, \end{aligned}$$

which leads to

$$\begin{aligned} C_2 = \frac{1}{\Lambda_1}\Big[&\gamma_3 - \frac{\mu(\lambda-1)}{\lambda\Gamma(\alpha-\beta-\gamma_1)}\int_0^T (T-s)^{\alpha-\beta-\gamma_1-1}x(s)ds \\ &- \frac{\mu}{\lambda\Gamma(\alpha-\gamma_1)}\int_0^T (T-s)^{\alpha-\gamma_1-1}\omega(s)ds \\ &- \frac{(1-\mu)(\lambda-1)}{\lambda\Gamma(\alpha-\beta-\gamma_2)}\int_0^T (T-s)^{\alpha-\beta-\gamma_2-1}x(s)ds \\ &- \frac{1-\mu}{\lambda\Gamma(\alpha-\gamma_2)}\int_0^T (T-s)^{\alpha-\gamma_2-1}\omega(s)ds\Big]. \end{aligned}$$

Substituting the value of the constant C_2 in (2.6), we deduce the integral equation (2.3). The converse follows by direct computation. This completes the proof. □

The following lemmas concerning the boundary value problems (1.1)-(1.3) and (1.1)-(1.4), are similar to that of Lemma 2.6. We omit the proofs.

Lemma 2.7. *The boundary value problem*

$$\begin{gathered}\big(\lambda D^{\alpha}+(1-\lambda)D^{\beta}\big)\,x(t)=\omega(t),\quad t\in(0,T),\\ x(0)=0,\quad \mu I^{\delta_1}x(T)+(1-\mu)I^{\delta_2}x(T)=\delta_3,\end{gathered}\tag{2.7}$$

is equivalent to the integral equation

$$\begin{aligned}x(t)&=\frac{\lambda-1}{\lambda\Gamma(\alpha-\beta)}\int_0^t(t-s)^{\alpha-\beta-1}x(s)ds+\frac{1}{\lambda\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\omega(s)ds\\ &\quad+\frac{t}{\Lambda_2}\Big(\delta_3-\frac{\mu(\lambda-1)}{\lambda\Gamma(\delta_1+\alpha-\beta)}\int_0^T(T-s)^{\delta_1+\alpha-\beta-1}x(s)ds\\ &\quad-\frac{\mu}{\lambda\Gamma(\delta_1+\alpha)}\int_0^T(T-s)^{\delta_1+\alpha-1}\omega(s)ds\\ &\quad-\frac{(1-\mu)(\lambda-1)}{\lambda\Gamma(\delta_2+\alpha-\beta)}\int_0^T(T-s)^{\delta_2+\alpha-\beta-1}x(s)ds\\ &\quad-\frac{1-\mu}{\lambda\Gamma(\delta_2+\alpha)}\int_0^T(T-s)^{\delta_2+\alpha-1}\omega(s)ds\Big),\quad t\in J:=[0,T],\end{aligned}\tag{2.8}$$

where the non-zero constant Λ_2 *is defined by*

$$\Lambda_2=\frac{\mu T^{1+\delta_1}}{\Gamma(2+\delta_1)}+\frac{(1-\mu)T^{1+\delta_2}}{\Gamma(2+\delta_2)}.\tag{2.9}$$

Lemma 2.8. *The boundary value problem*

$$\begin{gathered}\big(\lambda D^{\alpha}+(1-\lambda)D^{\beta}\big)\,x(t)=\omega(t),\quad t\in(0,T),\\ x(0)=0,\quad \mu D^{\gamma_1}x(T)+(1-\mu)I^{\delta_2}x(T)=\gamma_3,\end{gathered}\tag{2.10}$$

is equivalent to the integral equation

$$\begin{aligned}x(t)&=\frac{\lambda-1}{\lambda\Gamma(\alpha-\beta)}\int_0^t(t-s)^{\alpha-\beta-1}x(s)ds+\frac{1}{\lambda\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\omega(s)ds\\ &\quad+\frac{t}{\Lambda_3}\Big(\gamma_3-\frac{\mu(\lambda-1)}{\lambda\Gamma(\alpha-\beta-\gamma_1)}\int_0^T(T-s)^{\alpha-\beta-\gamma_1-1}x(s)ds\\ &\quad-\frac{\mu}{\lambda\Gamma(\alpha-\gamma_1)}\int_0^T(T-s)^{\alpha-\gamma_1-1}\omega(s)ds\\ &\quad-\frac{(1-\mu)(\lambda-1)}{\lambda\Gamma(\delta_2+\alpha-\beta)}\int_0^T(T-s)^{\delta_2+\alpha-\beta-1}x(s)ds\\ &\quad-\frac{1-\mu}{\lambda\Gamma(\delta_2+\alpha)}\int_0^T(T-s)^{\delta_2+\alpha-1}\omega(s)ds\Big),\quad t\in J:=[0,T],\end{aligned}\tag{2.11}$$

where the non zero constant Λ_3 *is defined by*

$$\Lambda_3=\frac{\mu T^{1-\gamma_1}}{\Gamma(2-\gamma_1)}+\frac{(1-\mu)T^{1+\delta_2}}{\Gamma(2+\delta_2)}.\tag{2.12}$$

3. Existence result for problem (1.1)-(1.2)

Let $\mathcal{C}:=C([0,T],\mathbb{R})$ denote the Banach space of all continuous functions from $[0,T]$ into $\mathbb{R}$ with the norm $\|x\|=\sup\{|x(t)|,\,t\in[0,T]\}$.

Our existence result for the problem (1.1)-(1.2) is based on Sadovskii's fixed point theorem. Before proceeding further, let us recall some auxiliary material.

Definition 3.1. Let M be a bounded set in metric space (X,d), then Kuratowskii measure of noncompactness, $\alpha(M)$ is defined as $\inf\{\epsilon : M$ covered by a finitely many sets such that the diameter of each set $\leq \epsilon\}$.

Definition 3.2 ([11])**.** Let $\Phi : \mathcal{D}(\Phi) \subseteq X \to X$ be a bounded and continuous operator on Banach space X. Then Φ is called a condensing map if $\alpha(\Phi(B)) < \alpha(B)$ for all bounded sets $B \subset \mathcal{D}(\Phi)$, where α denotes the Kuratowski measure of noncompactness.

Lemma 3.3 ([26, Example 11.7])**.** *The map $K + C$ is a k-set contraction with $0 \leq k < 1$, and thus also condensing, if*

(i) *$K, C : \mathcal{D} \subseteq X \to X$ are operators on the Banach space X;*

(ii) *K is k-contractive, i.e.,*
$$\|Kx - Ky\| \leq k\|x - y\|$$
for all $x, y \in \mathcal{D}$ and fixed $k \in [0, 1)$;

(iii) *C is compact.*

Lemma 3.4 ([21])**.** *Let B be a convex, bounded and closed subset of a Banach space X and $\Phi : B \to B$ be a condensing map. Then Φ has a fixed point.*

Theorem 3.5. *Let $f : J \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Assume that:*

(H1) *there exists a function $\nu \in C(J, \mathbb{R}_+)$ such that*
$$|f(t, u)| \leq \nu(t), \quad \text{for a.e. } t \in J, \text{ and each } u \in \mathbb{R}.$$

(H2)
$$\begin{aligned}\Omega_1 :=& \frac{T^{\alpha-\beta}|\lambda - 1|}{\lambda\Gamma(\alpha - \beta + 1)} + \frac{T^{\alpha-\beta-\gamma_1+1}\mu|\lambda - 1|}{\lambda\Lambda_1\Gamma(\alpha - \beta - \gamma_1 + 1)}\\ &+ \frac{T^{\alpha-\beta-\gamma_2+1}(1-\mu)|\lambda - 1|}{\lambda\Lambda_1\Gamma(\alpha - \beta - \gamma_2 + 1)} < 1.\end{aligned}$$

Then, problem (1.1)-(1.2) *has at least one solution on J.*

Proof. Let $B_r = \{x \in \mathcal{C} : \|x\| \leq r\}$ be a closed bounded and convex subset of $\mathcal{C}$, where r is a fixed constant. Consider the operator $\mathcal{P} : \mathcal{C} \to \mathcal{C}$ defined by
$$\begin{aligned}\mathcal{P}x(t) =& \frac{\lambda - 1}{\lambda\Gamma(\alpha - \beta)}\int_0^t (t - s)^{\alpha-\beta-1}x(s)ds + \frac{1}{\lambda\Gamma(\alpha)}\int_0^t (t - s)^{\alpha-1}f(s, x(s))ds\\ &+ \frac{t}{\Lambda_1}\Big(\gamma_3 - \frac{\mu(\lambda - 1)}{\lambda\Gamma(\alpha - \beta - \gamma_1)}\int_0^T (T - s)^{\alpha-\beta-\gamma_1-1}x(s)ds\\ &- \frac{\mu}{\lambda\Gamma(\alpha - \gamma_1)}\int_0^T (T - s)^{\alpha-\gamma_1-1}f(s, x(s))ds\\ &- \frac{(1-\mu)(\lambda - 1)}{\lambda\Gamma(\alpha - \beta - \gamma_2)}\int_0^T (T - s)^{\alpha-\beta-\gamma_2-1}x(s)ds\\ &- \frac{1-\mu}{\lambda\Gamma(\alpha - \gamma_2)}\int_0^T (T - s)^{\alpha-\gamma_2-1}f(s, x(s))ds\Big), \quad t \in J.\end{aligned} \tag{3.1}$$

Let us define $\mathcal{P}_1, \mathcal{P}_2 : \mathcal{C} \to \mathcal{C}$ by
$$(\mathcal{P}_1x)(t) = \frac{(\lambda - 1)}{\lambda\Gamma(\alpha - \beta)}\int_0^t (t - s)^{\alpha-\beta-1}x(s)ds$$

$$
\begin{aligned}
&- \frac{t}{\Lambda_1}\Big[\frac{\mu(\lambda-1)}{\lambda\Gamma(\alpha-\beta-\gamma_1)}\int_0^T (T-s)^{\alpha-\beta-\gamma_1-1}x(s)ds \\
&+ \frac{(1-\mu)(\lambda-1)}{\lambda\Gamma(\alpha-\beta-\gamma_2)}\int_0^T (T-s)^{\alpha-\beta-\gamma_2-1}x(s)ds\Big],
\end{aligned}
$$

and

$$
\begin{aligned}
(\mathcal{P}_2x)(t) &= \frac{1}{\lambda\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s,x(s))ds \\
&+ \frac{t}{\Lambda_1}\Big[\gamma_3 - \frac{\mu}{\lambda\Gamma(\alpha-\gamma_1)}\int_0^T (T-s)^{\alpha-\gamma_1-1}f(s,x(s))ds \\
&- \frac{(1-\mu)}{\lambda\Gamma(\alpha-\gamma_2)}\int_0^T (T-s)^{\alpha-\gamma_2-1}f(s,x(s))ds\Big].
\end{aligned}
$$

Clearly

$$
(\mathcal{P}x)(t) = (\mathcal{P}_1x)(t) + (\mathcal{P}_2x)(t),\ t\in J. \tag{3.2}
$$

Obviously the operator $\mathcal{P}$ has a fixed point is equivalent to $\mathcal{P}_1+\mathcal{P}_2$ has one, so it turns to prove that $\mathcal{P}_1+\mathcal{P}_2$ has a fixed point. We shall show that the operators $\mathcal{P}_1$ and $\mathcal{P}_2$ satisfy all conditions of Lemma 3.4. The proof will be given in several steps.

Step 1: $\mathcal{P}B_r\subset B_r$. Let us select $r\geq \frac{\|\nu\|\Omega_2+|\gamma_3|T/\Lambda_1}{1-\Omega_1}$ where Ω_1 defined by (H2) and

$$
\Omega_2 = \frac{T^\alpha}{\lambda\Gamma(\alpha+1)} + \frac{T^{\alpha-\gamma_1+1}\mu}{\lambda\Lambda_1\Gamma(\alpha-\gamma_1+1)} + \frac{T^{\alpha-\gamma_2+1}(1-\mu)}{\lambda\Lambda_1\Gamma(\alpha-\gamma_2+1)}. \tag{3.3}
$$

For any $x\in B_r$, we have

$$
\begin{aligned}
&\|\mathcal{P}x\| \\
&\leq \sup_{t\in J}\Big|\frac{(\lambda-1)}{\lambda\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1}x(s)ds + \frac{1}{\lambda\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s,x(s))ds \\
&\quad - \frac{t\mu(\lambda-1)}{\lambda\Lambda_1\Gamma(\alpha-\beta-\gamma_1)}\int_0^T (T-s)^{\alpha-\beta-\gamma_1-1}x(s)ds \\
&\quad - \frac{t(1-\mu)(\lambda-1)}{\lambda\Lambda_1\Gamma(\alpha-\beta-\gamma_2)}\int_0^T (T-s)^{\alpha-\beta-\gamma_2-1}x(s)ds \\
&\quad + \frac{t}{\Lambda_1}\Big[\gamma_3 - \frac{\mu}{\lambda\Gamma(\alpha-\gamma_1)}\int_0^T (T-s)^{\alpha-\beta-\gamma_1-1}f(s,x(s))ds \\
&\quad - \frac{(1-\mu)}{\lambda\Gamma(\alpha-\gamma_2)}\int_0^T (T-s)^{\alpha-\gamma_2-1}f(s,x(s))ds\Big]\Big| \\
&\leq \|x\|\Big[\frac{T^{\alpha-\beta}|\lambda-1|}{\lambda\Gamma(\alpha-\beta+1)} + \frac{T^{\alpha-\beta-\gamma_1+1}\mu|\lambda-1|}{\lambda\Lambda_1\Gamma(\alpha-\beta-\gamma_1+1)} \\
&\quad + \frac{T^{\alpha-\beta-\gamma_2+1}(1-\mu)|\lambda-1|}{\lambda\Lambda_1\Gamma(\alpha-\beta-\gamma_2+1)}\Big] + \frac{|\gamma_3|T}{\Lambda_1} \\
&\quad + \|\nu\|\Big[\frac{T^\alpha}{\lambda\Gamma(\alpha+1)} + \frac{T^{\alpha-\gamma_1+1}\mu}{\lambda\Lambda_1\Gamma(\alpha-\gamma_1+1)} + \frac{T^{\alpha-\gamma_2+1}(1-\mu)}{\lambda\Lambda_1\Gamma(\alpha-\gamma_2+1)}\Big] \\
&\leq r\Omega_1 + \|\nu\|\Omega_2 + \frac{|\gamma_3|T}{\Lambda} \leq r,
\end{aligned}
$$

which implies that $\mathcal{P}B_r\subset B_r$.

Step 2: $\mathcal{P}_2$ is compact. Observe that the operator $\mathcal{P}_2$ is uniformly bounded in view of Step 1. Let $t_1, t_2 \in J$ with $t_1 < t_2$ and $x \in B_r$. Then we obtain

$$\begin{aligned}
&|(\mathcal{P}_2 x)(t_2) - (\mathcal{P}_2 x)(t_1)| \\
&\le \frac{1}{\lambda\Gamma(\alpha)}\Big[\int_0^{t_1}[t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}]\nu(s)ds + \int_{t_1}^{t_2}(t_1-s)^{\alpha-1}\nu(s)ds\Big] \\
&\quad + \frac{|t_2-t_1|}{\Lambda_1}\Big[|\gamma_3| + \frac{\mu}{\lambda\Gamma(\alpha-\gamma_1)}\int_0^T (T-s)^{\alpha-\beta-\gamma_1-1}\nu(s)ds \\
&\quad + \frac{(1-\mu)}{\lambda\Gamma(\alpha-\gamma_2)}\int_0^T (T-s)^{\alpha-\gamma_2-1}\nu(s)ds\Big],
\end{aligned}$$

which is independent of x and tends to zero as $t_2 - t_1 \to 0$. Thus, $\mathcal{P}_2$ is equicontinuous. Hence, by the Arzelá-Ascoli Theorem, $\mathcal{P}_2(B_r)$ is a relatively compact set.

Step 3: $\mathcal{P}_1$ is γ-contractive. Let $x, y \in B_r$. Then, we have

$$\begin{aligned}
\|\mathcal{P}_1 x - \mathcal{P}_1 y\| &\le \frac{|\lambda-1|}{\lambda\Gamma(\alpha-\beta)}\int_0^T (T-s)^{\alpha-\beta-1}|x(s)-y(s)|ds \\
&\quad + \frac{T\mu|\lambda-1|}{\lambda\Lambda_1\Gamma(\alpha-\beta-\gamma_1)}\int_0^T (T-s)^{\alpha-\beta-\gamma_1-1}|x(s)-y(s)|ds \\
&\quad + \frac{T(1-\mu)|\lambda-1|}{\lambda\Lambda_1\Gamma(\alpha-\beta-\gamma_2)}\int_0^T (T-s)^{\alpha-\beta-\gamma_2-1}|x(s)-y(s)|ds \\
&\le \Big\{\frac{T^{\alpha-\beta}|\lambda-1|}{\lambda\Gamma(\alpha-\beta+1)} + \frac{T^{\alpha-\beta-\gamma_1+1}\mu|\lambda-1|}{\lambda\Lambda_1\Gamma(\alpha-\beta-\gamma_1+1)} \\
&\quad + \frac{T^{\alpha-\beta-\gamma_2+1}(1-\mu)|\lambda-1|}{\lambda\Lambda_1\Gamma(\alpha-\beta-\gamma_2+1)}\Big]\Big\}\|x-y\| \\
&= \Omega_1\|x-y\|,
\end{aligned}$$

which is γ-contractive, since $\Omega_1 < 1$.

Step 4: $\mathcal{P}$ is condensing. Since $\mathcal{P}_1$ is continuous, γ-contraction and $\mathcal{P}_2$ is compact, therefore, by Lemma 3.3, $\mathcal{P} : B_r \to B_r$ with $\mathcal{P} = \mathcal{P}_1 + \mathcal{P}_2$ is a condensing map on B_r.

From the above four steps, we conclude by Lemma 3.4 that the map $\mathcal{P}$ has a fixed point which, in turn, implies that the problem (1.1)-(1.2) has a solution. □

Setting two constants

$$\begin{aligned}
\Omega_3 &= \frac{|\lambda-1|T^{\alpha-\beta}}{\lambda\Gamma(\alpha-\beta+1)} + \frac{\mu|\lambda-1|T^{\delta_1+\alpha-\beta+1}}{\lambda\Lambda_2\Gamma(\delta_1+\alpha-\beta+1)} + \frac{(1-\mu)|\lambda-1|T^{\delta_2+\alpha-\beta+1}}{\lambda\Lambda_2\Gamma(\delta_2+\alpha-\beta+1)}, \\
\Omega_4 &= \frac{|\lambda-1|T^{\alpha-\beta}}{\lambda\Gamma(\alpha-\beta+1)} + \frac{\mu|\lambda-1|T^{\alpha-\beta-\gamma_1+1}}{\lambda\Lambda_3\Gamma(\alpha-\beta-\gamma_1+1)} + \frac{(1-\mu)|\lambda-1|T^{\delta_2+\alpha-\beta+1}}{\lambda\Lambda_3\Gamma(\delta_2+\alpha-\beta+1)}.
\end{aligned}$$

Theorem 3.6. *Let condition* (H1) *of Theorem 3.5 be satisfied. If* $\Omega_3 < 1$*, then problem* (1.1)-(1.3) *has at least one solution on* J.

Theorem 3.7. *Let condition* (H1) *of Theorem 3.5 be satisfied. If* $\Omega_4 < 1$*, then problem* (1.1)-(1.4) *has at least one solution on* J.

Remark 3.8. If $\lambda = 1$, then (1.1) is reduced to a single order fractional differential equation and also $\Omega_1 = \Omega_3 = \Omega_4 = 0$. In this case, only condition (H1) can be used for the existence of solutions for problems (1.1)-(1.2), (1.1)-(1.3) and (1.1)-(1.4).

Example 3.9. Let us consider the following two orders fractional differential equation with two orders fractional derivative boundary conditions

$$\frac{38}{43}D^{7/4}x(t) + \frac{5}{43}D^{5/4}x(t) = \frac{x(t)e^{2t}}{|x(t)|+1}\sin^2 x(t) + \frac{2}{3}, \quad t \in [0, 3/2], \tag{3.4}$$

$$x(0) = 0, \quad \frac{15}{32}D^{1/3}x\left(\frac{3}{2}\right) + \frac{17}{32}D^{1/4}x\left(\frac{3}{2}\right) = \frac{3}{4}. \tag{3.5}$$

Here $\lambda = 38/43$, $\alpha = 7/4$, $\beta = 5/4$, $\mu = 15/32$, $\gamma_1 = 1/3$, $\gamma_2 = 1/4$, $\gamma_3 = 3/4$, $T = 3/2$ and $f(t,x) = (xe^{2t}\sin^2 x)/(|x|+1) + (2/3)$. Observe that $0 < \gamma_1, \gamma_2 < (1/2) = \alpha - \beta$. It is obvious that

$$|f(t,x)| \le e^{2t} + \frac{2}{3} := v(t),$$

which satisfies condition (H1) of Theorem 3.5. In addition, we can find that

$$\Omega_1 = 0.3421779589 < 1.$$

Hence, by Theorem 3.5, the four orders fractional boundary value problem (3.4)-(3.5) has at least one solution on $[0, 3/2]$.

Example 3.10. Let us consider the two orders fractional differential equation (3.4) subject to two orders fractional boundary conditions

$$x(0) = 0, \quad \frac{7}{16}I^{3/4}x\left(\frac{3}{2}\right) + \frac{9}{16}I^{4/5}x\left(\frac{3}{2}\right) = \frac{1}{6}, \tag{3.6}$$

and mixed fractional derivative and integral boundary conditions

$$x(0) = 0, \quad \frac{13}{28}D^{1/5}x\left(\frac{3}{2}\right) + \frac{15}{28}I^{4/3}x\left(\frac{3}{2}\right) = \frac{3}{7}. \tag{3.7}$$

Problem I (3.4)-(3.6). In this case $\mu = 7/16$, $\delta_1 = 3/4$, $\delta_2 = 4/5$ and $\delta_3 = 1/6$. We can find that $\Lambda_2 = 1.249160013$ and $\Omega_3 = 0.4121621065 < 1$. Therefore, by applying Theorem 3.6, the two orders fractional derivatives and integrals boundary value problem (3.4)-(3.6) has at least one solution on $[0, 3/2]$.
Problem II (3.4)-(3.7). In the final case $\mu = 13/28$, $\gamma_1 = 1/5$, $\delta_2 = 4/3$ and $\gamma_3 = 3/7$. We can find that $\Lambda_3 = 1.186148831$ and $\Omega_4 = 0.3877544803 < 1$. Therefore, by using the conclusion in Theorem 3.7, the mixed type of fractional derivative and integral boundary value problem (3.4)-(3.7) has at least one solution on $[0, 3/2]$.

4. Existence result for problem (1.5)-(1.2)

First of all, we recall some basic concepts for multi-valued maps [9, 12, 22]. For a normed space $(X, \|\cdot\|)$, let $\mathcal{P}_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is closed}\}$, $\mathcal{P}_b(X) = \{Y \in \mathcal{P}(X) : Y \text{ is bounded}\}$, $\mathcal{P}_{cp}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact}\}$ and $\mathcal{P}_{cp,c}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact and convex}\}$.

A multi-valued map $G : X \to \mathcal{P}(X)$:

(i) is *convex (closed) valued* if $G(x)$ is convex (closed) for all $x \in X$;
(ii) is *bounded* on bounded sets if $G(\mathbb{B}) = \cup_{x\in\mathbb{B}}G(x)$ is bounded in X for all $\mathbb{B} \in \mathcal{P}_b(X)$ (i.e. $\sup_{x\in\mathbb{B}}\{\sup\{|y| : y \in G(x)\}\} < \infty$);

(iii) is called *upper semi-continuous (u.s.c.)* on X if for each $x_0 \in X$, the set $G(x_0)$ is a nonempty closed subset of X, and if for each open set N of X containing $G(x_0)$, there exists an open neighborhood $\mathcal{N}_0$ of x_0 such that $G(\mathcal{N}_0) \subseteq N$;
(iv) G is *lower semi-continuous (l.s.c.)* if the set $\{y \in X : G(y) \cap B \neq \emptyset\}$ is open for any open set B in E;
(v) is said to be *completely continuous* if $G(\mathbb{B})$ is relatively compact for every $\mathbb{B} \in \mathcal{P}_b(X)$;
(vi) is said to be *measurable* if for every $y \in \mathbb{R}$, the function
$$t \mapsto d(y, G(t)) = \inf\{|y - z| : z \in G(t)\}$$
is measurable;
(vii) *has a fixed point* if there is $x \in X$ such that $x \in G(x)$. The fixed point set of the multivalued operator G will be denoted by $FixG$.

Definition 4.1. A multivalued map $F : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$, $J := [0, T]$, is said to be Carathéodory if

(i) $t \mapsto F(t, x)$ is measurable for each $x \in \mathbb{R}$;
(ii) $x \mapsto F(t, x)$ is upper semicontinuous for almost all $t \in J$.

Further a Carathéodory function F is called L^1-Carathéodory if

(iii) for each $\alpha > 0$, there exists $\varphi_\alpha \in L^1(J, \mathbb{R}^+)$ such that
$$\|F(t, x)\| = \sup\{|v| : v \in F(t, x)\} \leq \varphi_\alpha(t)$$
for all $\|x\| \leq \alpha$ and for a.e. $t \in J$.

Recall that $\mathcal{C} := C(J, \mathbb{R})$. For each $x \in \mathcal{C}$, define the set of selections of F by
$$S_{F,x} := \{v \in L^1(J, \mathbb{R}) : v(t) \in F(t, x(t)) \text{ for a.e. } t \in J\}.$$

We define the graph of G to be the set $Gr(G) = \{(x, y) \in X \times Y, y \in G(x)\}$ and recall two useful results regarding closed graphs and upper-semicontinuity.

Lemma 4.2 ([9, Proposition 1.2])**.** *If $G : X \to \mathcal{P}_{cl}(Y)$ is u.s.c., then $Gr(G)$ is a closed subset of $X \times Y$; i.e., for every sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ and $\{y_n\}_{n\in\mathbb{N}} \subset Y$, if when $n \to \infty$, $x_n \to x_*$, $y_n \to y_*$ and $y_n \in G(x_n)$, then $y_* \in G(x_*)$. Conversely, if G is completely continuous and has a closed graph, then it is upper semi-continuous.*

Lemma 4.3 ([14])**.** *Let X be a Banach space. Let $F : J \times \mathbb{R} \to \mathcal{P}_{cp,c}(X)$ be an L^1- Carathéodory multivalued map and let Θ be a linear continuous mapping from $L^1(J, X)$ to $C(J, X)$. Then the operator*
$$\Theta \circ S_F : C(J, X) \to \mathcal{P}_{cp,c}(C(J, X)), \quad x \mapsto (\Theta \circ S_F)(x) = \Theta(S_{F,x,y})$$
is a closed graph operator in $C(J, X) \times C(J, X)$.

To prove our main result in this section, we use the following form of the nonlinear alternative for contractive maps [18, Corollary 3.8].

Theorem 4.4. *Let X be a Banach space, and D a bounded neighborhood of $0 \in X$. Let $Z_1 : X \to \mathcal{P}_{cp,c}(X)$ and $Z_2 : \bar{D} \to \mathcal{P}_{cp,c}(X)$ two multi-valued operators satisfying*

(a) *Z_1 is contraction, and*
(b) *Z_2 is upper semi-continuous and compact.*

Then, if $Q = Z_1 + Z_2$, either

(i) *Q has a fixed point in $\bar{D}$ or*
(ii) *there is a point $u \in \partial D$ and $\lambda \in (0,1)$ with $u \in \lambda Q(u)$.*

Definition 4.5. A function $x \in C^2(J,\mathbb{R})$ is a solution of problem (1.5)-(1.2) if $x(0)=0$, $\mu D^{\gamma_1}x(T)+(1-\mu)D^{\gamma_2}x(T)=\gamma_3$, and there exists function $v \in L^1(J,\mathbb{R})$ such that $v(t) \in F(t,x(t))$ a.e. on J and

$$
\begin{aligned}
x(t) &= \frac{\lambda-1}{\lambda\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1}x(s)ds + \frac{1}{\lambda\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}v(s)ds \\
&\quad + \frac{t}{\Lambda_1}\Big(\gamma_3 - \frac{\mu(\lambda-1)}{\lambda\Gamma(\alpha-\beta-\gamma_1)}\int_0^T (T-s)^{\alpha-\beta-\gamma_1-1}x(s)ds \\
&\quad - \frac{\mu}{\lambda\Gamma(\alpha-\gamma_1)}\int_0^T (T-s)^{\alpha-\gamma_1-1}v(s)ds \\
&\quad - \frac{(1-\mu)(\lambda-1)}{\lambda\Gamma(\alpha-\beta-\gamma_2)}\int_0^T (T-s)^{\alpha-\beta-\gamma_2-1}x(s)ds \\
&\quad - \frac{1-\mu}{\lambda\Gamma(\alpha-\gamma_2)}\int_0^T (T-s)^{\alpha-\gamma_2-1}v(s)ds\Big), \quad t \in J,
\end{aligned} \tag{4.1}
$$

where $\Lambda_1 \neq 0$ is defined by (2.4).

Theorem 4.6. *Assume that* (H2) *holds. In addition we assume that:*

(H3) *$F : J \times \mathbb{R} \to \mathcal{P}_{cp,c}(\mathbb{R})$ is L^1-Carathéodory;*
(H4) *there exists a continuous nondecreasing function $\Phi : [0,\infty) \to (0,\infty)$ and a function $p \in L^1(J,\mathbb{R}^+)$ such that*
$$\|F(t,x)\|_{\mathcal{P}} := \sup\{|y| : y \in F(t,x)\} \le p(t)\Phi(\|x\|) \quad \text{for each } (t,x) \in J \times \mathbb{R};$$
(H5) *there exists a constant $M > 0$ such that*
$$\frac{(1-\Omega_1)M}{\Phi(M)\Psi_1 + |\gamma_3|T/\Lambda_1} > 1, \tag{4.2}$$
where
$$
\begin{aligned}
\Psi_1 &= \frac{1}{\lambda\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}p(s)ds + \frac{T}{\Lambda_1}\Big[\frac{\mu}{\lambda\Gamma(\alpha-\gamma_1)}\int_0^T (T-s)^{\alpha-\gamma_1-1}p(s)ds \\
&\quad + \frac{1-\mu}{\lambda\Gamma(\alpha-\gamma_2)}\int_0^T (T-s)^{\alpha-\gamma_2-1}p(s)ds\Big].
\end{aligned}
$$

Then the boundary value problem (1.5)-(1.2) *has at least one solution on J.*

Proof. To transform problem (1.5)-(1.2) into a fixed point problem, we define an operator $\mathcal{N} : \mathcal{C} \to \mathcal{P}(\mathcal{C})$ by

$$
\begin{aligned}
\mathcal{N}(x) = \Big\{h \in \mathcal{C} : h(t) &= \frac{\lambda-1}{\lambda\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1}x(s)ds \\
&\quad + \frac{1}{\lambda\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}v(s)ds \\
&\quad + \frac{t}{\Lambda_1}\Big(\gamma_3 - \frac{\mu(\lambda-1)}{\lambda\Gamma(\alpha-\beta-\gamma_1)}\int_0^T (T-s)^{\alpha-\beta-\gamma_1-1}x(s)ds \\
&\quad - \frac{\mu}{\lambda\Gamma(\alpha-\gamma_1)}\int_0^T (T-s)^{\alpha-\gamma_1-1}v(s)ds
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{(1-\mu)(\lambda-1)}{\lambda\Gamma(\alpha-\beta-\gamma_2)}\int_0^T (T-s)^{\alpha-\beta-\gamma_2-1}x(s)ds\\
&- \frac{1-\mu}{\lambda\Gamma(\alpha-\gamma_2)}\int_0^T (T-s)^{\alpha-\gamma_2-1}v(s)ds\Big)\Big\}
\end{aligned}
$$

for $v \in S_{F,x}$.

Next we introduce the operator $\mathcal{A}: \mathcal{C} \to \mathcal{C}$ by

$$
\begin{aligned}
\mathcal{A}x(t) &= \frac{(\lambda-1)}{\lambda\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1}x(s)ds\\
&\quad - \frac{t}{\Lambda_1}\Big[\frac{\mu(\lambda-1)}{\lambda\Gamma(\alpha-\beta-\gamma_1)}\int_0^T (T-s)^{\alpha-\beta-\gamma_1-1}x(s)ds\\
&\quad + \frac{(1-\mu)(\lambda-1)}{\lambda\Gamma(\alpha-\beta-\gamma_2)}\int_0^T (T-s)^{\alpha-\beta-\gamma_2-1}x(s)ds\Big],
\end{aligned}
$$

and the multi-valued operator $\mathcal{B}: \mathcal{C} \to \mathcal{P}(\mathcal{C})$ by

$$
\begin{aligned}
\mathcal{B}x(t) = \Big\{h \in \mathcal{C} : h(t) &= \frac{1}{\lambda\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}v(s)ds\\
&\quad + \frac{t}{\Lambda_1}\Big[\gamma_3 - \frac{\mu}{\lambda\Gamma(\alpha-\gamma_1)}\int_0^T (T-s)^{\alpha-\gamma_1-1}v(s)ds\\
&\quad - \frac{(1-\mu)}{\lambda\Gamma(\alpha-\gamma_2)}\int_0^T (T-s)^{\alpha-\gamma_2-1}v(s)ds\Big]\Big\}
\end{aligned}
$$

for $v \in S_{F,x}$. Observe that $\mathcal{N} = \mathcal{A} + \mathcal{B}$. We shall show that the operators $\mathcal{A}$ and $\mathcal{B}$ satisfy all the conditions of Theorem 4.4 on J. First, we show that the operators $\mathcal{A}$ and $\mathcal{B}$ define the multivalued operators $\mathcal{A}, \mathcal{B} : B_r \to \mathcal{P}_{cp,c}(\mathcal{C})$ where $B_r = \{x \in \mathcal{C} : \|x\| \le r\}$ is a bounded set in $\mathcal{C}$. First we prove that $\mathcal{B}$ is compact-valued on B_r. Note that the operator $\mathcal{B}$ is equivalent to the composition $\mathcal{L} \circ S_F$, where $\mathcal{L}$ is the continuous linear operator on $L^1(J,\mathbb{R})$ into $\mathcal{C}$, defined by

$$
\begin{aligned}
&\mathcal{L}(v)(t)\\
&= \frac{1}{\lambda\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}v(s)ds + \frac{t}{\Lambda_1}\Big[\gamma_3 - \frac{\mu}{\lambda\Gamma(\alpha-\gamma_1)}\int_0^T (T-s)^{\alpha-\gamma_1-1}v(s)ds\\
&\quad - \frac{(1-\mu)}{\lambda\Gamma(\alpha-\gamma_2)}\int_0^T (T-s)^{\alpha-\gamma_2-1}v(s)ds\Big].
\end{aligned}
$$

Suppose that $x \in B_r$ is arbitrary and let $\{v_n\}$ be a sequence in $S_{F,x}$. Then, by definition of $S_{F,x}$, we have $v_n(t) \in F(t,x(t))$ for almost all $t \in J$. Since $F(t,x(t))$ is compact for all $t \in J$, there is a convergent subsequence of $\{v_n(t)\}$ (we denote it by $\{v_n(t)\}$ again) that converges in measure to some $v(t) \in S_{F,x}$ for almost all $t \in J$. On the other hand, $\mathcal{L}$ is continuous, so $\mathcal{L}(v_n)(t) \to \mathcal{L}(v)(t)$ pointwise on J.

To show that the convergence is uniform, we have to show that $\{\mathcal{L}(v_n)\}$ is an equi-continuous sequence. Let $t_1, t_2 \in J$ with $t_1 < t_2$. Then, we have

$$
\begin{aligned}
&|\mathcal{L}(v_n)(t_2) - \mathcal{L}(v_n)(t_1)|\\
&\le \Big|\frac{1}{\lambda\Gamma(\alpha)}\Big[\int_0^{t_2} (t_2-s)^{\alpha-1}v_n(s)ds - \int_0^{t_1} (t_1-s)^{\alpha-1}v_n(s)ds\Big]\\
&\quad + \frac{|t_2-t_1|}{\Lambda_1}\Big[|\gamma_3| + \frac{\mu}{\lambda\Gamma(\alpha-\gamma_1)}\int_0^T (T-s)^{\alpha-\beta-\gamma_1-1}v_n(s)ds
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{(1-\mu)}{\lambda\Gamma(\alpha-\gamma_2)} \int_0^T (T-s)^{\alpha-\gamma_2-1} v_n(s) ds \Big] \Big| \\
&\le \frac{\Phi(r)}{\lambda\Gamma(\alpha)} \Big[\int_0^{t_1} [t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}] p(s) ds + \int_{t_1}^{t_2} (t_1-s)^{\alpha-1} p(s) ds \Big] \\
&+ \Phi(r) \frac{|t_2-t_1|}{\Lambda_1} \Big[|\gamma_3| + \frac{\mu}{\lambda\Gamma(\alpha-\gamma_1)} \int_0^T (T-s)^{\alpha-\beta-\gamma_1-1} p(s) ds \\
&+ \frac{(1-\mu)}{\lambda\Gamma(\alpha-\gamma_2)} \int_0^T (T-s)^{\alpha-\gamma_2-1} p(s) ds \Big].
\end{aligned}
$$

We see that the right-hand side of the above inequality tends to zero as $t_2 \to t_1$. Thus, the sequence $\{\mathcal{L}(v_n)\}$ is equi-continuous and by using the Arzelá-Ascoli theorem, we obtain that there is a uniformly convergent subsequence. So, there is a subsequence of $\{v_n\}$ (we denote it again by $\{v_n\}$) such that $\mathcal{L}(v_n) \to \mathcal{L}(v)$. Note that, $\mathcal{L}(v) \in \mathcal{L}(S_{F,x})$. Hence, $\mathcal{B}(x) = \mathcal{L}(S_{F,x})$ is compact for all $x \in B_r$. So $\mathcal{B}(x)$ is compact.

Now, we show that $\mathcal{B}(x)$ is convex for all $x \in \mathcal{C}$. Let $h_1, h_2 \in \mathcal{B}(x)$. We select $v_1, v_2 \in S_{F,x}$ such that

$$
\begin{aligned}
&h_i(t) \\
&= \frac{1}{\lambda\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v_i(s) ds + \frac{t}{\Lambda_1} \Big[\gamma_3 - \frac{\mu}{\lambda\Gamma(\alpha-\gamma_1)} \int_0^T (T-s)^{\alpha-\gamma_1-1} v_i(s) ds \\
&\quad - \frac{(1-\mu)}{\lambda\Gamma(\alpha-\gamma_2)} \int_0^T (T-s)^{\alpha-\gamma_2-1} v_i(s) ds \Big], \quad i = 1, 2,
\end{aligned}
$$

for almost all $t \in J$. Let $0 \le \theta \le 1$. Then, we have

$$
\begin{aligned}
&[\theta h_1 + (1-\theta) h_2](t) \\
&= \frac{1}{\lambda\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} [\theta v_1(s) + (1-\theta) v_2(s)] ds \\
&\quad + \frac{t}{\Lambda_1} \Big[\gamma_3 - \frac{\mu}{\lambda\Gamma(\alpha-\gamma_1)} \int_0^T (T-s)^{\alpha-\gamma_1-1} [\theta v_1(s) + (1-\theta) v_2(s)] ds \\
&\quad - \frac{(1-\mu)}{\lambda\Gamma(\alpha-\gamma_2)} \int_0^T (T-s)^{\alpha-\gamma_2-1} [\theta v_1(s) + (1-\theta) v_2(s)] ds \Big].
\end{aligned}
$$

Since F has convex values, so $S_{F,u}$ is convex and $\theta v_1(s) + (1-\theta) v_2(s) \in S_{F,x}$. Thus

$$
\theta h_1 + (1-\theta) h_2 \in \mathcal{B}(x).
$$

Consequently, $\mathcal{B}$ is convex-valued. Obviously, $\mathcal{A}$ is compact and convex-valued.

The rest of the proof consists of several steps and claims.

Step 1: $\mathcal{A}$ is a contraction on $\mathcal{C}$. This was proved in Step 3 of Theorem 3.5.

Step 2: $\mathcal{B}$ is compact and upper semi-continuous. This will be established in several claims.

Claim I: $\mathcal{B}$ maps bounded sets into bounded sets in $\mathcal{C}$. Let $B_r = \{x \in \mathcal{C} : \|x\| \le r\}$ be a bounded set in $\mathcal{C}$. Then, for each $h \in \mathcal{B}(x), x \in B_r$, there exists $v \in S_{F,x}$ such that

$$
h(t)
$$

$$
\begin{aligned}
&= \frac{1}{\lambda\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v(s)ds + \frac{t}{\Lambda_1}\Big[\gamma_3 - \frac{\mu}{\lambda\Gamma(\alpha-\gamma_1)} \int_0^T (T-s)^{\alpha-\gamma_1-1} v(s)ds \\
&\quad - \frac{(1-\mu)}{\lambda\Gamma(\alpha-\gamma_2)} \int_0^T (T-s)^{\alpha-\gamma_2-1} v(s)ds\Big].
\end{aligned}
$$

Then, for $t \in J$, we have

$$
\begin{aligned}
|h(t)| &\le \Phi(r)\frac{1}{\lambda\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} p(s)ds \\
&\quad + \frac{T\Phi(r)}{\Lambda_1}\Big[|\gamma_3| + \frac{\mu}{\lambda\Gamma(\alpha-\gamma_1)} \int_0^T (T-s)^{\alpha-\gamma_1-1} p(s)ds \\
&\quad + \frac{(1-\mu)}{\lambda\Gamma(\alpha-\gamma_2)} \int_0^T (T-s)^{\alpha-\gamma_2-1} p(s)ds\Big].
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\|h\| &\le \Phi(r)\frac{1}{\lambda\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} p(s)ds \\
&\quad + \frac{T\Phi(r)}{\Lambda_1}\Big[|\gamma_3| + \frac{\mu}{\lambda\Gamma(\alpha-\gamma_1)} \int_0^T (T-s)^{\alpha-\gamma_1-1} p(s)ds \\
&\quad + \frac{(1-\mu)}{\lambda\Gamma(\alpha-\gamma_2)} \int_0^T (T-s)^{\alpha-\gamma_2-1} p(s)ds\Big].
\end{aligned}
$$

Claim II: $\mathcal{B}$ maps bounded sets into equi-continuous sets. Let $t_1, t_2 \in J$ with $t_1 < t_2$ and $x \in B_r$. Then, for each $h \in \mathcal{B}(x)$, we obtain

$$
\begin{aligned}
&|h(t_2) - h(t_1)| \\
&\le \frac{\Phi(r)}{\lambda\Gamma(\alpha)}\Big[\int_0^{t_1} [t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}]p(s)ds + \int_{t_1}^{t_2} (t_1-s)^{\alpha-1} p(s)ds\Big] \\
&\quad + \Phi(r)\frac{|t_2-t_1|}{\Lambda_1}\Big[|\gamma_3| + \frac{\mu}{\lambda\Gamma(\alpha-\gamma_1)} \int_0^T (T-s)^{\alpha-\beta-\gamma_1-1} p(s)ds \\
&\quad + \frac{(1-\mu)}{\lambda\Gamma(\alpha-\gamma_2)} \int_0^T (T-s)^{\alpha-\gamma_2-1} p(s)ds\Big].
\end{aligned}
$$

Obviously the right-hand side of the above inequality tends to zero independently of $x \in B_r$ as $t_2 - t_1 \to 0$. Therefore it follows by the Ascoli-Arzelá theorem that $\mathcal{B} : \mathcal{C} \to \mathcal{P}(\mathcal{C})$ is completely continuous.

Next we show that $\mathcal{B}$ is an upper semi-continuous multi-valued mapping. It is knowm by Lemma 4.2 that $\mathcal{B}$ will be upper semicontinuous if we establish that it has a closed graph, since already shown to be completely continuous. Thus we will prove that:

Claim III: $\mathcal{B}$ has a closed graph. Let $x_n \to x_*, h_n \in \mathcal{B}(x_n)$ and $h_n \to h_*$. Then we need to show that $h_* \in \mathcal{B}(x_*)$. Associated with $h_n \in \mathcal{B}(x_n)$, there exists $v_n \in S_{F,x_n}$ such that for each $t \in J$,

$$
\begin{aligned}
h_n(t) &= \frac{1}{\lambda\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v_n(s)ds + \frac{t}{\Lambda_1}\Big[\gamma_3 - \frac{\mu}{\lambda\Gamma(\alpha-\gamma_1)} \int_0^T (T-s)^{\alpha-\gamma_1-1} v_n(s)ds \\
&\quad - \frac{(1-\mu)}{\lambda\Gamma(\alpha-\gamma_2)} \int_0^T (T-s)^{\alpha-\gamma_2-1} v_n(s)ds\Big].
\end{aligned}
$$

Thus it suffices to show that there exists $v_* \in S_{F,x_*}$ such that for each $t \in J$,

$$\begin{aligned}
h_*(t) &= \frac{1}{\lambda\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v_*(s)ds \\
&\quad + \frac{t}{\Lambda_1}\Big[\gamma_3 - \frac{\mu}{\lambda\Gamma(\alpha-\gamma_1)} \int_0^T (T-s)^{\alpha-\gamma_1-1} v_*(s)ds \\
&\quad - \frac{(1-\mu)}{\lambda\Gamma(\alpha-\gamma_2)} \int_0^T (T-s)^{\alpha-\gamma_2-1} v_*(s)ds\Big].
\end{aligned}$$

Let us consider the linear operator $\Theta : L^1(J,\mathbb{R}) \to \mathcal{C}$ given by

$$\begin{aligned}
v \mapsto \Theta(v)(t) &= \frac{1}{\lambda\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v(s)ds \\
&\quad + \frac{t}{\Lambda_1}\Big[\gamma_3 - \frac{\mu}{\lambda\Gamma(\alpha-\gamma_1)} \int_0^T (T-s)^{\alpha-\gamma_1-1} v(s)ds \\
&\quad - \frac{(1-\mu)}{\lambda\Gamma(\alpha-\gamma_2)} \int_0^T (T-s)^{\alpha-\gamma_2-1} v(s)ds\Big].
\end{aligned}$$

Observe that

$$\begin{aligned}
\|h_n(t) - h_*(t)\| &= \Big\| \frac{1}{\lambda\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} (v_n(s) - v_*(s))ds \\
&\quad + \frac{t}{\Lambda_1}\Big[- \frac{\mu}{\lambda\Gamma(\alpha-\gamma_1)} \int_0^T (T-s)^{\alpha-\gamma_1-1} (v_n(s) - v_*(s))ds \\
&\quad - \frac{(1-\mu)}{\lambda\Gamma(\alpha-\gamma_2)} \int_0^T (T-s)^{\alpha-\gamma_2-1} (v_n(s) - v_*(s))ds\Big]\Big\| \to 0,
\end{aligned}$$

as $n \to \infty$.

Thus, it follows by Lemma 4.3 that $\Theta \circ S_F$ is a closed graph operator. Further, we have $h_n(t) \in \Theta(S_{F,x_n})$. Since $x_n \to x_*$, we have that

$$\begin{aligned}
&h_*(t) \\
&= \frac{1}{\lambda\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v_*(s)ds + \frac{t}{\Lambda_1}\Big[\gamma_3 - \frac{\mu}{\lambda\Gamma(\alpha-\gamma_1)} \int_0^T (T-s)^{\alpha-\gamma_1-1} v_*(s)ds \\
&\quad - \frac{(1-\mu)}{\lambda\Gamma(\alpha-\gamma_2)} \int_0^T (T-s)^{\alpha-\gamma_2-1} v_*(s)ds\Big],
\end{aligned}$$

for some $v_* \in S_{F,x_*}$. Hence $\mathcal{B}$ has a closed graph (and therefore has closed values). In consequence, the operator $\mathcal{B}$ is compact valued and upper semi-continuous.

Thus the operators $\mathcal{A}$ and $\mathcal{B}$ satisfy all the conditions of Theorem 4.4 and hence its conclusion implies either condition (i) or condition (ii) holds. We show that the conclusion (ii) is not possible. If $x \in \theta\mathcal{A}(x) + \theta\mathcal{B}(x)$ for $\theta \in (0,1)$, then there exist $v \in S_{F,x}$ such that

$$\begin{aligned}
x(t) &= \theta\frac{\lambda-1}{\lambda\Gamma(\alpha-\beta)} \int_0^t (t-s)^{\alpha-\beta-1} x(s)ds + \theta\frac{1}{\lambda\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v(s)ds \\
&\quad + \theta\frac{t}{\Lambda_1}\Big(\gamma_3 - \frac{\mu(\lambda-1)}{\lambda\Gamma(\alpha-\beta-\gamma_1)} \int_0^T (T-s)^{\alpha-\beta-\gamma_1-1} x(s)ds \\
&\quad - \frac{\mu}{\lambda\Gamma(\alpha-\gamma_1)} \int_0^T (T-s)^{\alpha-\gamma_1-1} v(s)ds
\end{aligned}$$

$$
\begin{aligned}
&- \frac{(1-\mu)(\lambda-1)}{\lambda\Gamma(\alpha-\beta-\gamma_2)} \int_0^T (T-s)^{\alpha-\beta-\gamma_2-1} x(s)ds \\
&- \frac{1-\mu}{\lambda\Gamma(\alpha-\gamma_2)} \int_0^T (T-s)^{\alpha-\gamma_2-1} v(s)ds\Big), \quad t \in J.
\end{aligned}
$$

By our assumptions, we obtain

$$
\begin{aligned}
|x(t)| \le \|x\| \Big[&\frac{T^{\alpha-\beta}|\lambda-1|}{\lambda\Gamma(\alpha-\beta+1)} + \frac{T^{\alpha-\beta-\gamma_1+1}\mu|\lambda-1|}{\lambda\Lambda_1\Gamma(\alpha-\beta-\gamma_1+1)} \\
&+ \frac{T^{\alpha-\beta-\gamma_2+1}(1-\mu)|\lambda-1|}{\lambda\Lambda_1\Gamma(\alpha-\beta-\gamma_2+1)} \Big] + \frac{|\gamma_3|T}{\Lambda_1} \\
&+ \Phi(\|x\|) \frac{1}{\lambda\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} p(s)ds \\
&+ \frac{T\Phi(\|x\|)}{\Lambda_1} \Big[\frac{\mu}{\lambda\Gamma(\alpha-\gamma_1)} \int_0^T (T-s)^{\alpha-\gamma_1-1} p(s)ds \\
&+ \frac{(1-\mu)}{\lambda\Gamma(\alpha-\gamma_2)} \int_0^T (T-s)^{\alpha-\gamma_2-1} p(s)ds \Big].
\end{aligned}
$$

Thus

$$
(1-\Omega_1)\|x\| \le \Phi(\|x\|)\Psi_1 + |\gamma_3|T/\Lambda_1. \tag{4.3}
$$

If condition (ii) of Theorem 4.4 holds, then there exists $\theta \in (0,1)$ and $x \in \partial B_M$ with $x = \theta\mathcal{N}(x)$. Then, x is a solution of (1.5)-(1.2) with $\|x\| = M$. Now, by the inequality (4.3), we obtain

$$
\frac{(1-\Omega_1)M}{\Phi(M)\Psi_1 + |\gamma_3|T/\Lambda_1} \le 1,
$$

which contradicts (4.2). Hence, $\mathcal{N}$ has a fixed point in J by Theorem 4.4, and consequently the problem (1.5)-(1.2) has a solution. This completes the proof. $\square$

In the above results, we define the following two constants

$$
\begin{aligned}
\Psi_2 = &\frac{1}{\lambda\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} p(s)ds + \frac{T}{\Lambda_2} \Big[\frac{\mu}{\lambda\Gamma(\delta_1+\alpha)} \int_0^T (T-s)^{\delta_1+\alpha-1} p(s)ds \\
&+ \frac{1-\mu}{\lambda\Gamma(\delta_2+\alpha)} \int_0^T (T-s)^{\delta_2+\alpha-1} p(s)ds \Big], \\
\Psi_3 = &\frac{1}{\lambda\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} p(s)ds + \frac{T}{\Lambda_3} \Big[\frac{\mu}{\lambda\Gamma(\alpha-\gamma_1)} \int_0^T (T-s)^{\alpha-\gamma_1-1} p(s)ds \\
&+ \frac{1-\mu}{\lambda\Gamma(\delta_2+\alpha)} \int_0^T (T-s)^{\delta_2+\alpha-1} p(s)ds \Big].
\end{aligned}
$$

Theorem 4.7. *Let $\Omega_3 < 1$. Assume that the conditions* (H3), (H4) *are satisfied. If there exists a positive constant M such that*

$$
\frac{(1-\Omega_3)M}{\Phi(M)\Psi_2 + |\delta_3|T/\Lambda_2} > 1,
$$

then problem (1.5)-(1.3) *has at least one solution on J.*

Theorem 4.8. *Let $\Omega_4 < 1$. Suppose that the conditions* (H3), (H4) *are satisfied. If there exists a positive constant M such that*

$$\frac{(1-\Omega_4)M}{\Phi(M)\Psi_3 + |\gamma_3|T/\Lambda_3} > 1,$$

then problem (1.5)-(1.4) *has at least one solution on J.*

Example 4.9. Let us consider the following two order fractional differential inclusion with two order fractional derivative boundary conditions

$$\begin{gathered} \frac{47}{54}D^{16/9}x(t) + \frac{7}{54}D^{10/9}x(t) \in F(t,x(t)), \quad t \in [0,1], \\ x(0) = 0, \quad \frac{9}{23}D^{7/15}x(1) + \frac{14}{23}D^{4/15}x(1) = \frac{1}{12}. \end{gathered} \tag{4.4}$$

where $F(t,x)$ is the multivalue function

$$F(t,x) = \Big[\Big(\frac{\sqrt{t}+1}{5}\Big)\Big(\frac{|x|\sin^2 x}{18(1+|x|)} + \frac{1}{4}\Big), \Big(\sqrt[3]{t} + \frac{1}{4}\Big)\Big(\frac{|x|\sin x}{15} + \frac{1}{2}\Big)\Big].$$

Here $\lambda = 47/54$, $\alpha = 16/9$, $\beta = 10/9$, $\mu = 9/23$, $\gamma_1 = 7/15$, $\gamma_2 = 4/15$, $\gamma_3 = 1/12$, $T = 1$. Observe that $0 < \gamma_1, \gamma_2 < 2/3 = \alpha - \beta$. We can find that $\Lambda_1 = 1.105743248$ and $\Omega_1 = 0.3147893857$. It is easy to see that

$$\|F(t,x)\|_{\mathcal{P}} = \sup\{|y| : y \in F(t,x)\} \le \Big(\sqrt[3]{t} + \frac{1}{4}\Big)\Big(\frac{|x|}{15} + \frac{1}{2}\Big).$$

Set $p(t) = \sqrt[3]{t} + (1/4)$ and $\Phi(x) = (x/15) + (1/2)$. By direct computation, we have $\Psi_1 = 1.410896861$. From the given data, we can prove that there exists a positive constant $M > 1.267866938$ satisfying inequality (4.2) of Theorem 4.6. Therefore, by applying Theorem 4.6, we deduce that the boundary value problem (4.4) has at least one solution on $[0,1]$.

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