Electronic Journal of Differential Equations, Vol. 2017 (2017), No. 101, pp. 1–20.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

ON THE DYNAMICS OF SECOND-ORDER LAGRANGIAN SYSTEMS

RONALD ADAMS, WILLIAM D. KALIES, ROBERT C. A. M. VANDERVORST

Communicated by Paul Rabinowitz

Abstract. In this article we are concerned with improving the twist condition for second-order Lagrangian systems. We characterize a local Twist property and demonstrate how results on the existence of simple closed characteristics can be extended in the case of the Swift-Hohenberg / extended Fisher-Kolmogorov Lagrangian. Finally, we describe explicit evolution equations for broken geodesic curves that could be used to investigate more general systems or closed characteristics.

1. Introduction

Second-order Lagrangian systems arise as fourth-order differential equations obtained variationally as the Euler-Lagrange equations of an action functional which depends on the second derivative of the state variable u as well as its lower derivatives. One important class of such differential equations is $u'''' - \beta u'' + f(u) = 0$, known as the Swift-Hohenberg equation for $\beta \le 0$ and the extended Fisher-Kolmogorov (eFK) equation for $\beta > 0$. There have been numerous results concerning these equations, see for instance [2, 7, 8, 10, 11, 12, 13, 14, 15, 18, 19].

As is the case here, for the Hamiltonian system that is induced from the second-order Lagrangian there is a natural two-dimensional section for which bounded trajectories of the Hamiltonian system must intersect in finitely or infinitely many distinct points. We can characterize this section as $\Sigma_E = N_E \times \mathbb{R}$, where N_E is the one dimensional set

$$N_E = \big\{(u, u'') : \frac{\partial L}{\partial u''} u'' - L(u, 0, u'') = E\big\},$$

and H is the Hamiltonian and $E \in \mathbb{R}$. The Hamiltonian flow induces a return map to Σ_E where closed trajectories correspond to fixed points of the iterates of this map. As in [17] the return map can often be viewed as an analogue of a monotone area-preserving Twist map, Lagrangian systems that allow these Twist maps are referred to as *Twist systems*.

In this article we are concerned with improving the condition for a second-order Lagrangian to possess a local version of the Twist condition, cf. assumption (A1)

2010 *Mathematics Subject Classification.* 37J45, 34C25, 47J30.
Key words and phrases. Second-order Lagrangian; closed characteristic; twist system; curve evolution.

Submitted October 6, 2016. Published April 11, 2017.

below. Through a version of the weak comparison principle we establish that it is only necessary that the Twist condition holds for a restricted class of laps which lie in a cone. Exploiting the properties of this cone we can derive explicit formulas for the vector field that describes the dynamics of the endpoints between laps. The vector field can then be estimated analytically or numerically.

The paper is organized as follows. In Section 2 we review the basic properties of second-order Lagrangian systems and define the Twist property. We then characterize a local Twist property and demonstrate how results on the existence of simple closed characteristics can be extended in the case of the SH/eFK Lagrangian. In Section 3 we prove this extension and give another example from the fifth-order KdV equation. In Section 4, we describe explicit evolution equations for broken geodesic curves that could be used to investigate other systems or non-simple closed characteristics, and in Section 5 we briefly describe a framework for an analytic or numerical study.

2. Preliminaries

A second-order Lagrangian system is defined by extremizing an action functional of the form

$$J[u] = \int_I L(u, u', u'')\, dt.$$

Computing the Euler-Lagrange equation yields

$$\frac{d^2}{dt^2}\frac{\partial L}{\partial u''} - \frac{d}{dt}\frac{\partial L}{\partial u'} + \frac{\partial L}{\partial u} = 0. \tag{2.1}$$

The Lagrangian action J is invariant under the $\mathbb{R}$ action $t \mapsto t + c$, which by Noether's Theorem yields the conservation law

$$\Big(\frac{\partial L}{\partial u'} - \frac{d}{dt}\frac{\partial L}{\partial u''}\Big)u' + \frac{\partial L}{\partial u''}u'' - L(u, u', u'') = E. \tag{2.2}$$

Under the natural hypothesis that L is convex in u'', that is $\partial_w^2 L(u, v, w) > 0$ for all (u, v, w), the Lagrangian system (L, dt) is equivalent to a Hamiltonian system on $\mathbb{R}^4$ with the standard symplectic coordinates $x = (u, v, p_u, p_v)$ endowed with the symplectic form ω given by $\omega = du \wedge dp_u + dv \wedge dp_v$. The Hamiltonian is

$$H = p_u v + wL_w - L,$$

where w and p_v are related by $p_v = L_w(u, v, w)$ and $p_u = L_v - p_v'$. Stationary points of J satisfy Equation (2.2), which is equivalent to $H(u, v, p_u, p_v) = E$. Thus, for the associated Hamiltonian system (H, ω) the three-dimensional energy manifold $M_E = \{(u, v, p_u, p_v) \in \mathbb{R}^4 : H(u, v, p_u, p_v) = E\}$ is invariant. If $\nabla H \neq 0$ on M_E, then E is a regular value, and M_E is a smooth non-compact manifold without boundary. Equivalently, an energy level is regular if and only if $L_w(u, 0, 0) \neq 0$ for all $u \in \mathbb{R}$ satisfying $L(u, 0, 0) + E = 0$.

A bounded characteristic of a Lagrangian system (L, dt) is a function $u \in C_b^2(\mathbb{R}, \mathbb{R})$ for which $\delta \int_I L(u, u', u'') = 0$ with respect to variations $\delta u \in C_c^2(I, \mathbb{R})$ for any compact interval $I \subset \mathbb{R}$. Since the Lagrangian is a C^2-function of (u, v, w), it follows from the Euler-Lagrange equations that $u \in C_b^3(\mathbb{R}, \mathbb{R})$, $L_w(\cdot) \in C_b^2(\mathbb{R}, \mathbb{R})$, and $(\frac{d}{dt}\frac{\partial L}{\partial w} - \frac{\partial L}{\partial v})(\cdot) \in C_b^2(\mathbb{R})$. Our main concern is to develop a new analytic method to study the existence and structure of bounded, closed characteristics on M_E, i.e. bounded characteristics which are periodic functions. We conclude this preliminary section with some relationships between closed characteristics and the geometry of M_E.

Given an arbitrary $(2n-1)$-dimensional manifold M embedded in $(\mathbb{R}^{2n},\omega)$, with ω being the standard symplectic form, one can construct a Hamiltonian H for which $M = H^{-1}(0)$. The choice of H is not instrinsic to finding periodic orbits, it turns out that the geometry of M is enough to describe them. The geometry of M and the symplectic 2-form ω define a characteristic line bundle,

$$\mathcal{E}_M = \{(x,\xi) \in T_xM \setminus \{0\} : w_x(\xi,\eta) = 0 \ \forall \eta \in T_xM\} \subset TM\,.$$

The vector field of any Hamiltonian H with $M = H^{-1}(0)$ is a section of $\mathcal{E}_M$. The trajectory of a periodic orbit can be viewed as a closed characteristic of the line bundle, i.e. an embedding $\gamma : S^1 \to M$ of the circle into M for which

$$T\gamma = \mathcal{E}_M\big|_\gamma.$$

This gives us a relationship between existence of periodic solutions to (2.1) and topological and geometric properties of its energy surfaces. These closed characteristics can be found as critical points of the action functional. Weinstein [21] conjectured in the 1970's, that any compact hypersurface $M \in (\mathbb{R}^{2n},\omega)$, with the additional assumption that

$$\alpha(\xi) \neq 0 \quad \text{for } 0 \neq \xi \in \mathcal{E}_M,$$

for some 1-form α with $d\alpha = \omega$, i.e. M is of *contact type* relative to ω, has at least one closed characteristic. This was proved later by Viterbo [20]. However this theory cannot be applied to energy manifolds determined by second-order Lagrangian systems, because these manifolds are always non-compact and they are not necessarily of contact type in $(\mathbb{R}^4,\omega)$, as was proved in [2].

The existence of closed characteristics for second-order Lagrangian systems has been studied in several contexts, and we will summarize prior results later in this section. Following ideas first introduced in [17], one method that has been used to study closed characteristics variationally is to divide $u(t)$ into monotone segments.

Definition 2.1. For $\ell < r$, an *increasing lap* u_+ from ℓ to r is a solution to the Euler-Lagrange equation (2.1) satisfying the boundary conditions $u(0) = \ell$, $u(A1) = r$, $u'(0) = u'(\tau) = 0$ and $u'(t) > 0$ for $0 < t < \tau$ with a similar definition of a *decreasing lap* u_-. A *simple closed characteristic of type* (ℓ, r) is a periodic solution which is composed of a single increasing lap from ℓ to r followed by a single decreasing lap from r to ℓ extended periodically.

The concatenation of an increasing lap u_+ and a decreasing lap u_- is analogous to a broken geodesic and is not necessarily a solution to (2.1) at the concatenation points, since the third derivatives need not agree there. Setting $v = 0$ in the Hamiltonian (2.2), solutions satisfy $wL_w - L = E$ at critical points. Let N denote this level set in the (u,w)-plane. Then every simple closed characteristic intersects N exactly twice. Moreover N is a section of M given by $M \cap \{v = 0\}$, and due to the convexity of L in the w variable, N consists of two graphs in the (u,w)-plane. That is, the projection π of N onto the u-axis can be described by $\pi N = \{u : L(u,0,0) + E \geq 0\}$ and the sets $N \cap \{(u,w) : u \geq 0\}$ and $N \cap \{(u,w) : u \leq 0\}$ are graphs over πN. A particular connected component of πN will be denoted by I, and referred to as an *interval component*.

Consider an interval component I, and define $B = \{(\ell, r) \in I^2 : \ell < r\}$. For simple closed characteristics one needs to find points $(\ell, r) \in B$ for which the concatenation of an increasing and a decreasing lap is a solution. For an increasing lap u_+ from ℓ to r we let p_ℓ^+ and p_r^+ be the p_u-values at the concatenation points, and also for a decreasing lap u_- we let p_ℓ^- and p_r^- be the corresponding p_u values. If u is the concatenation $u = u_+ \# u_-$, then necessary conditions for u to be a solution of (2.1) are $p_\ell^+ = p_\ell^-$ and $p_r^+ = p_r^-$ see [17]. Since u_+ and u_- are solutions to (2.1), their intersection with N determines the values of $u'' = p_v$ uniquely from ℓ and r, and we denote these values by $p_v(\ell)$ and $p_v(r)$. Thus the necessary compatibility conditions on the p_u-values are also sufficient.

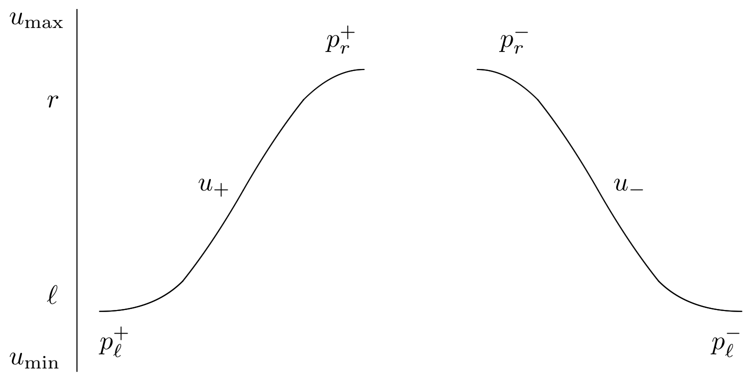

FIGURE 1. The p_v values at the endpoints of each lap are determined by the minimum and maximum values ℓ and r, but the p_u values are not, which gives a necessary and sufficient condition for the concatenation $u_- \# u_+$ to be a simple closed characteristic, $p_r^+ = p_r^-$ and $p_\ell^+ = p_\ell^-$.

In [17] this broken geodesic method is used to prove the existence of simple closed characteristics in systems where laps can be determined by minimization. In particular fix an energy level E, and let

$$J_{E,\tau}[u] = \int_0^\tau L(u, u', u'') + E\, dt.$$

Suppose that

(A1) for every pair $\ell \neq r$ there exists a unique minimizing lap $u(t; \ell, r)$ of J_E defined on the interval $[0, \tau(\ell, r)]$, and both u and τ are C^1 in ℓ and r.

This *twist property* allows a reduction of the variational problem to finite dimensions by plugging the minimizer into the functional to obtain the function $S_E(\ell, r) = J_E[u(t; \ell, r)]$.

$$S_E(\ell, r) = \inf_{u \in X_\tau, \tau \in \mathbb{R}^+} \int_0^\tau L(u, u', u'') + E\, dt, \tag{2.3}$$

where $X_\tau = X_\tau(\ell, r) = \{u \in C^2([0, \tau]) : u(0) = \ell, u(\tau) = r, u'(0) = u'(\tau) = 0, u'\big|_{(0,\tau)} > 0 \text{ if } \ell < r, \text{ and } u'\big|_{(0,\tau)} < 0 \text{ if } \ell > r\}$. One then shows the existence of a critical point of $S_E(\ell, r) + S_E(r, \ell)$ in order to obtain a simple closed characteristic.

The above results are extended in [12] where a degree theoretic argument is used to establish the existence of a simple closed characteristic in systems which do not necessarily satisfy the twist condition via continuation to a twist system. It is shown that for regular energy manifolds, the number of closed characteristics can be bounded below by the second Betti number of M_E, which in turn can be computed from the superlevel sets of the potential function $L(u,0,0)+E \geq 0$. This result is extended to singular energy levels in [14].

Results concerning the existence of more general closed characteristics are addressed in [7, 8]. For twist systems, a type of Conley-Morse theory is used on the space of braids to obtain forcing theorems for closed characteristics. It is shown that the set of closed characteristics can have a rich structure. The degree arguments of [12] do not extend these forcing results to systems which do not necessarily satisfy the twist condition, because the degree for many of these closed characteristics is trivial even though the Conley index is nontrivial.

Our methods begin by reconsidering [17, Lemma 9] which provides sufficient conditions for the twist property to hold on an interval component for a Lagrangian of the form $L(u,u',u'') = \frac{1}{2}|u''|^2 + K(u,u')$.

Lemma 2.2 ([17, Lemma 9]). *Let I_E be a connected component of $\pi^u N_E$. Assume that*

(a) $\frac{\partial K}{\partial v}v - K(u,v) - E \leq 0$ *for all* $u \in I_E$ *and* $v \in \mathbb{R}$*; and*
(b) $\frac{\partial^2 K}{\partial v^2}v^2 - \frac{5}{2}\{\frac{\partial K}{\partial v}v - K(u,v) - E\} \geq 0$ *for all* $u \in I_E$ *and* $v \in \mathbb{R}$.

Then for any pair $(\ell,r) \in I_E \times I_E \setminus \triangle$, Problem (2.3) *has a unique minimizer $(u,\tau) \in X_\tau \times \mathbb{R}^+$ (in fact the only critical point), and the minimizer $u(t;\ell,r)$ depends C^1-smoothly on (ℓ,r) for $(\ell,r) \in \operatorname{int}(I_E \times I_E \setminus \triangle)$.*

Fix a regular energy level E. We assume that $K(u,v)+E$ has the form

$$K(u,v) + E = \frac{\beta}{2}v^2 + F(u), \tag{2.4}$$

so that we consider Lagrangians of the eFK/Swift-Hohenberg type with

$$L(u,u',u'') + E = \frac{1}{2}|u''|^2 + \frac{\beta}{2}|u'|^2 + F(u).$$

From now on we will suppress the dependence on E. The arguments we present apply to more general form of the Lagrangian, as in Example 3.3 below, but we consider this more restrictive case for convenience and clarity of presentation.

In the next section we utilize a comparison principle to weaken the sufficient conditions of Lemma 2.2 and obtain the following stronger result.

Proposition 2.3. *For a Lagrangian of eFK/Swift-Hohenberg type, let I be a regular interval component and I_1, I_2 be subintervals of I. There is a constant $C(\ell,r) > 0$ such that if*

$$\beta \leq \frac{2F((r-\ell)x+l)}{(r-\ell)C^{4/3}(\ell,r)\sin(\pi x)} \quad \textit{for all } x \in (0,1) \tag{2.5}$$

for all $(\ell,r) \in I_1 \times I_2 \setminus \triangle$ with $\ell < r$, then the minimization problem in (2.3) *has a local minimizer $(u,\tau) \in X_\tau \times \mathbb{R}^+$ for each $(\ell,r) \in I_1 \times I_2 \setminus \triangle$. Moreover, the family $u(t;\ell,r)$ depends C^1- smoothly on $I_1 \times I_2 \setminus \triangle$.*

Note that the we need only consider $\ell < r$ in Proposition 2.3 because time-reversal of an increasing lap for (ℓ, r) yields a decreasing lap for (r, ℓ). The proof of the proposition is given in Section 3.

Proposition 2.3 allows the techniques in [17] to be applied to a larger class of Lagrangians. If Proposition 2.3 is applied over the whole interval component $I = [u_{\min}, u_{\max}]$, i.e. on $I \times I \setminus \triangle$, then one can obtain a stronger result than can be attained using Lemma 2.2. Indeed the hypotheses of Lemma 2.2 hold only for $\beta \leq 0$. However, for $\ell = u_{\max}$ and $r = u_{\min}$ we have that

$$\lim_{x \to 0^+} \frac{F((u_{\max} - u_{\min})x + u_{\min})}{\sin(\pi x)} \quad \text{and} \quad \lim_{x \to 1^-} \frac{F((u_{\max} - u_{\min})x + u_{\min})}{\sin(\pi x)}$$

are strictly positive when F has a simple zero at the endpoints of I_E, which occurs for generic potential functions $F(u)$, such as the double-well potential and others, see [17, Sections 2.2.1, 2.2.2, 2.2.3].

Therefore, in these cases

$$\min_{r, \ell \in I_E,\, x \in \Omega} \frac{F((r - \ell)x + \ell)}{\sin(\pi x)} > 0, \tag{2.6}$$

so that the local version of the twist property that is characterized in Proposition 2.3 holds for a range of positive values for β that can be estimated using (2.5). As an illustration we estimate the β bound for a specific potential function $F(u)$.

Example 2.4. Consider the double-well potential

$$F(u) = \frac{1}{4}(u^2 - 1)^2 - \frac{1}{8}.$$

A compact interval component is $I = [-\sqrt{1 - \frac{1}{\sqrt{2}}}, \sqrt{1 - \frac{1}{\sqrt{2}}}]$, and it suffices to compute

$$\inf_{[u_{\min}, u_{\max}]} \inf_{\Omega} \Big(\frac{\frac{1}{2}\left(((r - l)x + l)^2 - 1\right)^2 - \frac{1}{4}}{(r - \ell) C_2^{4/3}(\ell, r) \sin(\pi x)} \Big). \tag{2.7}$$

In Section 3.1 the following formula for the constant C_2 is given in equation (3.5)

$$C_2^{4/3}(\ell, r) = \sup_{\Omega} \sqrt{\frac{8F((r - \ell)x + \ell)}{3\pi^2 \sin^2(\pi x) + \pi^2}} = \sup_{\Omega} \sqrt{\frac{2\left(((r - l)x + l)^2 - 1\right)^2 - 1}{3\pi^2 \sin^2(\pi x) + \pi^2}}.$$

Using (2.7), we numerically estimate the upper bound on β given in equation (2.5) to be approximately 3.82.

The existence of a simple closed characteristic of L can then be proved using [17, Theorem 12]. Note that simply the existence of a closed characteristic for all β in the SH / eFK system follows from the results in [12]. However, for $\beta > 0$ this existence is obtained via a degree argument, and for the above range of positive β, we obtain existence variationally, which provides more information about the closed characteristics.

Corollary 2.5. *If condition* (2.6) *holds, then the hypotheses of Proposition 2.3 hold, and therefore there exists a simple closed characteristic of L on the interval component I.*

In Section 3 we prove Proposition 2.3. Then we define in Section 4 a type of curve evolution for which closed characteristics are stationary points of the dynamics. Finally, we briefly indicate how techniques from dynamical systems might be applied using these methods in Section 5.

3. Localizing the twist property

Fix a regular interval component $I = [u_{\min}, u_{\max}]$ with $F(u) > 0$ on $(u_{\min}, u_{\max})$, and assume $u_{\min}, u_{\max}$ are simple zeros of F. Throughout this section ℓ and r are arbitrary points with $u_{\min} \le \ell < r \le u_{\max}$.

Next we introduce two changes of variables. Let

$$z = v^{3/2} \quad \text{and} \quad x = \frac{u-\ell}{r-\ell}.$$

The action functional $J : H_0^1(\Omega) \times I^2 \to \mathbb{R}$ written in the three variables z, ℓ, r is

$$J[z,\ell,r] = \int_\Omega \frac{2}{9(r-\ell)} z_x^2 + \frac{(r-\ell)F((r-\ell)x+\ell)}{z^{2/3}} + \frac{(r-\ell)\beta}{2} z^{2/3} dx$$

where $\Omega = (0,1)$. For fixed values of $\ell < r$, the first change of variable ensures that the gradient flow of J in $H_0^1(\Omega)$ will be semilinear, and the second change of coordinates provides the uniform spatial domain $[0,1]$. Stationary functions of $J[z,\ell,r]$ satisfy the Euler -Lagrange equation

$$u'''' - \frac{d}{dt}\frac{\partial K}{\partial u'} + \frac{\partial K}{\partial u} = 0.$$

Using the Hamiltonian relation, solutions of this equation satisfy

$$-u'u''' + \frac{1}{2}(u'')^2 + \frac{\partial K}{\partial u'}u' - K(u,u') - E = 0.$$

Since z is a function of u on a lap, this can be expressed as

$$z_{uu} = \frac{3}{2}\frac{\frac{\partial K}{\partial u'}u' - K(u,u') - E}{z^{5/3}}, \tag{3.1}$$

and in terms of the variable x we have

$$\frac{4}{9} z_{xx} = \frac{(r-\ell)^2}{3}\Big(\frac{-2F((r-\ell)x+\ell)}{z^{5/3}} + \frac{\beta}{z^{1/3}}\Big). \tag{3.2}$$

for K in (2.4).

3.1. **Monotonicity of the nonlinearity.** To take full advantage of the comparison principle for elliptic operators, stated in Theorem 3.2 below, we need the following proposition.

Proposition 3.1. *If*

$$\beta \le \frac{10F((\ell-r)x+l)}{z^{4/3}} \quad \textit{for all } x \in \Omega, \tag{3.3}$$

then the nonlinearity

$$N[z] = \frac{2}{3}\frac{F((\ell-r)x+\ell)}{z^{5/3}} - \frac{\beta}{3z^{1/3}}$$

is nonincreasing in z.

Proof. The derivative with respect to z is

$$\frac{d}{dz}\Big(\frac{2}{3}\frac{F((\ell-r)x+l)}{z^{5/3}}-\frac{\beta}{3z^{1/3}}\Big)=\frac{-10}{9}z^{-8/3}F((\ell-r)x+l)+\frac{\beta}{9}z^{-4/3},$$

which is nonpositive when condition (3.3) is satisfied. □

3.2. **Lower and upper solutions.** To consider the behavior of the solutions of (3.2) near $\partial\Omega$, we need to analyze the singular operator

$$E[z]=\frac{4}{9}z_{xx}+\frac{(r-\ell)^2}{3}\Big(\frac{2F((r-\ell)x+\ell)}{z^{5/3}}-\frac{\beta}{z^{1/3}}\Big)$$

for fixed $\ell<r$. The correct scaling between the z_{xx} and $z^{-5/3}$ terms is $z\sim x^{3/4}$, and hence we define the conical shell

$$\mathcal{C}=\big\{z\in L^\infty(\Omega):C_1d(x)^{3/4}\le z(x)\le C_2(d(x)^{3/4}+d(x))\ \forall x\in\Omega\big\}$$

for some C_1, C_2 with $0<C_1<C_2$, where $d(x)=\operatorname{dist}(x,\partial\Omega)$.

We use the following version of the weak comparison principle from Cuesta and Takáč [5] with $p=2$, see also Fleckinger-Pelléand Takáč [6].

Theorem 3.2 (Weak comparison principle [5])**.** *Assume $f\le g$ in $L^{p/(p-1)}(\Omega)$, $\tilde f\le\tilde g$ in $W^{1-(1/p),p}(\partial\Omega)$, and u, $v\in W^{1,p}(\Omega)$ are any weak solutions of the Dirichlet problems*

$$\begin{gathered}-\operatorname{div}(\mathbf{a}(x,\nabla u))-b(x,u)=f(x)\ \text{ in }\Omega;\quad u=\tilde f\ \text{ on }\partial\Omega,\\ -\operatorname{div}(\mathbf{a}(x,\nabla v))-b(x,v)=g(x)\ \text{ in }\Omega;\quad v=\tilde g\ \text{ on }\partial\Omega.\end{gathered}$$

Then $u\le v$ hold almost everywhere in Ω provided $b(x,\cdot):\mathbb{R}\to\mathbb{R}$ is nonincreasing for a.e. $x\in\Omega$.

From Theorem 3.2 we can now construct lower and upper solutions to $E[z]=0$ following the arguments in Badra, Bal, and Giacomoni [3]. That is, we can find functions $\underline{z},\bar z\in H_0^1(\Omega)\cap\mathcal{C}$ such that $\underline{z}\le\bar z$ and $E[\underline{z}]\ge0$ and $E[\bar z]\le0$ in Ω.

Let ϕ_1 denote the normalized positive eigenfunction associated with the principal eigenvalue λ_1 of $-\Delta$ with homogeneous Dirichlet boundary conditions. In this case $\phi_1(x)=\sin(\pi x)$ and $\lambda_1=\pi^2$. Since $\phi_1\in C^1(\overline{\Omega})$, we have $\phi_1^{3/4}\in\mathcal{C}$. We define the lower and upper solutions by

$$\underline{z}=\eta\phi_1^{3/4}\quad\text{and}\quad\overline{z}=M\phi_1^{3/4}$$

for $\eta>0$ sufficiently small and $M>\eta$ sufficiently large. For $z=C\phi_1^{3/4}$ with $C>0$

$$-E[z]=\frac{C\pi^2\phi_1^{3/4}}{3}+\frac{C}{12}\phi_1^{-5/4}(\partial_x\phi_1)^2-\frac{2}{3}\frac{(r-\ell)^2F((r-\ell)x+\ell)}{z^{5/3}}+\frac{(r-\ell)^2\beta}{3z^{1/3}}.$$

Multiplying $-E[z]=0$ through by $12C^{5/3}\phi_1^{5/4}$ yields

$$C^{8/3}\left(4\pi^2\phi_1^2+(\partial_x\phi_1)^2\right)+C^{4/3}4(r-\ell)^2\beta\phi_1-8(r-\ell)^2F((r-\ell)x+\ell)=0.$$

Treating this as a quadratic in $C^{4/3}$ the nonnegative root is

$$\begin{aligned}&\Big(-2\beta(r-\ell)^2\sin(\pi x)\\&+2(r-\ell)\sqrt{(r-\ell)^2\beta^2\sin^2(\pi x)+2F((r-\ell)x+\ell)\cdot\big(3\pi^2\sin^2(\pi x)+\pi^2\big)}\Big)\\&\div\Big(3\pi^2\sin^2(\pi x)+\pi^2\Big).\end{aligned}\tag{3.4}$$

Therefore if we take $C_1^{4/3}(\ell,r,\beta)\cdot(r-\ell)$ and $C_2^{4/3}(\ell,r,\beta)\cdot(r-\ell)$ as lower and upper bounds for the expression (3.4) respectively, then for $\eta = C_1(\ell,r,\beta)\cdot(r-\ell)^{3/4}$ and $M = C_2(\ell,r,\beta)\cdot(r-\ell)^{3/4}$ we have that $\underline{z}$ and $\bar{z}$ are lower and upper solutions respectively, which leads us to a second cone

$$\mathsf{C}(\ell,r,\beta) = \{z \in H_0^1(\Omega) \mid \underline{z} \le z \le \bar{z}\}.$$

We can bound the function z above independently of β for $\beta > 0$.

$$\begin{aligned}
&\Big(-2\beta(r-\ell)^2\sin(\pi x) \\
&+ 2(r-\ell)\sqrt{(r-\ell)^2\beta^2\sin^2(\pi x) + 2F((r-\ell)x+\ell)\cdot\big(3\pi^2\sin^2(\pi x)+\pi^2\big)}\Big) \\
&\div \Big(3\pi^2\sin^2(\pi x)+\pi^2\Big) \\
&\le \frac{(r-\ell)\sqrt{8F((r-\ell)x+\ell)\cdot\big(3\pi^2\sin^2(\pi x)+\pi^2\big)}}{3\pi^2\sin^2(\pi x)+\pi^2} \\
&= (r-\ell)\sqrt{\frac{8F((r-\ell)x+\ell)}{3\pi^2\sin^2(\pi x)+\pi^2}}
\end{aligned} \tag{3.5}$$

We denote the maximum of $\sqrt{\frac{8F((r-\ell)x+\ell)}{3\pi^2\sin^2(\pi x)+\pi^2}}$ over Ω by $C_2^{4/3}(\ell,r)$.

For $\beta > 0$, restricting to the cone $\mathsf{C}(\ell,r,\beta)$ where $z \le \bar{z}$, the condition

$$\beta \le \frac{10F((r-\ell)x+l)}{(r-\ell)C_2^{4/3}(\ell,r)\sin(\pi x)} \quad \text{for all } x \in \Omega \tag{3.6}$$

guarantees that $N[z]$ is nonincreasing for $z \in \mathsf{C}(\ell,r,\beta)$. Note that this inequality is similar to (2.5) in Proposition 2.3. Inequality (3.6) is necessary, but not sufficient for the existence of local minimizers, see the proof of Proposition 2.3 below.

With this in mind, we compare (3.6) with Lemma 2.2, which provides sufficient conditions for the twist property to hold on an interval component I. Hypothesis (b) of Lemma 2.2 is

$$\frac{\partial^2 K}{\partial v^2}v^2 - \frac{5}{2}\Big\{\frac{\partial K}{\partial v}v - K(u,v) - E\Big\} \ge 0 \quad \text{for all } u \in I_E \text{ and } v \in \mathbb{R}.$$

In our setting for eFK/Swift-Hohenberg type Lagrangians, this hypothesis reduces to

$$v^2\beta \le 10F(u) \quad \text{for all } u \in I \text{ and } v \in \mathbb{R},$$

which holds only when $\beta \le 0$. Note that when we consider functions $z(x)$ with $z = v^{3/2}$ and $u = (r-\ell)x+\ell$, we obtain exactly condition (3.3), so that Hypothesis (b) of Lemma 2.2 is sufficient, but not necessary, for the monotonicity of the operator $N[z]$. Therefore while considering the cone $\mathsf{C}(\ell,r,\beta)$, it is natural to replace condition Hypothesis (b) of Lemma 2.2 with (3.6), which leads to the more general Proposition 2.3. In particular, as shown in Example 2.4, the inequality (3.6) estimates the range of positive β values for which the twist property holds, whereas Lemma 2.2 requires $\beta \le 0$. Note that the constant in (3.6) is 10 while the constant in (2.5) in Proposition 2.3 is 2. This discrepancy is explained in the following proof.

Proof of Proposition 2.3. For existence and uniqueness we can apply the results of Crandall, Rabinowitz, and Tartar [4], where the factor of 2 in (2.5) is necessary to ensure that $\frac{2F((r-\ell)x+\ell)}{z^{5/3}} - \frac{\beta}{z^{1/3}} \to +\infty$ as $z \to 0^+$ in Equation (3.2). The C^1 dependence on ℓ and r follows from the same arguments as in the proof of [17, Lemma 9]. $\square$

Example 3.3. *Fifth-order KdV equation.* Here we briefly describe another example of how the results from this section extend the known results concerning the existence of simple closed characteristics. Consider the Lagrangian

$$L(u,v,w) = \frac{1}{2}w^2 + \frac{1}{2}(\alpha + 2\mu u)\,v^2 + \frac{\kappa}{3}u^3 + \frac{\sigma}{2}u^2 \tag{3.7}$$

which describes traveling waves in a fifth-order Korteweg-de Vries equation (see [15])

$$u_t + \gamma u_{xxxxx} + \beta u_{xxx} - \alpha\{2uu_{xx} + (u_x)^2\}_x + 2\kappa u u_x + 3ru^2 u_x = 0.$$

The techniques from the proof of Proposition 2.3 can be adapted to include the Lagrangian (3.7) and possibly more general Lagrangians of the form $\mathcal{L}(u,v,w) = \frac{1}{2}w^2 + K(u,v)$ of which L is a special case, note that when $\mu = 0$ L reduces to the eFK/Swift-Hohenberg Lagrangian. Applying Lemma 2.2 to L gives the inequality

$$\alpha + 2\mu u \le 10\left(\frac{\kappa}{3}u + \frac{\sigma}{2}\right)\frac{u^2}{v^2} \quad \text{for } u \in I_E,\ v \in \mathbb{R}, \tag{3.8}$$

which would require that $\alpha + 2\mu u \le 0$ to obtain the existence of simple closed characteristics, cf. [17, Section 2.2.3].

Assume κ, $\mu > 0$ and consider an interval component in $\left(-\frac{3\sigma}{2\kappa}, 0\right)$, if we utilize Theorem 3.2 and construct upper and lower solutions for equation (3.1), then it is sufficient that inequality (3.8) hold for v in a cone defined by upper and lower solutions. Thus no longer requiring $\alpha + 2\mu u \le 0$, we would have an estimate of the form

$$\alpha + 2\mu\left((r-\ell)x + \ell\right) \le 10\left(\frac{\kappa}{3}\left((r-\ell)x+\ell\right) + \frac{\sigma}{2}\right)\frac{\left((r-\ell)x+\ell\right)^2}{(\underline{z}(x))^{4/3}} \quad \text{for } x \in \Omega,$$

where $I = [\ell, r]$, $u = (r-\ell)x + \ell$. Note that using the explicit form of the lower solution $\underline{z}$, the right hand side term can be bounded away from zero, which gives the existence of simple closed characteristics for a range of positive values for $\alpha + 2\mu u$.

4. Evolution equations

The twist condition, as discussed in the previous section, concerns the existence of laps for fixed endpoints ℓ, r, which are found as solutions to the degenerate elliptic equation

$$E[z] = \frac{4}{9}z_{xx} + \frac{(r-\ell)^2}{3}\Big(\frac{2F((r-\ell)x+\ell)}{z^{5/3}} - \frac{\beta}{z^{1/3}}\Big) = 0.$$

In this section, we embed the problem of finding a closed characteristic into a dynamical system where the endpoints ℓ, r evolve, which would allow tools from dynamical systems theory, such as the Conley index, to be applied. There are several ways of constructing such a dynamical system. We begin by considering the gradient dynamics generated by the action functional

$$J[z,\ell,r] = \int_\Omega \frac{2}{9(r-\ell)}z_x^2 + \frac{(r-\ell)F((r-\ell)x+\ell)}{z^{2/3}} + \frac{(r-\ell)\beta}{2}z^{2/3}\,dx. \tag{4.1}$$

The evolution equations arise by computing the total variation of $J : H_0^1(\Omega) \times I^2 \to \mathbb{R}$ as a function of three variables z, ℓ, r.

To avoid the singularity at $z = 0$ we consider the perturbed functional

$$J^\epsilon[z,\ell,r] = \int_\Omega \frac{2}{9(r-\ell)} z_x^2 + \frac{(r-\ell)F((r-\ell)x+\ell)}{(z+\epsilon)^{2/3}} + \frac{(r-\ell)\beta}{2}(z+\epsilon)^{2/3} dx \quad (4.2)$$

for $\epsilon > 0$ so that

$$(r-\ell)\delta_z J^\epsilon[z,\ell,r] = -\frac{4}{9} z_{xx} - \frac{(r-\ell)^2}{3}\Big(\frac{2F((r-\ell)x+\ell)}{(z+\epsilon)^{5/3}} + \frac{\beta}{(z+\epsilon)^{1/3}}\Big)\cdot \delta z \quad (4.3)$$

$$\begin{aligned}(r-\ell)^2\delta_\ell J^\epsilon[z,\ell,r] &= \frac{2}{9}\int_\Omega z_x^2 dx + (r-\ell)^2 \int_\Omega \frac{(r-\ell)(1-x)F' - F}{(z+\epsilon)^{2/3}} dx \\ &\quad - (r-\ell)^2 \int_\Omega \frac{\beta}{2}(z+\epsilon)^{2/3} dx \cdot \delta\ell,\end{aligned} \quad (4.4)$$

$$\begin{aligned}(r-\ell)^2\delta_r J^\epsilon[z,\ell,r] &= -\frac{2}{9}\int_\Omega z_x^2 dx + (r-\ell)^2 \int_\Omega \frac{(r-\ell)xF' + F}{(z+\epsilon)^{2/3}} dx \\ &\quad + (r-\ell)^2 \int_\Omega \frac{\beta}{2}(z+\epsilon)^{2/3} dx \cdot \delta r.\end{aligned} \quad (4.5)$$

From this we obtain the system

$$\begin{aligned} z_t &= \frac{4}{9} z_{xx} + \frac{(r-\ell)^2}{3}\Big(\frac{2F((r-\ell)x+\ell)}{(z+\epsilon)^{5/3}} - \frac{\beta}{(z+\epsilon)^{1/3}}\Big), \\ \ell_t &= -\frac{2}{9}\int_\Omega z_x^2 dx - (r-\ell)^2 \int_\Omega \frac{(r-\ell)(1-x)F' - F}{(z+\epsilon)^{2/3}} dx \\ &\quad + (r-\ell)^2 \int_\Omega \frac{\beta}{2}(z+\epsilon)^{2/3} dx, \\ r_t &= \frac{2}{9}\int_\Omega z_x^2 dx - (r-\ell)^2 \int_\Omega \frac{(r-\ell)xF' + F}{(z+\epsilon)^{2/3}} dx \\ &\quad - (r-\ell)^2 \int_\Omega \frac{\beta}{2}(z+\epsilon)^{2/3} dx. \end{aligned} \quad (4.6)$$

where the argument of F and F' is $(r-\ell)x+\ell$. Let $Q_T = \Omega\times(0,T)$ and $\Sigma_T = \partial\Omega\times(0,T)$. System (4.6) allows the possibility of studying laps and closed characteristics as stationary points of a type of curve evolution in the plane. Indeed the evolution can be cast as a curve shortening type flow under a Finsler form induced by the Lagrangian [1]. However, when $\epsilon = 0$, this system is degenerate when $z = 0$.

Theorem 4.1. *Given $(z_0,\ell_0,r_0) \in H_0^1(\Omega)\times I\times I$ with $r_0 \neq \ell_0$ and $z_0 > 0$ on Ω there exists $T(z_0,\ell_0,r_0,\epsilon) > 0$ such that a unique solution $(z_\epsilon,\ell_\epsilon,r_\epsilon)$ in the space $C^1([0,T];C^\infty(\Omega))\times C^1([0,T];I)\times C^1([0,T];I)$ to (4.6) exists, and $J^\epsilon[z_\epsilon(t),\ell_\epsilon(t),r_\epsilon(t)]$ is decreasing on $[0,T]$.*

Proof. Existence and uniqueness follow from standard semigroup theory, since the perturbed nonlinearity is Lipschitz for $z+\epsilon > 0$, see for instance [9]. Moreover, J^ϵ decreases along $(z_\epsilon,\ell_\epsilon,r_\epsilon)$ by construction

$$\frac{d}{dt}J^\epsilon[z,\ell,r] = -(r-\ell)(\nabla_z J^\epsilon)^2 - (r-\ell)^2(\nabla_r J^\epsilon)^2 - (r-\ell)^2(\nabla_\ell J^\epsilon)^2 < 0.$$

□

While it may be possible to analyze the dynamics of (4.6) directly, for example by scaling the system so that the z-dynamics is fast compared to the dynamics of

ℓ, r and use geometric singular perturbation theory, we take a different approach. We consider the finite-dimensional system obtained by removing the parabolic PDE and substituting a family of stationary solutions $z_{\epsilon,\infty}$ to the PDE into the evolution equations for ℓ, r to obtain

$$\begin{aligned} \ell_t &= -\frac{2}{9}\int_\Omega (\partial_x z_{\epsilon,\infty})^2 dx - (r-\ell)^2 \int_\Omega \frac{(r-\ell)(1-x)F' - F}{(z_{\epsilon,\infty}+\epsilon)^{2/3}} dx \\ &\quad + (r-\ell)^2 \int_\Omega \frac{\beta}{2}(z_{\epsilon,\infty}+\epsilon)^{2/3} dx, \\ r_t &= \frac{2}{9}\int_\Omega (\partial_x z_{\epsilon,\infty})^2 dx - (r-\ell)^2 \int_\Omega \frac{(r-\ell)xF' + F}{(z_{\epsilon,\infty}+\epsilon)^{2/3}}\, dx \\ &\quad - (r-\ell)^2 \int_\Omega \frac{\beta}{2}(z_{\epsilon,\infty}+\epsilon)^{2/3} dx. \end{aligned} \tag{4.7}$$

Stationary solutions to this two-dimensional system are necessarily stationary solutions to system (4.6).

To see that a smooth family of solutions $z_{\epsilon,\infty}$ exists, define $E_\epsilon[z] = E[z+\epsilon]$, and recall the lower and upper solutions $\underline{z}, \bar{z}$ of $E[z]$. Then $\underline{z} - \epsilon$ is a lower solution for the regularized system since

$$E_\epsilon[\underline{z}-\epsilon] = \frac{4}{9}(\underline{z}-\epsilon)_{xx} + \frac{(r-\ell)^2}{3}\left(\frac{2F((r-\ell)x+\ell)}{(\underline{z}-\epsilon+\epsilon)^{5/3}} - \frac{\beta}{(\underline{z}-\epsilon+\epsilon)^{1/3}}\right) = E[\underline{z}] \geq 0.$$

For β satisfying (3.6),

$$E_\epsilon[\bar{z}] = \frac{4}{9}\bar{z}_{xx} + N[\bar{z}+\epsilon] \leq \frac{4}{9}\bar{z}_{xx} + N[\bar{z}] = E[\bar{z}] \leq 0,$$

and hence $\bar{z}$ is an upper solution for all $\epsilon \geq 0$. With this in mind we can define a cone for the ϵ-perturbed flow

$$\mathsf{C}(\ell, r, \beta, \epsilon) = \{z \in H_0^1(\Omega) : \underline{z} - \epsilon \leq z \leq \bar{z}\}.$$

Then the methods used in [3] and [4] imply that such a family exists within the cones $\mathsf{C}(\ell, r, \beta, \epsilon)$, see Theorem 0.8 in [3].

These methods also establish the existence of time-dependent solutions for the family of parabolic PDE's $z_t = E_\epsilon[z]$ for $\epsilon \geq 0$, however, not necessarily for the coupled PDE-ODE system (4.6) when $\epsilon = 0$, since for $\epsilon = 0$ the PDE is singular.

Here we state the existence and regularity for the following singular parabolic equation where r, ℓ are fixed.

$$z_t = \frac{4}{9}z_{xx} + \frac{(r-\ell)^2}{3}\Big(\frac{2F((r-\ell)x+\ell)}{z^{5/3}} - \frac{\beta}{z^{1/3}}\Big). \tag{4.8}$$

Definition 4.2. Let

$$\mathbf{V}(Q_T) = \{z : z \in L^\infty(Q_T), z_t \in L^2(Q_T), z \in L^\infty(0,T; H_0^1(\Omega))\}.$$

A *weak solution* to (4.8) is a function $z \in \mathbf{V}(Q_T)$, satisfying

(1) $\operatorname{ess\,inf}_K z > 0$ for every compact $K \subset Q_T$;
(2) for every test function $\phi \in \mathbf{V}(Q_T)$,

$$\int_{Q_T} \Big(\phi\frac{\partial z}{\partial t} + \frac{4}{9}z_x\phi_x - \frac{(r-l)^2}{3}\Big(\frac{2F((r-l)x+l)\phi}{z^{5/3}} - \frac{\beta\phi}{z^{1/3}}\Big)\Big)\, dx\, dt = 0;$$

(3) $z(0,x) = z_0(x)$ a.e. in Ω.

Remark 4.3. If $1/z^{5/3} \in L^\infty(0,T;H^{-1}(\Omega))$ then the integral in (2) is well-defined.

Theorem 4.4. *For $z_0 \in H_0^1(\Omega)\cap\mathcal{C}$ and β satisfying* (3.6) *there exists a unique weak solution z to the parabolic equation* (4.8) *which satisfies $z(t) \in C((0,\infty); H_0^{\frac{5}{4}-\frac{5}{2}\eta}(\Omega))$ for all $0<\eta<3/8$.*

To obtain this regularity result we utilize the methods from [3, Theorem 4.2], which require the interpolation theory of Sobolev spaces (cf. Triebel [16]), and the Hardy Inequality. Recall that the operator $A = -\frac{4}{9}\frac{d^2}{dx^2}$ has domain $\mathcal{D}(A) = H_0^1(\Omega)\cap H^2(\Omega)$, hence for fractional powers of A, A^θ with domain $\mathcal{D}(A^\theta)$ in $L^2(\Omega)$ we have the following fact.

Proposition 4.5.

(i) $\mathcal{D}(A^\theta) = (\mathcal{D}(-A), L^2(\Omega))_{1-\theta,2}$;

(ii) $\mathcal{D}(A^\theta) = H_0^{2\theta}$ *if* $1/4<\theta<1$;

(iii) A^θ *is an isomorphism from* $\mathcal{D}(A)$ *onto* $\mathcal{D}(A^{1-\theta})$ *as well as from* $L^2(\Omega)$ *onto the dual space* $(\mathcal{D}(A^\theta))'$.

See [3, Proposition 4.1], also one can look in [16].

Lemma 4.6. *Let $\theta\in[0,1)$ and $q>\frac{2}{1-\theta}$. For $0\le t_0<T<\infty$ let L_T be the linear operator defined by $L_T(f)=u$ where u is the solution to*

$$\begin{gathered}
u_t - \frac{4}{9}u_{xx} = f \quad \text{in } Q_T,\\
u = 0 \quad \text{on } \Sigma_T,\\
u(0)=0 \quad \text{in } \Omega.
\end{gathered} \tag{4.9}$$

Then L_T is a bounded operator from $L^q(t_0,T;(\mathcal{D}(A^\theta))')$ into $\mathcal{X}_{q,\theta,T}$ as well as from $L^q(t_0,T;(\mathcal{D}(A^\theta))')$ into $C([t_0,T],\mathcal{D}(A^{1-\theta-\frac{2}{q}}))$.

For a proof of the above lemma, see [3, Lemma 4.4]. In particular we have the inequality

$$\|L_T f\|_{C([t_0,T],\mathcal{D}(A^{1-\theta-\frac{2}{q}}))} \le C\|f\|_{L^q(t_0,T;(\mathcal{D}(A^\theta))')}. \tag{4.10}$$

We make use of the following Hardy type inequality, see [3, Lemma 4.5] or [16, Lemma 3.2.6.1].

Lemma 4.7. *Let $s\in[0,2]$ such that $s\ne 1/2$ and $s\ne 3/2$. Then the following generalization of Hardy's inequality holds:*

$$\|d^{-s}g\|_{L^2(\Omega)} \le C\|g\|_{H^s(\Omega)} \quad \text{for all } g\in H_0^s(\Omega). \tag{4.11}$$

Proof of Theorem 4.4. The existence and uniqueness of weak solutions to (4.8) follows from a straightforward modification of the techniques in [3] to accommodate singular terms of the form $F(x)z^{-5/3}$ with $F(x)$ bounded and positive a.e., instead of simply $z^{-5/3}$, and the addition of a weaker singular term $\beta z^{-1/3}$. In particular see [3, Theorem 0.16].

To obtain regularity we proceed as follows. For $t>0$ choose $t_0,T>0$ such that $t_0<t<T$. Note that the solution z to P_t on $(0,T)$ satisfies

$$z(t) = e^{-At}z_0 + L_T\Big(\frac{(r-l))^2}{3}\Big(\frac{2F}{z^{5/3}} - \frac{\beta}{z^{1/3}}\Big)\Big).$$

Since $0 < \eta < \frac{3}{8}$, we have $\theta = \frac{5}{4} - \frac{5}{2}\eta > \frac{1}{4}$ as in Proposition 4.5, and since $z \in \mathcal{C}$, we know that $\frac{1}{z^{5/3}} = O\big(\frac{1}{d(x)^{5/4}}\big)$. Therefore by Lemma 4.7 we have

$$\frac{(r-l))^2}{3}\Big(\frac{2F}{z^{5/3}} - \frac{\beta}{z^{1/3}}\Big) \in C((0,T],(H_0^{\frac{3}{4}+\frac{\eta}{2}}(\Omega))').$$

Indeed setting $s = \frac{3}{4} + \frac{\eta}{2}$ in (4.11), for $g \in H_0^{\frac{3}{4}+\frac{\eta}{2}}$ we obtain

$$\begin{aligned}
\|\frac{1}{z^{5/3}}g\|^2_{L^2(\Omega)} &\le \int_\Omega \Big|\frac{g}{C_1 d(x)^{5/4}}\Big|^2 dx \\
&= \frac{1}{C_1^2}\int_\Omega \Big|\frac{d(x)^{3/4+\eta/2}}{d(x)^{5/4}}\frac{g}{d(x)^{3/4+\eta/2}}\Big|^2 dx \\
&\le \frac{1}{C_1^2}\|d^{-1/2+\eta/2}\|_{L^2(\Omega)}\|d(x)^{-(3/4+\eta/2)}g\|_{L^2(\Omega)} \\
&\le \frac{1}{C_1^2}\|d^{-1/2+\eta/2}\|_{L^2(\Omega)}\|g\|_{H_0^{3/4+\eta/2}}.
\end{aligned}$$

Furthermore, $C((0,T],(H_0^{\frac{3}{4}+\frac{\eta}{2}}(\Omega))') = C((0,T],(\mathcal{D}(A)^{\frac{3}{8}+\frac{\eta}{4}})')$ from Proposition 4.5. Applying Lemma 4.6 with $q = \frac{2}{\eta}$ and $\theta = \frac{3}{8} + \frac{\eta}{4}$ we see that

$$L_T\Big(\frac{(r-l))^2}{3}\Big(\frac{2F}{z^{5/3}} - \frac{\beta}{z^{1/3}}\Big)\Big) \in C([0,T],D(A^{1-\theta-\frac{2}{q}})) = C([0,T],D(A^{\frac{5}{8}-\frac{5}{4}\eta})).$$

Therefore,

$$L_T\Big(\frac{(r-l))^2}{3}\Big(\frac{2F}{z^{5/3}} - \frac{\beta}{z^{1/3}}\Big)\Big) \in C([0,T],H_0^{\frac{5}{4}-\frac{5}{2}\eta}(\Omega)),$$

and

$$t \to z(t) = e^{-At}z_0 + L_T\Big(\frac{(r-l)^2}{3}\Big(\frac{2F}{z^{5/3}} - \frac{\beta}{z^{1/3}}\Big)\Big) \in C((0,T],H_0^{\frac{5}{4}-\frac{5}{2}\eta}(\Omega)).$$

Since t is arbitrary, $z(t) \in C((0,\infty),H_0^{\frac{5}{4}-\frac{5}{2}\eta}(\Omega))$. □

We now obtain the following existence and stabilization result.

Theorem 4.8. *Fix $u_{\min} < \ell < r < u_{\max}$. If β satisfies the bound in* (3.6)*, then there exists a unique $z_{\epsilon,\infty} \in H_0^1(\Omega) \cap \mathsf{C}(\ell,r,\beta,\epsilon) \cap C_0(\overline{\Omega})$ satisfying*

$$E_\epsilon[z_{\epsilon,\infty}] = 0.$$

Moreover, for each $z_0 \in \mathsf{C}(\ell,r,\beta,\epsilon)$ there exists a unique solution $z(t)$ of

$$z_t = E_\epsilon[z] \quad \text{for } t \in (0,\infty) \text{ with } z(0) = z_0, \tag{4.12}$$

and $z(t) \to z_{\epsilon,\infty}$ as $t \to \infty$ in $H_0^1(\Omega)$. Furthermore, there exists $M(\ell,r,\beta) > 0$ such that

$$\|z_{\epsilon,\infty}\|_{H_0^{9/8}(\Omega)} \le M(\ell,r,\beta) \quad \text{for } \epsilon \ge 0.$$

Proof. First, as described in the proof of Theorem 4.4, we note that the techniques in [3] can be modified to apply to the type of nonlinearity we are considering here. In particular Theorem 0.8 in [3] establishes the existence and uniqueness of the function $z_\infty \in H_0^1 \cap \mathcal{C}$ with $E[z_\infty] = 0$, and [3, Theorem 0.15] can then be applied to obtain convergence of $z(t) \to z_\infty$ in $L^\infty(\Omega)$.

Now setting $\eta = 1/20$ in Theorem 4.4 yields $z(t) \in C((0,\infty), H_0^{9/8}(\Omega))$. Fix $\tau > 0$ and for each $t > 0$ choose $0 < t_0 < t < t_0 + \tau = T$. Then

$$t \to z(t) = e^{-tA} z_0 + L_T\Big(\frac{(r-\ell)^2}{3}\big(\frac{2F}{z^{5/3}} - \frac{\beta}{z^{1/3}}\big)\Big).$$

Since $\underline{z} \le z(t) \le \bar{z}$, we can apply Lemma 4.6

$$\begin{aligned}
&\|L_T\Big(\frac{(r-\ell)^2}{3}\big(\frac{2F}{z^{5/3}} - \frac{\beta}{z^{1/3}}\big)\Big)\|_{C([t_0,T],H_0^{9/8}(\Omega))}\\
&\le C\|\frac{(r-\ell)^2}{3}\big(\frac{2F}{z^{5/3}} - \frac{\beta}{z^{1/3}}\big)\|_{L^q(t_0,T;(H_0^{31/40}(\Omega))')}\\
&\le C\|\frac{(r-\ell)^2}{3}\frac{2F}{z^{5/3}}\|_{L^q(t_0,T;(H_0^{31/40}(\Omega))')}\\
&\quad + C\|\frac{(r-\ell)^2}{3}\frac{|\beta|}{z^{1/3}}\|_{L^q(t_0,T;(H_0^{31/40}(\Omega))')}\\
&\le C(\ell,r)\tau^{1/q}\|\frac{(r-\ell)^2}{3}\frac{2F}{z^{5/3}}\|_{L^q(t_0,T;(H_0^{31/40}(\Omega))')}\\
&\quad + C(\ell,r)\tau^{1/q}\|\frac{(r-\ell)^2}{3}\frac{|\beta|}{z^{1/3}}\|_{(H_0^{13/32}(\Omega))'} < \infty
\end{aligned} \tag{4.13}$$

Using the Hardy inequality as in the proof of Theorem 4.4 we see that the last terms are bounded independent of T, and hence $\sup_{[1,\infty]} \|z(t)\|_{H_0^{9/8}(\Omega)} < \infty$. From the compactness of the embedding of $H_0^{9/8}$ into H_0^1 it follows that for every $z_0 \in H_0^1 \cap \mathcal{C}$ we have $z(t) \to z_\infty$ as $t \to \infty$ in $H_0^1(\Omega)$.

For $\epsilon > 0$ we can repeat the above arguments for the perturbed parabolic equation

$$z_t = \frac{4}{9} z_{xx} + \frac{(r-\ell)^2}{3}\Big(\frac{2F((r-\ell)x+\ell)}{(z+\epsilon)^{5/3}} - \frac{\beta}{(z+\epsilon)^{1/3}}\Big) \tag{4.14}$$

as follows. Just as for the singular case $\epsilon = 0$ addressed in Theorem 4.4, the techniques of [3] can be suitably modified to establish the existence, uniqueness, and regularity to the parabolic PDE (4.14) for $\epsilon > 0$. Indeed, standard semigroup theory [9] also applies in this case since the nonlinearity is not singular. Recall that a lower solution for (4.14) is given by $\underline{z}_\epsilon = \underline{z} - \epsilon$, and the upper solution $\bar{z}$ for P_t is also an upper solution for (4.14). Therefore the arguments used in the proof of Theorem 4.4 can be applied to establish that $z_\epsilon(t) \in C((0,\infty), H_0^{9/8}(\Omega))$. Note that since $z_\epsilon(t) > -\epsilon$, the solution exists for all $t \ge 0$. Also recall the existence of a unique solution $z_{\epsilon,\infty} \in H_0^1 \cap \mathcal{C}$ to $E_\epsilon[z_{\epsilon,\infty}] = 0$ with $z_\epsilon(t) \to z_{\epsilon,\infty}$ in $L^\infty(\Omega)$ by Theorem 0.15 in [3].

Moreover since $\underline{z}_\epsilon \le z_\epsilon(t)$, we have

$$\frac{1}{(z_\epsilon(t)+\epsilon)^p} \le \frac{1}{(\underline{z}_\epsilon+\epsilon)^p} = \frac{1}{\underline{z}^p} \quad \text{for } p = 5/3 \text{ or } 1/3.$$

Hence the bound in equation (4.13) holds for the solutions to (4.14) as well so that $\sup_{[1,\infty]} \|z_\epsilon(t)\|_{H_0^{9/8}(\Omega)} < \infty$, and for every $z_0 \in H_0^1 \cap \mathcal{C}$ we have $z_\epsilon(t) \to z_{\epsilon,\infty}$ in $H_0^1(\Omega)$.

Finally, taking $z_0 = z_{\epsilon,\infty}$ so that $z_\epsilon(t) = z_{\epsilon,\infty}$ for all $t \ge 0$, and using the fact that $e^{-tA}z_0 \to 0$ in $H_0^{9/8}(\Omega)$, for each $\epsilon > 0$ there exists a time $t_1 > 0$ such that $\|e^{-tA}z_{\epsilon,\infty}\|_{H_0^{9/8}} \le 1$. Since the bound (4.13) on the nonlinearity is independent of ϵ, there exists $M(\ell,r,\beta) > 0$ such that $\|z_{\epsilon,\infty}\|_{H_0^{9/8}} \le M(\ell,r,\beta)$ for all $\epsilon \ge 0$. $\square$

Corollary 4.9. *$z_{\epsilon,\infty} \to z_\infty$ as $\epsilon \to 0$ in $H_0^1(\Omega)$. Hence the expressions for ℓ_t and r_t in equation (4.7) for $\epsilon > 0$ converge as $\epsilon \to 0$ to the corresponding expression for $\epsilon = 0$.*

Proof. For notational convenience let us write $z_\epsilon = z_{\epsilon,\infty}$. First we establish convergence in $L^\infty(\Omega)$. From the weak comparison principle, we get that if $0 \le \tilde\epsilon \le \epsilon$ then $z_\epsilon \le z_{\tilde\epsilon}$ and $z_{\tilde\epsilon} + \tilde\epsilon \le z_\epsilon + \epsilon$. To see this note that for $\xi_\epsilon = z_\epsilon + \epsilon$ and $\xi_{\tilde\epsilon} = z_{\tilde\epsilon} + \tilde\epsilon$ we have

$$\begin{gathered}
\frac{4}{9}\partial_{xx}^2(\xi_{\tilde\epsilon} - \xi_\epsilon) + (r-l)^2\frac{2}{3}F((r-l)x+l)\Big(\frac{1}{\xi_{\tilde\epsilon}^{5/3}} - \frac{1}{\xi_\epsilon^{5/3}}\Big) = 0 \\
-\frac{\beta}{3}\Big(\frac{1}{\xi_{\tilde\epsilon}^{1/3}} - \frac{1}{\xi_\epsilon^{1/3}}\Big) \quad \text{in } \Omega, \\
z_{\tilde\epsilon} - z_\epsilon \le 0 \quad \text{on } \partial\Omega.
\end{gathered}$$

From this it follows that $(z_\epsilon)_{\epsilon>0}$ is a Cauchy sequence in $L^\infty(\Omega)$ and there exists $z \in L^\infty(\Omega)$ satisfying $z_\epsilon \to z$ as $\epsilon \to 0$ in $L^\infty(\Omega)$ and $\underline{z} = \lim_{\epsilon\to 0^+} \underline{z}_\epsilon \le z \le \bar z$. From Theorem 4.8 we have that $z_{\epsilon,\infty} \to z$ in $H_0^1(\Omega)$. For $\varphi \in H_0^1(\Omega)$

$$\begin{aligned}
&\int_\Omega (z_\epsilon)_{xx}\,\varphi dx + \frac{(r-\ell)^2}{3}\int_\Omega \Big(\frac{2F((r-\ell)x+\ell)}{z_\epsilon^{5/3}} - \frac{\beta}{z_\epsilon^{1/3}}\Big) \\
&= -\int_\Omega (z_\epsilon)_x\,\varphi_x dx + \frac{(r-\ell)^2}{3}\int_\Omega \Big(\frac{2F((r-\ell)x+\ell)}{z_\epsilon^{5/3}} - \frac{\beta}{z_\epsilon^{1/3}}\Big) = 0.
\end{aligned}$$

Letting $\epsilon \to 0$ yields

$$-\int_\Omega z_x\varphi_x dx + \frac{(r-\ell)^2}{3}\int_\Omega \Big(\frac{2F((r-\ell)x+\ell)}{z^{5/3}} - \frac{\beta}{z^{1/3}}\Big) = 0.$$

Therefore z is a weak solution to the stationary equation for (4.8), and since z_∞ is the unique such solution, we have $z_{\epsilon,\infty} \to z_\infty$ as $\epsilon \to 0$. Since the right hand side of (4.7) converges pointwise a.e. in Ω, letting $\epsilon \to 0$ and applying the Dominated Convergence Theorem yields the convergence of ℓ_t and r_t. □

Next we establish that the maximal time interval of existence for (4.6) relies only on the dynamics of $\ell_\epsilon(t)$ and $r_\epsilon(t)$.

Theorem 4.10. *The solution $(z_\epsilon(t), l_\epsilon(t), r_\epsilon(t))$ to (4.6) stays in $\mathsf{C}(\ell, r, \beta, \epsilon) \times I^2$ for all $t \ge 0$ in its maximal interval of existence. Moreover, either the solution can be extended globally to $[0,\infty)$, or at the maximal time of existence T, either $\ell_\epsilon(T) = u_{\min}$ and ℓ is decreasing at time T or $r_\epsilon(T) = u_{\max}$ and r is increasing at time T, i.e. either the maximal interval of existence is infinite, or either ℓ_ϵ or r_ϵ leaves I in finite time.*

Proof. Suppose that $\ell_\epsilon(t)$, $r_\epsilon(t)$ remain in I over the time interval $[0,T]$. From Theorem 4.8 the set $\{(z_\epsilon(t), \ell_\epsilon(t), r_\epsilon(t))\}$ is compact in $C\left([T/2,T], H_0^1(\Omega)\right) \times I^2$. We can then choose a sequence $\{(z_\epsilon(t_n), \ell_\epsilon(t_n), r_\epsilon(t_n))\}_{n\ge1}$ such that $(z_\epsilon(t_n), \ell_\epsilon(t_n), r_\epsilon(t_n)) \to (z_\epsilon^*, \ell_\epsilon^*, r_\epsilon^*)$ for $t_n \to T$. Treating $(z_\epsilon^*, \ell_\epsilon^*, r_\epsilon^*)$ as new initial conditions, if ℓ is nondecreasing and r is nonincreasing at time T, the solution can be extended beyond time T. Therefore, if the maximal interval of existence is $[0,T]$ with $T < \infty$, we must have either $\ell(T) = u_{\min}$ with ℓ decreasing at T or $r(T) = u_{\max}$ with r increasing at T. □

The following proposition shows that in the regularized system with $\epsilon > 0$, the integral expressions for ℓ_t, r_t in equation (4.7) have a simpler pointwise formulation.

Proposition 4.11. *Let $z_{\epsilon,\infty}$ be the family of solutions of $E_\epsilon[z] = 0$ for $\epsilon > 0$. Recalling that $u = (r-\ell)x + \ell$, then*

$$\partial_\ell J^\epsilon[z,\ell,r] = \frac{2}{9}\left(\partial_u z_{\epsilon,\infty}(\ell)\right)^2 - \frac{F(\ell)}{\epsilon^{2/3}} - \frac{\beta}{2}\epsilon^{2/3}, \tag{4.15}$$

$$\partial_r J^\epsilon[z,\ell,r] = -\frac{2}{9}\left(\partial_u z_{\epsilon,\infty}(r)\right)^2 + \frac{F(r)}{\epsilon^{2/3}} + \frac{\beta}{2}\epsilon^{2/3}. \tag{4.16}$$

Proof. We start by reducing the expression

$$\begin{aligned}
&\int_\Omega \frac{2}{9(r-\ell)^2} z_x^2 + \frac{(r-\ell)(1-x)F' - F}{(z+\epsilon)^{2/3}} - \frac{\beta}{2}(z+\epsilon)^{2/3} dx \\
&= \int_\ell^r \frac{2}{9(r-\ell)} z_u^2 + \frac{F_\ell}{(z+\epsilon)^{2/3}} - \frac{F}{(r-\ell)(z+\epsilon)^{2/3}} - \frac{\beta(z+\epsilon)^{2/3}}{2(r-\ell)} du.
\end{aligned}$$

The term $\int_\ell^r \frac{F_\ell}{(z+\epsilon)^{2/3}} du$ can be further simplified

$$\begin{aligned}
&\int_\ell^r \frac{F_\ell}{(z+\epsilon)^{2/3}} du \\
&= \int_\ell^r \frac{1}{(z+\epsilon)^{2/3}} \frac{du}{d\ell}\frac{dF}{du} du = \frac{1}{(r-\ell)} \int_\ell^r \frac{r-u}{(z+\epsilon)^{2/3}} \frac{dF}{du} du \\
&= \frac{(r-u)}{(r-\ell)} \frac{F}{(z+\epsilon)^{2/3}}\Big|_\ell^r - \frac{1}{(r-\ell)} \int_\ell^r \frac{d}{du}\Big[\frac{r-u}{(z+\epsilon)^{2/3}}\Big] F du \\
&= -\frac{F(\ell)}{\epsilon^{2/3}} + \frac{1}{(r-\ell)} \int_\ell^r \frac{F}{(z+\epsilon)^{2/3}} du + \frac{2}{3(r-\ell)} \int_\ell^r \frac{(r-u)F}{(z+\epsilon)^{5/3}} z_u du.
\end{aligned}$$

Next note that

$$\begin{aligned}
&\frac{2}{3(r-\ell)} \int_\ell^r F \frac{(r-u)}{(z+\epsilon)^{5/3}} z_u du \\
&= -\frac{4}{9(r-\ell)} \int_\ell^r (r-u) z_{uu} z_u du + \frac{\beta}{3(r-\ell)} \int_\ell^r (r-u)(z+\epsilon)^{-1/3} z_u du \\
&= -\frac{2}{9(r-\ell)} \int_\ell^r (r-u) \frac{d}{du}(z_u^2) du + \frac{\beta(r-u)}{2(r-\ell)} (z+\epsilon)^{2/3}\Big|_\ell^r + \frac{\beta}{2(r-\ell)} \int_\ell^r (z+\epsilon)^{2/3} du \\
&= \frac{2}{9} z_u^2(\ell) - \frac{2}{9(r-\ell)} \int_\ell^r z_u^2 du - \frac{\beta}{2}\epsilon^{2/3} + \frac{\beta}{2(r-\ell)} \int_\ell^r (z+\epsilon)^{2/3} du.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\int_\Omega \frac{2}{9(r-\ell)^2} z_x^2 + \frac{(r-\ell)(1-x)F' - F}{(z+\epsilon)^{2/3}} - \frac{\beta}{2}(z+\epsilon)^{2/3} dx \\
&= \frac{2}{9} z_u^2(\ell) - \frac{F(\ell)}{(z+\epsilon)^{2/3}} - \frac{\beta}{2}(z+\epsilon)^{2/3} \\
&= \frac{2}{9} z_u^2(\ell) - \frac{F(\ell)}{\epsilon^{2/3}} - \frac{\beta}{2}\epsilon^{2/3},
\end{aligned}$$

the last equality follows from the boundary conditions $z(\ell) = z(r) = 0$. The proposition now follows from Equation (4.3). ∎

Note that Corollary 4.9 and Proposition 4.11 can be used together to establish the signs of ℓ_t and r_t in equation (4.7) at $\epsilon = 0$. In Corollary 2.5 we established the existence of simple closed characteristics in the more general setting of Proposition 2.3. We now use the results of this section to describe a framework to consider non-simple closed characteristics.

5. Final remarks: a framework for non-simple closed characteristics

In the previous sections we built a framework in which to study a gradient-like dynamics using the second-order Lagrangian action functional

$$J[u] = \int_0^\tau \frac{1}{2}(u'')^2 + \frac{\beta}{2}(u')^2 + F(u)\,dt \tag{5.1}$$

where we first considered an individual monotone lap u with $u(0) = \ell, u(\tau) = r$ and $u' > 0$ in $(0, \tau)$. Now we consider a periodic function u as a concatenation of finitely many laps, which yields a closed curve in the (u, u')-plane. Recall that for each increasing lap with $u(0) = \ell$, $u(\tau) = r$, and $\ell < r$, we define $z = |u'|^{3/2}$, and z is a function of u by monotonicity. Changing variables, z is taken to be a function of the variable x defined by $u = (r-\ell)x+\ell$. The action functional $J[z; \ell, r]$ in these variables is given in equation (4.1). In the case of a decreasing lap with $u(0) = r$, $u(\tau) = \ell$, and $\ell < r$, we still define $z = |u'|^{3/2}$ so that z is a function of u from ℓ to r, and hence the variable x is again defined by $u = (r - \ell)x + \ell$, and $J[z; r, \ell]$ in these variables is given again by equation (4.1). Indeed, since the Lagrangian L has the symmetry $L(u, -v, w) = L(u, v, w)$, we have that if $u(t)$ is an increasing lap from ℓ to r, then $u(\tau - t)$ is a decreasing lap from r to ℓ so that z is the same function in both cases, and hence $J[z; \ell, r] = J[z; r, \ell]$.

The gradient dynamics of J for functions with n laps can then be expressed as a coupled system of n parabolic equations and n ODE's by extending the system (4.6) defined in Section 4. The resulting dynamics can be viewed as a type of curve evolution in the (u, u')-plane. However, as forecast in the previous section, we reduce this dynamics by fixing the laps to be in the family of unique stationary laps determined by fixed endpoints in Theorem 4.8. This results an n-dimensional system of ODE's for the endpoints only, and stationary solutions of the reduced system must be stationary for the full system.

We now derive the system we consider in some detail. As described in Section 2, the method of broken geodesics has been applied to second-order Lagrangian systems in [17] in the case where unique laps exist as minimizers as follows. Let $u_1, u_2, \dots, u_n$ and $u_{n+1} = u_1$ be the local extreme values of u and define

$$S(u_k, u_{k+1}) = \min J[z; u_k, u_{k+1}] = J[(z_\infty)_k, u_k, u_{k+1}].$$

Then define $W(u_1, \dots, u_n) = \sum_{i=1}^n S(u_k, u_{k+1})$ which is the sum of the actions of the minimizing laps, i.e. the action of the broken geodesic. The k-th component of the gradient of W is simply $\partial_2 S(u_{k-1}, u_k) + \partial_1 S(u_k, u_{k+1})$. Properties of these derivatives can be exploited to prove the existence of critical points for W, which yield closed characteristics, see [17].

Here we let z_k be the variable of the k-th lap and consider the action of the broken geodesic which is given by

$$\widehat{W}[z_1, \dots, z_n; u_1, \dots, u_n] = \sum_{k=1}^n J[z_k, u_k, u_{k+1}].$$

Assume that $u_1 < u_2$ so that for k odd u_k is a minimum and for k even u_k is a maximum of the broken geodesic. Then from equation (4.3) we have

$$\begin{aligned}
&(-1)^{k+1}\partial_{u_k}\widehat{W} \\
&= \frac{2}{9(u_{k-1}-u_k)^2}\int_\Omega (\partial_x z_{k-1})^2\,dx + \frac{2}{9(u_{k+1}-u_k)^2}\int_\Omega (\partial_x z_k)^2\,dx \\
&+ \int_\Omega \frac{(u_{k-1}-u_k)\sigma_k(x)F'((u_{k-1}-u_k)x+u_k) - F((u_{k-1}-u_k)x+u_k)}{z_{k-1}^{2/3}}\,dx \\
&+ \int_\Omega \frac{(u_{k+1}-u_k)\sigma_k(x)F'((u_{k+1}-u_k)x+u_k) - F((u_{k+1}-u_k)x+u_k)}{z_k^{2/3}}\,dx \\
&- \frac{\beta}{2}\int_\Omega z_{k-1}^{2/3}\,dx - \frac{\beta}{2}\int_\Omega z_k^{2/3}\,dx
\end{aligned} \tag{5.2}$$

where $\sigma_k(x) = 1-x$ for k odd and $\sigma_k(x) = -x$ for k even.

Substituting $(z_\infty)_k$ into the gradient-like system derived from equation (5.2) yields a dynamics for the evolution of the endpoints of the closed curve, as in Section 4. One can simplify the terms involving $\int_\Omega (\partial_x z_k)^2\,dx$ in (5.2). Using the fact that $(z_\infty)_k$ is a solution to the stationary PDE (3.2) we have

$$\frac{2}{9(u_{k+1}-u_k)^2}\int_\Omega (\partial_x (z_\infty)_k)^2\,dx = \frac{1}{3}\int_\Omega \frac{F((u_{k+1}-u_k)x+u_k)}{(z_\infty)_k^{2/3}}\,dx - \frac{\beta}{6}\int_\Omega (z_\infty)_k^{2/3}\,dx.$$

Moreover, to obtain a pointwise estimate of the right-hand side of (5.2) one can consider the corresponding perturbed system and apply Proposition 4.11. This would lead to pointwise estimates for the unperturbed system after sending $\epsilon \to 0$ and applying Corollary 4.9 along with standard convergence theorems. Furthermore using the explicit upper and lower solutions from Section 3.2 the integral terms in (5.2) could be numerically investigated. This type of analysis could be helpful in constructing an index pair or isolating neighborhood for the classes of non-simple closed characteristics with certain profiles.

References

[1] Ronald Adams; *Curve Shortening in Second-Order Lagrangian Systems*, Ph.D. thesis, Florida Atlantic University, 2014.

[2] S. B. Angenent, J. B. van den Berg, R. C. A. M. Vandervorst; *Contact and non-contact type Hamiltonian systems generated by second-order Lagrangians*, Ergodic Theory Dynam. Systems **27** (2007), no. 1, 23–49. MR 2297085

[3] Mehdi Badra, Kaushik Bal, Jacques Giacomoni; *A singular parabolic equation: existence, stabilization*, J. Differential Equations **252** (2012), no. 9, 5042–5075. MR 2891356

[4] M. G. Crandall, P. H. Rabinowitz, L. Tartar; *On a Dirichlet problem with a singular nonlinearity*, Comm. Partial Differential Equations **2** (1977), no. 2, 193–222. MR 0427826

[5] Mabel Cuesta, Peter Takáč; *A strong comparison principle for positive solutions of degenerate elliptic equations*, Differential Integral Equations **13** (2000), no. 4-6, 721–746. MR 1750048

[6] Jacqueline Fleckinger-Pellé, Peter Takáč; *Uniqueness of positive solutions for nonlinear cooperative systems with the p-Laplacian*, Indiana Univ. Math. J. **43** (1994), no. 4, 1227–1253. MR 1322617

[7] R. W. Ghrist, J. B. Van den Berg, R. C. Vandervorst; *Morse theory on spaces of braids and Lagrangian dynamics*, Invent. Math. **152** (2003), no. 2, 369–432. MR 1974892

[8] Robert W. Ghrist, Jan Bouwe van den Berg, Robert C. Vandervorst; *Closed characteristics of fourth-order twist systems via braids*, C. R. Acad. Sci. Paris Sér. I Math. **331** (2000), no. 11, 861–865. MR 1806423

[9] Daniel Henry; *Geometric Theory of Semilinear Parabolic Equations*, Lecture Notes in Mathematics, vol. 840, Springer-Verlag, Berlin-New York, 1981. MR 610244
[10] W. D. Kalies, J. Kwapisz, J. B. VandenBerg, R. C. A. M. VanderVorst; *Homotopy classes for stable periodic and chaotic patterns in fourth-order Hamiltonian systems*, Comm. Math. Phys. **214** (2000), no. 3, 573–592. MR 1800862
[11] W. D. Kalies, J. Kwapisz, R. C. A. M. VanderVorst; *Homotopy classes for stable connections between Hamiltonian saddle-focus equilibria*, Comm. Math. Phys. **193** (1998), no. 2, 337–371. MR 1618147
[12] W. D. Kalies, R. C. A. M. Vander Vorst; *Closed characteristics of second-order Lagrangians*, Proc. Roy. Soc. Edinburgh Sect. A **134** (2004), no. 1, 143–158. MR 2039908
[13] W. D. Kalies, R. C. A. M. VanderVorst; *Multitransition homoclinic and heteroclinic solutions of the extended Fisher-Kolmogorov equation*, J. Differential Equations **131** (1996), no. 2, 209–228. MR 1419012
[14] William D. Kalies, Mark Wess; *Closed characteristics on singular energy levels of second-order Lagrangian systems*, J. Differential Equations **244** (2008), no. 3, 555–581. MR 2384484
[15] L. A. Peletier, W. C. Troy; *Spatial Patterns*, Progress in Nonlinear Differential Equations and their Applications, 45, Birkhäuser Boston, Inc., Boston, MA, 2001. MR 1839555
[16] Hans Triebel; *Interpolation Theory, Function Spaces, Differential Operators*, second ed., Johann Ambrosius Barth, Heidelberg, 1995. MR 1328645
[17] J. B. van den Berg, R. C. Vandervorst; *Second order Lagrangian twist systems: simple closed characteristics*, Trans. Amer. Math. Soc. **354** (2002), no. 4, 1393–1420 (electronic). MR 1873011
[18] J. B. van den Berg, R. C. Vandervorst; *Stable patterns for fourth-order parabolic equations*, Duke Math. J. **115** (2002), no. 3, 513–558. MR 1940411
[19] R. C. A. M. Vandervorst, J. B. van den Berg; *Periodic orbits for fourth order conservative systems and Morse type theory*, International Conference on Differential Equations, Vol. 1, 2 (Berlin, 1999), World Sci. Publ., River Edge, NJ, 2000, pp. 241–245. MR 1870130
[20] Claude Viterbo; *A proof of Weinstein's conjecture in* $\mathbf{R}^{2n}$, Ann. Inst. H. Poincaré Anal. Non Linéaire **4** (1987), no. 4, 337–356. MR 917741
[21] Alan Weinstein; *On the hypotheses of Rabinowitz' periodic orbit theorems*, J. Differential Equations **33** (1979), no. 3, 353–358. MR 543704

Ronald Adams
Department of Mathematics, Embry-Riddle Aeronautical University, Daytona Beach, FL 32114, USA
E-mail address: `adamsr25@erau.edu`

William D. Kalies
Department of Mathematics, Florida Atlantic University, Boca Raton, FL 33431, USA
E-mail address: `wkalies@fau.edu`

Robert C. A. M. Vandervorst
Department of Mathematics, VU University, Amsterdam, NL
E-mail address: `vdvorst@few.vu.nl`