Electronic Journal of Differential Equations, Vol. 2017 (2017), No. 120, pp. 1–13.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

LORENTZ ESTIMATES FOR ASYMPTOTICALLY REGULAR FULLY NONLINEAR ELLIPTIC EQUATIONS

YONGYONG WANG, JUNJIE ZHANG, SHENZHOU ZHENG

Communicated by Zhasheng Feng

Abstract. We prove a global Lorentz estimate of the Hessian of strong solutions to a class of asymptotically regular fully nonlinear elliptic equations over a $C^{1,1}$ smooth bounded domain. Here, the approach of the main proof is based on the Possion's transform from an asymptotically regular elliptic equation to the regular one.

1. Introduction

Let Ω be a bounded domain with $\partial\Omega \in C^{1,1}$ in $\mathbb{R}^n$ for $n \geq 2$. The main purpose of this paper is to attain a global Calderón-Zygmund type estimate in the scale of Lorentz spaces for the Hessian of strong solutions to the Dirichlet problem of asymptotical regular fully nonlinear elliptic equations of nondivergence form. The studied problem is

$$\begin{gathered} F(x, D^2u) = f \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where the real valued function $F(x, D^2u) : \Omega \times \mathcal{S}(n) \to \mathbb{R}$ is an asymptotically regular elliptic operator which $\mathcal{S}(n)$ is the space of real $n \times n$ symmetric matrices, and f is any given function in Lorentz spaces $L^{\gamma,q}(\Omega)$ with $\gamma > n$ and $0 < q \leq \infty$.

The Calderón-Zygmund estimate is a popular research to various elliptic and parabolic problems in recent decades. In the settings of discontinuous coefficients, an interior and boundary $W^{2,p}$ estimate for linear elliptic equations with VMO (Vanishing Mean Oscillations) discontinuous coefficients was first proved by Chiarenza, Frasca, and Longo [8, 9]. Since then, there have been a lot of research activities on the Calderón-Zygmund theory of elliptic and parabolic equations and systems in divergence or non-divergence form. Regarding fully nonlinear elliptic equations (1.1), an interior $W^{2,p}$ estimate was first obtained by Caffarelli in [6] if $f \in L^p$ with $p > n$ under the assumption of a small measure to the oscillation of $F(x, M)$ in the variable x uniformly for M. Later, Caffarelli and Huang [7] further showed that under the assumptions of $F(x, M)$ with a small multiplier of BMO in x and the prerequisite of Evans-Krylov estimates [12], if f belongs to the generalized Campamato-John-Nirenberg spaces, then D^2u correspondingly belongs to the same spaces as f.

2010 *Mathematics Subject Classification.* 35J60, 35B65, 35D30.
Key words and phrases. Nonlinear elliptic equations; Lorentz estimate; asymptotically regular; small BMO coefficients.

Submitted October 25, 2016. Published May 4, 2017.

Recently, Winter [21] also used a similar technique to establish the corresponding boundary estimate so as to get a global $W^{2,p}$-solvability of the associated boundary-value problem. All these papers showed that a small oscillation assumption in the L^∞ or L^n integral average sense is imposed on the operators $F(x, M)$ in x. Recently, Krylov et al [11, 14, 15] developed $W^{2,p}$-solvability for fully nonlinear elliptic and parabolic equations with VMO "coefficients" whose local oscillations are measured in a certain average sense allowing rather rough discontinuity. After that, Dong-Krylov-Li [11] demonstrated an interior solvability in $W^{2,p}$ for $p > n$ and $W_p^{2,1}$ for $p > n + 1$, respectively, to fully nonlinear elliptic and parabolic equations with VMO "coefficients" in bounded domains or cylinders. Moreover, Byun et al [4] also attained the global weighted $W^{2,p}$ estimates of the Hessian for fully nonlinear elliptic equations with small BMO "coefficients" in a bounded $C^{1,1}$ domain via rather different geometrical approaches.

On the other hand, Lorentz spaces are a two-parameter scale of spaces which refine Lebesgue spaces in some sense. It is an important observation that requiring the principle coefficients to have small mean oscillations in the integral average sense is sufficient to achieve higher integrability and Lorentz regularity. Since the pioneering work of Talenti [20] based on symmetrization, there were a large of literature on the topic of Lorentz regularity to elliptic and parabolic PDEs. Recently, Mengesha-Phuc in [16] used a kind of geometrical approach to prove the weighted Lorentz regularity of the gradient for quasilinear elliptic p-Laplacian equations, and Zhang-Zhou [24] extended their results to the setting of quasilinear $p(x)$-Laplacian. Meanwhile, Baroni in [1, 2] made use of so-called Large-M-inequality principle introduced by Acerbi-Mingione to show the Lorentz estimates of gradient for evolutionary p-Laplacian systems and obstacle parabolic p-Laplacian, respectively.

We would like to point out that another key ingredient is that $F(x, D^2u)$ is assumed to be an asymptotically regular. It was Chipot and Evans [10] to first introduce the notion of asymptotically regular in the elliptic framework, and Raymond [17] further considered a Lipschitz regularity to asymptotically regular problems with p-growth. Since then there are a large of literatures on the topic of asymptotically regular problems. In particular, Scheven and Schmidt in [18, 19] recently obtained a local higher integrability and a local partial Lipschitz continuity with a singular set of small positive measure for the gradient Du to the system which exhibits a certain kind of elliptic behavior near infinity, respectively. Furthermore, a global Lipschitz regularity result was extended by Foss in [13]. Very recently, Byun-Oh-Wang [5] proved global Calderón-Zygmund estimates for nonhomogeneous asymptotically regular elliptic and parabolic problems in divergence form in the Reifenberg flat domain by covering the given asymptotically regular problems to suitable regular problems. Furthermore, Byun-Cho-Oh [3] extended the same conclusions to the setting of nonlinear obstacle elliptic problems. Zhang-Zheng [22, 23] also further extended the work of Byun-Oh-Wang [5] to the case of obstacle parabolic problems in the scale of Lorentz spaces.

Inspired by those recent works mentioned above, in this paper we consider a global Lorentz estimate of the Hessian of strong solutions to the Dirichlet problem (1.1) for asymptotical regular fully nonlinear elliptic equations over a $C^{1,1}$ bounded domain. More precisely, our aim is to attain a global Lorentz estimate of the second derivative to the Dirichlet problem (1.1) with asymptotically regular nonlinearity. Indeed, it is a natural refined outgrowth of Byun et al's recent papers [5]. In particular, the Lebesgue space L^γ is a special case of Lorentz space $L^{\gamma,q}$ when $q=\gamma$. Before stating the main result, let us give some basic concepts and facts.

Let us first recall that the Lorentz space $L^{\gamma,q}(\Omega)$ with $1\le\gamma<\infty$ and $0<q<\infty$, which is the set of measurable function $g:\Omega\to\mathbb{R}$ such that

$$\|g\|^q_{L^{\gamma,q}(\Omega)} := q\int_0^\infty \Big(\mu^\gamma|\{\xi\in\Omega:|g(\xi)|>\mu\}|\Big)^{q/\gamma}\frac{d\mu}{\mu}<+\infty.$$

While the Lorentz space $L^{\gamma,\infty}$ for $1\le\gamma<\infty, q=\infty$ is set to be the usual Marcinkiewicz space $\mathcal{M}^\gamma(\Omega)$ with quasinorm

$$\|g\|_{L^{\gamma,\infty}}=\|g\|_{\mathcal{M}^\gamma(\Omega)}:=\sup_{\mu>0}\Big(\mu^\gamma|\{\xi\in\Omega:|g(\xi)|>\mu\}|\Big)^{\frac{1}{\gamma}}<+\infty.$$

The local variant of such spaces is defined in the usual way. Moreover, we note that by Fubini's theorem there holds

$$\|g\|^\gamma_{L^\gamma(\Omega)}=\gamma\int_0^\infty\Big(\mu^\gamma|\{\xi\in\Omega:|g(\xi)|>\mu\}|\Big)\frac{d\mu}{\mu}=\|g\|^\gamma_{L^{\gamma,\gamma}(\Omega)},$$

so that $L^\gamma(\Omega)=L^{\gamma,\gamma}(\Omega)$; cf. [1, 2]. In this context, we denote by $C(n,\lambda,\Lambda,\dots)$ a universal constant depending only on prescribed quantities and possibly varying from line to line.

In this article, we are interested in the case that $F(x,M)$ is asymptotically elliptic. This is to say that it is getting closer to some real-valued function $G(x,M)$ as $\|M\|$ goes to infinity, where $G(x,M)$ satisfies the following uniformly elliptic assumption.

Definition 1.1. (uniformly ellipticity) We say $G(x,M):\Omega\times\mathcal{S}(n)\to\mathbb{R}$ is uniformly elliptic if there exist constants $0<\lambda\le\Lambda<\infty$ such that for any $x\in\Omega$ and any $M\in\mathcal{S}(n)$, there holds

$$\lambda\|N\|\le G(x,M+N)-G(x,M)\le\Lambda\|N\|,\quad\forall N\ge 0. \tag{1.2}$$

Remark 1.2. (i) We write $N\ge0$ whenever N is a non-negative definite symmetric matrix. $\|N\|$ denotes the (L^2,L^2)-norm of N, that is, $\|N\|=\sup_{|x|=1}|Nx|$. Therefore $\|N\|$ is equal to the maximum eigenvalue of N whenever $N\ge0$.

(ii) The uniformly elliptic assumption implies that $G(x,M)$ is monotone increasing and Lipschitz in $M\in\mathcal{S}(n)$.

We next introduce the definition of asymptotically elliptic operators.

Definition 1.3. $F(x,M)$ is asymptotically elliptic if there exists a uniformly elliptic operator $G(x,M)$ and a bounded function $\omega:\mathbb{R}^+\to\mathbb{R}^+$ with $\lim_{r\to\infty}\omega(r)=0$ such that

$$0\le|F(x,M)-G(x,M)|\le\omega(\|M\|)(1+\|M\|) \tag{1.3}$$

for all $M\in\mathcal{S}(n)$ and any $x\in\Omega$.

By a direct calculation, we conclude that

$$\lim_{\|M\|\to\infty}\frac{F(x,M)-G(x,M)}{\|M\|}=0, \tag{1.4}$$

uniformly with respect to $x\in\Omega$.

We define function β_G to measure the oscillation of $G(x,M)$ in the variable x. Let $G:\Omega\times\mathcal{S}(n)\to\mathbb{R}$ and let $x_0\in\Omega$ be fixed. For $x\in\Omega$, we define

$$\beta_G(x,x_0):=\sup_{M\in\mathcal{S}(n)\setminus\{0\}}\frac{|G(x,M)-G(x_0,M)|}{\|M\|}. \tag{1.5}$$

Now, let us summarize our main results as follows.

Theorem 1.4. *Assume $\gamma>n$ and $0<q\le\infty$. Let $u\in W^{2,n}(\Omega)$ be a strong solution to (1.1) with $f\in L^{\gamma,q}(\Omega)$ and $\partial\Omega\in C^{1,1}$. Then there exists a small positive constant $\beta_0=\beta_0(n,\lambda,\Lambda,\gamma,q)$ such that if $F(x,D^2u)$ is asymptotically elliptic with $G(x,D^2u)$ satisfying uniformly elliptic condition and*

$$\Big(\frac{1}{|B_r(x_0)\cap\Omega|}\int_{B_r(x_0)\cap\Omega}\beta_G(x,x_0)^n dx\Big)^{1/n}\le\beta_0, \tag{1.6}$$

for any $x_0\in\Omega$ and $0<r<R_0$. We suppose that $G(x,D^2u)$ is convex and positive homogeneous of degree one in D^2u. Then we have $D^2u\in L^{\gamma,q}(\Omega)$, satisfying the estimate

$$\|D^2u\|_{L^{\gamma,q}(\Omega)}\le C(\|f\|_{L^{\gamma,q}(\Omega)}+1), \tag{1.7}$$

where $C=C(n,\lambda,\Lambda,\gamma,q,\Omega)$. In the case $q=\infty$ the constant C depends only on $n,\lambda,\Lambda,\gamma,\Omega$.

To realize our aim, some ideas from [5] are employed in our main proof. For example, to get the global Lorentz estimate we use an equivalent representation of Lorentz norm, the Hardy-Littlewood maximal functions, and the Poisson formula by constructing a regular problem from the given irregular problem.

The rest of this article is organized as follows. In section 2, we first prove the global Lorentz estimates of the corresponding regular problem, and then we give a proof of the main result by taking a transformation from given asymptotically regular problem to a suitable regular problem.

2. Proof of Theorem 1.4

We prove Theorem 1.4 by employing an appropriate transformation to construct a uniformly elliptic operator from a given asymptotically elliptic operator. To this end, we assume that real-valued function $F(x,M)$ is asymptotical to a uniformly elliptic $G(x,M)$, which satisfies convex and positive homogeneous of degree one in M and

$$\Big(\fint_{B_r(x_0)\cap\Omega}\beta_G(x,x_0)^n dx\Big)^{1/n}\le\beta_0,$$

for any $x_0\in\Omega$ and $0<r<R_0$, where $\beta_0>0$ will be determined later. By Definition 1.3, we have

$$\lim_{\|M\|\to\infty}\frac{F(x,M)-G(x,M)}{\|M\|}=0.$$

Now we write a real-valued function $H(x,M):\Omega\times\mathcal{S}(n)$ by

$$\|M\|H(x,M):=F(x,M)-G(x,M); \tag{2.1}$$

then there exists an $K=K(\beta_0)>1$ such that

$$\|M\|\ge K\Rightarrow|H(x,M)|\le\beta_0,\quad\forall x\in\Omega. \tag{2.2}$$

For any fixed point $x \in \Omega$, we next define a new real-valued function $\tilde{H}(x,M)$ by

$$\tilde{H}(x,M) := \begin{cases} H(x,M) & \text{if } \|M\| \geq K, \\ \frac{\|M\|}{K} H\big(x, \frac{K}{\|M\|}M\big) & \text{if } 0 < \|M\| < K, \\ 0 & \text{if } M = 0. \end{cases} \tag{2.3}$$

It follows that $\tilde{H}(x,M)$ is also convex in M, positive homogeneous of degree one in M, and

$$\tilde{H}(x,M) \leq \beta_0, \quad \forall M \in \mathcal{S}(n), \tag{2.4}$$

uniformly with respect to $x \in \Omega$.

Note that $\tilde{H}(x,M) = H(x,M)$ if $\|M\| \geq K$. Therefore, for $M \neq 0$ we have

$$\begin{aligned} F(x,M) &= G(x,M) + \|M\|H(x,M) \\ &= G(x,M) + \|M\|\tilde{H}(x,M) + \|M\|(H(x,M) - \tilde{H}(x,M)) \\ &= G(x,M) + \|M\|\tilde{H}(x,M) \\ &\quad + \|M\|_{\chi\{M\in\mathcal{S}(n):\|M\|<K\}}(H(x,M) - \tilde{H}(x,M)), \end{aligned} \tag{2.5}$$

where $\chi\{M \in \mathcal{S}(n) : \|M\| < K\}$ denotes the characteristic function on the set $\{M \in \mathcal{S}(n) : \|M\| < K\}$. In the setting of $M = 0$, we define $\|M\|H(x,M)|_{M=0} := F(x,0) - G(x,0)$, then the formula (2.5) still holds for all $M \in \mathcal{S}(n)$.

Let $u \in W^{2,n}$ be a strong solution of the Dirichlet problem (1.1). Define $\tilde{G} : \Omega \times \mathcal{S}(n) \to \mathbb{R}$ by

$$\tilde{G}(x,M) := G(x,M) + \|M\|\tilde{H}(x,D^2u). \tag{2.6}$$

Then, by (2.5) and (2.6), it yields

$$F(x,D^2u) = \tilde{G}(x,D^2u) + \|D^2u\|_{\chi\{\|D^2u\|<K\}}(H(x,D^2u) - \tilde{H}(x,D^2u)), \tag{2.7}$$

where $\chi\{\|D^2u\| < K\} = \chi\{x \in \Omega : \|D^2u\| < K\}$ denotes the characteristic function on the set $\{x \in \Omega : \|D^2u(x)\| < K\}$. Thus, from (1.1) it implies that u is a strong solution of

$$\tilde{G}(x,D^2u) = f + \|D^2u\|_{\chi\{\|D^2u\|<K\}}(\tilde{H}(x,D^2u) - H(x,D^2u)) := g, \quad x \in \Omega. \tag{2.8}$$

To prove Theorem 1.4, we also need to show that the new nonlinearity $\tilde{G}$ satisfies uniformly ellipticity and the oscillation condition (1.6) in the L^n integral average sense with small constant $3\beta_0$. More precisely, we have the following lemma.

Lemma 2.1. *Let $u \in W^{2,n}(\Omega)$ be a strong solution of the Dirichlet problem* (1.1). *Assume that $F(x,M)$ is asymptotically elliptic with $G(x,M)$ satisfying*

$$\Big(\fint_{B_r(x_0)\cap\Omega} \beta_G(x,x_0)^n dx\Big)^{1/n} \leq \beta_0, \tag{2.9}$$

for any $x_0 \in \Omega$ and $0 < r < R_0$. Then we have the following conclusions:

- (i) *If $0 < \beta_0 \leq \lambda/3$, then $\tilde{G}(x,M)$ is uniformly elliptic.*
- (ii) *For any $x_0 \in \Omega$ and $0 < r < R_0$, $\tilde{G}(x,M)$ satisfies*

$$\Big(\fint_{B_r(x_0)\cap\Omega} \beta_{\tilde{G}}(x,x_0)^n dx\Big)^{1/n} \leq 3\beta_0. \tag{2.10}$$

Proof. (i) Let $0 < \beta_0 \le \frac{\lambda}{3}$. For any $M, N \in \mathcal{S}(n)$ with $N \ge 0$, we have

$$\tilde{G}(x, M+N) - \tilde{G}(x, M) = G(x, M+N) - G(x, M) + (\|M+N\| - \|M\|)\tilde{H}(x, M)$$

because of (2.6). From (2.4) it follows that

$$\tilde{H}(x, D^2u) \le \beta_0, \tag{2.11}$$

uniformly with respect to $x \in \Omega$. We have the triangle inequality

$$|\ \|M+N\| - \|M\|\ | \le \|N\|\,. \tag{2.12}$$

Hence, by (1.2), (2.11) and (2.12), we find that

$$\tilde{G}(x, M+N) - \tilde{G}(x, M) \ge \lambda\|N\| - \beta_0\|N\| = (\lambda - \beta_0)\|N\| \ge \frac{2\lambda}{3}\|N\|,$$

and

$$\tilde{G}(x, M+N) - \tilde{G}(x, M) \le \Lambda\|N\| + \beta_0\|N\| = (\Lambda + \beta_0)\|N\| \le (\Lambda + \frac{\lambda}{3})\|N\|,$$

since $0 < \beta_0 \le \frac{\lambda}{3}$. Namely,

$$\tilde{\lambda}\|N\| \le \tilde{G}(x, M+N) - \tilde{G}(x, M) \le \tilde{\Lambda}\|N\|, \tag{2.13}$$

where $\tilde{\lambda} = \frac{2}{3}\lambda$ and $\tilde{\Lambda} = \Lambda + \frac{\lambda}{3}$. So the assertion (i) is proved.

(ii) Let $x_0 \in \Omega$ and $0 < r < R_0$. For any $x \in B_r(x_0) \cap \Omega$, it follows from (2.4) and (2.6) that

$$|\tilde{G}(x, M+N) - \tilde{G}(x, M)| \le |G(x, M) - G(x_0, M)| + 2\beta_0\|M\|, \tag{2.14}$$

which implies

$$\begin{aligned} \beta_{\tilde{G}}(x, x_0) &= \sup_{M \in \mathcal{S}(n)\setminus\{0\}} \frac{|\tilde{G}(x, M) - \tilde{G}(x_0, M)|}{\|M\|} \\ &\le \sup_{M \in \mathcal{S}(n)\setminus\{0\}} \frac{|G(x, M) - G(x_0, M)|}{\|M\|} + 2\beta_0 \\ &= \beta_G(x, x_0) + 2\beta_0. \end{aligned} \tag{2.15}$$

Therefore, by (2.9), (2.15) and the Minkowski inequality we obtain

$$\begin{aligned} &\Big(\frac{1}{|B_r(x_0) \cap \Omega|} \int_{B_r(x_0)\cap\Omega} \beta_{\tilde{G}}(x, x_0)^n dx\Big)^{1/n} \\ &\le \Big(\frac{1}{|B_r(x_0) \cap \Omega|} \int_{B_r(x_0)\cap\Omega} \beta_G(x, x_0)^n dx\Big)^{1/n} + 2\beta_0 \\ &\le \beta_0 + 2\beta_0 = 3\beta_0, \end{aligned} \tag{2.16}$$

which implies the assertion (ii). $\square$

We recall an interior Lorentz estimate of strong solutions to fully nonlinear uniformly elliptic equations, whose proof can be found in [23] by using the approach of large-M-inequality principle originated from Acerbi-Mingione's work. More precisely, let us consider the fully nonlinear uniformly elliptic equations

$$G(x, D^2u) = f(x), \quad \text{in } \Omega \tag{2.17}$$

with $f \in L^{\gamma,q}(\Omega)$.

Lemma 2.2 ([23, Corollary 1.4])**.** *Assume $\gamma > n$ and $0 < q \le \infty$. Let $u \in W^{2,n}(\Omega)$ be a strong solution to (2.17) satisfying uniformly ellipticity and $f \in L^{\gamma,q}(\Omega)$. If $G(x, D^2u)$ is convex and positive homogeneous of degree one in D^2u, then there exists a small positive $\beta_0 = \beta_0(n, \lambda, \Lambda, \gamma, q)$ such that if $G(x, D^2u)$ satisfies*

$$\Big(\frac{1}{|B_r(x_0) \cap \Omega|} \int_{B_r(x_0)\cap\Omega} \beta_G(x, x_0)^n dx\Big)^{1/n} \le \beta_0,$$

for every $x_0 \in \Omega$ and $0 < r < R_0$; then we have $D^2u \in L^{\gamma,q}_{loc}(\Omega)$. Moreover, there exists a radii $R_1 = R_1(n, \lambda, \Lambda, \gamma, q)$ such that for each ball $B_{2R}(x_0) \subset\subset \Omega$ and $0 < R \le R_1$ with the estimate

$$\|D^2u\|_{L^{\gamma,q}(B_R)} \le C(\|D^2u\|_{L^n(B_{2R})} + \|f\|_{L^{\gamma,q}(B_{2R})}), \tag{2.18}$$

where $C = C(n, \lambda, \Lambda, \gamma, q)$. In the case $q = \infty$ the constant C and R_1 above depend only on $n, \lambda, \Lambda, \gamma$.

Next, we establish a local boundary estimate in the scale of Lorentz spaces by using the idea of odd/even extensions over the flat boundary. Fixed a point $x_0 \in \partial\Omega$, without loss of generality let us write

$$\partial\Omega \text{ is flat near } x_0 \text{ lying in the plane } \{x^1 = 0\}.$$

Then we may assume there exists an open ball $B_{2R}(x_0)$ with center x_0 and radius $2R$ such that

$$B^+_{2R} := B_{2R}(x_0) \cap \{x^1 > 0\} \subset \overline{\Omega},$$
$$B^-_{2R} := B_{2R}(x_0) \cap \{x^1 < 0\} \subset \mathbb{R}^n - \overline{\Omega}.$$

We also set $\Gamma_{2R} = B_{2R}(x_0) \cap \{x^1 = 0\}$.

Lemma 2.3. *For $\gamma > n$ and $0 < q \le \infty$, let $u \in W^{2,n}(\Omega)$ be a strong solution of local boundary value problem*

$$\begin{gathered} G(x, D^2u) = f(x), \quad \text{in } B^+_{2R}, \\ u = 0, \quad \text{on } \Gamma_{2R} \end{gathered} \tag{2.19}$$

with $f \in L^{\gamma,q}(\Omega)$. Then there exist small positive constants δ and R_0 depending only on $n, \lambda, \Lambda, \gamma, q$ such that, $G(x, D^2u)$ satisfying uniformly elliptic, $G(x, D^2u)$ is convex and positive homogeneous of degree one in D^2u and

$$\Big(\fint_{B_r(x_0)\cap\Omega} \beta_G(x, x_0)^n dx\Big)^{1/n} \le \beta_0,$$

we have

$$\|D^2u\|_{L^{\gamma,q}(B^+_R)} \le C\left(\|D^2u\|_{L^n(B^+_{2R})} + \|f\|_{L^{\gamma,q}(B^+_{2R})}\right), \tag{2.20}$$

for each half ball B^+_{2R} with $0 < R \le R_0$, where $C = C(n, \lambda, \Lambda, \gamma, q)$. In the case $q = \infty$, the constant C above depends only on $n, \lambda, \Lambda, \gamma$.

Proof. Note that $G(x, D^2u)$ is convex and positive homogeneous of degree one in D^2u and

$$\Big(\fint_{B_r(x_0)\cap\Omega} \beta_G(x, x_0)^n dx\Big)^{1/n} \le \beta_0,$$

which implies

$$G(x, D^2u) = G_{D^2_{ij}u}(x, D^2u)D^2_{ij}u := a_{ij}(x)D^2_{ij}u.$$

Let us now define $\hat{u}$ in $B_{2R}(x_0)$ with x_0 on the flat boundary by

$$\hat{u}(x^1,x') = \begin{cases} u(x^1,x') & \text{if } x^1 \ge 0, \\ u(-x^1,x') & \text{if } x^1 < 0, \end{cases}$$

and extend $a_{ij}(x) = a_{ij}(x^1,x')$ from $\{x^1 \ge 0\}$ to $\{x^1 < 0\}$ by even or odd reflection, depending on the indices i and j. Specifically, when $x^1 \ge 0$, $\hat{a}_{ij}(x) = a_{ij}(x)$; when $x^1 < 0$,

$$\hat{a}_{ij}(x) = \begin{cases} a_{ij}(-x^1,x'), & \text{if } i=j=1 \text{ or } i,j \in \{2,\dots,n\}, \\ -a_{ij}(-x^1,x'), & \text{if } i \in \{2,\dots,n\} \text{ and } j=1. \end{cases}$$

Also set $\hat{a}_{1j} = \hat{a}_{j1}$. We see that the nonlinearity $\widehat{G}(x,D^2\hat{u})$ satisfy uniformly elliptic, convex and positive homogeneous of degree one in $D^2\hat{u}$ and

$$\Big(\fint_{B_r(x_0)\cap\Omega} \beta_{\widehat{G}}(x,x_0)^n dx\Big)^{1/n} \le \beta_0,$$

Let $\hat{f}$ be the odd extension of f with respect to x^1, then it is easy to check that $\hat{f} \in L^{\gamma,q}(B_{2R}(x_0))$. By Lemma 2.2 it implies that the extended $\hat{u}$ is a strong solution of $\widehat{G}(x,D^2\hat{u}) = \hat{f}$ in $B_{2R}(x_0)$, then it gives rise to the local Lorentz estimate (2.18). Therefore, the desired estimate (2.20) is obtained by restricting $\hat{u}$ from $B_{2R}(x_0)$ to B^+_{2R}. □

Using the standard flattening and covering arguments, we can derive a global Lorentz estimate as follows.

Theorem 2.4. *For $\gamma > n$ and $0 < q \le \infty$, let $u \in W^{2,n}(\Omega)$ be a strong solution to the following Dirichlet problem*

$$\begin{gathered} G(x,D^2u) = f(x), \quad \text{in } \Omega, \\ u = 0, \quad \text{on } \partial\Omega, \end{gathered} \tag{2.21}$$

$G(x,D^2u)$ satisfying uniformly elliptic, convex and positive homogeneous of degree one in D^2u with the following oscillation on "coefficients"

$$\Big(\fint_{B_r(x_0)\cap\Omega} \beta_G(x,x_0)^n dx\Big)^{1/n} \le \beta_0$$

for some small positive constant $\beta_0 = \beta_0(n,\lambda,\Lambda,\gamma,q)$. If $f \in L^{\gamma,q}(\Omega)$ and $\partial\Omega \in C^{1,1}$, then $D^2u \in L^{\gamma,q}(\Omega)$ and there exists a positive constant $C = C(n,\lambda,\Lambda,\gamma,q,\Omega)$ with the estimate

$$\|D^2u\|_{L^{\gamma,q}(\Omega)} \le C\|f\|_{L^{\gamma,q}(\Omega)}. \tag{2.22}$$

While $q = \infty$, the constant C depends only on $n,\lambda,\Lambda,\gamma,\Omega$.

Proof. (1) For fixed any point $x_0 \in \partial\Omega$, we now flatten the boundary near x_0 in order to apply the flat boundary estimates (2.20). Thanks to the assumption $\partial\Omega \in C^{1,1}$, there exists a neighborhood $\mathcal{N}_0 \ni x_0$ and a $C^{1,1}$-diffeomorphism $\Phi : \mathcal{N}_0 \to B_{2R}$ such that

$$\Phi(x_0) = 0, \quad \text{and} \quad \Phi(\mathcal{N}_0 \cap \Omega) = B^+_{2R}(0).$$

We write $y = \Phi(x), x \in \mathcal{N}_0 \cap \Omega$, and define $\Psi = \Phi^{-1}$, then $x = \Psi(y)$. Define $\tilde{u}(y) = u(\Psi(y)) = u(x)$ for $y \in B^+_{2R}$. Then it is readily checked that $\tilde{u} \in W^{2,n}(B^+_{2R})$ is a strong solution of the flat initial-boundary problem

$$\widetilde{F}(y,D^2\tilde{u}) = \tilde{f}(y) \quad \text{in } B^+_{2R},$$

$$\tilde{u} = 0 \quad \text{on } \Gamma_{2R} \cup B_{2R}^+,$$

where

$$\widetilde{F}(y, D^2\tilde{u}) := F\left(\Psi(y), \left(D\Phi^T \circ \Psi\right) D^2\tilde{u}\left(D\Phi \circ \Psi\right) + D\tilde{u}\left(D^2\Phi \circ \Psi\right)\right),$$
$$\tilde{f}(y) := f(\Psi(y)).$$

It is obvious that $\widetilde{F}$ is convex in $D^2\tilde{u}$ and $\widetilde{F}(y,0) = 0$. Moreover, we readily see that $\beta_{\widetilde{F}}(y, y_0) \le C(\Phi)\beta_F(\Psi(y), \Psi(y_0))$ for any $y, y_0 \in B_{2R}^+$; and $\widetilde{F}$ satisfies the similar assumptions of Lemma 2.3 with different positive constants. Therefore, it yields

$$\|D^2\tilde{u}\|_{L^{\gamma,q}(B_R^+)} \le C\left(\|D^2\tilde{u}\|_{L^n(B_{2R}^+)} + \|\tilde{f}\|_{L^{\gamma,q}(B_{2R}^+)}\right).$$

Converting back to the original x-variables, we conclude

$$\|D^2u\|_{L^{\gamma,q}(\Psi(B_R^+))} \le C\left(\|D^2u\|_{L^n(\Psi(B_{2R}^+))} + \|f\|_{L^{\gamma,q}(\Psi(B_{2R}^+))}\right), \tag{2.23}$$

From this estimate, along with the interior bound (2.18) in Lemma 2.2, the standard covering arguments lead to

$$\|D^2u\|_{L^{\gamma,q}(\Omega)} \le C\left(\|D^2u\|_{L^n(\Omega)} + \|f\|_{L^{\gamma,q}(\Omega)}\right), \tag{2.24}$$

for some positive constant C depending on $n, \lambda, \Lambda, \gamma, q, \Omega$, which implies

$$\|u\|_{W^2L^{\gamma,q}(\Omega)} \le C\left(\|Du\|_{L^{\gamma,q}(\Omega)} + \|D^2u\|_{L^n(\Omega)} + \|f\|_{L^{\gamma,q}(\Omega)}\right). \tag{2.25}$$

(2) At this point, the desired estimate (2.22) follows from the uniqueness property of the homogeneous equation. Indeed, if (2.22) is not true, there exists a sequence $\{u_k\}_{k=1}^\infty$ and $\{f_k\}_{k=1}^\infty$ such that u_k for each k is a strong solution of the problem

$$F(x, D^2u_k) = f_k(x) \quad \text{in } \Omega,$$
$$u_k = 0 \quad \text{on } \partial\Omega,$$

with the estimate

$$\|u_k\|_{W^2L^{\gamma,q}(\Omega)} > k\|f_k\|_{L^{\gamma,q}(\Omega)}, \quad \text{for all } k \ge 1. \tag{2.26}$$

Without loss of generality, we may suppose that

$$\|u_k\|_{W^2L^{\gamma,q}(\Omega)} = 1. \tag{2.27}$$

Then it follows from (2.26) that

$$\|f_k\|_{L^{\gamma,q}(\Omega)} < \frac{1}{k} \to 0, \quad \text{as } k \to \infty. \tag{2.28}$$

Since $\{u_k\}_{k=1}^\infty$ is uniformly bounded in $W^2L^{\gamma,q}(\Omega)$, there exists a subsequence, which be still denoted by $\{u_k\}_{k=1}^\infty$, and a function $u_0 \in W^2L^{\gamma,q}(\Omega)$, such that

$$u_k \rightharpoonup u_0 \text{ weakly in } W^2L^{\gamma,q}(\Omega), \quad u_k \to u_0 \text{ in } L^{\gamma,q}(\Omega), \quad \text{as } k \to \infty. \tag{2.29}$$

It is easy to check that u_0 is a strong solution of

$$\begin{gathered} F(x, D^2u_0) = 0, \quad \text{in } \Omega, \\ u_0 = 0, \quad \text{on } \partial\Omega. \end{gathered} \tag{2.30}$$

Accordingly, $u_0 = 0$ due to the uniqueness of strong solutions to zero initial-boundary problem (2.30), so it follows from (2.28) and (2.29) that

$$\begin{gathered} f_k \to 0 \quad \text{in } L^{\gamma,q}(\Omega), \\ u_k \rightharpoonup 0 \quad \text{weakly in } W^2L^{\gamma,q}(\Omega), \\ u_k \to 0 \quad \text{in } L^{\gamma,q}(\Omega), \end{gathered} \tag{2.31}$$

as $k \to \infty$. Note that $W^2L^{\gamma,q}(\Omega) \hookrightarrow W^2L^n(\Omega)$ because $\gamma > n$, hence

$$\|u_k\|_{L^n(\Omega)} \to 0, \quad \|Du_k\|_{L^n(\Omega)} \to 0, \quad \text{as } k \to \infty. \tag{2.32}$$

Moreover, letting the measure $\nu = dx$, we see that

$$Du_k \to 0 \quad \nu\text{-a.e. in } \Omega \quad \text{as } k \to \infty \quad \text{(up to subsequence)},$$

which implies

$$|\{x \in \Omega : |Du_k| > \mu\}| \to 0 \quad \text{for all } \mu > 0 \quad \text{as } k \to \infty,$$

so by the Lebesgue Dominated Convergence Theorem we obtain

$$Du_k \to 0 \quad \text{in } L^{\gamma,q}(\Omega) \quad \text{as } k \to \infty. \tag{2.33}$$

Combining (2.25), (2.27), (2.31), (2.32) and (2.33), it yields

$$1 \le C\left(\|Du_k\|_{L^{\gamma,q}(\Omega)} + \|D^2u_k\|_{L^n(\Omega)} + \|f_k\|_{L^{\gamma,q}(\Omega)}\right) \to 0 \quad \text{as } k \to \infty,$$

which is a contradiction. This completes the proof. $\square$

In view of the lemma 2.1, the asymptotically elliptic equation (1.1) turns out to be a uniformly elliptic equation (2.8). For the asymptotically elliptic equation, lemma 2.1 and the existing theory for fully nonlinear, uniformly elliptic equation, see Lemma 2.2, will be employed to finally derive the required estimate. We are now ready to prove our main result.

Proof of theorem 1.4. From (2.8), for any given positive constant β_0, as in Lemma 2.2, we define a new data $\beta_1 := \min\{\frac{\lambda}{3}, 1\}$, and set

$$\tilde{\beta}_0 := \frac{1}{3}\min\{\beta_0, \beta_1\} > 0.$$

Then there exists a real-valued function $\tilde{H}(x, D^2u)$ such that

$$|\tilde{H}(x, D^2u)| \le \beta_0 < 1 \quad \text{for all } x \in \Omega.$$

Now let $u \in W^{2,p}$ be a strong solution of the problem (1.1) and assume that $F(x, D^2u)$ is asymptotically elliptic with $G(x, D^2u)$ which satisfies

$$\Big(\frac{1}{|B_r(x_0) \cap \Omega|}\int_{B_r(x_0)\cap\Omega} \beta_G(x, x_0)^n dx\Big)^{1/n} \le \tilde{\beta}_0,$$

for any $x_0 \in \Omega$ and $0 < r < R_0$. Then by Lemma 2.2, $\tilde{G}(x, D^2u)$ is uniformly elliptic and satisfies

$$\Big(\frac{1}{|B_r(x_0) \cap \Omega|}\int_{B_r(x_0)\cap\Omega} \beta_{\tilde{G}}(x, x_0)^n dx\Big)^{1/n} \le 3\tilde{\beta}_0 \le \beta_0,$$

for any $x_0 \in \Omega$ and $0 < r < R_0$.

According to $f \in L^{\gamma,q}(\Omega)$ and equality (2.8), we have

$$|g(x)| \le |f(x)| + 2|D^2u(x)|_{\chi\{\|D^2u\| < K\}}.$$

which implies that

$$\begin{aligned}
&|\{x \in \Omega : |g(x) > \mu|\}| \\
&\le |\{x \in \Omega : |f(x)| > \frac{\mu}{2}\}| + |\{x \in \Omega : 2|D^2u(x)|_{\chi\{\|D^2u\|<K\}} > \frac{\mu}{2}\}|.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|g\|^q_{L^{\gamma,q}(\Omega)} &\le q\int_0^\infty \Big(\mu^\gamma|\{x \in \Omega : |f(x)| > \frac{\mu}{2}\}|\Big)^{q/\gamma}\,\frac{d\mu}{\mu} \\
&\quad + q\int_0^\infty \Big(\mu^\gamma|\{x \in \Omega : 2|D^2u(x)|\chi_{\{\|D^2u\|<K\}} > \frac{\mu}{2}\}|\Big)^{q/\gamma}\,\frac{d\mu}{\mu} \\
&= 2^q q\int_0^\infty (\mu^\gamma|\{x \in \Omega : |f(x)| > \mu\}|)^{q/\gamma}\,\frac{d\mu}{\mu} \\
&\quad + 2^q q\int_0^\infty \big(\mu^\gamma|\{x \in \Omega : 2|D^2u(x)|\chi_{\{\|D^2u\|<K\}} > \mu\}|\big)^{q/\gamma}\,\frac{d\mu}{\mu}.
\end{aligned}$$

Since

$$|\{x \in \Omega : 2|Du^2(x)|\chi_{\{\|D^2u\|<K\}} > \mu\}| \le |\{x \in \Omega : 2K > \mu\}|,$$

we can derive that

$$\begin{aligned}
\|g\|^q_{L^{\gamma,q}(\Omega)} &\le 2^q\|f\|^q_{L^{\gamma,q}(\Omega)} + 2^q q\int_0^\infty (\mu^\gamma|\{x \in \Omega : 2K > \mu\}|)^{q/\gamma}\,\frac{d\mu}{\mu} \\
&= 2^q\|f\|^q_{L^{\gamma,q}(\Omega)} + 2^q q\int_0^{2K} (\mu^\gamma|\{x \in \Omega : 2K > \mu\}|)^{q/\gamma}\,\frac{d\mu}{\mu} \\
&\quad + 2^q q\int_{2K}^\infty (\mu^\gamma|\{x \in \Omega : 2K > \mu\}|)^{q/\gamma}\,\frac{d\mu}{\mu} \\
&\le 2^q\|f\|^q_{L^{\gamma,q}(\Omega)} + 2^q q\int_0^{2K} (\mu^\gamma|\Omega|)^{q/\gamma}\,\frac{d\mu}{\mu} + 0 \\
&= 2^q\|f\|^q_{L^{\gamma,q}(\Omega)} + 2^q q|\Omega|^{q/\gamma}\int_0^{2K} \mu^{q-1}d\mu \\
&= 2^q\|f\|^q_{L^{\gamma,q}(\Omega)} + 2^q|\Omega|^{q/\gamma}(2K)^q \\
&\le C(\|f\|^q_{L^{\gamma,q}(\Omega)} + 1).
\end{aligned}$$

Thus, we have

$$\|g\|_{L^{\gamma,q}(\Omega)} \le C(\|f\|_{L^{\gamma,q}(\Omega)} + 1),$$

for some positive constant $C = C(\beta_0, K, n, \gamma, q, |\Omega|)$.

Considering

$$H(x, M) = \frac{F(x, M) - G(x, M)}{\|M\|} \ge 0$$

for all $M \in \mathcal{S}(n)$ and $x \in \Omega$, we have $\tilde{H}(x, M) \ge 0$ for all $x \in \Omega$ due to the definition of $\tilde{H}$. This shows that $\tilde{H}(x, D^2u) \ge 0$ for all $x \in \Omega$. On the other hand, we know $G(x, M)$ and $\|M\|$ are convex in M; therefore $\tilde{F}(x, M) = G(x, M) + \tilde{H}(x, D^2u)\|M\|$ is also convex with respect to M. We then apply Theorem 2.4 to $g \in L^{\gamma,q}(\Omega)$ and $\tilde{G}(x, D^2u)$ to discover $u \in W^{2,n}$ with the estimate

$$\|D^2u\|_{L^{\gamma,q}(\Omega)} \le C\|g\|_{L^{\gamma,q}(\Omega)} \le C(\|f\|_{L^{\gamma,q}(\Omega)} + 1), \tag{2.34}$$

where $C = C(n, \lambda, \Lambda, R_0, \Omega, \gamma, q)$ is a positive constant. This completes the proof. $\square$

Acknowledgements. This work is supported by the NSF of China under grants No. 11371050 and No. 11401165.

References

[1] Baroni, P.; *Lorentz estimates for obstacle parabolic problems*, Nonlinear Anal., **96** (2014), 167-188.
[2] Baroni, P.; *Lorentz estimates for degenerate and singular evolutionary systems*, J. Differ. Equ., **255** (2013), 2927-2951.
[3] Byun, S. S.; Cho, Y.; Oh, J.; *Global Calderón-Zygmund theory for nonlinear elliptic obstacle problems with asymptotically regular nonlinearities*, Nonlinear Anal., 123-124 (2015), 150-157.
[4] Byun, S. S.; Lee, M.; Palagachev, D. K.; *Hessian estimates in weighted Lebesgue spaces for fully nonlinear elliptic equations*, J. Differ. Equ., **260** (2016), 4550-4571.
[5] Byun, S. S.; Oh, J.; Wang, L.; *Global Calderón-Zygmund theory for asymptotically regular nonlinear elliptic and parabolic equations.* Int. Math. Res. Not. IMRN, **2015**(17) (2015), 8289-8308.
[6] Caffarelli, L. A.; *Interior a priori estimates for solutions of fully nonlinear equations*, Ann. Math. **130** (2) (1989), 189-213.
[7] Caffarelli, L. A.; Huang, Q.; *Estimates in the generalized Campamato-John-Nirenberg spaces for fully nonlinear elliptic equations*, Duke Math. J., **118** (1) (2003), 1-17.
[8] Chiarenza, F.; Frasca, M.; Longo, P.; *Interior $W^{2,p}$ estimates for nondivergence elliptic equations with discontinuous coefficients*, Ricerche Mat., **40** (1991), 149-168.
[9] Chiarenza, F.; Frasca, M.; Longo, P.; *$W^{2,p}$ solvability of the Dirichlet problem for nonlinear elliptic equations with VMO coefficients*, Trans. Amer. Math. Soc., **336** (2) (1993), 841-853.
[10] Chipot, M.; Evans, L. C.; *Linearisation at infinity and Lipschitz estimates for certain problems in the calculus of variations*, Proc. Roy. Soc. Edinburgh Sect. A, **102**(3-4) (1986), 291-303.
[11] Dong, H.; Krylov, N. V.; Li, X.; *On fully nonlinear elliptic and parabolic equations with VMO coefficients in domains*, St. Petersburg Math. J., **24** (1) (2013), 39-69.
[12] Evans, L. C.; *Classical solutions of fully nonlinear, convex, second-order elliptic equations*, Comm. Pure Appl. Math. **25** (1982), 333-363.
[13] Foss, M.; *Global regularity for almost minimizers of nonconvex variational problems*, Ann. Mat. Pura Appl., (4) **187**(2) (2008), 263-321.
[14] Krylov, N. V.; *Some L^p-estimates for elliptic and parabolic operators with measurable coefficients*, Discrete Contin. Dyn. Syst. Ser. B, **17** (6) (2012), 2073-2090.
[15] Krylov, N. V.; *On the existence of $W^{2,p}$ solutions for fully nonlinear elliptic equations under relaxed convexity assumptions*, Comm. Part. Diff. Equ., **38** (2013), 687-710.
[16] Mengesha, T.; Phuc, N. C.; *Global estimates for quasilinear elliptic equations on Reifenberg flat domains*, Arch. Rational Mech. Anal., **203**(2012), 189-216.
[17] Raymond, J. P.; *Lipschitz regularity of solutions of some asymptotically convex problems*, Proc. Roy. Soc. Edinburgh Sect. A, **117**(1-2)(1991), 59-73.
[18] Scheven, C.; Schmidt, T.; *Asymptotically regular problems I: Higher integrability*, J. Differ. Equ., **248** (2010), 745-791.
[19] Scheven, C.; Schmidt, T.; *Asymptotically regular problems II: Partial Lipschitz continuity and a singular set of positive measure*, Ann. Sc. Norm. Super. Pisa Cl. Sci., (5) **8**(3) (2009), 469-507.
[20] Talenti, G; *Elliptic equations and rearrangements*, Ann Scuola Norm. Sup. Pisa Cl. Sci. (IV), **3** (1976), 697-718.
[21] Winter, N.; *$W^{2,p}$ and $W^{1,p}$-estimates at the boundary for solutions of fully nonlinear, uniformly elliptic equations*, Z. Anal. Anwend., **28** (2) (2009), 129-164.
[22] Zhang, J.; Zheng, S.; *Lorentz estimates for asymptotically regular elliptic equations in quasiconvex domains*, Electron. J. Differential Equations, **2016** (142) (2016), 1-13.
[23] Zhang, J.; Zheng, S.; *Lorentz estimates for fully nonlinear parabolic and elliptic equations*, Nonlinear Anal., **148** (2017), 106-125.
[24] Zhang, C.; Zhou, S. L.; *Global weighted estimates for quasilinear elliptic equations with non-standard growth*, J. Funct. Anal. **267** (2014), 605-642.

Yongyong Wang
Department of Mathematics, Beijing Jiaotong University, Beijing 100044, China
E-mail address: yongyongwang@bjtu.edu.cn

Junjie Zhang
Department of Mathematics, Beijing Jiaotong University, Beijing 100044, China
E-mail address: junjiezhang@bjtu.edu.cn

Shenzhou Zheng (Corresponding author)
Department of Mathematics, Beijing Jiaotong University, Beijing 100044, China
E-mail address: shzhzheng@bjtu.edu.cn