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PROPERTIES OF THE LINEAR MULTIPLIER OPERATOR FOR THE WEINSTEIN TRANSFORM AND APPLICATIONS

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Abstract. In this article, we use the theory of reproducing kernels to study the Weinstein multiplier operators on Sobolev type spaces. Some applications are given and an associated Hörmander type theorem on L^p-boundedness is established.

1. Introduction

In this article, we consider the Weinstein operator $\Delta_W^{\alpha,d}$ defined on $\mathbb{R}_+^{d+1} = \mathbb{R}^d \times]0,+\infty[$, by

$$\Delta_W^{\alpha,d} = \sum_{i=1}^{d+1} \frac{\partial^2}{\partial x_i^2} + \frac{2\alpha+1}{x_{d+1}} \frac{\partial}{\partial x_{d+1}} = \Delta_d + L_\alpha, \quad \alpha > -\frac{1}{2}, \tag{1.1}$$

where Δ_d is the Laplacian for the d first variables and L_α is the Bessel operator for the last variable defined on $]0,+\infty[$ by

$$L_\alpha u = \frac{\partial^2 u}{\partial x_{d+1}^2} + \frac{2\alpha+1}{x_{d+1}} \frac{\partial u}{\partial x_{d+1}} = \frac{1}{x_{d+1}^{2\alpha+1}} \frac{\partial}{\partial x_{d+1}} \big[x_{d+1}^{2\alpha+1} \frac{\partial u}{\partial x_{d+1}}\big].$$

The Weinstein operator $\Delta_W^{\alpha,d}$, mostly referred to as the Laplace-Bessel differential operator is now known as an important operator in analysis, because of its applications in pure and applied Mathematics, especially in Fluid Mechanics [10]. The relevant harmonic analysis associated with the Bessel differential operator L_α goes back to Bochner, Delsarte, Levitan and has been studied by many other authors such as Löfström and Peetre [19], Kipriyanov [17], Stempak [28], Trimèche [29], Aliev and Rubin [1].

The Weinstein transform generalizing the usual Fourier transform, is given for $f \in L_\alpha^1(\mathbb{R}_+^{d+1})$ and $\lambda \in \mathbb{R}_+^{d+1}$, by

$$\mathcal{F}_W^{\alpha,d}(f)(\lambda) = \int_{\mathbb{R}_+^{d+1}} f(x)\Lambda_{\alpha,d}(x,\lambda)d\mu_{\alpha,d}(x),$$

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where

$$d\mu_{\beta,d}(x) = \frac{x_{d+1}^{2\beta+1}}{(2\pi)^{d/2}2^{\beta}\Gamma(\beta+1)}dx \text{ and } \Lambda_{\alpha,d} \text{ is given later by (2.2).}$$

In this article, we deal with the theory of multiplier operators in the Weinstein settings. For $s \in \mathbb{R}$, we consider the Sobolev type spaces $\mathcal{H}^s_{\alpha,\beta}$, when $\alpha \geq \beta > -1/2$, consisting of all $f \in S'_*(\mathbb{R}^{d+1})$, (the space of tempered distributions, even with respect to the last variable), such that $\mathcal{F}^{\alpha,d}_W(f)$ is a function and

$$(1+|z|^2)^{s/2}\mathcal{F}^{\alpha,d}_W(f) \in L^2(d\mu_{\beta,d}).$$

The space $\mathcal{H}^s_{\alpha,\beta}$ is an Hilbert space when endowed with the inner product

$$\langle f, g\rangle_{\mathcal{H}^s_{\alpha,\beta}} = \int_{\mathbb{R}^{d+1}_+} \mathcal{F}^{\alpha,d}_W(f)(z)\overline{\mathcal{F}^{\alpha,d}_W(g)(z)}d\mu^s_{\beta,d}(z),$$

and it is continuously embedded in $L^2(d\mu_{\alpha,d})$, when $s \geq \alpha - \beta$, where $d\mu^s_{\beta,d}(z) = (1+|z|^2)^s d\mu_{\beta,d}(z)$.

For $m \in L^\infty(d\mu_{\beta,d})$, we define the Weinstein multiplier operators $T_{\alpha,\beta,m}$, for $f \in \mathcal{H}^s_{\alpha,\beta}$, by

$$T_{\alpha,\beta,m}f(x) = (\mathcal{F}^{\beta,d}_W)^{-1}(m\mathcal{F}^{\alpha,d}_W(f))(x), \quad x \in \mathbb{R}^{d+1}_+.$$

These operators are a generalization of the usual linear multiplier operator T_m associated with a bounded function m and given by $T_m(f) = \mathcal{F}^{-1}(m\mathcal{F}(f))$, where $\mathcal{F}(f)$ denotes the ordinary Fourier transform on $\mathbb{R}^n$. These operators gained the interest of many Mathematicians and they were generalized in many settings, (see for instance [3, 9, 15, 14, 24]).

For $m \in L^\infty(d\mu_{\beta,d})$ the operator $T_{\alpha,\beta,m}$ is shown to be a bounded operator from $\mathcal{H}^0_{\alpha,\beta}$ onto $L^2(d\mu_{\beta,d})$, and for $f \in \mathcal{H}^0_{\alpha,\beta}$ we have

$$\|T_{\alpha,\beta,m}f\|_{L^2(d\mu_{\beta,d})} \leq \|m\|_{L^\infty(d\mu_{\beta,d})}\|f\|_{\mathcal{H}^0_{\alpha,\beta}}.$$

Furthermore, if f is ϵ-concentrated on E and $\mathcal{F}^{\alpha,d}_W(f)$ is ν-concentrated on S, where E and S are two measurable subsets on $\mathbb{R}^{d+1}_+$. Using Donoho-Stark uncertainty principle for the Weinstein transform, we obtain the following estimation

$$(\mu_{\alpha,d}(E))^{1/2}(\mu_{\beta,d}(S))^{1/2} \geq \frac{c_{\alpha,d}}{c_{\beta,d}}(1-\nu-\epsilon),$$

where $c_{\alpha,d}$ is the constant given by

$$c_{\alpha,d} = \frac{1}{(2\pi)^{d/2}2^{\alpha}\Gamma(\alpha+1)}. \tag{1.2}$$

As in [20, 26, 30], the theory of reproducing kernels is used to give the best approximation of the operator $T_{\alpha,\beta,m}$ on the Sobolev-Weinstein spaces $\mathcal{H}^s_{\alpha,\beta}$. More precisely, for all $\eta > 0$ and $g \in L^2(d\mu_{\beta,d})$, we show that there exists a unique function $f^*_{\eta g}$, where the infimum

$$\inf_{f\in\mathcal{H}^s_{\alpha,\beta}}\{\eta\|f\|^2_{\mathcal{H}^s_{\alpha,\beta}} + \|g - T_{\alpha,\beta,m}f\|^2_{L^2(\mu_{\beta,d})}\}$$

is attained.

The function $f^*_{\eta g}$ is called the extremal function and it is given by

$$f^*_{\eta g} = \langle g, T_{\alpha,\beta,m}(K_s(\cdot,y))\rangle_{L^2(d\mu_{\beta,d})},$$

where K_s is the reproducing kernel of the space $(\mathcal{H}^s_{\alpha,\beta}, \langle\cdot,\cdot\rangle_{\eta,\mathcal{H}^s_{\alpha,\beta}})$.

When $\alpha = \beta$, as in [2], we develop the original Hörmander's technique to establish an analogous of the well-known Hörmander theorem (see [16]), which gives a sufficient condition on m guaranteeing the boundedness of T_m on $L^p(\mathbb{R}^n)$, for $1 < p < \infty$.

This paper is organized as follows. In section 2, we recall some basic Harmonic Analysis results related with the Weinstein operator developed in [4, 5] and [6]. In the third section, the Weinstein multiplier operators are studied on the space $\mathcal{H}^s_{\alpha,\beta}$, for $\alpha \geq \beta > -1/2$. In the fourth section, the extremal function associated with the Weinstein operators is given using the theory of the reproducing kernel and we list some of its properties in Corollary 4.4 and Corollary 4.5. In the last section, we prove the Hörmander multiplier theorem for the operators $T_{\alpha,\beta,m}$, when $\alpha = \beta > -1/2$.

2. Harmonic analysis and the Weinstein-Laplace operator

In this section, we shall collect some results and definitions from the theory of the harmonic analysis associated with the Weinstein operator $\Delta^{\alpha,d}_W$ defined on $\mathbb{R}^{d+1}_+$ by the relation (1.1). Main references are [4, 5, 6, 7, 12, 13, 21, 22].

Let us begin by the following result, which gives the eigenfunction $\Psi^{\alpha,d}_\lambda$ of the Weinstein operator $\Delta^{\alpha,d}_W$.

Proposition 2.1. *For all $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_{d+1}) \in \mathbb{R}^{d+1}_+$, the system*

$$\begin{gathered} \frac{\partial^2 u}{\partial x_j^2}(x) = -\lambda_j^2 u(x), \quad \text{if } 1 \leq j \leq d \\ L_\alpha u(x) = -\lambda_{d+1}^2 u(x), \\ u(0) = 1, \quad \frac{\partial u}{\partial x_{d+1}}(0) = 0, \quad \frac{\partial u}{\partial x_j}(0) = -i\lambda_j, \quad \text{if } 1 \leq j \leq d \end{gathered} \tag{2.1}$$

has a unique solution $\Psi^{\alpha,d}_\lambda$ given by

$$\Psi^{\alpha,d}_\lambda(z) = e^{-i\langle z',\lambda'\rangle} j_\alpha(\lambda_{d+1} z_{d+1}), \quad \forall z \in \mathbb{C}^{d+1}, \tag{2.2}$$

where $z = (z', z_{d+1})$, $z' = (z_1, z_2, \dots, z_d)$ and j_α is the normalized Bessel function of index α, defined by

$$j_\alpha(\xi) = \Gamma(\alpha+1) \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+\alpha+1)} (\frac{\xi}{2})^{2n} \quad \forall \xi \in \mathbb{C}.$$

Remark 2.2. The Weinstein kernel $\Lambda_{\alpha,d} : (\lambda, z) \mapsto \Psi^{\alpha,d}_\lambda(z)$ has a unique extension to $\mathbb{C}^{d+1} \times \mathbb{C}^{d+1}$ and can be written in the form

$$\Lambda_{\alpha,d}(x,y) = a_\alpha e^{-i\langle x',y'\rangle} \int_0^1 (1-t^2)^{\alpha-\frac{1}{2}} cos(t x_{d+1} y_{d+1}) dt \quad \forall x, y \in \mathbb{C}^{d+1}, \tag{2.3}$$

where $x = (x', x_{d+1})$, $x' = (x_1, x_2, \dots, x_d)$ and a_α is the constant given by

$$a_\alpha = \frac{2\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}.$$

The following result summarizes some of the Weinstein kernel's properties.

Proposition 2.3. *(i) For all* $\lambda,\ z \in \mathbb{C}^{d+1}$ *and* $t \in \mathbb{R}$, *we have*

$$\Lambda_{\alpha,d}(\lambda,0)=1,\ \Lambda_{\alpha,d}(\lambda,z)=\Lambda_{\alpha,d}(z,\lambda),\quad \Lambda_{\alpha,d}(\lambda,tz)=\Lambda_{\alpha,d}(t\lambda,z).$$

(ii) For all $\nu \in \mathbb{N}^{d+1}$, $x \in \mathbb{R}_+^{d+1}$ *and* $z \in \mathbb{C}^{d+1}$, *we have*

$$|D_z^\nu \Lambda_{\alpha,d}(x,z)| \le \|x\|^{|\nu|}\exp(\|x\|\|\operatorname{Im} z\|), \tag{2.4}$$

where

$$D_z^\nu = \frac{\partial^\nu}{\partial z_1^{\nu_1}\dots\partial z_{d+1}^{\nu_{d+1}}}$$

and $|\nu| = \nu_1 + \dots + \nu_{d+1}$. *In particular*

$$|\Lambda_{\alpha,d}(x,y)| \le 1, \forall x,y \in \mathbb{R}_+^{d+1}. \tag{2.5}$$

In this article, we use the following notation:

- $C_*(\mathbb{R}^{d+1})$, the space of continuous functions on $\mathbb{R}^{d+1}$, even with respect to the last variable.
- $C_{*,c}(\mathbb{R}^{d+1})$, the space of continuous functions on $\mathbb{R}^{d+1}$ with compact support, even with respect to the last variable.
- $C_*^p(\mathbb{R}^{d+1})$, the space of functions of class C^p on $\mathbb{R}^{d+1}$, even with respect to the last variable.
- $\mathcal{E}_*(\mathbb{R}^{d+1})$, the space of C^∞-functions on $\mathbb{R}^{d+1}$, even with respect to the last variable.
- $\mathcal{S}_*(\mathbb{R}^{d+1})$, the Schwartz space of rapidly decreasing functions on $\mathbb{R}^{d+1}$, even with respect to the last variable.
- $\mathcal{D}_*(\mathbb{R}^{d+1})$, the space of C^∞-functions on $\mathbb{R}^{d+1}$ which are of compact support, even with respect to the last variable.
- $\mathcal{S'}_*(\mathbb{R}^{d+1})$, the space of temperate distributions on $\mathbb{R}^{d+1}$, even with respect to the last variable. It is the topological dual of $\mathcal{S}_*(\mathbb{R}^{d+1})$.
- $L^p(d\mu_{\alpha,d})$, $1 \le p \le +\infty$, the space of measurable functions on $\mathbb{R}_+^{d+1}$ such that

$$\|f\|_{L^\infty(d\mu_{\alpha,d})} = \operatorname{ess\,sup}_{x\in\mathbb{R}_+^{d+1}} |f(x)| < +\infty,$$

$$\|f\|_{L^p(d\mu_{\alpha,d})} = \Big[\int_{\mathbb{R}_+^{d+1}} |f(x)|^p d\mu_{\alpha,d}(x)\Big]^{1/p} < +\infty, \quad \text{if } 1 \le p < +\infty,$$

where $\mu_{\alpha,d}$ is the measure on $\mathbb{R}_+^{d+1}$ given by

$$d\mu_{\alpha,d}(x) = c_{\alpha,d}x_{d+1}^{2\alpha+1}dx, \tag{2.6}$$

dx is the Lebesgue measure on $\mathbb{R}^{d+1}$, and $c_{\alpha,d}$ is the constant given by relation (1.2)

- $\mathcal{H}_*(\mathbb{C}^{d+1})$, the space of entire functions on $\mathbb{C}^{d+1}$, even with respect to the last variable, rapidly decreasing and of exponential type.

Definition 2.4. The Weinstein transform for $f \in L_\alpha^1(\mathbb{R}_+^{d+1})$ is

$$\mathcal{F}_W^{\alpha,d}(f)(\lambda) = \int_{\mathbb{R}_+^{d+1}} f(x)\Lambda_{\alpha,d}(x,\lambda)d\mu_{\alpha,d}(x), \forall\lambda \in \mathbb{R}_+^{d+1}, \tag{2.7}$$

where $\mu_{\alpha,d}$ is the measure on $\mathbb{R}_+^{d+1}$ given by relation (2.6).

Some basic properties of the transform $\mathcal{F}_W^{\alpha,d}$ are summarized in the following results. For the proofs, we refer to [6, 7, 8].

Proposition 2.5. *(i) For all $f \in L^1(d\mu_{\alpha,d})$, we have*
$$\|\mathcal{F}_W^{\alpha,d}(f)\|_{L^\infty(d\mu_{\alpha,d})} \le \|f\|_{L^1(d\mu_{\alpha,d})}. \tag{2.8}$$
(ii) For $m \in \mathbb{N}$ and $f \in \mathcal{S}_*(\mathbb{R}^{d+1})$, we have
$$\mathcal{F}_W^{\alpha,d}\big[(\Delta_W^{\alpha,d})^m f\big](y) = (-1)^m |y|^{2m} \mathcal{F}_W^{\alpha,d}(f)(y), \quad \forall y \in \mathbb{R}_+^{d+1}. \tag{2.9}$$
(iii) For all f in $\mathcal{S}_*(\mathbb{R}^{d+1})$ and $m \in \mathbb{N}$, we have
$$\big(\Delta_W^{\alpha,d}\big)^m \big[\mathcal{F}_W^{\alpha,d}(f)\big](\lambda) = \mathcal{F}_W^{\alpha,d}(P_m f)(\lambda), \quad \forall \lambda \in \mathbb{R}_+^{d+1}, \tag{2.10}$$
where $P_m(\lambda) = (-1)^m \|\lambda\|^{2m}$.

Theorem 2.6. *(i) The Weinstein transform $\mathcal{F}_W^{\alpha,d}$ is a topological isomorphism from $\mathcal{S}_*(\mathbb{R}^{d+1})$ onto itself, from $\mathcal{D}_*(\mathbb{R}^{d+1})$ onto $\mathcal{H}_*(\mathbb{C}^{d+1})$ and from $\mathcal{S}'_*(\mathbb{R}^{d+1})$ onto itself.*

(ii) Let $f \in \mathcal{S}_*(\mathbb{R}^{d+1})$. The inverse transform $(\mathcal{F}_W^{\alpha,d})^{-1}$ is given by
$$\big(\mathcal{F}_W^{\alpha,d}\big)^{-1}(f)(x) = \mathcal{F}_W^{\alpha,d}(f)(-x), \quad \forall x \in \mathbb{R}_+^{d+1},. \tag{2.11}$$

(iii) Let $f \in L^1(d\mu_{\alpha,d})$. If $\mathcal{F}_W^{\alpha,d}(f) \in L^1(d\mu_{\alpha,d})$, then we have
$$f(x) = \int_{\mathbb{R}_+^{d+1}} \mathcal{F}_W^{\alpha,d}(f)(y)\Lambda_{\alpha,d}(-x,y)d\mu_{\alpha,d}(y), \quad a.e.\ x \in \mathbb{R}_+^{d+1}. \tag{2.12}$$

Theorem 2.7. *(i) For all $f, g \in \mathcal{S}_*(\mathbb{R}^{d+1})$, we have the Parseval formula*
$$\int_{\mathbb{R}_+^{d+1}} f(x)\overline{g(x)}d\mu_{\alpha,d}(x) = \int_{\mathbb{R}_+^{d+1}} \mathcal{F}_W^{\alpha,d}(f)(\lambda)\overline{\mathcal{F}_W^{\alpha,d}(g)(\lambda)}d\mu_{\alpha,d}(\lambda). \tag{2.13}$$
(ii) (Plancherel formula) For all $f \in \mathcal{S}_*(\mathbb{R}^{d+1})$, we have
$$\int_{\mathbb{R}_+^{d+1}} |f(x)|^2 d\mu_{\alpha,d}(x) = \int_{\mathbb{R}_+^{d+1}} |\mathcal{F}_W^{\alpha,d}(f)(\lambda)|^2 d\mu_{\alpha,d}(\lambda). \tag{2.14}$$

(iii) (Plancherel theorem) The transform $\mathcal{F}_W^{\alpha,d}$ extends uniquely to an isometric isomorphism on $L^2(d\mu_{\alpha,d})$.

Definition 2.8. The translation operator τ_x^α, $x \in \mathbb{R}_+^{d+1}$, associated with the Weinstein operator $\Delta_W^{\alpha,d}$ is defined on $C_*(\mathbb{R}^{d+1})$, for all $y \in \mathbb{R}_+^{d+1}$, by
$$\begin{aligned} \tau_x^\alpha f(y) &= \frac{a_\alpha}{2}\int_0^\pi f\Big(x'+y', \sqrt{x_{d+1}^2 + y_{d+1}^2 + 2x_{d+1}y_{d+1}\cos\theta}\Big)(\sin\theta)^{2\alpha}d\theta, \\ &= \int_{\mathbb{R}_+} f(x'+y', u) q_\alpha(x_{d+1}, y_{d+1}, u)u^{2\alpha+1}du, \end{aligned} \tag{2.15}$$
where $x' + y' = (x_1 + y_1, \dots, x_d + y_d)$, and
$$q_\alpha(v,w,u) = 2^{2\alpha-1}\frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+1/2)}\frac{\Upsilon(v,w,u)^{2\alpha-1}}{(uvw)^{2\alpha}}\mathbf{1}_{[|v-w|,v+w]}(z),$$
with
$$\Upsilon(v,w,u) = \frac{1}{4}\sqrt{(v+w+u)(v+w-u)(v-w+u)(w+u-v)}.$$

We should note that for all $v, w > 0$, we have

$$\int_{\mathbb{R}_+} q_\alpha(v,w,u)u^{2\alpha+1}du = 1. \tag{2.16}$$

The following proposition summarizes some properties of the Weinstein translation operator.

Proposition 2.9. *(i) For $f \in C_*(\mathbb{R}^{d+1})$, we have*

$$\tau_x^\alpha f(y) = \tau_y^\alpha f(x) \text{ and } \tau_0^\alpha f = f, \quad \forall x, y \in \mathbb{R}_+^{d+1}.$$

(ii) For all $f \in \mathcal{E}_*(\mathbb{R}^{d+1})$ and $y \in \mathbb{R}_+^{d+1}$, the function $x \mapsto \tau_x^\alpha f(y)$ belongs to $\mathcal{E}_*(\mathbb{R}^{d+1})$.

(iii) We have

$$\Delta_W^{\alpha,d} \circ \tau_x^\alpha = \tau_x^\alpha \circ \Delta_W^{\alpha,d}, \quad \forall x \in \mathbb{R}_+^{d+1}.$$

(iv) Let $f \in L^p(d\mu_{\alpha,d})$, $1 \le p \le +\infty$ and $x \in \mathbb{R}_+^{d+1}$. Then $\tau_x^\alpha f$ belongs to $L^p(d\mu_{\alpha,d})$ and we have

$$\|\tau_x^\alpha f\|_{L^p(d\mu_{\alpha,d})} \le \|f\|_{L^p(d\mu_{\alpha,d})}.$$

(v) The function $\Lambda_{\alpha,d}(\cdot,\lambda)$, $\lambda \in \mathbb{C}^{d+1}$, on $\mathbb{R}_+^{d+1}$ satisfies the product formula

$$\Lambda_{\alpha,d}(x,\lambda)\Lambda_{\alpha,d}(y,\lambda) = \tau_x^\alpha[\Lambda_{\alpha,d}(.,\lambda)](y), \quad \forall y \in \mathbb{R}_+^{d+1}. \tag{2.17}$$

(vi) Let $f \in L^p(d\mu_{\alpha,d})$, $p = 1$ or 2 and $x \in \mathbb{R}_+^{d+1}$, we have

$$\mathcal{F}_W^{\alpha,d}(\tau_x^\alpha f)(y) = \Lambda_{\alpha,d}(x,y)\mathcal{F}_W^{\alpha,d}(f)(y), \quad \forall y \in \mathbb{R}_+^{d+1}. \tag{2.18}$$

(vii) The space $\mathcal{S}_*(\mathbb{R}^{d+1})$ is invariant under the operators τ_x^α, with $x \in \mathbb{R}_+^{d+1}$.

Definition 2.10. The Weinstein convolution product of $f, g \in C_*(\mathbb{R}^{d+1})$ is given by:

$$f *_W g(x) = \int_{\mathbb{R}_+^{d+1}} \tau_x^\alpha f(-y)g(y)d\mu_{\alpha,d}(y), \quad \forall x \in \mathbb{R}_+^{d+1}. \tag{2.19}$$

Proposition 2.11. *(i) Let $p, q, r \in [1, +\infty]$ be such that $\frac{1}{p} + \frac{1}{q} - \frac{1}{r} = 1$. Then for all $f \in L^p(d\mu_{\alpha,d})$ and $g \in L^q(d\mu_{\alpha,d})$, the function $f *_W g$ belongs to $L^r(d\mu_{\alpha,d})$ and*

$$\|f *_W g\|_{L^r(d\mu_{\alpha,d})} \le \|f\|_{L^p(d\mu_{\alpha,d})}\|g\|_{L^q(d\mu_{\alpha,d})}. \tag{2.20}$$

(ii) For all $f, g \in L^1(d\mu_{\alpha,d})$, (resp. $\mathcal{S}_*(\mathbb{R}^{d+1})$), $f *_W g \in L^1(d\mu_{\alpha,d})$ (resp. $\mathcal{S}_*(\mathbb{R}^{d+1})$) and

$$\mathcal{F}_W^{\alpha,d}(f *_W g) = \mathcal{F}_W^{\alpha,d}(f)\mathcal{F}_W^{\alpha,d}(g). \tag{2.21}$$

3. Weinstein multiplier operators on a Sobolev type space

Throughout this section α and β denote two real numbers satisfying $\alpha \ge \beta > -\frac{1}{2}$. Let $s \in \mathbb{R}$, we define the Sobolev-Weinstein type space of order s, denoted by $\mathcal{H}_{\alpha,\beta}^s$, as the set of all $f \in \mathcal{S}'_*(\mathbb{R}^{d+1})$ such that $\mathcal{F}_W^{\alpha,d}(f)$ is a function and

$$(1 + |z|^2)^{\frac{s}{2}}\mathcal{F}_W^{\alpha,d}(f) \in L^2(d\mu_{\alpha,d}).$$

The space $\mathcal{H}_{\alpha,\beta}^s$ is endowed with the inner product

$$\langle f, g\rangle_{\mathcal{H}_{\alpha,\beta}^s} = \int_{\mathbb{R}_+^{d+1}} \mathcal{F}_W^{\alpha,d}(f)(z)\overline{\mathcal{F}_W^{\alpha,d}(g)(z)}d\mu_{\beta,d}^s(z)$$

and the norm

$$\|f\|^2_{\mathcal{H}^s_{\alpha,\beta}} = \int_{\mathbb{R}^{d+1}_+} |\mathcal{F}^{\alpha,d}_W(f)(z)|^2 d\mu^s_{\beta,d}(z),$$

where $d\mu^s_{\beta,d}(z) = (1+|z|^2)^s d\mu_{\beta,d}(z)$.

Lemma 3.1. *Let $s \in \mathbb{R}$, the space $\mathcal{H}^s_{\alpha,\beta}$ is an Hilbert space.*

Proof. Let $(f_n)_n$ be a Cauchy sequence on $\mathcal{H}^s_{\alpha,\beta}$. It is easy to see that $(\mathcal{F}^{\alpha,d}_W(f_n))_n$ is a Cauchy sequence of $L^2(\mu^s_{\beta,d})$, which is a complete space. Therefore there exists a function $g \in L^2(\mu^s_{\beta,d})$ satisfying

$$\lim_{n\to+\infty} \|\mathcal{F}^{\alpha,d}_W(f_n) - g\|_{L^2(\mu^s_{\beta,d})} = 0.$$

Then $g \in \mathcal{S}'_*(\mathbb{R}^{d+1})$ and if we denote f the distribution given by $f = (\mathcal{F}^{\alpha,d}_W)^{-1}(g)$, according to Theorem2.6, we deduce that $f \in \mathcal{S}'_*(\mathbb{R}^{d+1}_+)$ and $\mathcal{F}^{\alpha,d}_W(f) = g \in L^2(\mu^s_{\beta,d})$, which proves that $f \in \mathcal{H}^s_{\alpha,\beta}$. Furthermore,

$$\lim_{n\to+\infty} \|f_n - f\|_{\mathcal{H}^s_{\alpha,\beta}} = \lim_{n\to+\infty} \|\mathcal{F}^{\alpha,d}_W(f_n) - g\|_{L^2(\mu^s_{\beta,d})} = 0.$$

This proves that $\mathcal{H}^s_{\alpha,\beta}$ is a complete space. □

Remark 3.2. (i) For $s \geq \alpha - \beta$ and $f \in \mathcal{H}^s_{\alpha,\beta}$, the Plancherel Theorem associated with the Weinstein transform leads to

$$\|f\|^2_{L^2(\mu_{\alpha,d})} \leq \frac{c_{\alpha,d}}{c_{\beta,d}} \int_{\mathbb{R}^{d+1}_+} \frac{|\mathcal{F}^{\alpha,d}_W(f)(z)|^2}{(1+|z|^2)^s} z^{2(\alpha-\beta)}_{d+1} d\mu^s_{\beta,d}(z).$$

Also for $s \geq \alpha - \beta$, we have

$$\frac{z^{2(\alpha-\beta)}_{d+1}}{(1+|z|^2)^s} \leq \frac{|z|^{2(\alpha-\beta)}}{(1+|z|^2)^s} \leq 1.$$

So, we deduce that

$$\|f\|_{L^2(\mu_{\alpha,d})} \leq (\frac{c_{\alpha,d}}{c_{\beta,d}})^{1/2} \|f\|_{\mathcal{H}^s_{\alpha,\beta}}. \tag{3.1}$$

Consequently, the space $\mathcal{H}^s_{\alpha,\beta}$ is continuously contained in $L^2(d\mu_{\alpha,d})$.

(ii) Let $s > 2\alpha - \beta + \frac{d}{2} + 1$. If $f \in \mathcal{H}^s_{\alpha,\beta}$, then $\mathcal{F}^{\alpha,d}_W(f) \in L^1(d\mu_{\alpha,d})$ and

$$\|\mathcal{F}^{\alpha,d}_W(f)\|_{L^1(\mu_{\alpha,d})} \leq C_{\alpha,\beta} \|f\|_{\mathcal{H}^s_{\alpha,\beta}},$$

where

$$C_{\alpha,\beta} = (\frac{c_{\alpha,d}}{c_{\beta,d}} \int_{\mathbb{R}^{d+1}_+} z^{2(\alpha-\beta)}_{d+1} d\mu^{-s}_{\alpha,d}(z))^{1/2}.$$

Using (3.1), we deduce that the function $\mathcal{F}^{\alpha,d}_W(f) \in L^1(d\mu_{\alpha,d}) \cap L^2(d\mu_{\alpha,d})$ and

$$f(x) = \int_{\mathbb{R}^{d+1}_+} \mathcal{F}^{\alpha,d}_W(f)(z) \Lambda_{\alpha,d}(-x,z) d\mu_{\alpha,d}(z); \quad \text{a.e. } x \in \mathbb{R}^{d+1}_+.$$

Definition 3.3. Let m be a function in $L^\infty(d\mu_{\beta,d})$, we define the Weinstein multiplier operator $T_{\alpha,\beta,m}$ on $\mathcal{H}^s_{\alpha,\beta}$, by

$$T_{\alpha,\beta,m} f(x) = (\mathcal{F}^{\beta,d}_W)^{-1}(m \mathcal{F}^{\alpha,d}_W(f))(x), \quad x \in \mathbb{R}^{d+1}_+.$$

Using Plancherel Theorem associated with the Weinstein transform, we get the following result.

Theorem 3.4. *Let $m \in L^\infty(d\mu_{\beta,d})$ and $f \in \mathcal{H}^0_{\alpha,\beta}$, then we have*

$$\|T_{\alpha,\beta,m} f\|_{L^2(d\mu_{\beta,d})} \le \|m\|_{L^\infty(d\mu_{\beta,d})} \|f\|_{\mathcal{H}^0_{\alpha,\beta}}.$$

Definition 3.5. (i) Let E be a measurable subset of $\mathbb{R}^{d+1}_+$, we say that the function $f \in \mathcal{H}^0_{\alpha,\beta}$ is ϵ-concentrated on E, if

$$\|f - \chi_E f\|_{\mathcal{H}^0_{\alpha,\beta}} \le \epsilon \|f\|_{\mathcal{H}^0_{\alpha,\beta}},$$

where χ_E is the indicator function of the set E.

(ii) Let S be a measurable subset of $\mathbb{R}^{d+1}_+$ and let $f \in \mathcal{H}^0_{\alpha,\beta}$. We say that $\mathcal{F}^{\alpha,d}_W(f)$ is ν-concentrated on S, if

$$\|\mathcal{F}^{\alpha,d}_W(f) - \chi_S \mathcal{F}^{\alpha,d}_W(f)\|_{L^2(d\mu_{\beta,d})} \le \nu \|f\|_{\mathcal{H}^0_{\alpha,\beta}}.$$

The following theorem can be obtained from Donoho-Stark Uncertainty Principle for the Weinstein transform (see [23]).

Theorem 3.6. *Let $f \in \mathcal{H}^0_{\alpha,\beta}$, if f is ϵ-concentrated on E and $\mathcal{F}^{\alpha,d}_W(f)$ is ν-concentrated on S, then*

$$(\mu_{\alpha,d}(E))^{1/2} (\mu_{\beta,d}(S))^{1/2} \ge \frac{c_{\alpha,d}}{c_{\beta,d}} (1 - \nu - \epsilon).$$

4. Extremal functions for the operator $T_{\alpha,\beta,m}$

The theory of extremal functions and reproducing kernels on Hilbert spaces, is an important tool in this section concerning the study of the extremal function associated with the Weinstein multiplier operators $T_{\alpha,\beta,m}$. In this section, s denotes a real number satisfying $s > 2\alpha - \beta + \frac{d}{2} + 1$.

Let $\eta > 0$. We denote by $\langle \cdot,\cdot \rangle_{\eta,\mathcal{H}^s_{\alpha,\beta}}$ the inner product on the space $\mathcal{H}^s_{\alpha,\beta}$ by

$$\langle f, g\rangle_{\eta,\mathcal{H}^s_{\alpha,\beta}} = \eta \langle f, g\rangle_{\mathcal{H}^s_{\alpha,\beta}} + \langle T_{\alpha,\beta,m} f, T_{\alpha,\beta,m} g\rangle_{L^2(d\mu_{\beta,d})},$$

and $\|\cdot\|_{\eta,\mathcal{H}^s_{\alpha,\beta}}$ the associated norm.

We remark that the two norms $\|\cdot\|_{\eta,\mathcal{H}^s_{\alpha,\beta}}$ and $\|\cdot\|_{\mathcal{H}^s_{\alpha,\beta}}$ are equivalent, therefore the pair $(\mathcal{H}^s_{\alpha,\beta}, \langle\cdot,\cdot\rangle_{\eta,\mathcal{H}^s_{\alpha,\beta}})$ is an Hilbert space with a reproducing kernel given in the following theorem.

Theorem 4.1. *Let $\eta > 0$ and $m \in L^\infty(d\mu_{\beta,d})$. The space $(\mathcal{H}^s_{\alpha,\beta}, \langle\cdot,\cdot\rangle_{\eta,\mathcal{H}^s_{\alpha,\beta}})$ has the reproducing kernel*

$$K_s(x,y) = \frac{c_{\alpha,d}}{c_{\beta,d}} \int_{\mathbb{R}^{d+1}_+} \frac{\Lambda_{\alpha,d}(-x,z)\Lambda_{\alpha,d}(y,z)}{|m(z)|^2 + \eta(1+|z|^2)^s} z_{d+1}^{2(\alpha-\beta)} d\mu_{\alpha,d}(z);$$

that is

(i) *For all $y \in \mathbb{R}^{d+1}_+$, the function $K_s(\cdot,y)$ belongs to $\mathcal{H}^s_{\alpha,\beta}$.*

(ii) *The reproducing property: For all $f \in \mathcal{H}^s_{\alpha,\beta}$ and $y \in \mathbb{R}^{d+1}_+$,*

$$\langle f, K_s(\cdot,y)\rangle_{\eta,\mathcal{H}^s_{\alpha,\beta}} = f(y).$$

Proof. (i) Let $y \in \mathbb{R}^{d+1}_+$. Using the relation (2.5), one can see that the function

$$\phi_y : z \mapsto \frac{c_{\alpha,d}}{c_{\beta,d}} \frac{\Lambda_{\alpha,d}(y,z)}{|m(z)|^2 + \eta(1+|z|^2)^s} z_{d+1}^{2(\alpha-\beta)}$$

belongs to $L^1(d\mu_{\alpha,d}) \cap L^2(d\mu_{\alpha,d})$. Then the function K_s is well defined and

$$K_s(x,y) = (\mathcal{F}_W^{\alpha,d})^{-1}(\phi_y)(x), x \in \mathbb{R}_+^{d+1}. \tag{4.1}$$

Plancherel Theorem for the Weinstein transform and relation (2.5) give

$$(1+|z|^2)^s |\mathcal{F}_W^{\alpha,d}(K_s(\cdot,y))(z)|^2 \le (\frac{c_{\alpha,d}}{c_{\beta,d}})^2 \frac{z_{d+1}^{4(\alpha-\beta)}}{\eta^2(1+|z|^2)^s},$$

So

$$\begin{aligned} \|K_s(\cdot,y)\|_{\mathcal{H}^s_{\alpha,\beta}} &\le \frac{c_{\alpha,d}}{c_{\beta,d}} \Big(\int_{\mathbb{R}_+^{d+1}} \frac{z_{d+1}^{4(\alpha-\beta)}}{\eta^2(1+|z|^2)^s} d\mu_{\beta,d}(z) \Big)^{1/2} \\ &\le \frac{1}{\eta} c_{\alpha,\beta} < \infty. \end{aligned}$$

This proves that for all $y \in \mathbb{R}_+^{d+1}$ the function $K_s(\cdot,y)$ belongs to $\mathcal{H}^s_{\alpha,\beta}$.

(ii) Let $f \in \mathcal{H}^s_{\alpha,\beta}$ and $y \in \mathbb{R}_+^{d+1}$, using relation (4.1), we have

$$\begin{aligned} &\langle f, K_s(\cdot,y)\rangle_{\eta,\mathcal{H}^s_{\alpha,\beta}} \\ &= \eta \langle f, (\mathcal{F}_W^{\alpha,d})^{-1}(\phi_y)\rangle_{\mathcal{H}^s_{\alpha,\beta}} + \langle T_{\alpha,\beta,m} f, T_{\alpha,\beta,m}(\mathcal{F}_W^{\alpha,d})^{-1}(\phi_y)\rangle_{L^2(d\mu_{\beta,d})}. \end{aligned}$$

Using Parseval formula for the Weinstein transform,

$$\begin{aligned} &\langle f, K_s(\cdot,y)\rangle_{\eta,\mathcal{H}^s_{\alpha,\beta}} \\ &= \eta \int_{\mathbb{R}_+^{d+1}} \mathcal{F}_W^{\alpha,d}(f)(z)\overline{\phi_y(z)} d\mu^s_{\beta,d}(z) + \int_{\mathbb{R}_+^{d+1}} |m(z)|^2 \mathcal{F}_W^{\alpha,d}(f)(z)\overline{\phi_y(z)} d\mu_{\beta,d}(z) \\ &= \int_{\mathbb{R}_+^{d+1}} \mathcal{F}_W^{\alpha,d}(f)(z) \frac{c_{\alpha,d}}{c_{\beta,d}} \Lambda_{\alpha,d}(y,z) z_{d+1}^{2(\alpha-\beta)} d\mu_{\beta,d}(z) \\ &= \int_{\mathbb{R}_+^{d+1}} \mathcal{F}_W^{\alpha,d}(f)(z) \Lambda_{\alpha,d}(y,z) d\mu_{\alpha,d}(z) = f(y). \end{aligned}$$

Hence $K_s(\cdot,y)$ is a reproducing kernel. □

Remark 4.2. The space $\mathcal{H}^s_{\alpha,\beta}$ has the reproducing kernel

$$K_s(x,y) = \frac{c_{\alpha,d}}{c_{\beta,d}} \int_{\mathbb{R}_+^{d+1}} \Lambda_{\alpha,d}(-x,z)\Lambda_{\alpha,d}(y,z) z_{d+1}^{2(\alpha-\beta)} d\mu^{-s}_{\alpha,d}(z).$$

Theorem 4.3. *Let $m \in L^\infty(d\mu_{\alpha,d})$, for any function $g \in L^2(d\mu_{\beta,d})$ and for any $\eta > 0$, there exists a unique function $f^*_{\eta,g}$, where the infimum*

$$\inf_{f \in \mathcal{H}^s_{\alpha,\beta}} \{\eta \|f\|^2_{\mathcal{H}^s_{\alpha,\beta}} + \|g - T_{\alpha,\beta,m} f\|^2_{L^2(\mu_{\beta,d})}\} \tag{4.2}$$

is attained. Moreover, the extremal function $f^*_{\eta,g}$ is given by

$$f^*_{\eta,g}(y) = \int_{\mathbb{R}_+^{d+1}} g(x) Q_{\alpha,\beta}(x,y) d\mu_{\beta,d}(x),$$

where

$$Q_{\alpha,\beta}(x,y) = \int_{\mathbb{R}_+^{d+1}} \frac{\overline{m(z)}\Lambda_{\alpha,d}(x,z)\Lambda_{\alpha,d}(-y,z)}{|m(z)|^2 + \eta(1+|z|^2)^s} d\mu_{\alpha,d}(z).$$

Proof. The existence and the uniqueness of the extremal function $f^*_{\eta,g}$ satisfying (4.2) is obtained in [11, 20, 25]. Especially, using the reproducing kernel of the space $(\mathcal{H}^s_{\alpha,\beta}, \|\cdot\|_{\eta,\mathcal{H}^s_{\alpha,\beta}})$, $f^*_{\eta,g}$ is given by

$$f^*_{\eta,g} = \langle g, T_{\alpha,\beta,m}(K_s(\cdot,y))\rangle_{L^2(d\mu_{\beta,d})}. \tag{4.3}$$

The relation (4.1) leads to

$$\begin{aligned} T_{\alpha,\beta,m}(K_s(\cdot,y))(x) &= \int_{\mathbb{R}^{d+1}_+} m(z)\mathcal{F}^{\alpha,d}_W(K_s(\cdot,y))(z)\Lambda_{\beta,d}(-x,z)d\mu_{\beta,d}(z) \\ &= \int_{\mathbb{R}^{d+1}_+} m(z)\frac{\Lambda_{\alpha,d}(-x,z)\Lambda_{\alpha,d}(y,z)}{|m(z)|^2+\eta(1+|z|^2)^s}d\mu_{\alpha,d}(z). \end{aligned}$$

Which gives the result. □

Corollary 4.4. *Let $g \in L^2(d\mu_{\beta,d})$ and $\eta > 0$. The extremal function $f^*_{\eta,g}$ satisfies:*

$$|f^*_{\eta,g}| \le \frac{c_{\alpha,d}}{2\sqrt{\eta}}\|g\|_{L^2(d\mu_{\beta,d})}, \tag{4.4}$$

$$\|f^*_{\eta,g}\|_{L^2(d\mu_{\alpha,d})} \le \frac{c_{\alpha,\alpha}}{2\sqrt{\eta}}\Big(\int_{\mathbb{R}^{d+1}_+}|g(x)|^2e^{\frac{|x|^2}{2}}d\mu_{\beta,d}(x)\Big)^{1/2}; \tag{4.5}$$

$$f^*_{\eta,g}(y) = \int_{\mathbb{R}^{d+1}_+}\frac{\Lambda_{\alpha,d}(-y,z)\overline{m(z)}}{|m(z)|^2+\eta(1+|z|^2)^s}\mathcal{F}^{\beta,d}_W(g)(z)d\mu_{\alpha,d}(z). \tag{4.6}$$

Proof. To prove (4.4), have

$$f^*_{\eta,g}(y) = \langle g, T_{\alpha,\beta,m}(K_s(\cdot,y))\rangle_{L^2(d\mu_{\beta,d})}.$$

This leads to

$$\begin{aligned} |f^*_{\eta,g}(y)| &\le \|g\|_{L^2(d\mu_{\beta,d})}\|m\mathcal{F}^{\alpha,d}_W(K_s(\cdot,y))\|_{L^2(d\mu_{\beta,d})} \\ &\le \|g\|_{L^2(d\mu_{\beta,d})}\Big(\int_{\mathbb{R}^{d+1}_+}|m(z)|^2|\phi_y(z)|^2d\mu_{\beta,d}(z)\Big)^{1/2} \\ &\le \|g\|_{L^2(d\mu_{\beta,d})}\Big(\int_{\mathbb{R}^{d+1}_+}\frac{c_{\alpha,d}}{c_{\beta,d}}\frac{z_{d+1}^{2(\alpha-\beta)}|m(z)|^2}{[|m(z)|^2+\eta(1+|z|^2)^s]^2}d\mu_{\alpha,d}(z)\Big)^{1/2}. \end{aligned}$$

Using the inequality

$$\big[|m(z)|^2+\eta(1+|z|^2)^s\big]^2 \ge 4\eta(1+|z|^2)^s|m(z)|^2,$$

we obtain

$$\begin{aligned} |f^*_{\eta,g}(y)| &\le \|g\|_{L^2(d\mu_{\beta,d})}\Big(\frac{c_{\alpha,d}}{c_{\beta,d}}\int_{\mathbb{R}^{d+1}_+}\frac{z_{d+1}^{2(\alpha-\beta)}}{4\eta(1+|z|^2)^s}d\mu_{\alpha,d}(z)\Big)^{1/2} \\ &\le \frac{c_{\alpha,\beta}}{2\sqrt{\eta}}\|g\|_{L^2(d\mu_{\beta,d})}. \end{aligned}$$

To prove (4.5) we write

$$f^*_{\eta,g}(y) = \int_{\mathbb{R}^{d+1}_+}e^{-\frac{|x|^2}{4}}e^{\frac{|x|^2}{4}}g(x)Q_{\alpha,\beta}(x,y)d\mu_{\beta,d}(x).$$

Hölder inequality gives

$$|f^*_{\eta,g}(y)|^2 \le \int_{\mathbb{R}^{d+1}_+}e^{\frac{|x|^2}{2}}|g(x)|^2|Q_{\alpha,\beta}(x,y)|^2d\mu_{\beta,d}(x).$$

Now, applying Fubini-Tonelli's Theorem we obtain

$$\|f^*_{\eta,g}(y)\|^2_{L^2(d\mu_{\alpha,d})} \leq \int_{\mathbb{R}^{d+1}_+} e^{\frac{|x|^2}{2}}|g(x)|^2\|Q_{\alpha,\beta}(x,\cdot)\|^2_{L^2(d\mu_{\alpha,d})}d\mu_{\beta,d}(x).$$

Let ψ_x be the function defined by

$$\psi_x(z) = \frac{\overline{m(z)}\Lambda_{\alpha,d}(x,z)}{[|m(z)|^2+\eta(1+|z|^2)^s]^2}.$$

Since $s > 2\alpha-\beta+\frac{d}{2}+1$, $\psi_x \in L^1(d\mu_{\alpha,d})\cap L^2(d\mu_{\alpha,d})$ and $Q_{\alpha,\beta}(x,y) = (\mathcal{F}^{\alpha,d}_W)^{-1}(\psi_x)(y)$, we have

$$\begin{aligned}
\|Q_{\alpha,\beta}(x,\cdot)\|^2_{L^2(d\mu_{\alpha,d})} &= \|\mathcal{F}^{\alpha,d}_W(Q_{\alpha,\beta}(x\, cdot))\|^2_{L^2(d\mu_{\alpha,d})}\\
&\leq \int_{\mathbb{R}^{d+1}_+}\frac{|\overline{m(z)}|^2 d\mu_{\alpha,d}(z)}{[|m(z)|^2+\eta(1+|z|^2)^s]^2}\\
&\leq \int_{\mathbb{R}^{d+1}_+}\frac{d\mu_{\alpha,d}(z)}{4\eta(1+|z|^2)^s}\\
&\leq \frac{c_{\alpha,\alpha}}{2\sqrt{\eta}}.
\end{aligned}$$

Equality (4.6) follows from relation (4.3), Plancherel Theorem for the Weinstein transform and (4.1). □

Corollary 4.5. *For every $f \in \mathcal{H}^s_{\alpha,\beta}$ and $g = T_{\alpha,\beta,m}f$, we have*

(i)
$$\mathcal{F}^{\alpha,d}_W(f^*_{\eta,g})(z) = \frac{|m(z)|^2}{m(z)^2+\eta(1+|z|^2)^s}\mathcal{F}^{\alpha,d}_W(f)(z);$$

(ii) $\|f^*_{\eta,g}\|_{\mathcal{H}^s_{\alpha,\beta}} \leq \frac{\|m\|_{L^\infty(d\mu_{\beta,d})}}{2\sqrt{\eta}}\|f\|_{\mathcal{H}^s_{\alpha,\beta}}$;

(iii) $\lim_{\eta\to 0^+}\|f^*_{\eta,g}-f\|_{\mathcal{H}^s_{\alpha,\beta}} = 0$;

(iv) $\lim_{\eta\to 0^+}\|f^*_{\eta,g}-f\|_{L^\infty(d\mu_{\alpha,d})} = 0$.

Proof. (i) For every $f \in \mathcal{H}^s_{\alpha,\beta}$ and $g = T_{\alpha,\beta,m}f$, we have

$$\begin{aligned}
&\mathcal{F}^{\alpha,d}_W(f^*_{\eta,g})(z)\\
&= \int_{\mathbb{R}^{d+1}_+} g(x)\mathcal{F}^{\alpha,d}_W(Q_{\alpha,\beta}(x\, cdot))(z)d\mu_{\beta,d}(x)\\
&= \int_{\mathbb{R}^{d+1}_+} T_{\alpha,\beta,m}f(x)\psi_x(z)d\mu_{\beta,d}(x)\\
&= \frac{\overline{m(z)}}{m(z)^2+\eta(1+|z|^2)^s}\int_{\mathbb{R}^{d+1}_+}(\mathcal{F}^{\alpha,d}_W)^{-1}(m\mathcal{F}^{\alpha,d}_W(f))(x)\Lambda_{\alpha,d}(x,z)d\mu_{\beta,d}(x)\\
&= \frac{|m(z)|^2}{m(z)^2+\eta(1+|z|^2)^s}\mathcal{F}^{\alpha,d}_W(f)(z).
\end{aligned}$$

(ii) We have

$$\|f^*_{\eta,g}\|_{\mathcal{H}^s_{\alpha,\beta}} = \Big(\int_{\mathbb{R}^{d+1}_+}\frac{|m(z)|^4}{[m(z)^2+\eta(1+|z|^2)^s]^2}|\mathcal{F}^{\alpha,d}_W(f)(z)|^2 d\mu^s_{\beta,d}(x)\Big)^{1/2}$$

$$\begin{aligned}&\le \Big(\int_{\mathbb{R}_+^{d+1}} \frac{|m(z)|^2}{4\eta(1+|z|^2)^s}|\mathcal{F}_W^{\alpha,d}(f)(z)|^2 d\mu_{\beta,d}^s(x)\Big)^{1/2}\\&\le \frac{\|m\|_{L^\infty(d\mu_{\beta,d})}}{2\sqrt{\eta}}\|f\|_{\mathcal{H}_{\alpha,\beta}^s}.\end{aligned}$$

(iii) We have

$$\begin{aligned}\|f_{\eta,g}^*-f\|_{\mathcal{H}_{\alpha,\beta}^s}&=\Big(\int_{\mathbb{R}_+^{d+1}} \Big|\frac{|m(z)|^2}{m(z)^2+\eta(1+|z|^2)^s}-1\Big|^2|\mathcal{F}_W^{\alpha,d}(f)(z)|^2 d\mu_{\beta,d}^s(x)\Big)^{1/2}\\&=\Big(\int_{\mathbb{R}_+^{d+1}} \frac{\eta^2(1+|z|^2)^{2s}}{[m(z)^2+\eta(1+|z|^2)^s]^2}|\mathcal{F}_W^{\alpha,d}(f)(z)|^2 d\mu_{\beta,d}^s(x)\Big)^{1/2}.\end{aligned}$$

Since

$$\frac{\eta^2(1+|z|^2)^{2s}}{[m(z)^2+\eta(1+|z|^2)^s]^2}\le 1,$$

we obtain the result with the dominated convergence theorem.

(iv) It is easy to see that

$$\mathcal{F}_W^{\alpha,d}(f_{\eta,g}^*-f)(z)=\frac{-\eta(1+|z|^2)^s}{m(z)^2+\eta(1+|z|^2)^s}\mathcal{F}_W^{\alpha,d}(f)(z).$$

So

$$f_{\eta,g}^*(y)-f(y)=\int_{\mathbb{R}_+^{d+1}}\frac{-\eta(1+|z|^2)^s}{m(z)^2+\eta(1+|z|^2)^s}\mathcal{F}_W^{\alpha,d}(f)(z)\Lambda_{\alpha,d}(-y,z)d\mu_{\alpha,d}(z),$$

and

$$\sup_{y\in\mathbb{R}_+^{d+1}}|f_{\eta,g}^*(y)-f(y)|\le\int_{\mathbb{R}_+^{d+1}}\frac{\eta(1+|z|^2)^s}{m(z)^2+\eta(1+|z|^2)^s}|\mathcal{F}_W^{\alpha,d}(f)(z)|d\mu_{\alpha,d}(z).$$

Hence the needed result is a consequence of the dominated convergence theorem and the inequality

$$\frac{\eta(1+|z|^2)^s}{m(z)^2+\eta(1+|z|^2)^s}\le 1.$$

□

5. Hörmander multiplier theorem for the operator $T_{\alpha,\alpha,m}$

The aim of this section is to prove an analogue of the famous Hörmander multiplier theorem for the operator $T_{\alpha,\alpha,m}$, which will be denoted as $T_{\alpha,m}$. The theorem is stated as follows.

Theorem 5.1. *Let ℓ the least integer greater than $\alpha+1$ and m be a bounded $C_*^{2\ell}$-function on $\mathbb{R}^{d+1}\setminus\{0\}$, satisfying the Hörmander condition*

$$\Big(\int_{R/2\le|\xi|\le 2R}|\Delta_d^s(\frac{\partial^q m}{\partial\xi_{d+1}^q})(\xi)|^2 d\mu_{\alpha,d}(\xi)\Big)^{1/2}\le CR^{\alpha+\frac{d+1}{2}-(2s+q)},\tag{5.1}$$

for all $R>0$, where C is a constant independent of R and $s\in\{0,1,\dots,\ell\}$ and $q\in\{0,1,\dots,2\ell\}$. Then the operator $T_{\alpha,m}$ can be extended to a bounded operator from $L^p(d\mu_{\alpha,d})$ into itself for $1<p<\infty$.

We need to establish some results associated with the Weinstein analysis to prove Theorem5.1.

Theorem 5.2. *Let K be a measurable function on $\{(x,y)\in\mathbb{R}_+^{d+1}\times\mathbb{R}_+^{d+1}:|x|\neq|y|\}$ and T be a bounded operator from $L^2(d\mu_{\alpha,d})$ into itself such that*

$$T(f)(x)=\int_{\mathbb{R}_+^{d+1}}K(x,y)f(y)d\mu_{\alpha,d}(y), \tag{5.2}$$

for all compactly supported f in $L^2(d\mu_{\alpha,d})$ and for a.e. $x\in\mathbb{R}_+^{d+1}$, $|x|\notin|\operatorname{supp}(f)|=\{|y|,y\in\operatorname{supp}(f)\}$. If K satisfies

$$\int_{||x|-|y||>2|y-z|}|K(x,y)-K(x,z)|\,d\mu_{\alpha,d}(x)\le C\,,\quad \forall y,z\in\mathbb{R}_+^{d+1}, \tag{5.3}$$

then T extends to a bounded operator from $L^p(d\mu_{\alpha,d})$ into itself for $1<p\le 2$.

Proof. We proceed by the same manner as in the proof of the classical theorem of singular integral given in [27, Chp. I], by considering here the doubling measure $d\mu_{\alpha,d}$ and proving that T is a weak-type $(1,1)$. □

Before proving Theorem 5.1 we need the following lemmas.

Lemma 5.3. *Let $\varphi\in\mathcal{S}_*(\mathbb{R}_+^{d+1})$, then for all $x,y\in\mathbb{R}_+^{d+1}$ we have*

$$\|\tau_x^\alpha(\varphi)-\tau_y^\alpha(\varphi)\|_{L^1(d\mu_{\alpha,d})}\le C|x_{d+1}-y_{d+1}|\|\frac{\partial\varphi}{\partial z_{d+1}}\|_{L^1(d\mu_{\alpha,d})} \tag{5.4}$$

Proof. In view of (2.15) we have

$$\tau_x^\alpha\varphi(z)=\frac{a_\alpha}{2}\int_0^\pi\varphi(x'+z',\psi_\theta(x_{d+1},z_{z+1}))d\nu(\theta),$$

where $\psi_\theta(x_{d+1},z_{d+1})=\sqrt{x_{d+1}^2+z_{d+1}^2+2x_{d+1}z_{d+1}\cos\theta}$ and $d\nu(\theta)=(\sin\theta)^{2\alpha}d\theta$. By the mean value theorem, it follows that

$$\begin{aligned}&\tau_x^\alpha(\varphi)(z)-\tau_y^\alpha(\varphi)(z)\\&=\frac{a_\alpha}{2}(x_{d+1}-y_{d+1})\int_0^1\int_0^\pi\frac{z_t+z_{d+1}\cos\theta}{\Psi_\theta(z_t,z_{d+1})}\frac{\partial\varphi}{\partial z_{d+1}}(x'+z',\Psi_\theta(z_t,z))d\nu(\theta)dt,\end{aligned}$$

where $z_t=x_{d+1}+t(y_{d+1}-x_{d+1})$. Now, using the inequality

$$\frac{|z_t+z_{d+1}\cos\theta|}{\Psi_\theta(z_t,z_{d+1})}\le 1,$$

and making the change of variable $u\to\Psi_\theta(z_t,z_{d+1})$, we obtain

$$\begin{aligned}&|\tau_x^\alpha(\varphi)(z)-\tau_y^\alpha(\varphi)(z)|\\&\le\frac{a_\alpha}{2}|x_{d+1}-y_{d+1}|\int_0^1\int_0^\pi|\frac{\partial\varphi}{\partial z_{d+1}}(x'+z',\Psi_\theta(z_t,z))|d\nu(\theta)dt\\&\le\frac{a_\alpha}{2}|x_{d+1}-y_{d+1}|\int_0^1\int_{\mathbb{R}_+}|\frac{\partial\varphi}{\partial z_{d+1}}(x'+z',u)|q_\alpha(z_t,z_{d+1},u)u^{2\alpha+1}\,du\,dt.\end{aligned}$$

So, (5.4) follows from Fubini's theorem and (2.16). □

Lemma 5.4 (Berstein's lemma)**.** *Let $f\in L^1(d\mu_{\alpha,d})$ and $\lambda>0$ and suppose that $\operatorname{supp}(\mathcal{F}_W^{\alpha,d}(f))\subset\{x\in\mathbb{R}_+^{d+1},|x|<\lambda\}$. Then for all x, $y\in\mathbb{R}_+^{d+1}$, we have*

$$\|\tau_x^\alpha(f)-\tau_y^\alpha(f)\|_{L^1(d\mu_{\alpha,d})}\le C\lambda|x_{d+1}-y_{d+1}|\|f\|_{L^1(d\mu_{\alpha,d})}.$$

Proof. Choose $\phi \in \mathcal{S}_*(\mathbb{R}_+^{d+1})$ such that $|\mathcal{F}_W^{\alpha,d}(\phi)(x)| = 1$ for all $x \in \{x \in \mathbb{R}_+^{d+1}, |x| < 1\}$ and put $\phi_\lambda(x) = \lambda^{2\alpha+2+d}\phi(\lambda x)$. Then $|\mathcal{F}_W^{\alpha,d}(\phi_\lambda)(x)| = |\mathcal{F}_W^{\alpha,d}(\phi)(\frac{x}{\lambda})| = 1$ in $\{x \in \mathbb{R}_+^{d+1}, |x| < 1\}$ and we can write

$$\begin{aligned} \tau_x^\alpha(f)(z) - \tau_y^\alpha(f)(z) &= \phi_\lambda *_W (\tau_x^\alpha(f) - \tau_y^\alpha(f))(z) \\ &= f *_W (\tau_x^\alpha(\phi_\lambda) - \tau_y^\alpha(\phi_\lambda))(z). \end{aligned}$$

Using (5.4) we get

$$\begin{aligned} \|\tau_x^\alpha(f) - \tau_y^\alpha(f)\|_{1,\alpha} &\le C\|f\|_{L^1(d\mu_{\alpha,d})}\|\tau_x^\alpha(\phi_\lambda) - \tau_y^\alpha(\phi_\lambda)\|_{L^1(d\mu_{\alpha,d})} \\ &\le C\|f\|_{L^1(d\mu_{\alpha,d})}\|\tau_{x\lambda}^\alpha(\phi) - \tau_{y\lambda}^\alpha(\phi)\|_{L^1(d\mu_{\alpha,d})} \\ &\le C\lambda|x_{d+1} - y_{d+1}|\|f\|_{L^1(d\mu_{\alpha,d})}, \end{aligned}$$

which gives the desired result. □

Lemma 5.5. *If m satisfies* (5.1)*. Then there exists a locally integrable function k on $\mathbb{R}_+^{d+1} \setminus \{0\}$ such that for all $|x| \notin |\operatorname{supp}(f)|$,*

$$T_{\alpha,m}(f)(x) = \int_{\mathbb{R}_+^{d+1}} K(x,y)f(y)d\mu_{\alpha,d}(y),$$

where K is given on $\{(x,y) \in \mathbb{R}_+^{d+1} \times \mathbb{R}_+^{d+1} : |x| \ne |y|\}$, by

$$K(x,y) = \tau_x^\alpha(k)(-y', y_{d+1}). \tag{5.5}$$

Proof. Let $\varphi \in \mathcal{D}_*(\mathbb{R}^{d+1})$, supported in $\{\frac{1}{2} \le |\xi| \le 2\}$ and satisfying :

$$\sum_{j=-\infty}^{+\infty} \varphi(2^{-j}\xi) = 1, \quad \xi \ne 0.$$

Put $m_j(\xi) = m(\xi)\varphi(2^{-j}\xi)$ and $k_j = (\mathcal{F}_W^{\alpha,d})^{-1}(m_j)$. Let us prove first the estimate

$$\|(\Delta_W^{\alpha,d})^s m_j\|_{L^2(d\mu_{\alpha,d})} \le C_s 2^{j(\alpha+\frac{d+1}{2}-s)}, \quad s = 0,1,\dots,\ell. \tag{5.6}$$

In fact by induction we can show that there exist constants $b_{\alpha,r}, r \in \{0,1,\dots,s\}$, depending only on α, satisfying

$$(\Delta_W^{\alpha,d})^s m_j(\xi) = \sum_{j=0}^{s} C_s^j \sum_{r=1}^{2j} b_{\alpha r}\xi_{d+1}^{r-s}\Delta_d^{s-j}\frac{\partial^r m_j}{\partial \xi_{d+1}^r}(\xi), \quad \xi \ne 0. \tag{5.7}$$

Leibniz formula gives

$$\frac{\partial^r m_j}{\partial \xi_{d+1}^r}(\xi) = \sum_{q=0}^{r} C_r^q 2^{j(q-r)}\frac{\partial^q m}{\partial \xi_{d+1}^q}(\xi)\frac{\partial^{r-q}\varphi}{\partial \xi_{d+1}^{r-q}}(2^{-j}\xi).$$

Hence,

$$(\Delta_W^{\alpha,d})^s m_j(\xi) = \sum_{j=0}^{s} C_s^j \sum_{r=1}^{2j} b_{\alpha,r}\xi_{d+1}^{r-s}\sum_{q=0}^{r} C_r^q 2^{j(q-r)}\Delta_d^{s-j}\Big(\frac{\partial^q m}{\partial \xi_{d+1}^q}\frac{\partial^{r-q}\varphi}{\partial \xi_{d+1}^{r-q}}\Big).$$

Using (5.4), we obtain

$$\int_{\mathbb{R}_+^{d+1}} |\xi_{d+1}^{r-s}\Delta_d^p\frac{\partial^r m_j}{\partial \xi_{d+1}^r}(\xi)|^2 d\mu_{\alpha,d}(\xi)$$

$$\begin{aligned}&\leq C2^{2j(r-s)}\sum_{q=0}^{r}2^{2j(q+2p-r)}(C_r^q)^2\int_{2^{j-1}\leq|\xi|\leq 2^{j+1}}|\Delta_d^p(\frac{\partial^q m}{\partial\xi_{d+1}^q})(\xi)|^2 d\mu_{\alpha,d}(\xi)\\&\leq C_{r,s}\ 2^{2j(\alpha+\frac{d+1}{2}-s)}.\end{aligned}$$

So, the inequality (5.6) that we seek is a consequence of (5.7).

Now, applying Plancherel's theorem and using (5.6), we obtain

$$\|(-1)^s|x|^{2s}k_j(x)|_{L^2(d\mu_{\alpha,d})}=\|(\Delta_W^{\alpha,d})^s m_j\|_{L^2(d\mu_{\alpha,d})}\leq C_s 2^{j(\alpha+\frac{d+1}{2}-s)},$$

for $s=0,1,\dots,\ell$. Applying this formula with $s=0$ and $s=\ell$, we get that the series

$$\sum_{j=-\infty}^{-1}\|k_j(x)\|_{L^2(d\mu_{\alpha,d})},\quad \sum_{j=0}^{+\infty}\|(ix)^\ell k_j(x)\|_{L^2(d\mu_{\alpha,d})}$$

are convergent and the series $\sum_{j=-\infty}^{+\infty}|k_j(x)|$ is convergent for a.e. $x\neq 0$.

By the Cauchy-Schwarz inequality it follows that

$$\begin{aligned}&\int_{\mathbb{R}_+^{d+1}}|\tau_x^\alpha(f)(y)|\sum_{j=-\infty}^{-1}|k_j(-y',y_{d+1})|d\mu_{\alpha,d}(y)\\&\leq\|\tau_x^\alpha(f)\|_{L^2(d\mu_{\alpha,d})}\sum_{j=-\infty}^{-1}\|k_j\|_{L^2(d\mu_{\alpha,d})}<\infty,\\&\int_{\mathbb{R}_+^{d+1}}|\tau_x^\alpha(f)(y)|\sum_{j=0}^{\infty}|k_j(-y',y_{d+1})|d\mu_{\alpha,d}(y)\\&\leq\|\frac{\tau_x^\alpha(f)(y)}{y^\ell}\|_{L^2(d\mu_{\alpha,d})}\sum_{j=0}^{\infty}\|y^\ell k_j(-y',y_{d+1})\|_{L^2(d\mu_{\alpha,d})}<\infty,\end{aligned}$$

for $|x|\notin|\operatorname{supp}(f)|$ (which implies that $0\notin\operatorname{supp}(\tau_x(f))$). Thus we concluded that

$$\int_{\mathbb{R}_+^{d+1}}|\tau_x^\alpha(f)(y)|\sum_{j=-\infty}^{+\infty}|k_j(y)|d\mu_{\alpha,d}(y)<\infty,\quad |x|\notin|\operatorname{supp}(f)|.$$

This allows us to take $k=\sum_{j=-\infty}^{+\infty}k_j$ and one can write for $|x|\notin|\operatorname{supp}(f)|$,

$$T(f)(x)=\int_{\mathbb{R}_+^{d+1}}k(z)\tau_x^\alpha(f)(-z',z_{d+1})d\mu_{\alpha,d}(z)=\int_{\mathbb{R}_+^{d+1}}K(x,y)f(y)d\mu_{\alpha,d}(y).$$

Which completes the proof. □

Proof of Theorem 5.1. First, we note that the adjoint operator T_m^* is the multiplier operator associated with $\overline{m}$ and

$$T_m^*(f)(x)=\int_{\mathbb{R}_+^{d+1}}\overline{K(y,x)}f(y)d\mu_{\alpha,d}(y);\quad |x|\notin|\operatorname{supp}(f)|,$$

where K is given by (5.5). Using the duality argument, it is sufficient to show that the function K satisfies the condition (5.3) of the Theorem 5.2, which follows from

$$\sum_{j=-\infty}^{\infty}\int_{||x|-|y||>2|y-z|}|\tau_y^\alpha(k_j)(x)-\tau_z^\alpha(k_j)(x)|d\mu_{\alpha,d}(x)\leq C,\tag{5.8}$$

for all $y, z \in \mathbb{R}_+^{d+1}$. To prove (5.8) we need the following two estimates

$$\int_{\mathbb{R}_+^{d+1}} |k_j(x)| d\mu_\alpha(x) \le C, \quad \int_{|x|\ge t} |k_j(x)| d\mu_{\alpha,d}(x) \le C(2^j t)^{\alpha+\frac{d+1}{2}-s}, \quad t > 0. \tag{5.9}$$

Cauchy-Schwarz inequality, Plancherel's theorem and (2.9), give

$$\begin{aligned}\int_{\mathbb{R}_+^{d+1}} |k_j(x)| d\mu_\alpha(x) &\le \|(1+2^j|x|^2)^{-\ell}\|_{L^2(d\mu_{\alpha,d})} \|(1+2^j|x|^2)^{\ell}|k_j(x)|\|_{L^2(d\mu_{\alpha,d})} \\ &\le C 2^{-j(\alpha+\frac{d+1}{2})} \sum_{q=0}^{\ell} 2^{jq} C_\ell^q \|(\Delta_W^{\alpha,d})^q m_j\|_{L^2(d\mu_{\alpha,d})} \le C.\end{aligned}$$

Hence the first inequality of (5.9) is proved. The second inequality of (5.9) follows in similar way.

Let us remark that if $||x|-|y|| > 2|y-z|$ and $|u| > ||x|-|z||$ then $|u| > |y-z|$. Therefore, in view of (2.15), (2.16), (5.9) and Fubini-Tonelli's theorem

$$\begin{aligned}&\int_{||x|-|y||>2|y-z|} |\tau_y^\alpha(k_j)(x) - \tau_z^\alpha(k_j)(x)| d\mu_{\alpha,d}(x) \\ &\le \int_{||x|-|y||>2|y-z|} |\tau_y^\alpha(k_j)(x)| d\mu_{\alpha,d}(x) + \int_{||x|-|y||>2|y-z|} |\tau_z(k_j)(x)| d\mu_{\alpha,d}(x) \\ &\le 2\int_{|u|>|y-z|} |k_j(z)|| d\mu_{\alpha,d}(z) \\ &\le C(2^j|y-z|)^{\alpha+\frac{d+1}{2}-\ell}.\end{aligned}$$

On the other hand, Lemma 5.4 and (5.9) give

$$\begin{aligned}&\int_{||x|-|y||>2|y-z|} |\tau_y^\alpha(k_j)(x) - \tau_z^\alpha(k_j)(x)| d\mu_{\alpha,d}(x) \\ &\le \|\tau_y^\alpha(k_j) - \tau_z^\alpha(k_j)\|_{L^1(d\mu_{\alpha,d})} \\ &\le C2^j|y_{d+1} - z_{d+1}|.\end{aligned}$$

Thus to get (5.8), we write

$$\begin{aligned}&\sum_{j=-\infty}^{\infty} \int_{||x|-|y||>2|y-z|} |\tau_y^\alpha(k_j)(x) - \tau_z^\alpha(k_j)(x)| d\mu_{\alpha,d}(x) \\ &\le C\Big(\sum_{\{2^j|y-z|\ge 1\}} (2^j|y-z|)^{\alpha+\frac{d+1}{2}-\ell} + \sum_{\{2^j|y-z|<1\}} 2^j|y_{d+1} - z_{d+1}|\Big) \le C.\end{aligned}$$

This completes the proof of Theorem 5.1. □

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