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AN INVERSE SPECTRAL PROBLEM FOR STURM-LIOUVILLE OPERATOR WITH INTEGRAL DELAY

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In memory of M. G. Gasymov (1939-2008)

ABSTRACT. In this article, we study an inverse spectral problem for Sturm-Liouville operator with integral delay. We prove that the standard spectral asymptotic conditions are necessary and sufficient for unique solvability of the inverse problem.

1. INTRODUCTION

We consider inverse problem for the boundary-value problem (BVP) generated by the integro-differential equation

$$l_y := -y'' + q(x)y + \int_0^x M(x-t)y(t)dt = \lambda^2 y, \quad x \in (0,a) \cup (a,\pi) \tag{1.1}$$

with the Dirichlet boundary conditions

$$U(y) := y(0) = 0, \quad V(y) := y(\pi) = 0, \tag{1.2}$$

and the conditions at the point $x = a$:

$$I(y) := \begin{cases} y(a+0) = y(a-0) \equiv y(a), \\ y'(a+0) - y'(a-0) = 2\alpha\lambda y(a), \end{cases} \tag{1.3}$$

$q(x)$ and $M(x)$ are complex-valued functions, $q(x) \in L_2(0,\pi)$ and $(\pi - x)M(x) \in L_2(0,\pi)$, $\alpha \in \mathbb{C}$, $a \in (\frac{\pi}{2}, \pi)$ and λ is a spectral parameter.

Sturm-Liouville spectral problems with potentials depending on the spectral parameter (in case $K(x) \equiv 0$) arise in various models of quantum and classical mechanics. For instance, the evolution equations that are used to model interactions between colliding relativistic spineless particles can be reduced to the form (1.1). Then λ^2 is associated with the energy of the system (see [12, 13]).

Spectral problems of differential operators are studied in two main branches, namely, direct and inverse problems. Direct problems of spectral analysis consist in investigating the spectral properties of an operator. On the other hand, inverse problems aim at recovering operators from their spectral characteristics. Such problems often appear in mathematics, mechanics, physics, electronics, geophysics, meteorology and other branches of naturel sciences and engineering. Direct and

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inverse problems for the classical Sturm-Liouville operators have been extensively studied (see [5, 7, 11] and the references therein).

For integro-differential and other classes of nonlocal operators inverse problems are more difficult for investigation, and the classical methods either are not applicable to them or require essential modifications (see [1, 2, 3, 5, 6, 14, 15]). In this aspect, various inverse spectral problems for the (1.1), (1.3) BVP (special case $M(x) \equiv 0$) have been investigated in [8, 9, 10]).

In this article we establish uniqueness result for inverse spectral problem for Sturm-Liouville operator with integral delay.

2. Integral representations for solutions

In this section, we construct an integral representation of the solution $y(x,\lambda)$ of (1.1), (1.3), satisfying the initial conditions

$$y(0,\lambda)=1, \quad y'(0,\lambda)=i\lambda\,. \tag{2.1}$$

Also we study some properties of the solutions. Using the standard successive approximation methods (see [11]), we can prove the following theorem.

Theorem 2.1. *The solution $y(x,\lambda)$ has the form*

$$y(x,\lambda)=y_0(x,\lambda)+\int_{-x}^{x} A(x,t)e^{i\lambda t}dt, \tag{2.2}$$

where

$$y_0(x,\lambda)=\begin{cases} e^{ix\lambda}, & x<a \\ (1-i\alpha)e^{ix\lambda}+i\alpha e^{i\lambda}, & x>a \end{cases}$$

and the function $A(x,t)$ satisfies

$$\int_{-x}^{x} |A(x,t)|dt \le e^{C\sigma_0(x)}-1 \tag{2.3}$$

with

$$\sigma_0(x)=\int_0^x (x-t)[|q(t)|+\int_0^t |M(t-\tau)|d\tau]dt,$$

and $C=1+2|\alpha|$.

Proof. It is clear that when $\alpha=0$, if we consider the equation (1.1) separately on the intervals $(0,a)$ and (a,π), we can write the solutions as

$$e_0(x,\lambda)=e^{ix\lambda}+\int_{-x}^{x} K_0(x,t)e^{i\lambda t}dt, \quad 0\le x<a, \tag{2.4}$$

$$e_a(x,\lambda)=e^{i\lambda(x-a)}+\int_{-x+2a}^{x} K_a(x,t)e^{i\lambda(t-a)}dt, \quad x>a, \tag{2.5}$$

respectively. For the solutions of the above equations to solve the equation that has representation (2.5), the following equality must be satisfied:

$$\begin{aligned}
&\int_{-x+2a}^{x} K_a(x,t)e^{i\lambda(t-a)}dt \\
&=\frac{1}{\lambda}\int_a^x \sin\lambda(x-t)\Big\{q(t)\Big[e^{i\lambda(t-a)}+\int_{-t+2a}^{t} K_a(t,\tau)e^{i\lambda(\tau-a)}d\tau\Big] \\
&\quad+\int_0^t M(t-\tau)\Big[e^{i\lambda(\tau-a)}+\int_{-\tau+2a}^{\tau} K_a(\tau,s)e^{i\lambda(s-a)}ds\Big]d\tau\Big\}dt\,.
\end{aligned}$$

It is easy to obtain the integral equation

$$\begin{aligned} K_a(x,t) &= \frac{1}{2}\int_a^{\frac{x+t}{2}} q(u)du + \frac{1}{2}\int_a^x q(u)\int_{t-(x-u)}^{t+(x-u)} K_a(u,v)\,dv\,du \\ &\quad + \frac{1}{2}\int_a^x \int_{t-(x-u)}^{t+(x-u)} M(u-v)\,dx\,du \\ &\quad + \frac{1}{2}\int_a^x \int_0^u \int_{t-(x-u)}^{t+(x-u)} M(u-v)K_a(v,\xi)d\xi\,dx\,du. \end{aligned} \tag{2.6}$$

Since $e_a(x,-\lambda)$ is also the solution of (1.1), (1.3) on the interval $0 < x \le \pi$, the solution $y(x,\lambda)$ has the form

$$y(x,\lambda) = \begin{cases} e_0(x,\lambda), & 0 \le x < a, \\ c_1 e_a(x,\lambda) + c_2 e_a(x,-\lambda), & a < x \le \pi, \end{cases} \tag{2.7}$$

where the constants c_1, c_2 are defined from conditions (1.3). Hence, we have

$$y(x,\lambda) = \begin{cases} e_0(x,\lambda), & 0 \le x < a, \\ e_0(a,\lambda)\frac{(1-2i\alpha)e_a(x,\lambda)+(1+2i\alpha)e_a(x,-\lambda)}{2} & \\ +e_0'(a,\lambda)\frac{e_a(x,\lambda)-e_a(x,-\lambda)}{2i\lambda}, & 0 < x \le \pi. \end{cases} \tag{2.8}$$

Using (2.4), (2.5) and (2.8), after some simple computations, we find the following expression for $y(x,\lambda)$ $(a < x \le \pi)$,

$$y(x,\lambda) = e(x,\lambda) + \int_{-x+2a}^x K_a(x,t)e(t,\lambda)dt, \tag{2.9}$$

where

$$\begin{aligned} e(x,\lambda) &= e_0(a,\lambda)[\cos\lambda(x-a) + 2\alpha\sin\lambda(x-a)] + e_0'(a,\lambda)\frac{\sin\lambda(x-a)}{\lambda} \\ &= (1-i\alpha)e^{i\lambda x} + i\alpha e^{i\lambda(2a-x)} + \int_{-x}^x A_1(x,t)e^{i\lambda t}dt, \end{aligned} \tag{2.10}$$

$$A_1(x,t) = A_0 + \frac{1}{2}K_0(a,t+2a-x) + \frac{1}{2}K_0(a,t+x) + \frac{1}{2}\int_{t+2a-x}^{t+x} H(s)ds, \quad |t| < x,$$

$$A_0 = \begin{cases} \frac{1}{2}\int_0^a q(t)dt + \frac{1}{4}\int_{-a}^a q(\frac{a+t}{2})dt, & 2a-x < t < x, \\ 0, & -x < t < 2a-x, \end{cases}$$

$$H(s) = \frac{1}{2}\int_{\frac{a-s}{2}}^a K_0(\sigma, s-a+\sigma)q(\sigma)d\sigma + \frac{1}{2}\int_{\frac{a+s}{2}}^a K_0(\sigma, s+a-\sigma)q(\sigma)d\sigma. \tag{2.11}$$

Here, we assume that $K_0(a,t) \equiv 0$, $H(t) \equiv 0$, for $|t| > a$ and $A_1(x,t) = 0$ for $|t| > x$. Now using the expression (2.10) in (2.9), we have for $a < x \le \pi$ $(|t| < x)$

$$y(x,\lambda) = (1-i\alpha)e^{i\lambda x} + i\alpha e^{i\lambda(2a-x)} + \int_{-x}^x A_2(x,t)e^{i\lambda t}dt, \tag{2.12}$$

where

$$\begin{aligned} A_2(x,t) &= A_1(x,t) + (1-i\alpha)K_a(x,t) - i\alpha K_a(x,2a-t) \\ &\quad + \int_{2a-x}^x K_a(x,s)A_1(s,t)ds. \end{aligned} \tag{2.13}$$

From (2.4) and (2.12), we can write the formula (2.2) for the solution $y(x,\lambda)$, where

$$A(x,t)=\begin{cases}K_0(x,t), & \text{if } 0\le x\le a,\ |t|<x\\ A_2(x,t), & \text{if } a<x\le\pi,\ |t|<x.\end{cases} \tag{2.14}$$

From (2.6) it is easy to obtain

$$\int_{2a-x}^{x}|K_a(x,t)|dt\le e^{C_a\sigma_a(x)}-1, \tag{2.15}$$

where $C_a>0$ is a constant and

$$\sigma_a(x)=\int_a^x (x-t)\big[q(t)+\int_0^t |M(t-\tau)|d\tau\big]dt.$$

Using (2.15), from (2.11) and (2.13), we have the estimate

$$\int_{-x}^{x}|A_2(x,t)|dt\le e^{C\sigma_0(x)}-1 \tag{2.16}$$

for some constant $C>0$. Hence, from (2.14) and (2.16), we arrive at (2.3). $\square$

Let $s(x,\lambda)$ be a solution of (1.1) with initial conditions

$$s(0,\lambda)=0,\quad s'(0,\lambda)=1.$$

Because $y(x,\lambda)$ and $y(x,-\lambda)$ are two linearly independent solutions of (1.1), (1.3), then

$$s(x,\lambda)=\frac{y(x,\lambda)-y(x,-\lambda)}{2i}.$$

Using integral representation (2.2), we easily obtain

$$s(x,\lambda)=s_0(x,\lambda)+\int_0^x G(x,t)\frac{\sin\lambda t}{\lambda}dt, \tag{2.17}$$

where

$$s_0(x,\lambda)=\begin{cases}\frac{\sin\lambda x}{\lambda}, & x<a\\ (1-i\alpha)\frac{\sin\lambda x}{\lambda}+i\alpha\frac{\sin\lambda(2a-x)}{\lambda}, & x>a,\end{cases}$$

$G(x,t)=A(x,t)-A(x,-t)$ is a continuous function, and $G(x,0)=0$.

3. Properties of the spectral characteristics

In the section, we study properties of eigenvalues and eigenfunctions of (1.1). Let $y(x)$ and $z(x)$ be continuously differentiable functions on $(0,a)$ and (a,π). Denote $\langle y,z\rangle:=yz'-y'z$. If $y(x)$ and $z(x)$ satisfy the matching conditions (1.3), then

$$\langle y,z\rangle_{x=a-0}=\langle y,z\rangle_{x=a+0}, \tag{3.1}$$

i.e. the function $\langle y,z\rangle$ is continuous on $(0,\pi)$.

Denote $\Delta(\lambda)=s(\pi,\lambda)$. The eigenvalues $\{\lambda_n^2\}_{n\ge1}$ of the BVP (1.1) coincide with the zeros of the function $\Delta(\lambda)$.

Theorem 3.1. *The eigenvalues λ_n^2 and eigenfunctions $s(x,\lambda_n)$ of the BVP (1.1) satisfy the following asymptotic estimates for sufficiently large n,*

$$\lambda_n=\lambda_n^0+o\Big(\frac{1}{\lambda_n^0}\Big), \tag{3.2}$$

$$s(x,\lambda_n)=o\big(\frac{1}{\lambda^0}\big)+\begin{cases}\frac{\sin\lambda_n^0 x}{\lambda_n^0}, & x<a\\ (1-i\alpha)\frac{\sin\lambda_n^0 x}{\lambda_n^0}+i\alpha\frac{\sin\lambda_n^0(2a-x)}{\lambda_n^0}, & x>a,\end{cases} \tag{3.3}$$

where λ_n^0 *are the roots of* $\Delta_0(\lambda):=(1-i\alpha)\frac{\sin\lambda\pi}{\lambda}+i\alpha\frac{\sin\lambda(2a-\pi)}{\lambda}$ *and* $\lambda_n^0=n+h_n$, $h_n\in l_\infty$.

Proof. From (2.17), we have

$$\Delta(\lambda)=(1-i\alpha)\frac{\sin\lambda\pi}{\lambda}+i\alpha\frac{\sin\lambda(2a-\pi)}{\lambda}+\int_0^\pi G(\pi,t)\frac{\sin\lambda t}{\lambda}dt. \tag{3.4}$$

Denote $\Gamma_n:=\{\lambda:|\lambda|=\lambda_n^0+\delta\}$, $n=0,1,\dots,(\delta>0)$. Since $\Delta(\lambda)-\Delta_0(\lambda)=o(\frac{e^{|\operatorname{Im}\lambda|\pi}}{|\lambda|})$ and $|\Delta_0(\lambda)|\ge C_\delta\frac{e^{|\operatorname{Im}\lambda|\pi}}{|\lambda|}$ for all $\lambda\in\Gamma_n$, we establish by the Rouche's Theorem (see [4, p. 125]) that $\lambda_n=\lambda_n^0+\varepsilon_n$, where $\varepsilon_n=o(1)$. Moreover, $\varepsilon_n=o(\frac{1}{\lambda_n^0})$ is obtained from the equality $o=\Delta(\lambda_n)=(\Delta_0'(\lambda_n^0)+o(1))\varepsilon_n+o(\frac{1}{\lambda_n^0})$. This completes the proof of (3.2).

From (2.17) and (3.2), one can easily prove that the asymptotic formula (3.3) is true. □

Theorem 3.2. *The specification of the spectrum* $\{\lambda_n^2\}_{n\ge1}$ *uniquely determines the characteristic function* $\Delta(\lambda)$ *by the formula*

$$\Delta(\lambda)=[(1-i\alpha)\pi+i\alpha(2a-\pi)]\prod_{n=1}^\infty\frac{\lambda_n^2-\lambda^2}{(\lambda_n^0)^2}. \tag{3.5}$$

Proof. It follows from (3.4) and consequently by Hadamard's factorization theorem (see [4, p. 289]), $\Delta(\lambda)$ is uniquely determined up to a multiplicative constant by its zeros:

$$\Delta(\lambda)=C\prod_{n=1}^\infty\big(1-\frac{\lambda^2}{\lambda_n^2}\big). \tag{3.6}$$

Consider the function

$$\begin{aligned}\Delta_0(\lambda)&:=(1-i\alpha)\frac{\sin\lambda\pi}{\lambda}+i\alpha\frac{\sin\lambda(2a-\pi)}{\lambda}\\ &=[(1-i\alpha)\pi+i\alpha(2a-\pi)]\prod_{n=1}^\infty\big(1-\frac{\lambda^2}{(\lambda_n^0)^2}\big).\end{aligned}$$

Then

$$\frac{\Delta(\lambda)}{\Delta_0(\lambda)}=C\frac{1}{[(1-i\alpha)\pi+i\alpha(2a-\pi)]}\prod_{n=1}^\infty\frac{(\lambda_n^0)^2}{\lambda^2}\prod_{n=1}^\infty\Big(1+\frac{\lambda_n^2-(\lambda_n^0)^2}{(\lambda_n^0)^2-\lambda^2}\Big).$$

Taking (3.2) and (3.4) into account we calculate

$$\lim_{\lambda\to-\infty}\frac{\Delta(\lambda)}{\Delta_0(\lambda)}=1,\quad \lim_{\lambda\to-\infty}\prod_{n=1}^\infty\Big(1+\frac{\lambda_n^2-(\lambda_n^0)^2}{(\lambda_n^0)^2-\lambda^2}\Big)=1$$

and hence

$$C=[(1-i\alpha)\pi+i\alpha(2a-\pi)]\prod_{n=1}^\infty\frac{\lambda_n^2}{(\lambda_n^0)^2}.$$

Substituting this into account (3.6) we arrive at (3.5). □

4. Formulation of the inverse problem uniqueness theorem

In this section, we study inverse problem of recovering $M(x)$ from the given spectral characteristics. We denote the BVP (1.1)-(1.3) by $L = L(M)$. Together with $L = L(M)$ we consider a BVP $\widetilde{L} = L(\widetilde{M})$ of the same form, but with different kernel $\widetilde{M}$.

Inverse Problem: Given a function $q(x)$, numbers α, a, and the spectrum $\{\lambda_n\}_{n\geq 1}$, construct the function $M(x)$.

Let us prove the uniqueness theorem for the solution of the Inverse Problem. Everywhere below if a certain symbol e denotes an object to L, then the corresponding symbol $\widetilde{e}$ denotes the analogous object related to $\widetilde{L}$ and $\widehat{e} = e - \widetilde{e}$.

Theorem 4.1. *Fix $b \in (0, a)$. Let $\Lambda \subset \mathbb{N}$ be a subset of nonnegative integer numbers, and let $\Omega := \{\lambda_n^2\}_{n\in\Lambda}$ be a part of the spectrum of L such that the system of functions $\{\cos \lambda_n x\}_{n\in\Lambda}$ is complete in $L_2(0, \pi)$. Let $M(x) = \widetilde{M}(x)$ almost everywhere (a.e.) on (b, π), and $\Omega = \widetilde{\Omega}$. Then $M(x) = \widetilde{M}(x)$ a.e. on $(0, \pi)$.*

Proof. Let $\chi(x, \lambda)$ be the solution of the equation

$$l_z^* := -z'' + q(x)z + \int_x^\pi M(t-x)z(t)dt = \lambda^2 z, \quad x \in (0, a) \cup (a, \pi) \tag{4.1}$$

under the conditions $\chi(\pi, \lambda) = 0$, $\chi'(\pi, \lambda) = -1$ and the conditions at the point $x = a : \chi(a + 0, \lambda) = \chi(a - 0, \lambda) \equiv \chi(a, \lambda)$, $\chi'(a + 0, \lambda) - \chi'(a - 0, \lambda) = 2\alpha\lambda\chi(a, \lambda)$. Denote $\Delta^*(\lambda) = \chi(0, \lambda)$. Then by (3.1) we have

$$\begin{aligned}
&\int_0^\pi \chi(x,\lambda) \int_0^x \widehat{M}(x-t)\widetilde{s}(t,\lambda)\,dt\,dx \\
&= \int_0^\pi \chi(x,\lambda) l\widetilde{s}(x,\lambda)dx - \int_0^\pi \chi(x,\lambda)\widetilde{l}\widetilde{s}(x,\lambda)dx \\
&= \int_0^\pi l^*\chi(x,\lambda)\widetilde{s}(x,\lambda)dx - \int_0^\pi \chi(x,\lambda)\widetilde{l}\widetilde{s}(x,\lambda)dx \\
&\quad + [\widetilde{s}(x,\lambda)\chi'(x,\lambda) - \widetilde{s}'(x,\lambda)\chi(x,\lambda)](|_0^a + |_a^\pi) \\
&= \Delta^*(\lambda) - \widetilde{\Delta}(\lambda).
\end{aligned}$$

For $\widetilde{l} = l$ we have $\Delta^*(\lambda) \equiv \Delta(\lambda)$, and consequently

$$\int_0^\pi \chi(x,\lambda) \int_0^x \widehat{M}(x-t)\widetilde{s}(t,\lambda)\,dt\,dx = \widehat{\Delta}(\lambda). \tag{4.2}$$

We transform (4.2) into

$$\int_0^\pi \widehat{M}(x)\Big(\int_x^\pi \chi(t,\lambda)\widetilde{s}(t-x,\lambda)dt\Big)dx = \widehat{\Delta}(\lambda). \tag{4.3}$$

Denote $w(x, \lambda) = \chi(\pi - x, \lambda)$, $N(x) = M(\pi - x)$,

$$\varphi(x,\lambda) = \int_0^x w(t,\lambda)\widetilde{s}(x-t,\lambda)dt. \tag{4.4}$$

Then (4.2) takes the form

$$\int_0^\pi \widehat{N}(x)\varphi(x,\lambda)dx = \widehat{\Delta}(\lambda). \tag{4.5}$$

For $x \leq a$ the following representation holds [14],

$$\varphi(x,\lambda) = \frac{1}{2\lambda^2}\Big(- x\cos\lambda x + \int_0^x V(x,t)\cos\lambda t\,dt\Big), \tag{4.6}$$

where $V(x,t)$ is a continuous function which does not depend on λ. Since $\Omega = \widetilde{\Omega}$, we have by Theorem 3.2

$$\Delta(\lambda) \equiv \widetilde{\Delta}(\lambda) \Longrightarrow \widehat{\Delta}(\lambda) \equiv 0.$$

Then, substituting (4.6) into (4.5), we obtain

$$\int_0^b \Big(- x\widehat{N}(x) + \int_x^b V(t,x)\widehat{N}(t)\Big)\cos\lambda x\,dx \equiv 0,$$

and consequently,

$$-x\widehat{N}(x) + \int_x^b V(t,x)\widehat{N}(t)dt = 0 \quad \text{a.e. on } (0,b).$$

Since this homogeneous Volterra integral equation has only the trivial solution it follows that $\widehat{N}(x) = 0$ a.e. on $(0,b)$, i.e. $M(x) = \widetilde{M}(x)$ a.e. on $(0,\pi)$. □

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