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NONLOCAL STURM-LIOUVILLE PROBLEMS WITH INTEGRAL TERMS IN THE BOUNDARY CONDITIONS

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Abstract. We consider a new type Sturm-Liouville problems whose main feature is the nature of boundary conditions. Namely, we study the nonhomogeneous Sturm-Liouville equation

$$p(x)u''(x) + (q(x) - \lambda)u = f(x)$$

on two disjoint intervals $[-1, 0)$ and $(0, 1]$, subject to the nonlocal boundary-transmission conditions

$$\begin{aligned} &\alpha_k u^{(m_k)}(-1) + \beta_k u^{(m_k)}(-0) + \eta_k u^{(m_k)}(+0) + \gamma_k u^{(m_k)}(1) \\ &+ \sum_{j=1}^{n_k} \delta_{kj} u^{(m_k)}(x_{kj}) + \sum_{\upsilon=1}^{2} \sum_{j=0}^{m_k} \int_{\Omega_\upsilon} \mathcal{K}_{k\upsilon j}(t) u^{(j)}(t)dt = f_k, \quad k = 1, 2, 3, 4. \end{aligned}$$

where $\Omega_1 := [-1, 0)$, $\Omega_2 := (0, 1]$ and $x_{kj} \in (-1, 0) \cup (0, 1)$ are internal points. By using our own approaches we establish such important properties as Fredholmness, coercive solvability and isomorphism with respect to the spectral parameter λ.

1. Introduction

Various generalizations of classical Sturm-Liouville problems for ordinary linear differential equations have attracted a lot of attention because of the appearance of new important applications in physical sciences and applied mathematics. For instance, theoretical investigations have become interested in the discontinuous Sturm-Liouville problems for its application in physics. The discontinuity of the coefficients of the equations in the Sturm-Liouville problems corresponds to the fact that the heterogeneous media consists of two different materials. On the other hand, transmission problems appear frequently in various fields of physics such as in electrostatics, magnetostatics and in solid mechanic for discontinuous problems (in these regard see, [8, 21]). Solvability and some spectral properties of nonlocal Sturm-Liouville problems have been investigated by many authors; see for example, [3, 4, 12, 13, 14, 20, 27, 28]). An important special case of the nonlocal Sturm-Liouville problems are so-called multipoint Sturm-Liouville problems. Such

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problems have been extensively studied by many authors; see for example,[9, 10, 11] and references therein.

In general, the mathematical problems encountered in the study of boundary value transmission problems or nonclassical problems cannot be treated with the usual techniques within the standard framework of Sturm-Liouville problems. In classical theory of boundary-value problems for ordinary differential equations is usually considered for equations with continuous coefficients and for boundary conditions which contain only endpoints of the considered interval. This article deals with one nonclassical boundary-value problem for a second-order ordinary differential equation with discontinuous coefficients and boundary conditions containing not only endpoints of the considered interval, but also a finite number of internal points and integral terms. Namely, we consider the differential equation

$$L(\lambda)u := p(x)u''(x) + (q(x)-\lambda)u(x) = f(x), \quad x \in [-1,0)\cup(0,1] \tag{1.1}$$

together with new type boundary conditions

$$\begin{aligned} L_k u := {}& \alpha_k u^{(m_k)}(-1) + \beta_k u^{(m_k)}(-0) + \eta_k u^{(m_k)}(+0) + \gamma_k u^{(m_k)}(1) \\ & + \sum_{j=1}^{n_k} \delta_{kj} u^{(m_k)}(x_{kj}) + \sum_{\upsilon=1}^{2}\sum_{j=0}^{m_k}\int_{\Omega_\upsilon} \mathcal{K}_{k\upsilon j}(t)u^{(j)}(t)dt = f_k, \end{aligned} \tag{1.2}$$

for $k=1,2,3,4$, where $p(x)$ is piecewise constant function, $p(x)=p_1$ for $x\in[-1,0)$, $p(x)=p_2$ for $x\in(0,1]$; λ-complex parameter; p_i $(i=1,2)$, α_k, β_k, η_k, γ_k, δ_{ki} $(i=1,2,\ k=1,2,3,4)$ are complex coefficients; m_k $(k=1,2,3,4)$ are integers; $\Omega_1 := (-1,0)$, $\Omega_2 := (0,1)$; $\mathcal{K}_{k\upsilon j} \in W_q^{m_k}(-1,0)\dot{+}W_q^{m_k}(0,1)$; $x_{kj}\in(-1,0)\cup(0,1)$ are internal points and $q(x)$ is measurable function on $[-1,0)\cup(0,1]$. Naturally, we shall assume that, $p_1\neq 0$, $p_2\neq 0$ and $|\alpha_k|+|\beta_k|+|\eta_k|+|\gamma_k|\neq 0$ $(k=1,2,3,4)$. Some special cases of the considered Sturm-Liouville problem (1.1)–(1.2) arise after an application of the method of separation of variables to the varied assortment of physical problems, namely, in heat and mass transfer problems (see, for example, [19]), in diffraction problems (for example, [1]), in vibrating string problems, when the string loaded additionally with point masses (see, [29]) and etc. Some problems with transmission conditions which arise in mechanics were studied in [21, 29]. Investigation of various spectral properties of some nonlocal boundary-value problems can be found in some works of Imanbaev [12], Sadybekov [26], Shakhmurov [27], Aliyev [2] and Rasulov [25]. Note that some new type Sturm-Liouville problems with nonlocal boundary conditions were investigated by authors of this paper and some others [5, 6, 7, 15, 16, 24, 22, 23].

2. Homogeneous equation with nonhomogeneous transmission conditions

For convenience we denote

$$S_k u := \sum_{j=1}^{n_k}\delta_{kj}u^{(m_k)}(x_{kj}), \quad \mathcal{F}_k u := \sum_{\upsilon=1}^{2}\sum_{j=0}^{m_k}\int_{\Omega_\upsilon}\mathcal{K}_{k\upsilon j}(t)u^{(j)}(t)dt, \quad k=1,2,3,4\,.$$

We consider the homogeneous differential equation

$$L_0(\lambda)u := p(x)u''(x) - \lambda u(x) = 0 \tag{2.1}$$

with the nonlocal and nonhomogeneous boundary conditions

$$\begin{aligned} L_{k0}u &:= \alpha_k u^{(m_k)}(-1) + \beta_k u^{(m_k)}(-0) + \eta_k u^{(m_k)}(+0) \\ &\quad + \gamma_k u^{(m_k)}(1) + S_k u = f_k, \quad k = 1,2,3,4. \end{aligned} \tag{2.2}$$

For convenience we shall use the notation

$$\begin{gathered} \omega_1 := -(p_1^{-1}\lambda)^{1/2}, \quad \omega_2 := (p_1^{-1}\lambda)^{1/2}, \quad \omega_3 := -(p_2^{-1}\lambda)^{1/2}, \quad \omega_4 = (p_2^{-1}\lambda)^{1/2}, \\ \underline{\omega} := \min\{\arg p_1, \arg p_2\}, \quad \bar{\omega} := \max\{\arg p_1, \arg p_2\}, \\ \theta := \begin{vmatrix} \alpha_1\omega_1^{m_1} & \beta_1\omega_2^{m_1} & \eta_1\omega_3^{m_1} & \gamma_1\omega_4^{m_1} \\ \alpha_2\omega_1^{m_2} & \beta_2\omega_2^{m_2} & \eta_2\omega_3^{m_2} & \gamma_2\omega_4^{m_2} \\ \alpha_3\omega_1^{m_3} & \beta_3\omega_2^{m_3} & \eta_3\omega_3^{m_3} & \gamma_3\omega_4^{m_3} \\ \alpha_4\omega_1^{m_4} & \beta_4\omega_2^{m_4} & \eta_4\omega_3^{m_4} & \gamma_4\omega_4^{m_4} \end{vmatrix}, \\ B_\varepsilon(\underline{\omega}, \bar{\omega}) := \{\lambda \in \mathbb{C} : \pi + \bar{\omega} + \varepsilon < \arg\lambda < 3\pi + \underline{\omega} - \varepsilon\} \end{gathered}$$

for real $\varepsilon > 0$ small enough.

The direct sum of Sobolev spaces $W_q^k(-1,0)\dot{+}W_q^k(0,1)$ (for an integer $k \geq 0$ and real $q > 1$) is defined as Banach space of complex-valued functions $u = u(x)$ defined on $[-1,0) \cup (0,1]$ which belong to $W_q^k(-1,0)$ and $W_q^k(0,1)$ on intervals $(-1,0)$ and $(0,1)$ respectively, with the norm

$$\|u\|_{q,k} = \|u\|_{W_q^k(-1,0)} + \|u\|_{W_q^k(0,1)}.$$

Here, as usual, $W_q^k(a,b)$ is the Sobolev space, i.e. the Banach space consisting of all measurable functions $u(x)$ that have generalized derivatives on the interval (a,b) up to k-th order inclusive with the finite norm

$$\|u\|_{W_q^k(a,b)} = \sum_{i=0}^{k} \Big(\int_a^b |u^{(i)}(x)|^q dx \Big)^{1/q}.$$

Theorem 2.1. *If $\theta \neq 0$ then for any $\varepsilon > 0$ there exist $\rho_\varepsilon > 0$ such that for all $\lambda \in B_\varepsilon(\underline{\omega}, \bar{\omega})$ for which $|\lambda| > \rho_\varepsilon$, the problem (2.1)-(2.2) has a unique solution $u(x,\lambda)$ that belongs to $W_q^l(-1,0)\dot{+}W_q^l(0,1)$ for arbitrary $l \geq \max\{2, \max\{m_1, m_2, m_3, m_4\} + 1\}$ and for these λ the coercive estimate*

$$\sum_{k=0}^{l} |\lambda|^{l-k} \|u\|_{q,k} \leq C(\varepsilon) \sum_{j=0}^{4} |\lambda|^{l-m_j-\frac{1}{q}} |f_\upsilon| \tag{2.3}$$

is valid.

Proof. Let $\lambda = \mu^2$. Let us define four basic solutions $u_i = u_i(x,\mu)$ $(i = 1,2,3,4)$ of (2.1) as

$$u_i(x,\mu) := \begin{cases} \exp(\omega_i\mu(x - \xi_i)) & \text{for } x \in I_i \\ 0 & \text{for } x \notin I_i, \end{cases}$$

where, $\xi_1 = -1$, $\xi_2 = \xi_3 = 0$, $\xi_4 = 1$; $j = 1$ for $i = 1,2$ and $j = 2$ for $i = 3,4$; $I_1 = I_2 = [-1,0)$, $I_3 = I_4 = (0,1]$. Then the general solution of (2.1) can be written in the form

$$u(x,\mu) = \sum_{k=1}^{4} C_k u_k(x,\mu). \tag{2.4}$$

Substituting this expression into (2.2) yields the following system of linear homogeneous equations with respect to variables C_1, C_2, C_3, C_4:

$$\begin{aligned}&(\omega_1\mu)^{m_k}(\alpha_k+\beta_k e^{\omega_1\mu})C_1+(\omega_2\mu)^{m_k}(\alpha_k e^{-\omega_2\mu}+\beta_k)C_2\\&+(\omega_3\mu)^{m_k}(\eta_k+\gamma_k e^{\omega_3\mu})C_3+(\omega_4\mu)^{m_k}(\eta_k e^{-\omega_4\mu}+\gamma_k)C_4=f_k,\quad k=1,2,3,4.\end{aligned}\tag{2.5}$$

From $\lambda \in B_\varepsilon(\underline{\omega}, \bar{\omega})$ it follows that

$$\begin{gathered}\frac{\pi+\varepsilon}{2}<\arg(\omega_i\mu)<\frac{3\pi-\varepsilon}{2}\quad\text{for } i=1,3;\\-\frac{\pi-\varepsilon}{2}<\arg(\omega_i\mu)<\frac{\pi-\varepsilon}{2}\quad\text{for } i=2,4.\end{gathered}$$

Consequently, for these λ and for $\varepsilon > 0$ (small enough), we have

$$(-1)^{k+1}\operatorname{Re}(\omega_k\mu)\le -|\lambda||\omega_k|\sin\frac{\varepsilon}{2},\quad k=1,2,3,4.$$

Hence, the determinant of the system (2.5) has the form

$$\begin{aligned}\Delta(\lambda)&=\lambda^{\frac12\sum_{i=1}^4 m_i}\left(\begin{vmatrix}\alpha_1\omega_1^{m_1}&\beta_1\omega_2^{m_1}&\eta_1\omega_3^{m_1}&\gamma_1\omega_4^{m_1}\\\alpha_2\omega_1^{m_2}&\beta_2\omega_2^{m_2}&\eta_2\omega_3^{m_2}&\gamma_2\omega_4^{m_2}\\\alpha_3\omega_1^{m_3}&\beta_3\omega_2^{m_3}&\eta_3\omega_3^{m_3}&\gamma_3\omega_4^{m_3}\\\alpha_4\omega_1^{m_4}&\beta_4\omega_2^{m_4}&\eta_4\omega_3^{m_4}&\gamma_4\omega_4^{m_4}\end{vmatrix}\right.\\&\quad\left.+e^{\lambda^{1/2}\sum_{i=1}^4(-1)^{i+1}\omega_i}\begin{vmatrix}\beta_1\omega_1^{m_1}&\alpha_1\omega_2^{m_1}&\gamma_1\omega_3^{m_1}&\eta_1\omega_4^{m_1}\\\beta_2\omega_1^{m_2}&\alpha_2\omega_2^{m_2}&\gamma_2\omega_3^{m_2}&\eta_2\omega_4^{m_2}\\\beta_3\omega_1^{m_3}&\alpha_3\omega_2^{m_3}&\gamma_3\omega_3^{m_3}&\eta_3\omega_4^{m_3}\\\beta_4\omega_1^{m_4}&\alpha_4\omega_2^{m_4}&\gamma_4\omega_3^{m_4}&\eta_4\omega_4^{m_4}\end{vmatrix}\right)\\&=\lambda^m(\theta+r(\lambda))\end{aligned}$$

where $m = m_1 + m_2 + m_3 + m_4$ and $r(\lambda) \to 0$ as $|\lambda| \to \infty$ in the angle $B_\varepsilon(\underline{\omega}, \bar{\omega})$. Since $\theta \neq 0$, there exist $\rho_\varepsilon > 0$ such that for all complex numbers λ satisfying $\lambda \in B_\varepsilon(\underline{\omega}, \bar{\omega})$ and $|\lambda| > \rho_\varepsilon$ we have $\Delta(\lambda) \neq 0$. So, for these λ, system (2.5) has a unique solution

$$C_i(\lambda)=\frac{1}{\Delta(\lambda)}\sum_{k=1}^4\Delta_{ik}(\lambda)f_k,\quad i=1,2,3,4$$

where $\Delta_{ik}(\lambda)$ is an algebraic complement of (i,k)-th element of the determinant $\Delta(\lambda)$. It is easy to see that each of the determinant $\Delta_{ik}(\lambda)$ has the representation

$$\Delta_{ik}(\lambda)=(\theta_{ik}+r_{ik}(\lambda))\lambda^{m-m_k}$$

where θ_{ik} are complex numbers and $r_{ik} \to 0$ as $|\lambda| \to \infty$ in the angle $sB_\varepsilon(\underline{\omega}, \bar{\omega})$. Then we have

$$C_i(\lambda)=\sum_{k=1}^4\lambda^{-m_k}\frac{\theta_{ik}+r_{ik}(\lambda)}{\theta+r(\lambda)}f_k,\quad i=1,2,3,4.$$

Therefore, the solution of problem (2.1)-(2.2) has the form

$$u(x,\lambda)=\sum_{i=1}^4\sum_{k=1}^4\lambda^{-m_k}\frac{\theta_{ik}+r_{ik}(\lambda)}{\theta+r(\lambda)}f_k u_i(x,\lambda).$$

From this it follows that for each integer $l \geq 0$

$$\|u^{(l)}\|_{L_q(-1,1)} \leq C \sum_{k=1}^{4} \Big(|\lambda|^{l-m_k} |f_k| \sum_{i=1}^{4} \|u_i(.,\lambda)\|_{L_q(I_i)} \Big). \tag{2.6}$$

Further, by (2.4) we have the inequality

$$\begin{aligned}
\|u_1(.,\lambda)\|^q_{L_q(-1,0)} &= \int_{-1}^{0} e^{q\,\mathrm{Re}(\omega_1\lambda)(x+1)} dx \leq \int_{-1}^{0} e^{-q|\lambda||\omega_1|\sin\frac{\varepsilon}{2}(x+1)} dx \\
&= \big(-q|\lambda||\omega_1|\sin\frac{\varepsilon}{2}\big)^{-1} \big(e^{-q|\lambda||\omega_1|\sin\frac{\varepsilon}{2}} - 1\big) \\
&\leq C(\varepsilon)|\lambda|^{-1}
\end{aligned}$$

as $|\lambda| \to \infty$ in the angle $\lambda \in B_\varepsilon(\underline{\omega}, \bar{\omega})$. In a similar way we have

$$\|u_1(\cdot,\lambda)\|^q_{L_q(I_i)} \leq C(\varepsilon)|\lambda|^{-1}, \quad i = 2,3,4$$

as $|\lambda| \to \infty$ in the angle $\lambda \in B_\varepsilon(\underline{\omega}, \bar{\omega})$. Substituting these inequalities in (2.6) we have

$$\|u^{(l)}\|_{L_q(-1,1)} \leq C(\varepsilon) \sum_{k=1}^{4} |\lambda|^{l-m_k-\frac{1}{q}} |f_k|$$

which, in turn, gives us the needed estimation (2.3). The proof is complete. □

3. Fredholm property of problem with multipoint and functional conditions

Let us consider problem (1.1)-(1.2) and the operator $\mathcal{L}$ corresponding to this problem. Suppose that $l \geq \max\{2, \max\{m_1, m_2, m_3, m_4\} + 1\}$ and define a linear operator $\mathcal{L}$ from $W^l_q(-1,0)\dot{+}W^l_q(0,1)$ into $W^{l-2}_q(-1,0)\dot{+}W^{l-2}_q(0,1) + \mathbb{C}^4$ by action low

$$\mathcal{L}u = (L(\lambda)u, L_1u, L_2u, L_3u, L_4u).$$

Theorem 3.1. *Let the following conditions be satisfied:*

1. $p_1 \neq 0$, $p_2 \neq 0$*;*
2. *the functionals* $\mathcal{F}_k, k = 1,2,3,4$, *in* $W^{m_k}_q(-1,0)\dot{+}W^{m_k}_q(0,1)$ *are continuous;*
3. $q(x)$ *is measurable function on* $[-1,0) \cup (0,1]$.

Then the linear operator $\mathcal{L}$ *is bounded and Fredholm.*

Proof. The operator $\mathcal{L}$ can be rewritten in the form

$$\begin{aligned}
\mathcal{L}_0 u &= \big(L_0(\lambda)u, L_{10}u, L_{20}u, L_{30}u, L_{40}u\big), \\
\mathcal{L}_1 u &= \big(q(x)u + \lambda_0 u, \mathcal{F}_1 u, \mathcal{F}_2 u, \mathcal{F}_3 u, \mathcal{F}_4 u\big)
\end{aligned}$$

where $\lambda_0 \in B_\varepsilon(\underline{\omega}, \bar{\omega})$ is some complex number sufficiently large in modulus. By Theorem 2.1 the operator $\mathcal{L}_0$ is an isomorphism from $W^l_q(-1,0)\dot{+}W^l_q(0,1)$ onto $W^{l-2}_q(-1,0)\dot{+}W^{l-2}_q(0,1)\dot{+}\mathbb{C}^4$. Further, it is easy to see that the linear operator $\mathcal{L}_1$ acts compactly from $W^l_q(-1,0)\dot{+}W^l_q(0,1)$ onto $W^{l-2}_q(-1,0)\dot{+}W^{l-2}_q(0,1)\dot{+}\mathbb{C}^4$.

Consequently, we can apply the theorem of Fredholm operator perturbation [22, p. 238] to the operator $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1$, which follows that $\mathcal{L}$ is Fredholm. Moreover, it is obvious that the operator $\mathcal{L}$ is bounded. So, the proof of the theorem is complete. □

4. Isomorphism and coerciveness of the principal part of the problem

Consider problem (1.1)-(1.2) without internal points, namely,

$$L_0(\lambda)u := p(x)u''(x) - \lambda u(x) = f(x), \tag{4.1}$$

$$L_{k0}u := \alpha_k u^{(m_k)}(-1) + \beta_k u^{(m_k)}(-0) + \eta_k u^{(m_k)}(+0) + \gamma_k u^{(m_k)}(1) = f_k, \tag{4.2}$$

for $k = 1, 2, 3, 4$. The corresponding operator is

$$\widetilde{\mathcal{L}}_0 u = (L_0(\lambda)u, L_{10}u, L_{20}u, L_{30}u, L_{40}u).$$

Theorem 4.1. *Let the following conditions be satisfied:*

1. $\theta \neq 0$;
2. $l \geq \max\{2, \max\{m_1, m_2, m_3, m_4\} + 1\}$.

Then for each $\varepsilon > 0$ there exist $\rho_\varepsilon > 0$ such that for all complex numbers λ satisfying $\lambda \in B_\varepsilon(\underline{\omega}, \bar{\omega})$, $|\lambda| > \rho_\varepsilon$ the operator $\widetilde{\mathcal{L}}_0(\lambda)$ from $W_q^l(-1,0)\dot{+}W_q^l(0,1)$ onto $W_q^{l-2}(-1,0)\dot{+}W_q^{l-2}(0,1)\dot{+}\mathbb{C}^4$ is an isomorphism and for these λ the following inequality holds for the solution of (4.1)–(4.2),

$$\begin{aligned} &\sum_{k=0}^{l} |\lambda|^{\frac{l-k}{2}} \|u\|_{W_{q,k}} \\ &\leq C(\varepsilon)\Big(\|f\|_{W_{q,l-2}} + |\lambda|^{\frac{l-2}{2}} \|f\|_{L_{q,0}} + \sum_{\upsilon=1}^{4} |\lambda|^{(l-m_\upsilon-\frac{1}{q})/2} |f_\upsilon|\Big). \end{aligned} \tag{4.3}$$

Proof. It is obvious that the linear operator $\widetilde{\mathcal{L}}_0(\lambda)$ is continuous from the space $W_q^l(-1,0)\dot{+}W_q^l(0,1)$ to $W_q^{l-2}(-1,0)\dot{+}W_q^{l-2}(0,1)\dot{+}\mathbb{C}^4$. Let $(f(x), f_1, f_2, f_3, f_4) \in W_q^{l-2}(-1,0)\dot{+}W_q^{l-2}(0,1)\dot{+}\mathbb{C}^4$ be any element. We shall seek the solution $u(x,\lambda)$ of problem (4.1)-(4.2) in the form of the sum $u(x,\lambda) = u_1(x,\lambda) + u_2(x,\lambda)$ as follows. By $f_\upsilon(x)$ $(\upsilon = 1, 2)$ we shall denote the restriction of $f(x)$ on the interval Ω_υ. Let $\widetilde{f}_\upsilon(\cdot) \in W_q^{l-2}(\mathbb{R})$ be an extension of $f_\upsilon(\cdot) \in W_q^{l-2}(I_\upsilon)$ such that the extension operator $S_\upsilon f_\upsilon := \tilde{f}_\upsilon$ from $W_q^{l-2}(I_\upsilon)$ to $W_q^{l-2}(\mathbb{R})$ is bounded for $\upsilon = 1, 2$. [30, Lemma 1.7.6], where as usual $\mathbb{R} = (-\infty, \infty)$. First consider the equations

$$-p_\upsilon(x)u''(x) + \lambda u(x) = \widetilde{f}_\upsilon(x), x \in \mathbb{R}$$

for $\upsilon = 1, 2$. By applying the [30, Theorem 3.2.1] we see that this equation has a unique solution $\tilde{u}_{1\upsilon} = \tilde{u}_{1\upsilon}(\cdot, \lambda) \in W_q^l(\mathbb{R})$ and for $u_{1\upsilon}(x,\lambda)$ (i.e. the restriction of $\tilde{u}_{1\upsilon}(x,\lambda$ on interval) Ω_υ) the estimate

$$\sum_{k=0}^{l} |\lambda|^{\frac{l-k}{2}} \|u_{1\upsilon}\|_{W_q^k(I_{\Omega_\upsilon})} \leq C(\varepsilon)(\|f\|_{W_q^{l-2}(I_\upsilon)} + |\lambda|^{\frac{l-2}{2}} \|f\|_{L_q(\Omega_\upsilon)}), \tag{4.4}$$

for $\upsilon = 1, 2$, is valid for all complex numbers λ satisfying $\lambda \in B_\varepsilon(\underline{\omega}, \bar{\omega})$. Consequently, the function

$$u_1(x,\lambda) = \begin{cases} u_{11}(x,\lambda), & \text{for } x \in (-1,0) \\ u_{12}(x,\lambda), & \text{for } x \in (0,1) \end{cases}$$

satisfies equation (4.1). In terms of this solution, we construct the boundary-value problem

$$p(x)u''(x) - \lambda u(x) = 0, \quad x \in (-1,0) \cup (0,1),$$

$$L_{k0}u = f_k - L_{k0}u_1(.,\lambda), k = 1, 2, 3, 4.$$

By Theorem 2.1, this problem has a unique solution $u_2 = u_2(x,\lambda)$ that belongs to $W_q^l(-1,0)\dot{+}W_q^l(0,1)$ for all complex numbers λ satisfying $\lambda \in B_\varepsilon(\underline{\omega},\bar{\omega})$, sufficiently large in modulus, and for these λ the estimate

$$\sum_{k=0}^{l} |\lambda|^{\frac{l-k}{2}} \|u_2\|_{q,k} \le C(\varepsilon) \sum_{\upsilon=1}^{4} |\lambda|^{(l-m_\upsilon-\frac{1}{q})\frac{1}{2}} (|f_\upsilon| + |L_{\upsilon 0}u_1|) \tag{4.5}$$

holds. By applying the of Theorem 2.1 and taking into account [27, Theorem 1.7.7/2], we have that for all $\lambda \in B_\varepsilon(\underline{\omega},\bar{\omega})$ and $l \ge \max\{2, \max\{m_1,m_2,m_3,m_4\}+1\}$ that the following estimates hold.

$$\begin{aligned} |\lambda|^{(l-m_\upsilon-\frac{1}{q})/2}|L_{\upsilon 0}u_1| &\le C|\lambda|^{(l-m_\upsilon-\frac{1}{q})/2}\|u_1\|_{C^{m_\upsilon}[-1,0]+C^{m_\upsilon}[0,1]} \\ &\le C(|\lambda|^{\frac{l}{2}}\|u_1\|_{q,0} + \|u_1\|_{q,l}) \\ &\le C(\varepsilon)(\|f\|_{q,l-2} + |\lambda|^{\frac{l-2}{2}}\|f\|_{q,0}). \end{aligned} \tag{4.6}$$

From (4.5) and (4.6) we have the inequality

$$\begin{aligned} &\sum_{k=0}^{l} |\lambda|^{\frac{l-k}{2}} \|u_2\|_{q,k} \\ &\le C(\varepsilon)\Big(\|f\|_{q,l-2} + |\lambda|^{\frac{l-2}{2}}\|f\|_{q,0} + \sum_{\upsilon=1}^{4} |\lambda|^{(l-m_\upsilon-\frac{1}{2})/2}|f_\upsilon|\Big). \end{aligned} \tag{4.7}$$

It is easy to see that the function $u(x,\lambda)$ defined as $u(x,\lambda) = u_1(x,\lambda) + u_2(x,\lambda)$ is the solution of the considered problem (4.1)-(4.2). Taking into account the estimates (4.4) and (4.7), we see that for this solution the needed estimation (4.3) is valid. Moreover, from estimate (4.3) it follows the uniqueness of the solution. On the other hand by Theorem 3.1 the operator $\widetilde{\mathcal{L}}$ is Fredholm from $W_q^l(-1,0)\dot{+}W_q^l(0,1)$ to $W_q^{l-2}(-1,0)\dot{+}W_q^{l-2}(0,1)\dot{+}\mathbb{C}^4$. Now, isomorphism of this operator follows from the fact that it is a Fredholm and one-to-one operator. So,the proof of the theorem is complete. □

5. Solvability and coerciveness of the main problem with nonlocal boundary conditions

Now, we can study the main problem (1.1)-(1.2)

Theorem 5.1. *Let the following conditions be satisfied:*

1. $\theta \neq 0$;
2. $l \ge \max\{2, \max\{m_1,m_2,m_3,m_4\}+1\}$,
3. *The functionals* $\mathcal{F}_\upsilon$ *are continuous in* $W_q^{m_\upsilon}(-1,0)\dot{+}W_q^{m_\upsilon}(0,1)$.

Then for each $\varepsilon > 0$ *there exist* $\rho_\varepsilon > 0$ *such that for all complex numbers* $\lambda \in B_\varepsilon(\underline{\omega},\bar{\omega})$ *for which* $|\lambda| > \rho_\varepsilon$ *the operator*

$$\widetilde{\mathcal{L}}(\lambda)u := (L(\lambda)u, L_1u, L_2u, L_3u, L_4u)$$

is an isomorphism from $W_q^l(-1,0)\dot{+}W_q^l(0,1)$ *onto* $W_q^{l-2}(-1,0)\dot{+}W_q^{l-2}(0,1)\dot{+}\mathbb{C}^4$ *and for these* λ *the following coercive estimate holds for the solution of problem*

(1.1)-(1.2)

$$\sum_{k=0}^{l}|\lambda|^{\frac{l-k}{2}}\|u\|_{q,k}\le C(\varepsilon)\Big(\|f\|_{q,l-2}+|\lambda|^{\frac{l-2}{2}}\|f\|_{q,0}+\sum_{\upsilon=1}^{4}|\lambda|^{(l-m_\upsilon-\frac{1}{q})/2}|f_\upsilon|\Big) \tag{5.1}$$

where $C(\varepsilon)$ is a constant which depends only on ε.

Proof. Let $(f(x),f_1,f_2,f_3,f_4)$ be any element of $W_q^{l-2}(-1,0)\dot{+}W_q^{l-2}(0,1)\dot{+}\mathbb{C}^4$. Assume that there exists a solution $u=u(x,\lambda)$ of problem (1.1)-(1.2) corresponding to this element. Then this solution satisfies the equalities

$$L_0(\lambda)u=L(\lambda)u-q(x)u, \tag{5.2}$$
$$L_{k0}u=L_ku-S_ku-\mathcal{F}_ku,\ k=1,2,3,4. \tag{5.3}$$

By applying Theorem 4.1 to the problem (5.2)-(5.3) we have that for this solution the following a priory estimate hold

$$\begin{aligned}
\sum_{k=0}^{l}|\lambda|^{\frac{l-k}{2}}\|u\|_{q,k}&\le C(\varepsilon)\Big(\|L(\lambda)u-q(x)u\|_{q,l-2}+|\lambda|^{\frac{l-2}{2}}\|L(\lambda)u-q(x)u\|_{q,0}\\
&\quad+\sum_{\upsilon=1}^{4}|\lambda|^{(l-m_\upsilon-\frac{1}{q})/2}|L_\upsilon u-S_ku-\mathcal{F}_\upsilon u|\Big)\\
&\quad+C(\varepsilon)\Big(\|f\|_{q,l-2}+|\lambda|^{\frac{l-2}{2}}\|f\|_{q,0}+\|q(x)u\|_{q,l-2}\\
&\quad+|\lambda|^{\frac{l-2}{2}}\|q(x)u\|_{q,0}+\sum_{\upsilon=1}^{4}|\lambda|^{(l-m_\upsilon-\frac{1}{q})/2}|f_\upsilon|\\
&\quad+\sum_{\upsilon=1}^{4}|\lambda|^{(l-m_\upsilon-\frac{1}{q})/2}(|S_ku|+|\mathcal{F}_\upsilon u|)\Big)
\end{aligned} \tag{5.4}$$

Let δ be any real number satisfying

$$0<\delta<\min\big\{\frac{1}{2},1+x_{ki},|x_{ki}|,1-x_{ki}:k=1,2,3,4,\ i=1,2,\dots,n_k\big\}.$$

By applying the same approach as in [24, sec. 2.8.3] it is easy to construct a function $\psi_\delta(x)\in C_0^\infty[-1,1]$ such that

$$\psi_\delta(x)=1\quad\text{for } x\in[-1+\delta,-\delta]\cup[\delta,1-\delta]\,,$$
$$\psi_\delta(x)=0\quad\text{for } x\in[-1,-1+\frac{\delta}{2}]\cup[-\frac{\delta}{2},\frac{\delta}{2}]\cup[1-\frac{\delta}{2},1]$$

and $0\le\psi_\delta(x)\le1$ for all $x\in[-1,1]$. It is obvious that

$$|S_ku|\le C\|(\psi_\delta u)^{(m_k)}\|_{C[-1,1]}. \tag{5.5}$$

By [25, Theorem 3.10.4], for $u\in W_q^l(-1,0)\dot{+}W_q^l(0,1)$ the following estimate holds,

$$|\lambda|^{(l-m_\upsilon-\frac{1}{q})/2}\|u^{(m_\upsilon)}\|_{C[-1,1]}\le C(\|u\|_{q,l}+|\lambda|^{\frac{l}{2}}\|u\|_{q,0})\,. \tag{5.6}$$

By Theorem 5.1, from (5.5) and (5.6) it follows that for all $\lambda \in B_\varepsilon(\underline{\omega}, \bar{\omega})$ sufficiently large in modulus the following estimate holds,

$$\begin{aligned}
|\lambda|^{(l-m_\upsilon-\frac{1}{q})/2}|S_\upsilon u| &\le C|\lambda|^{(l-m_\upsilon-\frac{1}{q})/2}\|(\psi_\delta u)^{(m_\upsilon)}\|_{C[-1,1]} \\
&\le C\big(\|\psi_\delta u\|_{q,l} + |\lambda|^{\frac{l}{2}}\|\psi_\delta u\|_{q,0}\big) \\
&\le C(\varepsilon)\big(\|L_0(\lambda)(\psi_\delta u)\|_{q,l-2} + |\lambda|^{\frac{l-2}{2}}\|L_0(\lambda)(\psi_\delta u)\|_{q,0}\big) \\
&\le C(\varepsilon)\Big(\|L_0(\lambda)u\|_{q,l-2} + |\lambda|^{\frac{l-2}{2}}\|L_0(\lambda)u\|_{q,0} \\
&\quad + \|q(x)u\|_{q,l-2} + |\lambda|^{\frac{l-2}{2}}\|q(x)u\|_{q,0} + \sum_{k=0}^{l-1}|\lambda|^{\frac{l-1-k}{2}}\|u\|_{q,k}\Big) \\
&\le C(\varepsilon)\Big(\|f\|_{q,l-2} + |\lambda|^{\frac{l-2}{2}}\|f\|_{q,0} \\
&\quad + \|q(x)u\|_{q,l-2} + |\lambda|^{\frac{l-2}{2}}\|q(x)u\|_{q,0} + \sum_{k=0}^{l-1}|\lambda|^{\frac{l-1-k}{2}}\|u\|_{q,k}\Big)
\end{aligned} \tag{5.7}$$

By [5, Theorem 1.3.3] there is a positive constant C such that for all u in the set $W_q^l(-1,0)\dot{+}W_q^l(0,1)$ and for each $k=0,1,\dots,l-1$ the following inequality is valid

$$\|u\|_{q,k} \le C\|u\|_{q,k+1}^{\frac{k}{k+1}}\|u\|_{q,0}^{\frac{1}{k+1}}. \tag{5.8}$$

Applying the well-known Young inequality

$$ab \le \frac{1}{p}(\alpha a)^p + \frac{1}{q}(\frac{b}{\alpha})^q$$

where $a>0$, $b>0$, $\alpha>0$, $1<p,q<\infty$, $\frac{1}{p}+\frac{1}{q}=1$ to the right-hand of (5.7) for

$$a = \|u\|_{q,k+1}^{\frac{k}{k+1}}, \quad b = \|u\|_{q,0}^{\frac{1}{k+1}}, \quad p = \frac{k+1}{k},$$

we have

$$\|u\|_{q,k} \le C\Big(\frac{k}{k+1}\alpha^{\frac{k+1}{k}}\|u\|_{q,k+1} + \frac{1}{k+1}\alpha^{-(k+1)}\|u\|_{q,0}\Big)$$

for $k=0,1,\dots,l-1$. We denote

$$\begin{gathered}
A(\alpha) = \max\big\{C\frac{k}{k+1}\alpha^{\frac{k+1}{k}} : k=0,1,\dots,l-1\big\}, \\
B(\alpha) = \max\big\{C\frac{1}{k+1}\alpha^{-(k+1)} : k=0,1,\dots,l-1\big\}.
\end{gathered}$$

Then from inequality (5.6), we have

$$\begin{aligned}
|\lambda|^{(l-m_\upsilon-\frac{1}{q})/2}|S_\upsilon u| &\le C(\varepsilon)(\|f\|_{q,l-2} + |\lambda|^{\frac{l-2}{2}}\|f\|_{q,0}) \\
&\quad + C(\varepsilon)\sum_{k=0}^{l-1}|\lambda|^{\frac{l-1-k}{2}}(A(\alpha)\|u\|_{q,k+1} + B(\alpha)\|u\|_{q,0}) \\
&\le \big(C(\varepsilon)A(\alpha) + D(\varepsilon,\alpha)|\lambda|^{-1/2}\big)\sum_{k=0}^{l}|\lambda|^{\frac{l-k}{2}}\|u\|_{q,k}
\end{aligned} \tag{5.9}$$

where $D(\varepsilon,\alpha)$ is a constant which depends only on ε and α. In view of [30, Theorem 1.7.7/2], for any $\zeta > 0$ we obtain

$$\|u\|_{q,k} \le \zeta\|u\|_{q,k+1} + C(\zeta)\|u\|_{q,0}.$$

On the other hand, from [5, Lemma 1.8] and [25, Theorem 8.19] we have

$$\begin{aligned}
|\mathcal{F}_k u| &\le \sum_{j=0}^{m_k}\Big(|\int_{\Omega_1}\mathcal{K}_{k1j}(t)u^{(j)}(t)dt| + |\int_{\Omega_1}\mathcal{K}_{k2j}(t)u^{(j)}(t)dt|\Big)\\
&\le \sup_k\Big(\sum_{j=0}^{m_k}\int_{\Omega_1}|\mathcal{K}_{k1j}(t)u^{(j)}(t)|dt + \sum_{j=0}^{m_k}\int_{\Omega_1}|\mathcal{K}_{k2j}(t)u^{(j)}(t)|dt\Big)\\
&\le \sup_k\Big(\sum_{j=0}^{m_k}\int_{\Omega_1}|\mathcal{K}_{k1j}(t)u(t)|dt + \sum_{j=0}^{m_k}\int_{\Omega_1}|\mathcal{K}_{k2j}(t)u(t)|dt\Big)\\
&\le C_1\|u\|_{q,k} + C_2\|u\|_{q,k}\\
&\le C\|u\|_{q,k}.
\end{aligned} \tag{5.10}$$

From (5.8) and (5.9) we have

$$\begin{aligned}
&\|q(x)u\|_{q,l-2} + |\lambda|^{\frac{l-2}{2}}\|q(x)u\|_{q,0} + \sum_{\upsilon=1}^{4}|\lambda|^{(l-m_\upsilon-\frac{1}{q})/2}(|S_k u| + |\mathcal{F}_\upsilon u|)\\
&\le C(\varepsilon)(\|f\|_{q,l-2} + |\lambda|^{\frac{l-2}{2}}\|f\|_{q,0}) + \zeta(\|u\|_{q,l} + |\lambda|^{\frac{l-2}{2}}\|u\|_{q,0})\\
&\quad + C(\zeta)|\lambda|^{\frac{l-2}{2}}\|u\|_{q,0} + (C(\varepsilon)A(\alpha) + D(\varepsilon,\alpha)|\lambda|^{-1/2})\sum_{k=0}^{l}|\lambda|^{\frac{l-k}{2}}\|u\|_{q,k}\\
&\quad + C\sum_{\upsilon=1}^{4}|\lambda|^{(l-m_\upsilon-\frac{1}{q})/2}\|u\|_{q,k}\\
&\le C(\varepsilon)(\|f\|_{q,l-2} + |\lambda|^{\frac{l-2}{2}}\|f\|_{q,0})\\
&\quad + \big(C(\varepsilon)A(\alpha) + D(\varepsilon,\alpha)|\lambda|^{-\frac{1}{2q}}\big)\sum_{k=0}^{l}|\lambda|^{\frac{l-k}{2}}\|u\|_{q,k}
\end{aligned} \tag{5.11}$$

Substituting (5.10) into (5.4) we obtain

$$\begin{aligned}
\sum_{k=0}^{l}|\lambda|^{\frac{l-k}{2}}\|u\|_{q,k} &\le C(\varepsilon)\Big(\|f\|_{q,l-2} + |\lambda|^{\frac{l-2}{2}}\|f\|_{q,0} + \sum_{\upsilon=1}^{4}|\lambda|^{(l-m_\upsilon-\frac{1}{q})/2}|f_\upsilon|\Big)\\
&\quad + (C(\varepsilon)A(\alpha) + D(\varepsilon,\alpha)|\lambda|^{-\frac{1}{2q}})\sum_{k=0}^{l}|\lambda|^{\frac{l-k}{2}}\|u\|_{q,k}.
\end{aligned}$$

For a fixed $\varepsilon > 0$ we can choose $\alpha > 0$ so small, and $|\lambda|$ so large that

$$C(\varepsilon)A(\alpha) + D(\varepsilon,\alpha)|\lambda|^{-1/2q} < 1.$$

Thus, for $\lambda \in B_\varepsilon(\underline{\omega},\bar{\omega})$ sufficiently large in modulus we obtain a priori estimate (5.1). From this estimate it follows the uniqueness property of the solution of problem (1.1)-(1.2), i.e. the operator $\widetilde{\mathcal{L}}(\lambda)$ is one-to-one operator. Moreover, by Theorem 3.1 the operator $\widetilde{\mathcal{L}}(\lambda)$ from $W_q^l(-1,0)\dot{+}W_q^l(0,1)$ to $W_q^{l-2}(-1,0)\dot{+}W_q^{l-2}(0,1)\dot{+}\mathbb{C}^4$ is Fredholm. Consequently, the existence of a solution results in its uniqueness. So, the proof of the theorem is complete. $\square$

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