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FOURIER TRUNCATION METHOD FOR AN INVERSE SOURCE PROBLEM FOR SPACE-TIME FRACTIONAL DIFFUSION EQUATION

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Abstract. In this article, we study an inverse problem to determine an unknown source term in a space time fractional diffusion equation, whereby the data are obtained at a certain time. In general, this problem is ill-posed in the sense of Hadamard, so the Fourier truncation method is proposed to solve the problem. In the theoretical results, we propose a priori and a posteriori parameter choice rules and analyze them.

1. Introduction

In this work, we consider the inverse problem of finding the source function f in the problem

$$\begin{gathered}\frac{\partial^\beta}{\partial t^\beta}u(x,t) = -r^\beta(-\Delta)^{\frac{\alpha}{2}}u(x,t) + h(t)f(x), \quad (x,t)\in\Omega_T,\\ u(-1,t) = u(1,t) = 0, \quad 0<t<T,\\ u(x,0) = 0, \quad x\in\Omega,\\ u(x,T) = g(x), \quad x\in\Omega,\end{gathered} \tag{1.1}$$

where $\Omega_T = (-1,1)\times(0,T)$, $r>0$ is a parameter, $h\in C[0,T]$ is a given function, $\beta\in(0,1)$, $\alpha\in(1,2)$ are fractional order of the time and the space derivatives, respectively, and $T>0$ is a final time. The function $u=u(x,t)$ denotes a concentration of contaminant at a position x and time t. The symbol $\frac{\partial^\beta u}{\partial t^\beta}$ is the Caputo fractional derivative of order β for differentiable function u; it writes

$$\frac{\partial^\beta}{\partial t^\beta}u(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{u'(s)}{(t-s)^\beta}ds,$$

and $\Gamma(.)$ denotes the standard Gamma function. Note that if the fractional order β tends to unity, the fractional derivative $\frac{\partial^\beta}{\partial t^\beta}u$ converges to the first-order derivative $\frac{du}{dt}$ [6], and thus the problem reproduces the diffusion model. See, e.g., [6, 12] for the definition and properties of Caputo's derivative.

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It is known that the inverse source problem mentioned above is ill-posed in general, i.e., a solution does not always exist, and in the case of existence of a solution, it does not depend continuously on the given data. In fact, from a small noise of a physical measurement, for example (h, g) is noised by observation data $(h^\varepsilon, g^\varepsilon)$ with order of $\varepsilon > 0$

$$\|h^\varepsilon - h\|_{C([0,T])} + \|g^\varepsilon - g\|_{L^2(-1,1)} \le \varepsilon \tag{1.2}$$

where we denote $\|\theta\|_{C([0,T])} = \sup_{0\le t\le T}|\theta(t)|$ for any $\theta \in C([0,T]$. It is well-known that if ε is small, the sought solution f may have a large error. An example for illustrating this is given in Theorem 2.13. Hence some regularization method are required for stable computation of a sought solution.

The inverse source problem attracted many authors and its physical background can be found in [18]. Wei et al [20, 19, 21] studied an inverse source problem in a spatial fractional diffusion equation by quasi-boundary value and truncation methods. Recently, Kirane et al [7, 6] studied conditional well-posedness to determine a space dependent source in one-dimensional and two-dimensional time-fractional diffusion equations. Rundell et al [4, 13] considered an inverse problem for a one-dimensional time-fractional diffusion problem. However, the inverse source problem for both time and space fractional is limited. Recently, Tatar et al [17] considered Problem (1.1) with a general source $h(t,x)f(x)$. They show that the inverse source problem is well-posed in the sense of Hadamard except for a finite set of $r > 0$. However the source function is also unstable in L^2 norm (See Theorem 2.14 below). The topic in this paper is to finding approximate solution. Hence, our purpose is different and not contradict with the results in [17]. Motivated by above reasons, in this study, we apply the Fourier regularization method to establish a regularized solution. We estimate a convergence rate under an a priori bound assumption of the sought solution and a priori parameter choice rule. Because the a priori bound is difficult to obtain in practical application, so we also estimate a convergence rate under the a posteriori parameter choice rule which is independent on the a priori bound.

This article is organized as follows. In Section 2, we give a formula of the source function f and establish some lemmas and theorems which are useful to the next results. Moreover, the ill-posedness of the inverse source problem is also given in this section. In Section 2, we propose Fourier regularization method and give two convergence estimates under an a priori assumption for the exact solution and two regularization parameter choice rules.

2. Inverse source problem

2.1. **Formula of the source function.** First, we introduce a few properties of the eigenvalues of the operator $(-\Delta)^{\alpha/2}$, see [5, 12].

Theorem 2.1 (Eigenvalues of the fractional Laplacian operator)**.** *1. Each eigenvalues of $(-\Delta)^{\alpha/2}$ is real. The family of eigenvalues $\{\overline{\lambda_k}\}_{k=1}^\infty$ satisfy $0 \le \overline{\lambda_1} \le \overline{\lambda_2} \le \overline{\lambda_3} \le \dots$, and $\overline{\lambda_k} \to \infty$ as $k \to \infty$.*

2. We take $\{\overline{\lambda_k}, \phi_k\}$ the eigenvalues and corresponding eigenvectors of the fractional Laplacian operator in Ω with Dirichlet boundary conditions on $\partial\Omega$:

$$\begin{aligned} -\Delta\phi_k(x) &= \overline{\lambda_k}\phi_k(x), \quad x \in \Omega, \\ \phi_k(x) &= 0, \quad \text{on } \partial\Omega, \end{aligned} \tag{2.1}$$

for $k = 1, 2, \dots$.

Then we define the operator $(-\Delta)^{\frac{\alpha}{2}}$ by

$$(-\Delta)^{\alpha/2}u := \sum_{k=0}^{\infty} c_k\phi_k(x) = -\sum_{k=0}^{\infty} c_k\overline{\lambda}_k^{\alpha/2}\phi_k(x),$$

which maps $H_0^\alpha(\Omega)$ into $L^2(\Omega)$. Let $0 \neq \gamma < \infty$. By $H^\gamma(\Omega)$ we denote the space of all functions $g \in L^2(\Omega)$ with the property

$$\sum_{k=1}^{\infty}(1+\overline{\lambda_k})^{2\gamma}|g_k|^2 < \infty, \tag{2.2}$$

where $g_k = \int_\Omega g(x)\phi_k(x)dx$. Then we also define
$\|g\|_{H^\gamma(\Omega)} = \sqrt{\sum_{k=1}^{\infty}(1+\overline{\lambda_k})^{2\gamma}|g_k|^2}$. If $\gamma = 0$ then $H^\gamma(\Omega)$ is $L^2(\Omega)$.
Now we use the separation of variables to yield the solution of (1.1). Suppose that the solution of (1.1) is defined by Fourier series

$$u(x,t) = \sum_{k=1}^{\infty} u_k(t)\phi_k(x), \quad \text{with } u_k(t) = \langle u(.,t), \phi_k(x)\rangle. \tag{2.3}$$

Then the eigenfunction expansions can be defined by the Fourier method. That is, we multiply both sides of (1.1) by $\phi_k(x)$ and integrate the equation with respect to x. Using the Green formular and $\phi_k|_{\partial\Omega} = 0$, we obtain an uncouple system of initial value problem for the fractional differential equations for the unknown Fourier coefficient $u_k(t)$

$$\begin{gathered}\frac{\partial^\beta}{\partial t^\beta}u_k(t) = -r^\beta(-\Delta)^{\frac{\alpha}{2}}u_k(t) + h(t)f_k(x), \quad (x,t) \in \Omega \times (0,T), \\ u_k(0) = \langle u(x,0), \varphi_k(x)\rangle.\end{gathered} \tag{2.4}$$

From the result in [17], the formula of solution corresponding to the initial value problem for (2.4) is obtained as follows, from $u(x,0) = 0$.

$$u(x,t) = \sum_{k=1}^{\infty}\Big(\int_0^t \tau^{\beta-1}E_{\beta,\beta}\Big(-(\frac{k\pi}{2})^\alpha r^\beta\tau^\beta\Big)\langle f(x)h(t-\tau), \phi_k(x)\rangle d\tau\Big)\phi_k(x). \tag{2.5}$$

By a change variable in the integral, we can rewrite

$$u(x,t) = \sum_{k=1}^{\infty}\Big(\int_0^t (t-\tau)^{\beta-1}E_{\beta,\beta}\Big(-(\frac{k\pi}{2})^\alpha r^\beta(t-\tau)^\beta\Big)h(\tau)d\tau\Big)\langle f(x), \phi_k(x)\rangle\phi_k(x). \tag{2.6}$$

Letting $t = T$ in the latter equality, we obtain

$$f(x) = \sum_{k=1}^{\infty}\frac{\langle g(x), \phi_k(x)\rangle\phi_k(x)}{\int_0^T (T-\tau)^{\beta-1}E_{\beta,\beta}\big(-(\frac{k\pi}{2})^\alpha r^\beta(T-\tau)^\beta\big)h(\tau)d\tau}. \tag{2.7}$$

to abbreviate notation, we set

$$\Phi_\beta(\lambda_k^\alpha, \tau, r) = (T-\tau)^{\beta-1}E_{\beta,\beta}\Big(-(\frac{k\pi}{2})^\alpha r^\beta(T-\tau)^\beta\Big),$$

then the source function f is rewritten as

$$f(x) = \sum_{k=1}^{\infty}\frac{\langle g(x), \phi_k(x)\rangle\phi_k(x)}{\int_0^T \Phi_\beta(\lambda_k^\alpha, \tau, r)h(\tau)d\tau}. \tag{2.8}$$

Remark 2.2. Applying [17, Theorem 2.1], we obtain the existence and uniqueness of problem (1.1) such that $u \in L^2(0,T;H^\alpha(\Omega))$. The regularity estimate for u as in (2.6) is mentioned in [17] and so, we omit it here.

2.2. **Preliminary results.** Now, we consider the following definition and lemmas which are useful for our main results.

Definition 2.3 ([12])**.** The Mittag-Leffler function is

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)}, \quad z \in \mathbb{C}$$

where $\alpha > 0$ and $\beta \in \mathbb{R}$ are arbitrary constants.

Lemma 2.4 ([14])**.** *For $\lambda > 0$ and $0 < \beta < 1$, we have*

$$\frac{d}{dt} E_{\beta,1}(-\lambda t^\beta) = -\lambda t^{\beta-1} E_{\beta,\beta}(-\lambda t^\beta), \quad t > 0. \tag{2.9}$$

Lemma 2.5 ([12])**.** *For $\alpha > 0$ and $\beta \in \mathbb{R}$, we have*

$$E_{\alpha,\beta}(z) = z E_{\alpha,\alpha+\beta}(z) + \frac{1}{\Gamma(\beta)}, \quad z \in \mathbb{C}.$$

Lemma 2.6 ([14])**.** *The following equality holds for $\lambda > 0$, $\alpha > 0$ and $m \in \mathbb{N}$*

$$\frac{d^m}{dt^m} E_{\alpha,1}(-\lambda t^\alpha) = -\lambda t^{\alpha-m} E_{\alpha,\alpha-m+1}(-\lambda t^\alpha), \quad t > 0. \tag{2.10}$$

Lemma 2.7 ([17])**.** *If $\alpha \le 2$, β is arbitrary real number, μ is such that $\frac{\pi\alpha}{2} < \mu < \min\{\pi\alpha, \pi\}$, $\mu \le |arg(z)| \le \pi$ then there exists two constants $A_0 > 0$ and $A_1 > 0$ such that*

$$\frac{A_0}{1+|z|} \le |E_{\alpha,\beta}(z)| \le \frac{A_1}{1+|z|}. \tag{2.11}$$

Lemma 2.8 ([14])**.** *Let $E_{\beta,\beta}(-\eta) \ge 0$, $0 < \beta < 1$, we have*

$$\begin{aligned} \int_0^M \Big| t^{\beta-1} E_{\beta,\beta}(-\overline{\lambda_k} t^\beta) \Big| dt &= \int_0^M t^{\beta-1} E_{\beta,\beta}(-\overline{\lambda_k} t^\beta) dt \\ &= -\frac{1}{\overline{\lambda_k}} \int_0^M \frac{d}{dt} E_{\beta,1}(-\overline{\lambda_k} t^\beta) dt \\ &= \frac{1}{\overline{\lambda_k}} \Big(1 - E_{\beta,1}(-\overline{\lambda_k} M^\beta) \Big). \end{aligned} \tag{2.12}$$

Lemma 2.9 ([17])**.** *For any $\lambda_k^\alpha = (\frac{k\pi}{2})^\alpha$ satisfying $\lambda_k^\alpha \ge \lambda_1^\alpha$ there exists positive constant C depending on $\{\beta, T, \frac{\pi}{2}\}$ such that*

$$\frac{C}{r^\beta T^\beta \lambda_k^\alpha} \le E_{\beta,\beta+1}(-\lambda_k^\alpha r^\beta T^\beta) \le \frac{1}{r^\beta T^\beta \lambda_k^\alpha}. \tag{2.13}$$

Lemma 2.10. *Let $h : [0,T] \to \mathbb{R}^+$ be a continuous function such that $|\mathcal{I}(h)| = \inf_{t\in[0,T]} |h(t)| > 0$. Set $\|h\|_{C[0,T]} = \sup_{t\in[0,T]} |h(t)|$. Then we have*

$$\frac{|\mathcal{I}(h)|(1 - E_{\beta,1}(-\lambda_1^\alpha r^\beta T^\beta))}{\lambda_k^\alpha r^\beta} \le \int_0^T \Phi_\beta(\lambda_k^\alpha, \tau, r) h(\tau) d\tau \le \frac{\|h\|_{C[0,T]}}{\lambda_k^\alpha r^\beta}. \tag{2.14}$$

A proof of the above lemma can be found in [17].

Theorem 2.11. *Let $g \in H^\alpha(\Omega)$. Then the inverse source problem has the solution $f \in L^2(\Omega)$.*

Proof. The solution f exists if and only if the series in the right-hand side of (2.8) converges. Hence, we show this point. Indeed, using Lemma 2.10 and noting that $g \in H^\alpha(\Omega)$, we obtain

$$\begin{aligned} \sum_{k=1}^{\infty} \Big| \frac{\langle g(x), \phi_k(x)\rangle}{\int_0^T \Phi_\beta(\lambda_k^\alpha, \tau, r) h(\tau) d\tau} \Big|^2 &\geq \sum_{k=1}^{\infty} \frac{\lambda_k^{2\alpha} r^{2\beta} \langle g(x), \phi_k(x)\rangle^2}{\|h\|^2_{C[0,T]}} \\ &= \frac{r^{2\beta}}{\|h\|^2_{C[0,T]}} \|g\|^2_{H^\alpha(\Omega)}, \end{aligned} \tag{2.15}$$

and

$$\begin{aligned} \sum_{k=1}^{\infty} \Big| \frac{\langle g(x), \phi_k(x)\rangle}{\int_0^T \Phi_\beta(\lambda_k^\alpha, \tau, r) h(\tau) d\tau} \Big|^2 &\leq \sum_{k=1}^{\infty} \frac{\lambda_k^{2\alpha} r^{2\beta} \langle g(x), \phi_k(x)\rangle^2}{|\mathcal{I}(h)|^2 (1 - E_{\beta,1}(-\lambda_1^\alpha r^\beta T^\beta))^2} \\ &= \frac{r^{2\beta}}{|\mathcal{I}(h)|^2 (1 - E_{\beta,1}(-\lambda_1^\alpha r^\beta T^\beta))^2} \|g\|^2_{H^\alpha(\Omega)}. \end{aligned} \tag{2.16}$$

From two latter inequality, we conclude that the series $\sum_{k=1}^\infty \frac{\langle g(x), \phi_k(x)\rangle \phi_k(x)}{\int_0^T \Phi_\beta(\lambda_k^\alpha, \tau, r) h(\tau) d\tau}$ is convergent. The proof is complete. □

Theorem 2.12. *Let $R : [0,T] \to \mathbb{R}$ be as in Lemma 2.10, then the solution $(u(x,t), f(x))$ of Problem (1) is unique.*

Proof. Let f_1 and f_2 be the source functions corresponding to the final values g_1 and g_2 respectively. Suppose that $g_1 = g_2$ then we prove that $f_1 = f_2$. In fact, it is well-known that $E_{\beta,\beta}(-(\frac{k\pi}{2})^\alpha r^\beta (t-\tau)^\beta) \geq 0$ for $\tau \leq t$. Since $\|h\|_{C[0,T]} \geq |\mathcal{I}(h)| > 0$ for $t \in [0,T]$, we have

$$\begin{aligned} &\int_0^T \Phi_\beta(\lambda_k^\alpha, \tau, r) h(\tau) d\tau \\ &\geq |\mathcal{I}(h)| \int_0^T (T-\tau)^{\beta-1} E_{\beta,\beta}\Big(-(\frac{k\pi}{2})^\alpha r^\beta (T-\tau)^\beta \Big) d\tau \\ &= |\mathcal{I}(h)| T^\beta E_{\beta,\beta+1}\Big(-(\frac{k\pi}{2})^\alpha (rT)^\beta \Big) > 0. \end{aligned} \tag{2.17}$$

We have the estimate

$$f_1(x) - f_2(x) = \sum_{k=1}^{\infty} \frac{\langle g_1(x) - g_2(x), \phi_k(x)\rangle \phi_k(x)}{\int_0^T \Phi_\beta(\lambda_k^\alpha, \tau, r) h(\tau) d\tau} = 0. \tag{2.18}$$

The proof is complete. □

Theorem 2.13. *The inverse source problem of finding the function f is ill-posed in the Hadamard sense in the L^2 norm.*

Proof. Let us Define a linear operator $\mathcal{P} : L^2(\Omega) \to L^2(\Omega)$ as follows

$$\begin{aligned} \mathcal{P} f(x) &= \int_\Omega p(x,\omega) f(\omega) d\omega \\ &= \sum_{k=1}^{\infty} \Big[\int_0^T \Phi_\beta(\lambda_k^\alpha, \tau, r) h(\tau) d\tau \Big] \langle f(x), \phi_k(x)\rangle \phi_k(x) \end{aligned} \tag{2.19}$$

where

$$p(x,\omega)=\sum_{k=1}^{\infty}\Big[\int_0^T \Phi_\beta(\lambda_k^\alpha,\tau,r)h(\tau)d\tau\Big]\phi_k(x)\phi_k(\omega).$$

Because $p(x,\omega)=p(\omega,x)$ we know that $\mathcal{K}$ is self-adjoint operator. Next, we prove its compactness. Defining the finite rank operators $\mathcal{K}_N$ as follows

$$\mathcal{P}_N f(x)=\sum_{k=1}^{N}\Big[\int_0^T \Phi_\beta(\lambda_k^\alpha,\tau,r)h(\tau)d\tau\Big]\langle f(x),\phi_k(x)\rangle\phi_k(x). \tag{2.20}$$

Then, from (2.19) and (2.20), we have

$$\begin{aligned}\|\mathcal{P}_N f-\mathcal{P}f\|^2_{L^2(\Omega)} &= \sum_{k=N+1}^{\infty}\Big[\int_0^T \Phi_\beta(\lambda_k^\alpha,\tau,r)h(\tau)d\tau\Big]^2|\langle f(x),\phi_k(x)\rangle|^2\\ &\le \sum_{k=N+1}^{\infty}\frac{\|h\|^2_{C[0,T]}}{\lambda_k^{2\alpha}}|\langle f(x),\phi_k(x)\rangle|^2\\ &\le \frac{\|h\|^2_{C[0,T]}}{\lambda_N^{2\alpha}}\sum_{k=N+1}^{\infty}|\langle f(x),\phi_k(x)\rangle|^2.\end{aligned} \tag{2.21}$$

This implies

$$\|\mathcal{P}_N f-\mathcal{P}f\|_{L^2(\Omega)}\le\Big(\frac{\|h\|^2_{C[0,T]}}{\lambda_N^{2\alpha}}\|f\|^2_{L^2(\Omega)}\Big)^{1/2}=\frac{\|h\|_{C[0,T]}}{\lambda_N^{\alpha}}\|f\|_{L^2(\Omega)}. \tag{2.22}$$

Therefore, $\|\mathcal{P}_N-\mathcal{P}\|\to 0$ in the sense of operator norm in $L(L^2(\Omega);L^2(\Omega))$ as $N\to\infty$. Also, $\mathcal{P}$ is a compact operator. Next, the singular values for the linear self-adjoint compact operator $\mathcal{P}$ are

$$\begin{aligned}\psi_k&=\int_0^T (T-\tau)^{\beta-1}E_{\beta,\beta}\Big(-(\frac{k\pi}{2})^\alpha r^\beta(T-\tau)^\beta\Big)h(\tau)d\tau\\ &=\int_0^T \Phi_\beta(\lambda_k^\alpha,\tau,r)h(\tau)d\tau,\end{aligned} \tag{2.23}$$

and corresponding eigenvectors is ϕ_k which is known as an orthonormal basis in $L^2(\Omega)$. From (2.19), the inverse source problem we introduced above can be formulated as an operator equation.

$$\mathcal{P}f(x)=g(x) \tag{2.24}$$

and by Kirsch [8], we conclude that it is ill-posed. To illustrate an ill-posed problem, we present an example. Let us choose the input final data $g^m(x)=\frac{\phi_m(x)}{\sqrt{r^{2\beta}\lambda_m^\alpha}}$. Following (2.9), we know $\lambda_m^\alpha=(\frac{m\pi}{2})^\alpha$. By (2.8), the source term corresponding to g^m is

$$\begin{aligned}f^m(x)&=\sum_{k=1}^{\infty}\frac{\langle g^m(x),\phi_k(x)\rangle\phi_k(x)}{\int_0^T \Phi_\beta(\lambda_k^\alpha,\tau,r)h(\tau)d\tau}\\ &=\sum_{k=1}^{\infty}\frac{\langle\frac{\phi_m(x)}{\sqrt{r^{2\beta}(\frac{m\pi}{2})^\alpha}},\phi_k(x)\rangle\phi_k(x)}{\int_0^T \Phi_\beta(\lambda_k^\alpha,\tau,r)h(\tau)d\tau}\\ &=\frac{\phi_m(x)}{\sqrt{r^{2\beta}(\frac{m\pi}{2})^\alpha}\int_0^T \Phi_\beta(\lambda_m^\alpha,\tau,r)h(\tau)d\tau}.\end{aligned} \tag{2.25}$$

Let us choose another input final data $g = 0$. By (2.7), the source term corresponding to g is $f = 0$. An error in L^2 norm between two input final data is

$$\|g^m - g\|_{L^2(\Omega)} = \|\frac{\phi_m(x)}{|r^\beta|\sqrt{(\frac{m\pi}{2})^\alpha}}\|_{L^2(\Omega)} = \frac{1}{|r^\beta|\sqrt{(\frac{m\pi}{2})^\alpha}}, \tag{2.26}$$

where $\alpha \in (1, 2)$. Therefore

$$\lim_{m\to+\infty} \|g^m - g\|_{L^2(\Omega)} = \lim_{m\to+\infty} \frac{1}{|r^\beta|\sqrt{(\frac{m\pi}{2})^\alpha}} = 0. \tag{2.27}$$

And an error in $L^2(-1, 1)$ norm between two corresponding source term is

$$\begin{aligned} \|f^m - f\|_{L^2(\Omega)} &= \|\frac{\phi_m(x)}{|r^\beta|\sqrt{(\frac{m\pi}{2})^\alpha}\int_0^T \Phi_\beta(\lambda_k^\alpha, \tau, r)h(\tau)d\tau}\|_{L^2(\Omega)} \\ &= \frac{1}{|r^\beta|\sqrt{(\frac{m\pi}{2})^\alpha}\int_0^T \Phi_\beta(\lambda_k^\alpha, \tau, r)h(\tau)d\tau}, \end{aligned} \tag{2.28}$$

which we note that $\beta \in (0, 1)$ and r is positive number. From (2.28) and using the inequality as in Lemma 2.10, we obtain

$$\|f^m - f\|_{L^2(\Omega)} \ge \frac{\sqrt{(\frac{m\pi}{2})^\alpha}}{\|h\|_{C[0,T]}}. \tag{2.29}$$

This leads to

$$\lim_{m\to+\infty} \|f^m - f\|_{L^2(\Omega)} > \lim_{m\to+\infty} \frac{\sqrt{(\frac{m\pi}{2})^\alpha}}{\|h\|_{C[0,T]}} = +\infty. \tag{2.30}$$

Combining (2.27) with (2.30), we conclude that the inverse source problem is ill-posed. □

Theorem 2.14 (A conditional stability estimate)**.** *Assume that there exists* $\gamma > 0$ *such that* $\|f\|_{H^{\alpha\gamma}(\Omega)} \le M$ *for* $M > 0$*. Then*

$$\|f\|_{L^2(\Omega)} \le \mathcal{K}_{\alpha,\beta}(h, r, T) M^{\frac{1}{\gamma+1}} \|g\|_{L^2(\Omega)}^{\frac{\gamma}{\gamma+1}}. \tag{2.31}$$

where

$$\mathcal{K}_{\alpha,\beta}(h, r, T) = \frac{(r^\beta)^{\frac{\gamma}{\gamma+1}}}{|\mathcal{I}(h)|^{\frac{\gamma}{\gamma+1}}\ big(1 - E_{\alpha,1}(-\lambda_1^\alpha r^\beta T^\beta)\big)^{\frac{\gamma}{\gamma+1}}}. \tag{2.32}$$

Proof. According (2.8), by Hölder's inequality, we have

$$\begin{aligned}
\|f\|^2_{L^2(\Omega)} &= \sum_{k=1}^{\infty}\Big|\frac{\langle g(x),\phi_k(x)\rangle\phi_k(x)}{\int_0^T \Phi_\beta(\lambda_k^\alpha,\tau,r)h(\tau)d\tau}\Big|^2 \\
&\le \sum_{k=1}^{\infty}\frac{|\langle g(x),\phi_k(x)\rangle|^{\frac{2}{\gamma+1}}|\langle g(x),\phi_k(x)\rangle|^{\frac{2\gamma}{\gamma+1}}}{\Big|\int_0^T \Phi_\beta(\lambda_k^\alpha,\tau,r)h(\tau)d\tau\Big|^2} \\
&\le \Big(\sum_{k=1}^{\infty}\frac{|\langle g(x),\phi_k(x)\rangle|^2}{\big|\int_0^T \Phi_\beta(\lambda_k^\alpha,\tau,r)h(\tau)d\tau\big|^{2(\gamma+1)}}\Big)^{\frac{1}{\gamma+1}} \\
&\quad\times\Big(\sum_{k=1}^{\infty}|\langle g(x),\phi_k(x)\rangle|^2\Big)^{\frac{\gamma}{\gamma+1}} \\
&\le \Big(\sum_{k=1}^{\infty}\frac{|\langle f(x),\phi_k(x)\rangle|^2}{\big|\int_0^T \Phi_\beta(\lambda_k^\alpha,\tau,r)h(\tau)d\tau\big|^{2\gamma}}\Big)^{\frac{1}{\gamma+1}}\|g\|^{\frac{2\gamma}{\gamma+1}}_{L^2(\Omega)}.
\end{aligned} \tag{2.33}$$

Here we have used the fact that

$$\langle g(x),\phi_k(x)\rangle = \langle f(x),\phi_k(x)\rangle\Big|\int_0^T \Phi_\beta(\lambda_k^\alpha,\tau,r)h(\tau)d\tau\Big|^2.$$

Using Lemma 2.10, we have

$$\Big|\int_0^T \Phi_\beta(\lambda_k^\alpha,\tau,r)h(\tau)d\tau\Big|^{2\gamma} \ge \frac{|\mathcal{I}(h)|^{2\gamma}}{\lambda_k^{2\alpha\gamma}r^{2\beta\gamma}}\big(1-E_{\beta,1}(-\lambda_1^\alpha r^\beta T^\beta)\big)^{2\gamma} \tag{2.34}$$

and this inequality leads to

$$\begin{aligned}
&\sum_{k=1}^{\infty}\frac{|\langle f(x),\phi_k(x)\rangle|^2}{\big|\int_0^T \Phi_\beta(\lambda_k^\alpha,\tau,r)h(\tau)d\tau\big|^{2\gamma}} \\
&\le \sum_{k=1}^{\infty}\frac{r^{2\beta\gamma}\lambda_k^{2\alpha\gamma}|\langle f(x),\phi_k(x)\rangle|^2}{|\mathcal{I}(h)|^{2\gamma}\big(1-E_{\beta,1}(-\lambda_1^\alpha r^\beta T^\beta)\big)^{2\gamma}} \\
&\le \frac{r^{2\beta\gamma}}{|\mathcal{I}(h)|^{2\gamma}\big(1-E_{\beta,1}(-\lambda_1^\alpha r^\beta T^\beta)\big)^{2\gamma}}\sum_{k=1}^{\infty}\lambda_k^{2\alpha\gamma}|\langle f(x),\phi_k(x)\rangle|^2 \\
&= \frac{r^{2\beta\gamma}\|f\|^2_{H^{\alpha\gamma}(\Omega)}}{|\mathcal{I}(h)|^{2\gamma}\big(1-E_{\beta,1}(-\lambda_1^\alpha r^\beta T^\beta)\big)^{2\gamma}}.
\end{aligned} \tag{2.35}$$

Combining (2.33) with (2.35), we obtain

$$\begin{aligned}
\|f\|^2_{L^2(\Omega)} &\le \frac{r^{\frac{2\beta\gamma}{\gamma+1}}\|f\|^{\frac{2}{\gamma+1}}_{H^{\alpha\gamma}(\Omega)}}{|\mathcal{I}(h)|^{\frac{2\gamma}{\gamma+1}}\big(1-E_{\beta,1}(-\lambda_1^\alpha r^\beta T^\beta)\big)^{\frac{2\gamma}{\gamma+1}}}\|g\|^{\frac{2\gamma}{\gamma+1}}_{L^2(\Omega)} \\
&\le |\mathcal{K}_{\alpha,\beta}(h,r,T)|^2 M^{\frac{2}{\gamma+1}}\|g\|^{\frac{2\gamma}{\gamma+1}}_{L^2(\Omega)}.
\end{aligned} \tag{2.36}$$

□

3. Fourier truncation regularization and error estimate

In this section, we eliminate all the components of large k from the solution and define the truncation regularized solution as follows:

$$f^{\epsilon,N}(x) = \sum_{k=1}^{N} \frac{\langle g^\epsilon(x), \phi_k(x)\rangle \phi_k(x)}{\int_0^T \Phi_\beta(\lambda_k^\alpha, \tau, r) h^\epsilon(\tau) d\tau} \tag{3.1}$$

where the positive integer N plays the role of regularization parameter. Next, we consider an a *a-priori* and an a *a-posteriori* choice to find the regularization parameter. Under each choice of the regularization parameter, the error estimates between the exact solution f given by (2.7) and the regularized approximation solution $f^{\epsilon,N}$ given by (3.1) can be obtained.

3.1. An a priori parameter choice.

Afterwards, we will give an error estimation for $\|f(x) - f^{\epsilon,N}(x)\|_{L^2(\Omega)}$ and show convergence rate under a suitable choice for the regularization parameter.

Theorem 3.1. *Let $f^{\epsilon,N}$ be the regularized solution for problem* (1.1) *with noisy data g^ϵ and $f(x)$ be the exact solution for problem* (1.1)*. Let us choose parameter regularization $N = [\mu]$, where $[\mu]$ denotes the largest integer less than or equal to μ. Then we have the following:*

- *If $0 < \gamma \le 1$ then choose $\mu = \frac{2}{\pi}\left(\frac{M}{\epsilon}\right)^{\frac{1}{\alpha(\gamma+1)}}$, we have the estimate*

$$\|f(x) - f^{\epsilon,N}(x)\|_{L^2(\Omega)} \le \varepsilon^{\frac{\gamma}{\gamma+1}} M^{\frac{1}{\gamma+1}} \mathcal{D}_{\alpha,\beta}(f, h, h^\varepsilon, r, T). \tag{3.2}$$

- *If $\gamma > 1$, choose $\mu = \frac{2}{\pi}\left(\frac{M}{\epsilon}\right)^{\frac{1}{2\alpha}}$, we obtain the error estimate*

$$\|f(x) - f^{\epsilon,N}(x)\|_{L^2(\Omega)} \le \varepsilon^{\frac{1}{2}} M^{\frac{1}{2}} \mathcal{D}_{\alpha,\beta}(f, h, h^\varepsilon, r, T), \tag{3.3}$$

where

$$\mathcal{D}_{\alpha,\beta}(f, h, h^\varepsilon, r, T) = \Big[1 + \max\Big\{\frac{r^\beta}{|\mathcal{I}(h)|\big(1 - E_{\beta,1}(-\lambda_1^\alpha r^\beta T^\beta)\big)}, \frac{\|f\|_{L^2(\Omega)}}{|\mathcal{I}(h^\varepsilon)|}\Big\}\Big]. \tag{3.4}$$

Remark 3.2. If the function h depends on x and t, i.e. $h = h(t, x)$ then we can not represent f as the Fourier series as (2.8). Hence, we can not use some usual regularization methods. The regularized problem is open and difficult when h depends on x and t.

Proof of Theorem 3.1. Using (2.7) and (3.1) and the triangle inequality, we have

$$\begin{aligned} f(x) - f^{\epsilon,N}(x) &= \sum_{k=1}^{\infty} \frac{\langle g(x), \phi_k(x)\rangle \phi_k(x)}{\int_0^T \Phi_\beta(\lambda_k^\alpha, \tau, r) h(\tau) d\tau} - \sum_{k=1}^{N} \frac{\langle g^\epsilon(x), \phi_k(x)\rangle \phi_k(x)}{\int_0^T \Phi_\beta(\lambda_k^\alpha, \tau, r) h^\epsilon(\tau) d\tau} \\ &= \sum_{k=1}^{\infty} \frac{\langle g(x), \phi_k(x)\rangle \phi_k(x)}{\int_0^T \Phi_\beta(\lambda_k^\alpha, \tau, r) h(\tau) d\tau} - \sum_{k=1}^{N} \frac{\langle g(x), \phi_k(x)\rangle \phi_k(x)}{\int_0^T \Phi_\beta(\lambda_k^\alpha, \tau, r) h(\tau) d\tau} \\ &\quad + \sum_{k=1}^{N} \frac{\langle g(x), \phi_k(x)\rangle \phi_k(x)}{\int_0^T \Phi_\beta(\lambda_k^\alpha, \tau, r) h(\tau) d\tau} - \sum_{k=1}^{N} \frac{\Big\langle g^\epsilon(x), \phi_k(x)\Big\rangle \phi_k(x)}{\int_0^T \Phi_\beta(\lambda_k^\alpha, \tau, r) h^\epsilon(\tau) d\tau}. \end{aligned} \tag{3.5}$$

Hence

$$
\begin{aligned}
&f(x) - f^{\epsilon,N}(x)\\
&= \underbrace{\sum_{k=N+1}^{\infty} \frac{\langle g(x), \phi_k(x)\rangle \phi_k(x)}{\int_0^T \Phi_\beta(\lambda_k^\alpha, \tau, r) h(\tau) d\tau}}_{\mathcal{Q}_1} + \underbrace{\sum_{k=1}^{N} \frac{\langle g(x) - g^\epsilon(x), \phi_k(x)\rangle \phi_k(x)}{\int_0^T \Phi_\beta(\lambda_k^\alpha, \tau, r) h^\epsilon(\tau) d\tau}}_{\mathcal{Q}_2} \\
&\quad + \underbrace{\sum_{k=1}^{N} \frac{\langle g(x), \phi_k(x)\rangle \phi_k(x)}{\int_0^T \Phi_\beta(\lambda_k^\alpha, \tau, r) h(\tau) d\tau} \times \sum_{k=1}^{N} \frac{\int_0^T \Phi_\beta(\lambda_k^\alpha, \tau, r)(h^\epsilon(\tau) - h(\tau)) d\tau}{\int_0^T \Phi_\beta(\lambda_k^\alpha, \tau, r) h^\epsilon(\tau) d\tau}}_{\mathcal{Q}_3}.
\end{aligned}
\tag{3.6}
$$

First, we have the following estimate

$$
\begin{aligned}
\|\mathcal{Q}_1\|^2_{L^2(\Omega)} &= \sum_{k=N+1}^{\infty} \frac{|\langle g(x), \phi_k(x)\rangle|^2}{\left|\int_0^T \Phi_\beta(\lambda_k^\alpha, \tau, r) h(\tau) d\tau\right|^2} \\
&= |\langle f(x), \phi_k(x)\rangle|^2 \\
&\le \sum_{k=N+1}^{\infty} (1+\lambda_k)^{-2\alpha\gamma} (1+\lambda_k)^{2\alpha\gamma} |\langle f(x), \phi_k(x)\rangle|^2 \\
&\le (1+\lambda_N)^{-2\alpha\gamma} M^2.
\end{aligned}
\tag{3.7}
$$

Hence, we obtain

$$
\|\mathcal{Q}_1\|_{L^2(\Omega)} \le (1+\lambda_N)^{-\alpha\gamma} M. \tag{3.8}
$$

Second, the term $\|\mathcal{Q}_2\|_{L^2(\Omega)}$ is bounded by

$$
\begin{aligned}
\|\mathcal{Q}_2\|^2_{L^2(\Omega)} &\le \sum_{k=1}^{N} \frac{|< g(x) - g^\epsilon(x), \phi_k(x) >|^2}{\left|\int_0^T \Phi_\beta(\lambda_k^\alpha, \tau, r) h^\epsilon(\tau) d\tau\right|^2} \\
&\le \sum_{k=1}^{N} \frac{\lambda_k^{2\alpha} r^{2\beta} |< g(x) - g^\varepsilon(x), \phi_k(x) >|^2}{|\mathcal{I}(h^\varepsilon)|^2 \big(1 - E_{\beta,1}(-\lambda_1^\alpha r^\beta T^\beta)\big)^2} \\
&\le \sup_{1\le k\le N} \frac{\lambda_k^{2\alpha} r^{2\beta}}{|\mathcal{I}(h^\varepsilon)|^2 \big(1 - E_{\beta,1}(-\lambda_1^\alpha r^\beta T^\beta)\big)^2} \\
&\quad \times \sum_{k=1}^{N} |\langle g(x) - g^\epsilon(x), \phi_k(x)\rangle|^2 \\
&\le \frac{\lambda_N^{2\alpha} r^{2\beta}}{|\mathcal{I}(h^\varepsilon)|^2 \big(1 - E_{\beta,1}(-\lambda_1^2 r^\beta T^\beta)\big)^2} \|g^\varepsilon - g\|^2_{L^2(\Omega)} \\
&\le \frac{\lambda_N^{2\alpha} r^{2\beta}}{|\mathcal{I}(h^\varepsilon)|^2 \big(1 - E_{\beta,1}(-\lambda_1^2 r^\beta T^\beta)\big)^2} \epsilon^2.
\end{aligned}
\tag{3.9}
$$

Hence

$$
\|\mathcal{Q}_2\|_{L^2(\Omega)} \le \frac{\lambda_N^{\alpha} r^{\beta}}{|\mathcal{I}(h^\varepsilon)| \big(1 - E_{\beta,1}(-\lambda_1^2 r^\beta T^\beta)\big)} \epsilon. \tag{3.10}
$$

Finally, the term $\|\mathcal{Q}_3\|_{L^2(\Omega)}$ can be estimated as follows

$$
\|\mathcal{Q}_3\|^2_{L^2(\Omega)}
$$

$$
\begin{aligned}
&\le \Big[\sum_{k=1}^{N}\Big|\frac{\langle g(x),\phi_k(x)\rangle\phi_k(x)}{\int_0^T\Phi_\beta(\lambda_k^\alpha,\tau,r)h(\tau)d\tau}\Big|^2\Big]\Big[\sum_{k=1}^{N}\Big|\frac{\int_0^T\Phi_\beta(\lambda_k^\alpha,\tau,r)(h^\epsilon(\tau)-h(\tau))d\tau}{\int_0^T\Phi_\beta(\lambda_k^\alpha,\tau,r)h^\epsilon(\tau)d\tau}\Big|^2\Big]\\
&\le \Big[\sum_{k=1}^{N}\frac{\big|\int_0^T\Phi_\beta(\lambda_k^\alpha,\tau,r)(h^\varepsilon(\tau)-h(\tau))d\tau\big|^2}{\big|\int_0^T\Phi_\beta(\lambda_k^\alpha,\tau,r)h^\varepsilon(\tau)d\tau\big|^2}\Big]\Big[\sum_{k=1}^{N}\frac{|\langle g(x),\phi_k(x)\rangle|^2}{\big|\int_0^T\Phi_\beta(\lambda_k^\alpha,\tau,r)h(\tau)d\tau\big|^2}\Big]\\
&\le \frac{\|h^\varepsilon-h\|_{C[0,T]}^2}{|\mathcal{I}(h^\varepsilon)|^2}\sum_{k=1}^{\infty}\frac{|\langle g(x),\phi_k(x)\rangle|^2}{\big|\int_0^T\Phi_\beta(\lambda_k^\alpha,\tau,r)h(\tau)d\tau\big|^2} \qquad (3.11)\\
&= \frac{\|h^\varepsilon-h\|_{C[0,T]}^2}{|\mathcal{I}(h^\varepsilon)|^2}\|f\|_{L^2(\Omega)}^2\\
&\le \frac{\varepsilon^2\|f\|_{L^2(\Omega)}^2}{|\mathcal{I}(h^\varepsilon)|^2}.
\end{aligned}
$$

Hence

$$
\|\mathcal{Q}_3\|_{L^2(\Omega)} \le \frac{\varepsilon\|f\|_{L^2(\Omega)}}{|\mathcal{I}(h^\varepsilon)|}. \qquad (3.12)
$$

Combining (3.7), (3.9) and (3.11), it yields

$$
\begin{aligned}
&\|f(x)-f^{\epsilon,N}(x)\|_{L^2(\Omega)}\\
&\le (1+\lambda_N)^{-\alpha\gamma}M+\varepsilon\frac{\|f\|_{L^2(\Omega)}}{|\mathcal{I}(h^\varepsilon)|}+\varepsilon\frac{\lambda_N^\alpha r^\beta}{|\mathcal{I}(h^\varepsilon)|\big(1-E_{\beta,1}(-\lambda_1^2 r^\beta T^\beta)\big)}.
\end{aligned} \qquad (3.13)
$$

This and the fact that $N\le\mu\le N+1$ give

$$
\begin{aligned}
&\|f(x)-f^{\epsilon,N}(x)\|_{L^2(\Omega)}\\
&\le \Big(\frac{\mu\pi}{2}\Big)^{-\gamma\alpha}M+\varepsilon\Big(\frac{\mu\pi}{2}\Big)^{\alpha}\max\Big\{\frac{r^\beta}{|\mathcal{I}(h^\varepsilon)|\big(1-E_{\beta,1}(-\lambda_1^\alpha r^\beta T^\beta)\big)},\frac{\|f\|_{L^2(\Omega)}}{|\mathcal{I}(h^\varepsilon)|}\Big\}\\
&\le \varepsilon^{\frac{\gamma}{\gamma+1}}M^{\frac{1}{\gamma+1}}\Big[1+\max\ \Big\{\frac{r^\beta}{|\mathcal{I}(h^\varepsilon)|\big(1-E_{\beta,1}(-\lambda_1^\alpha r^\beta T^\beta)\big)},\frac{\|f\|_{L^2(\Omega)}}{|\mathcal{I}(h^\varepsilon)|}\Big\}\Big].
\end{aligned}
$$

□

3.2. **An a posteriori parameter choice.** In this subsection, we consider an a posteriori regularization parameter choice by the discrepancy principle. Define

$$
F_N g^\epsilon=\sum_{k=1}^{N}\langle g(x),\phi_k(x)\rangle\phi_k(x). \qquad (3.14)
$$

By the discrepancy principle, we take $K=K(\varepsilon,g^\varepsilon)$ as the solution of

$$
\|(I-F_N)g^\varepsilon\|_{L^2(\Omega)}\le m\epsilon\le\|(I-F_{N-1})g^\varepsilon\|_{L^2(\Omega)},\quad m>1. \qquad (3.15)
$$

Lemma 3.3. *We have*

$$
N\le\frac{2}{\pi}\Big(\frac{\|h\|_{C[0,T]}M}{r^\beta(m-1)\epsilon}\Big)^{\frac{1}{\alpha(\gamma+1)}}. \qquad (3.16)
$$

Proof. From $\|g^\varepsilon-g\|_{L^2(\Omega)}\le\varepsilon$ and (3.15), we have

$$
\begin{aligned}
\|F_{N-1}g-g\|_{L^2(\Omega)}&=\|(F_{N-1}-I)g^\varepsilon-(I-F_{N-1})(g-g^\varepsilon)\|_{L^2(\Omega)}\\
&\ge\|(F_{N-1}-I)g^\varepsilon\|_{L^2(\Omega)}-\|(I-F_{N-1})(g-g^\varepsilon)\|_{L^2(\Omega)}\\
&\ge(m-1)\varepsilon.
\end{aligned} \qquad (3.17)
$$

On the other hand, for $k \geq N$, we obtain

$$\begin{aligned}
\Big|\int_0^T \Phi_\beta(\lambda_k^\alpha,\tau,r)h(\tau)d\tau\Big| &\leq \|h\|_{C[0,T]}\int_0^T \Phi_\beta(\lambda_k^\alpha,\tau,r)d\tau \\
&= \|h\|_{C[0,T]}\frac{\big(1-E_{\beta,1}(-\lambda_k^\alpha r^\beta T^\beta)\big)}{\lambda_k^\alpha r^\beta} \\
&\leq \frac{\|h\|_{C[0,T]}}{\lambda_N^\alpha r^\beta}.
\end{aligned} \tag{3.18}$$

This implies

$$\begin{aligned}
&\|F_{N-1}g-g\|^2_{L^2(\Omega)} \\
&= \sum_{k=N}^\infty |\langle g(x),\phi_k(x)\rangle|^2 \\
&= \sum_{k=N}^\infty \Big|\int_0^T \Phi_\beta(\lambda_k^\alpha,\tau,r)h(\tau)d\tau\langle f(x),\phi_k(x)\rangle\Big|^2 \\
&\leq \frac{\|h\|^2_{C[0,T]}}{\lambda_N^{2\alpha}r^{2\beta}}\sum_{k=N}^\infty |\langle f(x),\phi_k(x)\rangle|^2 \\
&\leq \frac{\|h\|^2_{C[0,T]}}{\lambda_N^{2\alpha}r^{2\beta}}\sum_{k=N}^\infty (1+\lambda_k)^{-2\alpha\gamma}(1+\lambda_k)^{2\alpha\gamma}|\langle f(x),\phi_k(x)\rangle|^2 \\
&\leq \frac{\|h\|^2_{C[0,T]}}{\lambda_N^{2\alpha}r^{2\beta}\lambda_N^{2\alpha\gamma}}\sum_{k=N}^\infty (1+\lambda_k)^{2\alpha\gamma}|\langle f(x),\phi_k(x)\rangle|^2 \\
&\leq \frac{\|h\|^2_{C[0,T]}}{\lambda_N^{2\alpha}r^{2\beta}\lambda_N^{2\alpha\gamma}}\|f\|^2_{H^{\alpha\gamma}(\Omega)} \\
&\leq \|h\|^2_{C[0,T]}\frac{M^2}{r^{2\beta}}\frac{1}{\lambda_N^{2\alpha(\gamma+1)}}.
\end{aligned} \tag{3.19}$$

Hence,

$$\|F_{N-1}g-g\|_{L^2(\Omega)} \leq \frac{M}{r^\beta}\frac{\|h\|_{C[0,T]}}{\lambda_N^{\alpha(\gamma+1)}}. \tag{3.20}$$

From (3.17) and (3.20), we have

$$(m-1)\epsilon \leq \frac{M}{r^\beta}\frac{\|h\|_{C[0,T]}}{\lambda_N^{\alpha(\gamma+1)}}. \tag{3.21}$$

It follows from $\lambda_k^\alpha = (\frac{k\pi}{2})^\alpha$ and (3.21) that

$$N \leq \frac{2}{\pi}\Big(\frac{\|h\|_{C[0,T]}M}{r^\beta(m-1)\epsilon}\Big)^{\frac{1}{\alpha(\gamma+1)}}. \tag{3.22}$$

□

Next we present an error estimate for the approximate solution of problem (1.1).

Theorem 3.4. *Let $f^{\epsilon,N}$ and f be as in Theorem 3.1. Then we have*

$$\begin{aligned}&\|f(x)-f^{\varepsilon,N}(x)\|_{L^2(\Omega)}\\&\le \varepsilon^{\frac{\gamma}{\gamma+1}}M^{\frac{1}{\gamma+1}}\Big[\mathcal{L}_\beta(f,h^\varepsilon,h,r,m,T)+\mathcal{K}_{\alpha,\beta}(h,r,T)(m+1)^{\frac{\gamma}{\gamma+1}}\Big].\end{aligned}\tag{3.23}$$

where

$$\begin{gathered}\mathcal{L}_\beta(f,h^\varepsilon,h,r,m,T)\\=\Big(\frac{\|h\|_{C[0,T]}}{r^\beta(m-1)|\mathcal{I}(h^\varepsilon)|^{\gamma+1}}\Big)^{\frac{1}{\gamma+1}}\max\Big\{\|f\|_{L^2(\Omega)},\frac{r^\beta}{\big(1-E_{\beta,1}(-\lambda_1^\alpha r^\beta T^\beta)\big)}\Big\},\\ \mathcal{K}_{\alpha,\beta}(h,r,T)=\frac{(r^\beta)^{\frac{\gamma}{\gamma+1}}}{|\mathcal{I}(h)|^{\frac{\gamma}{\gamma+1}}\big(1-E_{\alpha,1}(-\lambda_1^\alpha r^\beta T^\beta)\big)^{\frac{\gamma}{\gamma+1}}}.\end{gathered}$$

Proof. Using the triangle inequality,

$$\|f(x)-f^{\varepsilon,N}(x)\|_{L^2(\Omega)}\le\|f(x)-f^N(x)\|_{L^2(\Omega)}+\|f(x)-f^{\epsilon,N}(x)\|_{L^2(\Omega)}.\tag{3.24}$$

We split the proof into three steps.

Step 1: Estimate $\|f(\cdot)-f^N(\cdot)\|_{L^2(\Omega)}$.

$$\begin{aligned}\|f(x)-f^N(x)\|_{H^\gamma(\Omega)}&\le\|\sum_{k=N+1}^\infty\langle f(x),\phi_k(x)\rangle\phi_k(x)\|\\&=\Big(\sum_{k=N+1}^\infty\big(1+\lambda_k^\alpha\big)^{2\gamma}|\langle f(x),\phi_k(x)\rangle|^2\Big)^{1/2}\le M.\end{aligned}\tag{3.25}$$

By triangle inequality and (3.15),

$$\begin{aligned}\|Af(x)-Af^N(x)\|_{L^2(\Omega)}&\le\|(I-F_N)g\|\\&\le\|(I-F_N)g^\epsilon+(I-F_N)(g-g^\epsilon)\|\\&\le\|(I-F_N)g^\epsilon\|+\|(I-F_N)(g-g^\varepsilon)\|\\&\le(m+1)\varepsilon.\end{aligned}\tag{3.26}$$

Therefore, by the conditional stability (2.31), we have

$$\|f(\cdot)-f^N(\cdot)\|_{L^2(\Omega)}\le\mathcal{K}_{\alpha,\beta}(h,r,T)((m+1)\varepsilon)^{\frac{\gamma}{\gamma+1}}.\tag{3.27}$$

Next, we obtain

$$
\begin{aligned}
&f^N(x)-f^{\epsilon,N}(x)\\
&=\sum_{k=1}^N \frac{\langle g(x),\phi_k(x)\rangle\phi_k(x)}{\int_0^T \Phi_\beta(\lambda_k^\alpha,\tau,r)h(\tau)d\tau}-\sum_{k=1}^N \frac{\langle g^\varepsilon(x),\phi_k(x)\rangle\phi_k(x)}{\int_0^T \Phi_\beta(\lambda_k^\alpha,\tau,r)h^\varepsilon(\tau)d\tau}\\
&\le \sum_{k=1}^N \frac{\langle g(x),\phi_k(x)\rangle\phi_k(x)}{\int_0^T \Phi_\beta(\lambda_k^\alpha,\tau,r)h(\tau)d\tau}-\sum_{k=1}^N \frac{\langle g(x),\phi_k(x)\rangle\phi_k(x)}{\int_0^T \Phi_\beta(\lambda_k^\alpha,\tau,r)h^\varepsilon(\tau)d\tau}\\
&\quad+\sum_{k=1}^N \frac{\langle g(x)-g^\varepsilon(x),\phi_k(x)\rangle\phi_k(x)}{\int_0^T \Phi_\beta(\lambda_k^\alpha,\tau,r)h^\varepsilon(\tau)d\tau}\\
&\le \underbrace{\sum_{k=1}^N \frac{\int_0^T \Phi_\beta(\lambda_k^\alpha,\tau,r)(h(\tau)-h^\varepsilon(\tau))d\tau}{\int_0^T \Phi_\beta(\lambda_k^\alpha,\tau,r)h^\varepsilon(\tau)d\tau}\sum_{k=1}^N \frac{\langle g(x),\phi_k(x)\rangle\phi_k(x)}{\int_0^T \Phi_\beta(\lambda_k^\alpha,\tau,r)h(\tau)d\tau}}_{:=\mathcal{Q}_3}\\
&\quad+\underbrace{\sum_{k=1}^N \frac{\langle g(x)-g^\varepsilon(x),\phi_k(x)\rangle\phi_k(x)}{\int_0^T \Phi_\beta(\lambda_k^\alpha,\tau,r)h^\varepsilon(\tau)d\tau}}_{:=\mathcal{Q}_4}.
\end{aligned}
\tag{3.28}
$$

Using (3.12), we obtain

$$
\|\mathcal{Q}_3\|_{L^2(\Omega)} \le \varepsilon\frac{\|f\|_{L^2(\Omega)}}{|\mathcal{I}(h^\varepsilon)|}. \tag{3.29}
$$

We now estimate the norm of $\mathcal{Q}_4$. Using Lemma 2.8, we have

$$
\begin{aligned}
\|\mathcal{Q}_4\|^2_{L^2(\Omega)} &= \sum_{k=1}^N \Big|\frac{\langle g(x)-g^\varepsilon(x),\phi_k(x)\rangle}{\int_0^T \Phi_\beta(\lambda_k^\alpha,\tau,r)h^\varepsilon(\tau)d\tau}\Big|^2\\
&\le \sum_{k=1}^N \frac{\langle g(x)-g^\varepsilon(x),\phi_k(x)\rangle^2}{\frac{|\mathcal{I}(h_\varepsilon)|^2\big(1-E_{\beta,1}(-\lambda_1^\alpha r^\beta T^\beta)\big)^2}{\lambda_k^{2\alpha}r^{2\beta}}}\\
&\le \frac{\lambda_N^{2\alpha}r^{2\beta}}{|\mathcal{I}(h_\varepsilon)|^2\big(1-E_{\beta,1}(-\lambda_1^\alpha r^\beta T^\beta)\big)^2}\sum_{k=1}^N \langle g(x)-g^\varepsilon(x),\phi_k(x)\rangle^2\\
&\le \frac{\varepsilon^2\lambda_N^{2\alpha}r^{2\beta}}{|\mathcal{I}(h_\varepsilon)|^2\big(1-E_{\beta,1}(-\lambda_1^\alpha r^\beta T^\beta)\big)^2}.
\end{aligned}
\tag{3.30}
$$

Hence

$$
\|\mathcal{Q}_4\|_{L^2(\Omega)} \le \Big(\frac{N\pi}{2}\Big)^\alpha \frac{\epsilon}{|\mathcal{I}(h^\varepsilon)|}\frac{r^\beta}{\big(1-E_{\beta,1}(-\lambda_1^\alpha r^\beta T^\beta)\big)}. \tag{3.31}
$$

From above observations, we deduce that

$$
\begin{aligned}
&\|f^N(x)-f^{\epsilon,N}(x)\|_{L^2(\Omega)}\\
&\le \Big(\frac{N\pi}{2}\Big)^\alpha \frac{\varepsilon}{|\mathcal{I}(h^\varepsilon)|}\max\Big\{\|f\|_{L^2(\Omega)},\frac{r^\beta}{\big(1-E_{\beta,1}(-\lambda_1^\alpha r^\beta T^\beta)\big)}\Big\}.
\end{aligned}
\tag{3.32}
$$

Substituting (3.22) in (3.32), we obtain

$$
\|f^N(x)-f^{\epsilon,N}(x)\|_{L^2(\Omega)} \le \varepsilon^{\frac{\gamma}{\gamma+1}}M^{\frac{1}{\gamma+1}}\mathcal{L}_\beta(f,h^\varepsilon,h,r,m,T). \tag{3.33}
$$

Combining (3.27) with (3.32), we obtain the final estimate as follows:

$$\begin{aligned} &\|f(x) - f^{\epsilon,N}(x)\|_{L^2(\Omega)} \\ &\le \varepsilon^{\frac{\gamma}{\gamma+1}} M^{\frac{1}{\gamma+1}} \big[\mathcal{L}_\beta(f, h^\varepsilon, h, r, m, T) + \mathcal{K}_{\alpha,\beta}(h, r, T)(m+1)^{\frac{\gamma}{\gamma+1}}\big]. \end{aligned} \tag{3.34}$$

hereby

$$\begin{aligned} &\mathcal{L}_\beta(f, h^\varepsilon, h, r, m, T) \\ &= \Big(\frac{\|h\|_{C[0,T]}}{r^\beta(m-1)|\mathcal{I}(h^\varepsilon)|^{\gamma+1}}\Big)^{\frac{1}{\gamma+1}} \max\Big\{\|f\|_{L^2(\Omega)}, \frac{r^\beta}{\big(1 - E_{\beta,1}(-\lambda_1^\alpha r^\beta T^\beta)\big)}\Big\}, \end{aligned}$$

$$\mathcal{K}_{\alpha,\beta}(h, r, T) = \frac{(r^\beta)^{\frac{\gamma}{\gamma+1}}}{|\mathcal{I}(h)|^{\frac{\gamma}{\gamma+1}}\big(1 - E_{\alpha,1}(-\lambda_1^\alpha r^\beta T^\beta)\big)^{\frac{\gamma}{\gamma+1}}}.$$

This completes the proof. □

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