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NULL CONTROLLABILITY OF POPULATION DYNAMICS WITH INTERIOR DEGENERACY

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Abstract. In this article, we study the null controllability of population model with an interior degenerate diffusion. To this end, we proved first a new Carleman estimate for the full adjoint system and then we deduce a suitable observability inequality which will be needed to establish the existence of a control acting on a subset of the space which lead the population to extinction in a finite time.

1. Introduction

Consider the system

$$\begin{gathered}\frac{\partial y}{\partial t}+\frac{\partial y}{\partial a}-(k(x)y_x)_x+\mu(t,a,x)y=\vartheta\chi_\omega \quad \text{in } Q,\\ y(t,a,1)=y(t,a,0)=0 \quad \text{on } (0,T)\times(0,A),\\ y(0,a,x)=y_0(a,x) \quad \text{in } Q_A,\\ y(t,0,x)=\int_0^A \beta(t,a,x)y(t,a,x)da \quad \text{in } Q_T,\end{gathered}\tag{1.1}$$

where $Q=(0,T)\times(0,A)\times(0,1)$, $Q_A=(0,A)\times(0,1)$, $Q_T=(0,T)\times(0,1)$ and we will denote $q=(0,T)\times(0,A)\times\omega$, where $\omega=(x_1,x_2)\subset\subset(0,1)$ is the region where the control ϑ is acting. This control corresponds to an external supply or to removal of individuals on the subdomain ω. Since (1.1) models the dispersion of gene of a given population, x represents the gene type and $y(t,a,x)$ is the distribution of individuals of age a at time t and of gene type x. The parameters $\beta(t,a,x)$ and $\mu(t,a,x)$ are respectively the natural fertility and mortality rates, A is the maximal age of life and k is the gene dispersion coefficient. $y_0\in L^2(Q_A)$ is the initial distribution of population. Finally, $\int_0^A\beta(t,a,x)y(t,a,x)da$ is the distribution of the newborns of the population that are of gene type x at time t. As usual, we will suppose that no individual reaches the maximal age A. We note that in the most works concerned with the diffusion population dynamics models, x is viewed as the space variable.

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The population models in their different aspects attracted many authors that investigated them from many sides (see for example [18, 20, 21, 25, 28, 29]). Among those questions, we find the null controllability problem or in general the controllability problems for age and space structured population dynamics models which were studied in a intensive literature basing, in general, on the references interested on the controllability of heat equation (see for instance [12, 13, 14, 15, 16, 19] for a different controllability problems of heat equation). In this context, we can cite the pioneering items of Barbu and al. [7], Ainseba and Anita [1, 2, 3, 4]. In [7], the authors proved the null controllability for a population dynamics model without diffusion both in the cases of migration and birth control for $T \geq A$ showing directly an appropriate observability inequality for the associated adjoint system and they concluded that in the case of the migration control, only a classes of age was controlled in contrary with the birth control which allows to steer all population to extinction. In [1, 2, 3, 4], the diffusion was taken into account in a age-space structured model and the null controllability of (1.1) for classes of age was established in the case where $k = 1$ and for any dimension n by means of a weighted estimates called Carleman estimates and exploiting the results gotten for heat equation in [26]. Ainseba et al. [5] studied a more general case allowing the dispersion coefficient to depend on the variable x and satisfies $k(0) = 0$ (i.e, the coefficient of dispersion k degenerates at 0). The authors tried to obtain (1.2) in such a situation with $\beta \in L^\infty$ basing on the work done in [6] for the degenerate heat equation to establish a new Carleman estimate for the full adjoint system (2.3) and afterwards his observability inequality. However, the null controllability property of this paper was showed under the condition $T \geq A$ (as in [7]) and this constitutes a restrictiveness on the "optimality" of the control time T since it means, for example, that for a pest population whose the maximal age A may equal to a many days (may be many months or years) we need much time to bring the population to the zero equilibrium. In the same trend and to overcome the condition $T \geq A$, Maniar et al. [17] suggested the fixed point technique in which the birth rate β must be in $C^2(Q)$ specially in the proof of [17, Proposition 4.2]. Such a technique consists briefly to demonstrate in a first time the null controllability for an intermediate system with a fertility function $b \in L^2(Q_T)$ instead of $\int_0^A \beta(t,a,x)y(t,a,x)da$ and to achieve the task via a Leray-Schauder Theorem.

Thereby, the main goal of this article is to sutdy the null controllability property with a minimum of regularity of β (see (2.2)) and a positive small control time T taking into account that k depends on the gene type and degenerates at a point $x_0 \in \omega$, i.e $k(x_0) = 0$, e.g $k(x) = |x - x_0|^\alpha$. To be more accurate, for a fixed $T \in (0, \delta)$ with $\delta \in (0, A)$ small enough, we investigate the existence of a suitable control $\vartheta \in L^2(q)$ which depends on y_0 and δ and such that the associated solution y of (1.1) satisfies

$$y(T, a, x) = 0, \quad \text{a.e. in } (\delta, A) \times (0, 1). \tag{1.2}$$

If $k(x_0) = 0$ in a point $x_0 \in \omega$, we say that (1.1) is a population dynamics model with interior degeneracy. Genetically speaking, the meaning of $k(x_0) = 0$ is that the gene of type $x_0 \in (0, 1)$ can not be transmitted from the studied population to its offspring. This objective will be attained via the classical procedure following the strategy in [22]. On other words, we will establish an appropriate observability inequality for the full adjoint system of (1.1) which is an outcome of a suitable Carleman estimate. We highlight that such a result can be shown if we replace the homogeneous Dirichlet boundary conditions by the ones of Neumann, i.e., $y_x(t,a,0) = y_x(t,a,1) = 0, (t,a) \in (0,T) \times (0,A)$ using the same way done in [10]. Another interesting null controllability problem of (1.1) can be elaborate using the work of Fragnelli et al. [23] arising in the case when the potential term admits an interior singularity belonging to gene type domain.

The remainder of this paper is organized as follows: in Section 2, we will provide the well-posedness of (1.1) and give the proof of the Carleman estimate of its adjoint system. The Section 3 will be devoted to the observability inequality and hence we obtain the null controllability result (1.2). The last section will take the form of an appendix where we will bring out a Caccioppoli's inequality which plays an important role to show the desired Carleman estimate.

2. Well-posedness and Carleman estimate results

2.1. **Well-posedness.** For this section and for the sequel, we assume that the dispersion coefficient k satisfies

$$\begin{gathered}\exists x_0 \in (0,1),\ k \in C([0,1]) \cap C^1([0,1]\backslash\{x_0\}),\ k > 0 \text{ in } [0,1]\backslash\{x_0\} \text{ and } k(x_0) = 0,\\ \exists \gamma \in [0,1) : (x - x_0)k'(x) \le \gamma k(x),\ x \in [0,1]\backslash\{x_0\}.\end{gathered} \tag{2.1}$$

It is well-known in the literature of degenerate problems that there exist two kinds of degeneracy namely the weakly degenerate and the strong degenerate problems, in our study we will restrict ourselves to the first one and this fact explains the choice of $\gamma \in [0,1)$ which in fact are associated to the Dirichlet boundary conditions (see [22, Hypothesis 1.1]). On the other hand, the last hypothesis on k means in the case of $k(x) = |x - x_0|^\alpha$ that $0 \le \alpha < 1$.

The investigation of (1.2) needs also the following assumptions on the natural rates β and μ:

$$\begin{gathered}\mu, \beta \in L^\infty(Q), \quad \beta(t,a,x), \mu(t,a,x) \ge 0, \quad \text{a.e. in } Q,\\ \beta(\cdot,0,\cdot) \equiv 0 \quad \text{a.e. in } (0,T) \times (0,1).\end{gathered} \tag{2.2}$$

The last assumption in (2.2) is natural since the newborns are not fertile. Also, it is worth mentioning to point out that, as in [5] we do not need to require that μ satisfies an hypotheses like $\int_0^A \mu(t-s, A-s, x)ds = +\infty, \quad (t,x) \in [0,T] \times [0,1]$ since it does not play any role on the well-posedness result and the computations concerning the proofs of our controllability result as well. However, we will suppose that no individual can reach the maximal age A as mentioned in the introduction. In the same context, we emphasize that in [17], the $L^\infty-$regularity of β is sufficient to prove the well posedness of the studied model which is exactly our case. To this end, we introduce the following weighted Sobolev spaces:

$$\begin{aligned} H^1_k(0,1) := \big\{u \in L^2(0,1) : u &\text{ is abs. cont. on } [0,1] :\\ &\sqrt{k}u_x \in L^2(0,1), u(1) = u(0) = 0\big\},\\ H^2_k(0,1) := \big\{u \in H^1_k(0,1) &: k(x)u_x \in H^1(0,1)\big\}, \end{aligned}$$

endowed respectively with the norms

$$\begin{gathered}\|u\|^2_{H^1_k(0,1)} := \|u\|^2_{L^2(0,1)} + \|\sqrt{k}u_x\|^2_{L^2(0,1)}, \quad u \in H^1_k(0,1),\\ \|u\|^2_{H^2_k} := \|u\|^2_{H^1_k(0,1)} + \|(k(x)u_x)_x\|^2_{L^2(0,1)}, \quad u \in H^2_k(0,1).\end{gathered}$$

We recall from [24, Theorem 2.2] that the operator $(P, D(P))$ defined by $Pu := (k(x)u_x)_x$, $u \in D(P) = H_k^2(0,1)$, is closed self-adjoint and negative with dense domain in $L^2(0,1)$. Consequently, from [32, Theorem 5] the operator $\mathcal{A} := -\frac{\partial}{\partial a} + P$ generates a C_0-semigroup on the space $L^2((0,A)\times(0,1))$. Then, the following well-posedness result holds.

Theorem 2.1. *Under assumptions* (2.1) *and* (2.2)*, for all* $\vartheta \in L^2(Q)$ *and* y_0 *in* $L^2(Q_A)$*, system* (1.1) *admits a unique solution* y*. This solution belongs to*

$$\begin{aligned} E :=& C([0,T], L^2((0,A)\times(0,1))) \cap C([0,A], L^2((0,T)\times(0,1))) \\ &\cap L^2((0,T)\times(0,A), H_k^1(0,1)). \end{aligned}$$

Moreover, the solution of (1.1) *satisfies*

$$\begin{aligned} &\sup_{t\in[0,T]} \|y(t)\|^2_{L^2(Q_A)} + \sup_{a\in[0,A]} \|y(a)\|^2_{L^2(Q_T)} + \int_0^1\int_0^A\int_0^T (\sqrt{k(x)}y_x)^2\,dt\,da\,dx \\ &\le C\Big(\int_q \vartheta^2 + \int_{Q_A} y_0^2\,dx\,dx\Big). \end{aligned}$$

2.2. **Carleman estimates.** As we said in the introduction, we will show the main key of this paper namely the Carleman type inequality. In general, it is well-known that to prove a controllability result of a studied model through this a priori estimate, we must show this last for the associated adjoint system. In our case, this adjoint system takes the form

$$\begin{gathered} \frac{\partial w}{\partial t} + \frac{\partial w}{\partial a} + (k(x)w_x)_x - \mu(t,a,x)w = -\beta(t,a,x)w(t,0,x), \\ w(t,a,1) = w(t,a,0) = 0, \\ w(T,a,x) = w_T(a,x), \\ w(t,A,x) = 0, \end{gathered} \tag{2.3}$$

where $T > 0$ and assume that $w_T \in L^2(Q_A)$. Of course, assumptions (2.1) and (2.2) on k, μ and β are perpetuated. To attaint our goal, we will introduce the weight functions

$$\begin{gathered} \varphi(t,a,x) := \Theta(t,a)\psi(x), \\ \Theta(t,a) := \frac{1}{(t(T-t))^4 a^4}, \\ \psi(x) := c_1\Big(\int_{x_0}^x \frac{r-x_0}{k(r)}dr - c_2\Big). \end{gathered} \tag{2.4}$$

For the moment, we will assume that $c_2 > \max\{\frac{(1-x_0)^2}{k(1)(2-\gamma)}, \frac{x_0^2}{k(0)(2-\gamma)}\}$ and $c_1 > 0$. A more precise restriction on c_1 will be given later. On the other hand, using the relation satisfied by c_2 and with the aid of [22, Lemma 2.1] one can prove that $\psi(x) < 0$ for all $x \in [0,1]$. Observe also that $\Theta(a,t) \to +\infty$ as $t \to T^-, 0^+$ and $a \to 0^+$. To demonstrate our Carleman estimate, we require that k fulfills, besides (2.1) the following hypothesis

$$\begin{gathered} \exists\theta \in (0,\gamma] \text{ such that } x \mapsto \frac{k(x)}{|x-x_0|^\theta} \text{ is non-increasing on the left of} \\ x = x_0 \text{ and nondecreasing on the right of } x = x_0, \end{gathered} \tag{2.5}$$

where γ is defined by (2.1). The first Carleman estimate result is the following result.

Proposition 2.2. *Consider the following two systems with $h \in L^2(Q)$,*

$$\begin{gathered}
\frac{\partial w}{\partial t} + \frac{\partial w}{\partial a} + (k(x)w_x)_x = h, \\
w(a,t,1) = w(a,t,0) = 0, \\
w(T,a,x) = w_T(a,x), \\
w(t,A,x) = 0,
\end{gathered} \tag{2.6}$$

and

$$\begin{gathered}
\frac{\partial w}{\partial t} + \frac{\partial w}{\partial a} + (k(x)w_x)_x - \mu(t,a,x)w = h, \\
w(t,a,1) = w(t,a,0) = 0, \\
w(T,a,x) = w_T(a,x), \\
w(t,A,x) = 0.
\end{gathered} \tag{2.7}$$

Then, there exist two positive constants C and s_0, such that every solution of (2.6) and (2.7) satisfy, for all $s \geq s_0$, the inequality

$$\begin{aligned}
&s^3 \int_Q \Theta^3 \frac{(x-x_0)^2}{k(x)} w^2 e^{2s\varphi}\,dt\,da\,dx + s\int_Q \Theta k(x) w_x^2 e^{2s\varphi}\,dt\,da\,dx \\
&\leq C\Big(\int_Q \mid h \mid^2 e^{2s\varphi}\,dt\,da\,dx + s\int_0^A\int_0^T [k\Theta e^{2s\varphi}(x-x_0)w_x^2]_{x=0}^{x=1}\,dt\,da\Big).
\end{aligned} \tag{2.8}$$

Proof. Firstly, we prove (2.8) for system (2.6) and replacing h by $h+\mu w$ we will get the same inequality for (2.7). So, let w be the solution of (2.6) and put

$$\nu(t,a,x) := e^{s\varphi(t,a,x)} w(t,a,x).$$

Then, ν satisfies the system

$$\begin{gathered}
L_s^+\nu + L_s^-\nu = e^{s\varphi(t,a,x)}h, \\
\nu(t,a,1) = \nu(t,a,0) = 0, \\
\nu(T,a,x) = \nu(0,a,x) = 0, \\
\nu(t,A,x) = \nu(t,0,x) = 0,
\end{gathered} \tag{2.9}$$

where

$$\begin{gathered}
L_s^+\nu := (k(x)\nu_x)_x - s(\varphi_a + \varphi_t)\nu + s^2\varphi_x^2 k(x)\nu, \\
L_s^-\nu := \nu_t + \nu_a - 2sk(x)\varphi_x\nu_x - s(k(x)\varphi_x)_x\nu.
\end{gathered}$$

Passing to the norm in (2.9), one has

$$\|L_s^+\nu\|^2_{L^2(Q)} + \|L_s^-\nu\|^2_{L^2(Q)} + 2\langle L_s^+\nu, L_s^-\nu\rangle = \|e^{s\varphi(a,t,x)}h\|^2_{L^2(Q)},$$

where $\langle\cdot,\cdot\rangle$ denotes here the inner product in $L^2(Q)$. Then, the proof of step one is based on the calculus of the inner product $\langle L_s^+\nu, L_s^-\nu\rangle$ whose a first expression is given in the following lemma.

Lemma 2.3. *The following identity holds*

$$\langle L_s^+\nu, L_s^-\nu\rangle = S_1 + S_2,$$

with

$$
\begin{aligned}
S_1 &= s\int_Q (k(x)\nu_x)^2\varphi_{xx}\,dt\,da\,dx - s^3\int_Q (k(x)\varphi_x)_x k(x)\varphi_x^2\nu^2\,dt\,da\,dx\\
&\quad + s^2\int_Q (\varphi_a+\varphi_t)(k(x)\varphi_x)_x\nu^2\,dt\,da\,dx\\
&\quad + s\int_Q k(x)\nu_x((k(x)\varphi_x)_{xx}\nu + (k(x)\varphi_x)_x\nu_x)\ ,dt\,da\,dx\\
&\quad + s^3\int_Q (k^2\varphi_x^3)_x\nu^2\,dt\,da\,dx - s^2\int_Q (k(x)(\varphi_a+\varphi_t)\varphi_x)_x\nu^2\,dt\,da\,dx\\
&\quad + \frac{s}{2}\int_Q (\varphi_{at}+\varphi_{tt})\nu^2\,dt\,da\,dx - \frac{s^2}{2}\int_Q (\varphi_x^2)_t k(x)\nu^2\,dt\,da\,dx\\
&\quad + \frac{s}{2}\int_Q (\varphi_{at}+\varphi_{aa})\nu^2\,dt\,da\,dx - \frac{s^2}{2}\int_Q (\varphi_x^2)_a k(x)\nu^2\,dt\,da\,dx,
\end{aligned}
\tag{2.10}
$$

and

$$
\begin{aligned}
&S_2\\
&= \int_0^A\int_0^T [k(x)\nu_x\nu_a]_0^1\,dt\,da + \int_0^A\int_0^T [k(x)\nu_x\nu_t]_0^1\,dt\,da\\
&\quad + s^2\int_0^A\int_0^T [k(x)\varphi_x(\varphi_a+\varphi_t)\nu^2]_0^1\,dt\,da - s^3\int_0^A\int_0^T [k^2(x)\varphi_x^3\nu^2]_0^1\,dt\,da\\
&\quad - s\int_0^A\int_0^T [k(x)\nu\nu_x(k(x)\varphi_x)_x]_0^1\,dt\,da - s\int_0^A\int_0^T [(k(x)\nu_x)^2\varphi_x]_0^1\,dt\,da.
\end{aligned}
\tag{2.11}
$$

For the proof of Lemma 2.3, see [17, Lemma 3.2]. The previous expressions of S_1 and S_2 can be simplified using the functions φ and ψ given in (2.4) and also the homogeneous Dirichlet boundary conditions satisfied by ν. Hence, one has

$$
\begin{aligned}
S_1 &= \frac{s}{2}\int_Q (\Theta_{aa}+\Theta_{tt})\psi\nu^2 dx\,dt\,da + s\int_Q \Theta_{ta}\psi\nu^2\,dt\,da\,dx\\
&\quad + sc_1\int_Q \Theta(2k(x)-(x-x_0)k'(x))\nu_x^2\,dt\,da\,dx\\
&\quad - 2s^2\int_Q \Theta c_1^2\frac{(x-x_0)^2}{k(x)}(\Theta_a+\Theta_t)\nu^2\,dt\,da\,dx\\
&\quad + s^3\int_Q \Theta^3 c_1^3\Big(\frac{x-x_0}{k(x)})^2(2k(x)-(x-x_0)k'(x)\Big)\nu^2\,dt\,da\,dx,
\end{aligned}
\tag{2.12}
$$

and

$$
S_2 = -sc_1\int_0^A\int_0^T [k\Theta e^{2s\varphi}(x-x_0)\nu_x^2]_{x=0}^{x=1}\,dt\,da.
$$

Accordingly,

$$
\begin{aligned}
&\langle L_s^+\nu, L_s^-\nu\rangle \\
&= \frac{s}{2}\int_Q (\Theta_{aa}+\Theta_{tt})\psi\nu^2 dx\,dt\,da + s\int_Q \Theta_{ta}\psi\nu^2\,dt\,da\,dx \\
&\quad + sc_1\int_Q \Theta(2k(x)-(x-x_0)k'(x))\nu_x^2\,dt\,da\,dx \\
&\quad - 2s^2\int_Q \Theta c_1^2\frac{(x-x_0)^2}{k(x)}(\Theta_a+\Theta_t)\nu^2\,dt\,da\,dx \\
&\quad + s^3\int_Q \Theta^3 c_1^3\big(\frac{x-x_0}{k(x)}\big)^2(2k(x)-(x-x_0)k'(x))\nu^2\,dt\,da\,dx \\
&\quad - sc_1\int_0^A\int_0^T [k\Theta e^{2s\varphi}(x-x_0)w_x^2]_{x=0}^{x=1}\,dt\,da.
\end{aligned}
\tag{2.13}
$$

Thanks to the third assumption in (2.1), we have

$$
\begin{aligned}
S_1 \geq{}& \frac{s}{2}\int_Q (\Theta_{aa}+\Theta_{tt})\psi\nu^2\,dt\,da\,dx + s\int_Q \Theta_{ta}\psi\nu^2\,dt\,da\,dx \\
&+ sc_1\int_Q \Theta k(x)\nu_x^2\,dt\,da\,dx \\
&- 2s^2\int_Q \Theta c_1^2\frac{(x-x_0)^2}{k(x)}(\Theta_a+\Theta_t)\nu^2\,dt\,da\,dx \\
&+ s^3\int_Q \Theta^3 c_1^3\frac{(x-x_0)^2}{k(x)}\nu^2\,dt\,da\,dx.
\end{aligned}
\tag{2.14}
$$

Observe that $|\Theta(\Theta_a+\Theta_t)| \leq c\Theta^3$; for s large we infer that

$$
\begin{aligned}
&\big|-2s^2\int_Q \Theta c_1^2\frac{(x-x_0)^2}{k(x)}(\Theta_a+\Theta_t)\nu^2\,dt\,da\,dx\big| \\
&\leq 2s^2c_1^2c\int_Q \frac{(x-x_0)^2}{k(x)}\Theta^3\nu^2\,dt\,da\,dx \\
&\leq \frac{c_1^3}{4}s^3\int_Q \frac{(x-x_0)^2}{k(x)}\Theta^3\nu^2\,dt\,da\,dx.
\end{aligned}
\tag{2.15}
$$

On the other hand, the mapping $r \mapsto \frac{|r-x_0|^\gamma}{k(r)}$ is nondecreasing at the right of x_0. Then

$$
\begin{aligned}
|\psi(x)| = |c_1 l(x) - c_1c_2| &\leq c_1\Big|\int_{x_0}^x \frac{r-x_0}{k(r)}dr\Big| + c_1c_2 \\
&\leq c_1c_2 + c_1\frac{(1-x_0)^2}{k(1)(2-\gamma)} \leq \frac{c_1}{(2-\gamma)k(1)} + c_1c_2.
\end{aligned}
\tag{2.16}
$$

A simple computations shows that $\Theta_{aa}+\Theta_{tt}+|\Theta_{ta}|\leq C_1\Theta^{3/2}$. This yields

$$\begin{aligned} &\Big|\frac{s}{2}\int_Q(\Theta_{aa}+\Theta_{tt})\psi\nu^2\,dt\,da\,dx+s\int_Q\Theta_{ta}\psi\nu^2\,dt\,da\,dx\Big|\\ &\leq s\Big(\frac{c_1}{(2-\gamma)k(1)}+c_1c_2\Big)\int_Q\Big(\frac{\Theta_{aa}+\Theta_{tt}}{2}+|\Theta_{ta}|\Big)\nu^2\,dt\,da\,dx\\ &\leq Ms\Big(\frac{c_1}{(2-\gamma)k(1)}+c_1c_2\Big)\int_Q\Theta^{3/2}\nu^2\,dt\,da\,dx. \end{aligned}\tag{2.17}$$

It remains now to bound the term $|\int_Q\Theta^{3/2}\nu^2\,dt\,da\,dx|$. Using the generalized Young inequality we obtain

$$\begin{aligned} \Big|\int_0^1\Theta^{3/2}\nu^2dx\Big| &= \Big|\int_0^1\Big(\Theta\frac{k^{1/3}}{|x-x_0|^{\frac{2}{3}}}\nu^2\Big)^{3/4}\Big(\Theta^3\frac{|x-x_0|^2}{k}\nu^2\Big)^{1/4}dx\Big|\\ &\leq\frac{3\epsilon}{4}\int_0^1\Theta\frac{k^{1/3}}{|x-x_0|^{\frac{2}{3}}}\nu^2dx+\frac{1}{4\epsilon}\int_0^1\Theta^3\frac{|x-x_0|^2}{k}\nu^2dx \end{aligned}\tag{2.18}$$

Put

$$p(x)=(k(x)|x-x_0|^4)^{1/3}.\tag{2.19}$$

By hypothesis (2.5), one can check that $x\mapsto\frac{p(x)}{|x-x_0|^q}$, with $q:=\frac{4+\theta}{3}\in(1,2)$ is nonincreasing on the left of $x=x_0$ and nondecreasing on the right of $x=x_0$. Furthermore, we have $\frac{k^{1/3}}{|x-x_0|^{\frac{2}{3}}}=\frac{p(x)}{(x-x_0)^2}$ and there exists $C_2>0$ such that $p(x)<C_2k(x)$. Hence, by Hardy-Poincaré inequality (see [22, Proposition 2.3]),

$$\begin{aligned} \int_0^1\Theta\frac{k^{1/3}}{|x-x_0|^{\frac{2}{3}}}\nu^2dx &= \int_0^1\Theta\frac{p(x)}{(x-x_0)^2}\nu^2dx\\ &\leq C\int_0^1\Theta p\nu_x^2dx\\ &\leq CC_2\int_0^1\Theta k\nu_x^2dx, \end{aligned}\tag{2.20}$$

where $C>0$ is the constant of Hardy-Poincaré. Combining (2.18) and (2.20), we obtain

$$\Big|\int_0^1\Theta^{3/2}\nu^2dx\Big|\leq\frac{3\epsilon}{4}C_3\int_Q\Theta k\nu_x^2\,dt\,da\,dx+\frac{1}{4\epsilon}\int_Q\Theta^3\frac{|x-x_0|^2}{k}\nu^2\,dt\,da\,dx.\tag{2.21}$$

Hence, (2.17) and (2.21) lead to

$$\begin{aligned} &\Big|\frac{s}{2}\int_Q(\Theta_{aa}+\Theta_{tt})\psi\nu^2\,dt\,da\,dx+s\int_Q\Theta_{ta}\psi\nu^2\,dt\,da\,dx\Big|\\ &\leq sc_1C_4\epsilon\int_Q\Theta k\nu_x^2\,dt\,da\,dx+\frac{sc_1C_5}{4\epsilon}\int_Q\Theta^3\frac{|x-x_0|^2}{k}\nu^2\,dt\,da\,dx. \end{aligned}\tag{2.22}$$

Taking ϵ small enough and s large, we conclude that

$$\begin{aligned} &\Big|\frac{s}{2}\int_Q(\Theta_{aa}+\Theta_{tt})\psi\nu^2\,dt\,da\,dx+s\int_Q\Theta_{ta}\psi\nu^2\,dt\,da\,dx\Big|\\ &\leq\frac{sc_1}{4}\int_Q\Theta k\nu_x^2\,dt\,da\,dx+\frac{s^3c_1^3}{4}\int_Q\Theta^3\frac{|x-x_0|^2}{k}\nu^2\,dt\,da\,dx. \end{aligned}\tag{2.23}$$

Taking into account relations (2.14) and (2.15) we arrive at

$$S_1 \geq K_1 s^3 \int_Q \Theta^3 \frac{|x-x_0|^2}{k} \nu^2 \, dt\, da\, dx + K_2 s \int_Q \Theta k \nu_x^2 \, dt\, da\, dx \tag{2.24}$$

Hence,

$$\begin{aligned} 2\langle L_s^+ \nu, L_s^- \nu\rangle &\geq m\Big(s^3 \int_Q \Theta^3 \frac{|x-x_0|^2}{k} \nu^2 \, dt\, da\, dx + s \int_Q \Theta k \nu_x^2 \, dt\, da\, dx\Big) \\ &\quad - 2sc_1 \int_0^A \int_0^T [k\Theta e^{2s\varphi}(x-x_0)\nu_x^2]_{x=0}^{x=1} \, dt\, da \end{aligned} \tag{2.25}$$

This yields to the following Carleman estimate satisfied by the solution ν of (2.9),

$$\begin{aligned} &s^3 \int_Q \Theta^3 \frac{|x-x_0|^2}{k} \nu^2 \, dt\, da\, dx + s \int_Q \Theta k \nu_x^2 \, dt\, da\, dx \\ &\leq C_6\Big(\int_Q h^2 e^{2s\varphi} \, dt\, da\, dx + s \int_0^A \int_0^T [k\Theta e^{2s\varphi}(x-x_0)\nu_x^2]_{x=0}^{x=1} \, dt\, da\Big) \end{aligned} \tag{2.26}$$

By the definition of ν we infer that

$$\nu_x = s\varphi_x e^{s\varphi} w + e^{s\varphi} w_x, \quad e^{2s\varphi} w_x^2 \leq 2(\nu_x^2 + s^2 \varphi_x^2 \nu^2). \tag{2.27}$$

Finally, the Carleman estimate (2.8) of (2.6) is obtained.

Now, If we apply the same inequality of Hardy-Poincaré in a similar way as before to the function $\nu := e^{s\varphi} w$, taking into account the hypothesis on μ assumed in (2.2), using the Carleman type inequality (2.8) for the function $h+\mu w$ and taking s quite we achieve the Proposition 2.2. □

With the aid of the estimate (2.8) and Caccioppoli's inequality (4.1), we can now show a ω-local Carleman estimate for the system (2.7). This result will be useful to show our main Carleman estimate replacing the second term h by $-\beta(t,a,x)w(t,0,x)$. To this end, we introduce the weight functions

$$\begin{gathered} \Phi(t,a,x) := \Theta(t,a)\Psi(x), \\ \Psi(x) = e^{\kappa\sigma(x)} - e^{2\kappa\|\sigma\|_\infty}, \end{gathered} \tag{2.28}$$

where Θ is given by (2.4), $\kappa > 0$, and σ is the function given by

$$\begin{gathered} \sigma \in C^2([0,1]), \quad \sigma(x) > 0 \quad \text{in } (0,1), \quad \sigma(0) = \sigma(1) = 0, \\ \sigma_x(x) \neq 0 \quad \text{in } [0,1]\backslash\omega_0, \end{gathered} \tag{2.29}$$

where $\omega_0 \Subset \omega$ is an open subset. The existence of the function σ is proved in [26]. On the other hand by the definition of φ (2.4) and taking

$$c_1 \geq \max\Big(\frac{k(1)(2-\gamma)(e^{2\kappa\|\sigma\|_\infty}-1)}{c_2 k(1)(2-\gamma) - (1-x_0)^2}, \frac{k(0)(2-\gamma)(e^{2\kappa\|\sigma\|_\infty}-1)}{c_2 k(0)(2-\gamma) - x_0^2}\Big), \tag{2.30}$$

one can prove that

$$\varphi \leq \Phi. \tag{2.31}$$

Our main theorem is stated as follows.

Theorem 2.4. *Assume that assumptions (2.1), (2.2) and (2.5) hold. Let $A > 0$ and $T > 0$ be given. Then, there exist positive constants C and s_0 such that for all $s \geq s_0$, every solution w of (2.7) satisfies*

$$\begin{aligned}
&\int_Q (s\Theta k w_x^2 + s^3\Theta^3 \frac{(x-x_0)^2}{k} w^2) e^{2s\varphi}\, dt\, da\, dx \\
&\leq C\left(\int_Q h^2 e^{2s\Phi}\, dt\, da\, dx + \int_q s^3\Theta^3 w^2 e^{2s\Phi}\, dt\, da\, dx\right),
\end{aligned} \tag{2.32}$$

To prove this theorem, we need the following result which represents the Carleman estimate of nondegenerate population dynamics systems. The inequality is stated as follows.

Proposition 2.5. *Let us consider the system*

$$\begin{gathered}
\frac{\partial z}{\partial t} + \frac{\partial z}{\partial a} + (k(x)z_x)_x - c(t,a,x)z = h \quad \text{in } Q_b, \\
z(t,a,b_1) = z(t,a,b_2) = 0 \quad \text{on } (0,T)\times(0,A),
\end{gathered} \tag{2.33}$$

where $Q_b := (0,T)\times(0,A)\times(b_1,b_2)$, $(b_1,b_2) \subset [0,x_0)$, or $(b_1,b_2) \subset (x_0,1]$, $h \in L^2(Q_b)$, $k \in C^1([0,1])$ is a strictly positive function and $c \in L^\infty(Q_b)$. Then, there exist two positive constants C and s_0, such that for any $s \geq s_0$, z satisfies

$$\begin{aligned}
&\int_{Q_b} (s^3\phi^3 z^2 + s\phi z_x^2) e^{2s\Phi}\, dt\, da\, dx \\
&\leq C\Big(\int_{Q_b} h^2 e^{2s\Phi}\, dt\, da\, dx + \int_\omega \int_0^A \int_0^T s^3\phi^3 z^2 e^{2s\Phi}\, dt\, da\, dx\Big),
\end{aligned} \tag{2.34}$$

where

$$\phi(t,a,x) = \Theta(t,a)e^{\kappa\sigma(x)}, \tag{2.35}$$

Θ and Φ are defined by (2.28), and σ by (2.29).

Before giving the proof of Theorem 2.4, we note that a similar result was demonstrated in [2, Lemma 2.1] in the case when k is a positive constant, for any dimension n without the source term h and with the weight function $\Theta(t,a) = \frac{1}{t(T-t)a}$. By careful computations, the same proof can be adapted to (2.34) where k is a positive general nondegenerate coefficient, with our weight function $\Theta(t,a) = \frac{1}{t^4(T-t)^4a^4}$ and the source term h.

Proof of Theorem 2.4. Let us introduce the smooth cut-off function $\xi : \mathbb{R} \to \mathbb{R}$ defined by

$$\begin{gathered}
0 \leq \xi(x) \leq 1, \quad x \in [0,1], \\
\xi(x) = 1, \quad x \in [\lambda_1, \lambda_2], \\
\xi(x) = 0, \quad x \in [0,1]\backslash\omega,
\end{gathered} \tag{2.36}$$

where $\lambda_1 = \frac{x_1+2x_0}{3}$ and $\lambda_2 = \frac{x_0+2x_2}{3}$. Let w be the solution of (2.7) and define $v := \xi w$. Then, v satisfies the system

$$\begin{gathered}
\frac{\partial v}{\partial t} + \frac{\partial v}{\partial a} + (k(x)v_x)_x - \mu(t,a,x)v = \overline{h}, \\
v(t,a,1) = v(t,a,0) = 0, \\
v(T,a,x) = \xi w_T(a,x), \\
v(t,A,x) = 0,
\end{gathered} \tag{2.37}$$

where $\overline{h} := \xi h + (k(x)\xi_x w)_x + k(x)w_x\xi_x$. Using Carleman estimate (2.8) and the definition of ξ, one has

$$\int_Q (s\Theta k v_x^2 + s^3\Theta^3 \frac{(x-x_0)^2}{k} v^2)e^{2s\varphi}\,dt\,da\,dx \le C\int_Q \overline{h}^2 e^{2s\varphi}\,dt\,da\,dx. \tag{2.38}$$

On the other hand, using again the definition of ξ we can check readily that

$$\begin{aligned}&\int_{\lambda_1}^{\lambda_2}\int_0^A\int_0^T (s\Theta k v_x^2 + s^3\Theta^3 \frac{(x-x_0)^2}{k} v^2)e^{2s\varphi}\,dt\,da\,dx\\ &= \int_{\lambda_1}^{\lambda_2}\int_0^A\int_0^T (s\Theta k w_x^2 + s^3\Theta^3 \frac{(x-x_0)^2}{k} w^2)e^{2s\varphi}\,dt\,da\,dx.\end{aligned} \tag{2.39}$$

Therefore, combining (2.38) and (2.39) we have

$$\begin{aligned}&\int_{\lambda_1}^{\lambda_2}\int_0^A\int_0^T (s\Theta k w_x^2 + s^3\Theta^3 \frac{(x-x_0)^2}{k} w^2)e^{2s\varphi}\,dt\,da\,dx\\ &\le C\int_Q \overline{h}^2 e^{2s\varphi}\,dt\,da\,dx.\end{aligned} \tag{2.40}$$

Hence by Caccioppoli's inequality (4.1) and (2.40), we conclude that

$$\begin{aligned}&\int_{\lambda_1}^{\lambda_2}\int_0^A\int_0^T (s\Theta k w_x^2 + s^3\Theta^3 \frac{(x-x_0)^2}{k} w^2)e^{2s\varphi}\,dt\,da\,dx\\ &\le C\Big(\int_Q h^2 e^{2s\varphi}\,dt\,da\,dx + \int_q s^2\Theta^2 w^2 e^{2s\varphi}\,dt\,da\,dx\Big),\end{aligned} \tag{2.41}$$

where ω' of Lemma 4.1 here is exactly $(x_1,\lambda_1)\cup(\lambda_2,x_2)$.

Now, let $z := \eta w$, with η is the smooth cut-off function defined by

$$\begin{gathered}0 \le \eta(x) \le 1, \quad x\in[0,1],\\ \eta(x) = 0, \quad x \in [0, \frac{\lambda_3+2\lambda_2}{3}],\\ \eta(x) = 1, \quad x\in[\lambda_2,1],\end{gathered} \tag{2.42}$$

where $\lambda_3 = \frac{x_2+2x_0}{3}$. We can observe easily that $\lambda_3 < \frac{\lambda_3+2\lambda_2}{3} < \lambda_2$. Then, z satisfies the population dynamics equation

$$\begin{gathered}\frac{\partial z}{\partial t} + \frac{\partial z}{\partial a} + k(x)z_{xx} + k'(x)z_x - \mu(t,a,x)z = \widetilde{h}, \quad \text{in } (\lambda_3, 1)\\ z(t,a,1) = z(t,a,\lambda_3) = 0, \quad \text{in }, (0,T)\times(0,A),\end{gathered} \tag{2.43}$$

where $\widetilde{h} := \eta h + (k(x)\eta_x w)_x + k(x)w_x\eta_x$. By assumption on k, we have $k(x) > 0$, $x\in(\lambda_3,1)$. Hence, (2.43) is a non-degenerate model. In this case, applying Proposition 2.5 to the function $\widetilde{h}$ with $b_1 = \lambda_3$ and $b_2 = 1$ and using again

Caccioppoli's inequality (4.1), we infer that

$$
\begin{aligned}
&\int_{\lambda_3}^1\int_0^A\int_0^T (s^3\phi^3z^2+s\phi z_x^2)e^{2s\Phi}\,dt\,da\,dx\\
&\le C\Big(\int_{\lambda_3}^1\int_0^A\int_0^T (\eta h+(k\eta_x w)_x+k\eta_x w_x)^2e^{2s\Phi}\,dt\,da\,dx\\
&\quad+\int_\omega\int_0^A\int_0^T s^3\Theta^3w^2e^{2s\Phi}\,dt\,da\,dx\Big)\\
&\le \widetilde{C}\Big(\int_Q h^2e^{2s\Phi}+((k\eta_x w)_x+k\eta_x w_x)^2e^{2s\Phi}\,dt\,da\,dx\\
&\quad+\int_\omega\int_0^A\int_0^T s^3\Theta^3w^2e^{2s\Phi}\,dt\,da\,dx\Big)\\
&\le \widetilde{C}(\int_Q h^2e^{2s\Phi}\,dt\,da\,dx+\int_{\omega'}\int_0^A\int_0^T (8(k\eta_x)^2w_x^2+2((k\eta_x)_x)^2w^2)e^{2s\Phi}\,dt\,da\,dx\\
&\quad+\int_\omega\int_0^A\int_0^T s^3\Theta^3w^2e^{2s\Phi}\,dt\,da\,dx)\\
&\le \widetilde{C}_1\Big(\int_Q h^2e^{2s\Phi}\,dt\,da\,dx+\int_{\omega'}\int_0^A\int_0^T (w_x^2+w^2)e^{2s\Phi}\,dt\,da\,dx\\
&\quad+\int_\omega\int_0^A\int_0^T s^3\Theta^3w^2e^{2s\Phi}\,dt\,da\,dx\Big)\\
&\le \widetilde{C}_2\Big(\int_Q h^2e^{2s\Phi}\,dt\,da\,dx+\int_\omega\int_0^A\int_0^T s^3\Theta^3w^2e^{2s\Phi}\,dt\,da\,dx\Big),
\end{aligned}
\tag{2.44}
$$

with $\omega' := (\frac{\lambda_3+2\lambda_2}{3},\lambda_2)$. By the restriction (2.31) there exists $c_3>0$ such that, for $(t,a,x)\in[0,T]\times[0,A]\times[\lambda_3,1]$, we have

$$
\Theta k(x)e^{2s\varphi}\le c_3\phi e^{2s\Phi},\quad \Theta^3\frac{(x-x_0)^2}{k(x)}e^{2s\varphi}\le c_3\phi^3e^{2s\Phi}.
$$

Then

$$
\begin{aligned}
&\int_{\lambda_3}^1\int_0^A\int_0^T (s\Theta kz_x^2+s^3\Theta^3\frac{(x-x_0)^2}{k}z^2)e^{2s\varphi}\,dt\,da\,dx\\
&\le c_3\int_{\lambda_3}^1\int_0^A\int_0^T (s^3\phi^3z^2+s\phi z_x^2)e^{2s\Phi}\,dt\,da\,dx.
\end{aligned}
\tag{2.45}
$$

This inequality and (2.44) lead to

$$
\begin{aligned}
&\int_{\lambda_3}^1\int_0^A\int_0^T (s\Theta kz_x^2+s^3\Theta^3\frac{(x-x_0)^2}{k}z^2)e^{2s\varphi}\,dt\,da\,dx\\
&\le \widetilde{c_3}\Big(\int_Q h^2e^{2s\Phi}\,dt\,da\,dx+\int_\omega\int_0^A\int_0^T s^3\Theta^3w^2e^{2s\Phi}\,dt\,da\,dx\Big).
\end{aligned}
\tag{2.46}
$$

Taking into account the definition of η (2.42), we can say that

$$\begin{aligned}&\int_{\lambda_2}^1\int_0^A\int_0^T(s\Theta kz_x^2+s^3\Theta^3\frac{(x-x_0)^2}{k}z^2)e^{2s\varphi}\,dt\,da\,dx\\&=\int_{\lambda_2}^1\int_0^A\int_0^T(s\Theta kw_x^2+s^3\Theta^3\frac{(x-x_0)^2}{k}w^2)e^{2s\varphi}\,dt\,da\,dx.\end{aligned}\tag{2.47}$$

Hence,

$$\begin{aligned}&\int_{\lambda_2}^1\int_0^A\int_0^T(s\Theta kw_x^2+s^3\Theta^3\frac{(x-x_0)^2}{k}w^2)e^{2s\varphi}\,dt\,da\,dx\\&\le\tilde{c_3}\Big(\int_Q h^2e^{2s\Phi}\,dt\,da\,dx+\int_\omega\int_0^A\int_0^T s^3\Theta^3w^2e^{2s\Phi}\,dt\,da\,dx\Big),\end{aligned}\tag{2.48}$$

as a consequence of (2.46) and (2.47). Arguing in the same way for $(0,\lambda_1)$, one can show that

$$\begin{aligned}&\int_0^{\lambda_1}\int_0^A\int_0^T(s\Theta kw_x^2+s^3\Theta^3\frac{(x-x_0)^2}{k}w^2)e^{2s\varphi}\,dt\,da\,dx\\&\le\tilde{c_3}\Big(\int_Q h^2e^{2s\Phi}\,dt\,da\,dx+\int_\omega\int_0^A\int_0^T s^3\Theta^3w^2e^{2s\Phi}\,dt\,da\,dx\Big),\end{aligned}\tag{2.49}$$

Finally, summing the inequalities (2.41), (2.48) and (2.49) side by side, taking s large and using again the restriction on c_1, (2.31), we arrive at

$$\begin{aligned}&\int_0^1\int_0^A\int_0^T(s\Theta kw_x^2+s^3\Theta^3\frac{(x-x_0)^2}{k}w^2)e^{2s\varphi}\,dt\,da\,dx\\&\le\tilde{c_3}\Big(\int_Q h^2e^{2s\Phi}\,dt\,da\,dx+\int_\omega\int_0^A\int_0^T s^3\Theta^3w^2e^{2s\Phi}\,dt\,da\,dx\Big),\end{aligned}\tag{2.50}$$

and this is exactly the desired estimate (2.32). □

Before providing the main Carleman estimate, we make the following remarks.

Remark 2.6. 1. The proof of our distributed-Carleman estimate (2.32) is based on the cut-off functions and given by two different weighted functions φ and Φ, in addition by (2.31) there is no positive constant C such that

$$e^{2s\Phi}\le Ce^{2s\varphi}.$$

2. Our proof is not based on the reflection method used for the proof of [22, Lemma 4.1] which is needed to eliminate the boundary term arising in the classical Carleman estimate for nondegenerate heat equation.

By the Carleman estimate (2.32), we are able to show the following ω-Carleman estimate for the full adjoint system (2.3).

Theorem 2.7. *Assume that* (2.1), (2.2) *and* (2.5) *hold. Let $A>0$ and $T>0$ be given such that $0<T<\delta$, where $\delta\in(0,A)$ small enough. Then, there exist positive constants C and s_0 such that for all $s\ge s_0$, every solution w of* (2.7) *satisfies*

$$\begin{aligned}&\int_Q(s\Theta kw_x^2+s^3\Theta^3\frac{(x-x_0)^2}{k}w^2)e^{2s\varphi}\,dt\,da\,dx\\&\le C\Big(\int_q s^3\Theta^3w^2e^{2s\Phi}\,dt\,da\,dx+\int_0^1\int_0^\delta w_T^2(a,x)\,dx\,dx\Big),\end{aligned}\tag{2.51}$$

for all $s \geq s_0$ and δ verifying (2.2).

Proof. Applying inequality (2.32) to the function $h(t,a,x) = -\beta(t,a,x)w(t,0,x)$, we have the existence of two positive constants C and s_0 such that, for all $s \geq s_0$, the following inequality holds

$$
\begin{aligned}
& s^3 \int_Q \Theta^3 \frac{(x-x_0)^2}{k(x)} w^2 e^{2s\varphi}\,dt\,da\,dx + s\int_Q \Theta k(x) w_x^2 e^{2s\varphi}\,dt\,da\,dx \\
& \leq C\Big(\int_Q \beta^2 w^2(t,0,x) e^{2s\Phi}\,dt\,da\,dx + \int_q s^3\Theta^3 w^2 e^{2s\Phi}\,dt\,da\,dx\Big) \\
& \leq C\Big(A\|\beta\|_\infty^2 \int_0^1\int_0^T w^2(t,0,x)\,dt\,dx + \int_q s^3\Theta^3 w^2 e^{2s\Phi}\,dt\,da\,dx\Big),
\end{aligned}
\tag{2.52}
$$

using (2.2). On the other hand, integrating over the characteristics lines and after a careful calculus we obtain the following implicit formula of w, the solution of (2.3),

$$
w(t,a,\cdot) = \begin{cases} \int_0^{A-a} S(A-a-l)\beta(t,A-l,\cdot)w(t,0,\cdot)dl \\ \quad \text{if } a > t + (A-T) \\ S(T-t)w_T(T+(a-t),\cdot) + \int_t^T S(l-t)\beta(l,a,\cdot)w(l,0,\cdot)dl \\ \quad \text{if } a \leq t + (A-T), \end{cases}
\tag{2.53}
$$

where $(S(t))_{t\geq 0}$ is the semi-group generated by the operator $A_2 w = (kw_x)_x - \mu w$. Thus,

$$
w(t,0,\cdot) = S(T-t)w_T(T-t,\cdot), \tag{2.54}
$$

using the last hypothesis in (2.2) on β. Inserting this formula in (2.52) and using the fact $(S(t))_{t\geq 0}$ is a bounded semi-group, we obtain

$$
\begin{aligned}
& s^3 \int_Q \Theta^3 \frac{(x-x_0)^2}{k(x)} w^2 e^{2s\varphi}\,dt\,da\,dx + s\int_Q \Theta k(x) w_x^2 e^{2s\varphi}\,dt\,da\,dx \\
& \leq \widehat{C}\Big(\int_0^1\int_0^T w_T^2(T-t,x)\,dt\,dx + \int_q s^3\Theta^3 w^2 e^{2s\Phi}\,dt\,da\,dx\Big) \\
& \leq \widehat{C}\Big(\int_0^1\int_0^T w_T^2(m,x)\,dm\,dx + \int_q s^3\Theta^3 w^2 e^{2s\Phi}\,dt\,da\,dx\Big) \\
& \leq \widehat{C}\Big(\int_0^1\int_0^\delta w_T^2(m,x)\,dm\,dx + \int_q s^3\Theta^3 w^2 e^{2s\Phi}\,dt\,da\,dx\Big),
\end{aligned}
\tag{2.55}
$$

since $T \in (0,\delta)$ with $\delta \in (0,A)$ small enough and this achieves the proof of (2.51). □

3. Observability inequality and null controllability

3.1. **Observability inequality.** The objective of this paragraph is to reach the observability inequality of the adjoint system (2.3). To attain this purpose, we combine the Carleman estimate (2.51) with the Hardy-Poincaré inequality stated in [22, Proposition 2.3] and arguing in a similar way as in [2]. Our observability inequality is given by the following proposition.

Proposition 3.1. *Assume that* (2.1), (2.2) *and* (2.5) *hold. Let $A > 0$ and $T > 0$ be given such that $0 < T < \delta$, where $\delta \in (0,A)$ is small enough. Then, there* *exists a positive constant C_δ such that for every solution w of (2.3), the following observability inequality holds*

$$\int_0^1\int_0^A w^2(0,a,x)\,dx\,dx \le C_\delta\Big(\int_q w^2\,dt\,da\,dx+\int_0^1\int_0^\delta w_T^2(a,x)\,dx\,dx\Big). \tag{3.1}$$

Proof. Let w be a solution of (2.3). Then for $\kappa>0$ to be defined later, let $\widetilde{w}=e^{\kappa t}w$ be a solution of

$$\begin{aligned}\frac{\partial \widetilde{w}}{\partial t}+\frac{\partial \widetilde{w}}{\partial a}+(k(x)\widetilde{w}_x)_x-(\mu(t,a,x)+\kappa)\widetilde{w}&=-\beta\widetilde{w}(t,0,x),\\ \widetilde{w}(t,a,1)=\widetilde{w}(t,a,0)&=0,\\ \widetilde{w}(T,a,x)&=e^{\kappa T}w_T(a,x),\\ \widetilde{w}(t,A,x)&=0.\end{aligned} \tag{3.2}$$

We point out that the parameter κ considered here is not the same as in (2.28). Multiplying the first equation of (3.2) by $\widetilde{w}$ and integrating by parts on $Q_t=(0,t)\times(0,A)\times(0,1)$. Then, one obtains

$$\begin{aligned}&-\frac12\int_{Q_A}\widetilde{w}^2(t,a,x)\,dx\,dx+\frac12\int_{Q_A}w^2(0,a,x)\,dx\,dx+\frac12\int_0^1\int_0^t\widetilde{w}^2(\tau,0,x)d\tau dx\\ &+\kappa\int_0^1\int_0^A\int_0^t\widetilde{w}^2(\tau,a,x)d\tau\,dx\,dx\\ &\le\frac{\|\beta\|_\infty^2}{4\epsilon'}\int_0^1\int_0^A\int_0^t\widetilde{w}^2(\tau,a,x)d\tau\,dx\,dx+\epsilon' A\int_0^1\int_0^t\widetilde{w}^2(\tau,0,x)d\tau dx.\end{aligned} \tag{3.3}$$

Thus, for $\kappa=\frac{\|\beta\|_\infty^2}{4\epsilon'}$ and $\epsilon'<\frac{1}{2A}$, integrating over $(\frac{T}{4},\frac{3T}{4})$ we obtain

$$\int_{Q_A}w^2(0,a,x)\,dx\,dx\le C_{12}e^{2\kappa T}\int_{Q_A}\int_{T/4}^{3T/4}w^2(t,a,x)\,dx\,dx. \tag{3.4}$$

On the other hand, let us prove that there exists a positive constant C_δ such that

$$\int_0^1\int_0^{\delta-\frac{3T}{4}}\int_{T/4}^{3T/4}w^2(t,a,x)\,dt\,da\,dx\le C_\delta\int_0^1\int_0^\delta w_T^2(a,x)\,dx\,dx. \tag{3.5}$$

For this purpose, we use the implicit formula of w defined by (2.53) and we discuss the two cases: when $a>t+(A-T)$ and when $a\le t+(A-T)$. In fact, if $a>t+(A-T)$ one has

$$\begin{aligned}w(t,a,\cdot)&=\int_0^{A-a}S(A-a-l)\beta(t,A-l,\cdot)w(t,0,\cdot)dl\\ &=\int_0^{A-a}S(A-a-l)\beta(t,A-l,\cdot)S(T-t)w_T(T-t,\cdot)dl,\end{aligned} \tag{3.6}$$

using (2.54). Since $(S(t))_{t\ge0}$ is a bounded semi-group and $\beta\in L^\infty(Q)$, one can see that for $T\in(0,\delta)$

$$\begin{aligned}\int_0^1\int_0^{\delta-\frac{3T}{4}}\int_{T/4}^{3T/4}w^2(t,a,x)\,dt\,da\,dx&\le\widetilde{C}_{10}\int_0^1\int_{T/4}^{3T/4}w_T^2(T-t,x)\,dt\,dx\\ &\le\widetilde{C}_{10}\int_0^1\int_0^\delta w_T^2(m,x)\,dm\,dx,\end{aligned} \tag{3.7}$$

Now, if $a \leq t + (A - T)$ one has

$$
\begin{aligned}
w(t,a,\cdot) &= S(T-t)w_T(T+(a-t),\cdot) + \int_t^T S(l-t)\beta(l,a,\cdot)w(l,0,\cdot)dl \\
&= S(T-t)w_T(T+(a-t),\cdot) \\
&\quad + \int_t^T S(l-t)\beta(l,a,\cdot)S(T-l)w_T(T-l,\cdot)dl.
\end{aligned} \tag{3.8}
$$

Thanks to the same argument employed to get (3.7), we conclude that

$$
\begin{aligned}
&\int_0^1 \int_0^{\delta-\frac{3T}{4}} \int_{T/4}^{3T/4} w^2(t,a,x)\,dt\,da\,dx \\
&\leq 2\widetilde{C}_{11}\Big(\int_0^1 \int_0^{\delta-\frac{3T}{4}} \int_{T/4}^{3T/4} w_T^2(T+(a-t),x)\,dt\,da\,dx \\
&\quad + \int_0^1 \int_t^T w_T^2(T-l,x)\,dl\,dx\Big).
\end{aligned} \tag{3.9}
$$

On the one hand, we can check that

$$
\begin{aligned}
&\int_0^1 \int_0^{\delta-\frac{3T}{4}} \int_{T/4}^{3T/4} w_T^2(T+(a-t),x)\,dt\,da\,dx \\
&\leq \int_0^1 \int_0^{\delta-\frac{3T}{4}} \int_{T/4}^{3T/4} w_T^2(a+m,x)dm\,dx\,dx \\
&\leq \int_0^1 \int_0^{\delta-\frac{3T}{4}} \int_{a+\frac{T}{4}}^{a+\frac{3T}{4}} w_T^2(z,x)dz\,dx\,dx \\
&\leq \int_0^1 \int_0^{\delta-\frac{3T}{4}} \int_0^{\delta} w_T^2(z,x)dz\,dx\,dx \\
&\leq \delta \int_0^1 \int_0^{\delta} w_T^2(z,x)\,dz\,dx.
\end{aligned} \tag{3.10}
$$

On the other hand, we have the inequality

$$
\begin{aligned}
\int_0^1 \int_t^T w_T^2(T-l,x)\,dl\,dx &= \int_0^1 \int_0^{T-t} w_T^2(z,x)\,dz\,dx \\
&\leq \int_0^1 \int_0^{\delta} w_T^2(z,x)\,dz\,dx.
\end{aligned} \tag{3.11}
$$

Combining inequalities (3.9), (3.10) and (3.11) we obtain

$$
\int_0^1 \int_0^{\delta-\frac{3T}{4}} \int_{T/4}^{3T/4} w^2(t,a,x)\,dt\,da\,dx \leq \widetilde{C}_{12} \int_0^1 \int_0^{\delta} w_T^2(z,x)\,dz\,dx. \tag{3.12}
$$

Subsequently, (3.5) occurs in both studied cases. Therefore, in light of inequality (3.4) we conclude that

$$
\begin{aligned}
&\int_{Q_A} w^2(0,a,x)\,dx\,dx \\
&\leq \widetilde{C}_{13} \int_0^1 \int_0^{\delta} w_T^2(a,x)\,dx\,dx + \frac{2e^{2\kappa T}}{T} \int_0^1 \int_{\delta-\frac{3T}{4}}^{A} \int_{T/4}^{3T/4} w^2(t,a,x)\,dt\,da\,dx.
\end{aligned} \tag{3.13}
$$

Now, let p defined by (2.19). Then, using the hypotheses (2.1) on k the function $x \mapsto \frac{(x-x_0)^2}{p(x)}$ is non-increasing at the left of x_0 and nondecreasing at the right of x_0. Hence, applying Hardy-Poincaré inequality (see [22, Proposition 2.3]) and taking into account the definition of φ stated in (2.4) we have

$$\begin{aligned}\int_{Q_A} w^2(0,a,x)\,dx\,dx \leq \widetilde{C}_{13} \int_0^1 \int_0^\delta w_T^2(a,x)\,dx\,dx \\ + C_\delta^{13} \int_0^1 \int_{\delta-\frac{3T}{4}}^A \int_{T/4}^{3T/4} s\Theta k(x) w_x^2(t,a,x) e^{2s\varphi}\,dt\,da\,dx.\end{aligned}$$

Therefore, using Carleman estimate (2.51) we infer that

$$\int_{Q_A} w^2(0,a,x)\,dx\,dx \leq \widetilde{C}_\delta^{15} \Big(\int_q s^3\Theta^3 w^2 e^{2s\Phi}\,dt\,da\,dx + \int_0^1 \int_0^\delta w_T^2(a,x)\,dx\,dx \Big),$$

and then the proof is complete using the fact that $\sup_{(t,a,x)\in Q} s^d\Theta^d e^{2s\Phi} < +\infty$ for all $d\ in \mathbb{R}$. $\square$

3.2. **Null controllability.** In the previous paragraph, we obtained the observability inequality of system (2.3). Such a tool will be very useful to prove the null controllability of the model (1.1) in the case where $T \in (0,\delta)$ as we emphasized in the introduction. Our main result is provided in the following theorem.

Theorem 3.2. *Assume that the dispersion coefficient k satisfies* (2.1) *and the natural rates β and μ verify* (2.2). *Let $A, T > 0$ be given such that $0 < T < \delta$, where $\delta \in (0, A)$ small enough. For all $y_0 \in L^2(Q_A)$, there exists a control $\vartheta \in L^2(q)$ such that the associated solution of* (1.1) *satisfies*

$$y(T,a,x) = 0, \quad a.e. \ in\ (\delta, A) \times (0,1). \tag{3.14}$$

Furthermore, there exists a positive constant C_{10} which depends on δ such that ϑ satisfies the inequality

$$\int_q \vartheta^2(t,a,x)\,dt\,da\,dx \leq C_{10} \int_{Q_A} y_0^2(a,x)\,dx\,dx. \tag{3.15}$$

The constant C_{10} is called the control cost.

Before proving , we make the following remark.

Remark 3.3. Inequality (3.15) shows us clearly that the control that we want depends on δ and the initial distribution y_0.

Proof Theorem 3.2. Let $\varepsilon > 0$ and consider the cost function

$$J_\varepsilon(\vartheta) = \frac{1}{2\varepsilon} \int_0^1 \int_\delta^A y^2(T,a,x)\,dx\,dx + \frac{1}{2} \int_q \vartheta^2(t,a,x)\,dt\,da\,dx.$$

We can prove that J_ε is continuous, convex and coercive. Then, it admits at least one minimizer ϑ_ε and we have

$$\vartheta_\varepsilon = -w_\varepsilon(t,a,x)\chi_\omega(x) \quad \text{in } Q, \tag{3.16}$$

with w_ε is the solution of the system

$$\begin{gathered}\frac{\partial w_\varepsilon}{\partial t}+\frac{\partial w_\varepsilon}{\partial a}+(k(x)(w_\varepsilon)_x)_x-\mu(t,a,x)w_\varepsilon=-\beta w_\varepsilon(t,0,x)\quad\text{in } Q,\\ w_\varepsilon(t,a,1)=w_\varepsilon(t,a,0)=0\quad\text{on } (0,T)\times(0,A),\\ w_\varepsilon(T,a,x)=\frac{1}{\varepsilon}y_\varepsilon(T,a,x)\chi_{(\delta,A)}(a)\quad\text{in } Q_A,\\ w_\varepsilon(t,A,x)=0\quad\text{in } Q_T,\end{gathered}\tag{3.17}$$

and y_ε is the solution of the system (1.1) associated to the control ϑ_ε. Multiplying (3.17) by y_ε, integrating over Q, using (3.16) and the Young inequality we obtain

$$\begin{aligned}&\frac{1}{\varepsilon}\int_0^1\int_\delta^A y_\varepsilon^2(T,a,x)\,dx\,dx+\int_q\vartheta_\varepsilon^2(t,a,x)\,dt\,da\,dx\\ &=\int_{Q_A}y_0(a,x)w_\varepsilon(0,a,x)\,dx\,dx\\ &\le\frac{1}{4C_\delta}\int_{Q_A}w_\varepsilon^2(0,a,x)\,dx\,dx+C_\delta\int_{Q_A}y_0^2(a,x)\,dx\,dx,\end{aligned}\tag{3.18}$$

with C_δ is the constant of the observability inequality (3.1). This again leads to

$$\begin{aligned}&\frac{1}{\varepsilon}\int_0^1\int_\delta^A y_\varepsilon^2(T,a,x)\,dx\,dx+\int_q\vartheta_\varepsilon^2(t,a,x)\,dt\,da\,dx\\ &\le\frac{1}{4}\int_q w^2\,dt\,da\,dx+C_\delta\int_{Q_A}y_0^2(a,x)\,dx\,dx.\end{aligned}\tag{3.19}$$

Keeping in the mind (3.16), we conclude that

$$\frac{1}{\varepsilon}\int_0^1\int_\delta^A y_\varepsilon^2(T,a,x)\,dx\,dx+\frac{3}{4}\int_q\vartheta_\varepsilon^2(t,a,x)\,dt\,da\,dx\le C_\delta\int_{Q_A}y_0^2(a,x)\,dx\,dx.$$

Hence, it follows that

$$\begin{aligned}&\int_0^1\int_\delta^A y_\varepsilon^2(T,a,x)\,dx\,dx\le\varepsilon C_\delta\int_{Q_A}y_0^2(a,x)\,dx\,dx\\ &\int_q\vartheta_\varepsilon^2(t,a,x)\,dt\,da\,dx\le\frac{4C_\delta}{3}\int_{Q_A}y_0^2(a,x)\,dx\,dx.\end{aligned}\tag{3.20}$$

Then, we can extract two subsequences of y_ε and ϑ_ε denoted also by ϑ_ε and y_ε that converge weakly towards ϑ and y in $L^2(q)$ and $L^2((0,T)\times(0,A);H_k^1(0,1))$ respectively. Now, by a variational technic, we prove that y is a solution of (1.1) corresponding to the control ϑ and, by the first estimate of (3.20), y satisfies (3.14) for $T\in(0,\delta)$ and this shows our claim. □

4. Appendix

As we said in the introduction, this Appendix concerns a result which plays an important role to show the ω-Carleman estimate associated to the full adjoint system (2.3) namely the Caccioppoli's inequality which is stated in the following lemma.

Lemma 4.1. *Let $\omega' \subset\subset \omega$ and w be the solution of (2.7). Suppose that $x_0 \notin \overline{\omega'}$. Then, there exists a positive constant C such that w satisfies*

$$\begin{aligned} &\int_{\omega'}\int_0^A\int_0^T w_x^2 e^{2s\varphi}\,dt\,da\,dx \\ &\leq C\Big(\int_q s^2\Theta^2 w^2 e^{2s\varphi}\,dt\,da\,dx + \int_q h^2 e^{2s\varphi}\,dt\,da\,dx\Big). \end{aligned} \tag{4.1}$$

Proof. Define the smooth cut-off function $\zeta : \mathbb{R} \to \mathbb{R}$ by

$$\begin{gathered} 0 \leq \zeta(x) \leq 1, \quad x \in \mathbb{R}, \\ \zeta(x) = 0, \quad x < x_1 \text{ and } x > x_2, \\ \zeta(x) = 1, \quad x \in \omega'. \end{gathered} \tag{4.2}$$

For the solution w of (2.7), we have

$$\begin{aligned} 0 &= \int_0^T \frac{d}{dt}\Big[\int_0^1\int_0^A \zeta^2 e^{2s\varphi} w^2\,dx\,dx\Big]dt \\ &= 2s\int_0^1\int_0^A\int_0^T \zeta^2\varphi_t w^2 e^{2s\varphi}\,dt\,da\,dx + 2\int_0^1\int_0^A\int_0^T \zeta^2 w w_t e^{2s\varphi}\,dt\,da\,dx \\ &= 2s\int_0^1\int_0^A\int_0^T \zeta^2\varphi_t w^2 e^{2s\varphi}\,dt\,da\,dx \\ &\quad + 2\int_0^1\int_0^A\int_0^T \zeta^2 w(-(kw_x)_x - w_a + h + \mu w)e^{2s\varphi}\,dt\,da\,dx. \end{aligned} \tag{4.3}$$

Then, integrating by parts we obtain

$$\begin{aligned} &2\int_Q k\zeta^2 e^{2s\varphi} w_x^2\,dt\,da\,dx \\ &= -2s\int_Q \zeta^2 w^2\psi(\Theta_a + \Theta_t)e^{2s\varphi}\,dt\,da\,dx - 2\int_Q \zeta^2 w h e^{2s\varphi}\,dt\,da\,dx \\ &\quad - 2\int_Q \zeta^2\mu w^2 e^{2s\varphi}\,dt\,da\,dx + \int_Q (k(\zeta^2 e^{2s\varphi})_x)_x w^2\,dt\,da\,dx. \end{aligned}$$

On the other hand, by the definitions of ζ, ψ and Θ, thanks to Young inequality, taking s quite large and using the fact that $x_0 \notin \overline{\omega'}$, one can prove the existence of a positive constant c such that

$$\begin{gathered} 2\int_Q k\zeta^2 e^{2s\varphi} w_x^2\,dt\,da\,dx \geq 2\min_{x\in\omega'} k(x)\int_{\omega'}\int_0^A\int_0^T w_x^2 e^{2s\varphi}\,dt\,da\,dx, \\ \int_Q (k(\zeta^2 e^{2s\varphi})_x)_x w^2\,dt\,da\,dx \leq c\int_\omega\int_0^A\int_0^T s^2\Theta^2 w^2 e^{2s\varphi}\,dt\,da\,dx, \\ -2s\int_Q \zeta^2 w^2\psi(\Theta_a + \Theta_t)e^{2s\varphi}\,dt\,da\,dx \leq c\int_\omega\int_0^A\int_0^T s^2\Theta^2 w^2 e^{2s\varphi}\,dt\,da\,dx, \\ \begin{aligned} &-2\int_Q \zeta^2 w h e^{2s\varphi}\,dt\,da\,dx \\ &\leq c\Big(\int_\omega\int_0^A\int_0^T s^2\Theta^2 w^2 e^{2s\varphi}\,dt\,da\,dx + \int_\omega\int_0^A\int_0^T h^2 e^{2s\varphi}\,dt\,da\,dx\Big), \end{aligned} \end{gathered}$$

$$-2\int_Q \zeta^2 \mu w^2 e^{2s\varphi}\,dt\,da\,dx \le c\int_\omega \int_0^A \int_0^T s^2\Theta^2 w^2 e^{2s\varphi}\,dt\,da\,dx.$$

This implies that there is $C>0$ such that

$$\int_{\omega'}\int_0^A\int_0^T w_x^2 e^{2s\varphi}\,dt\,da\,dx \le C\Big(\int_q s^2\Theta^2 w^2 e^{2s\varphi}\,dt\,da\,dx + \int_q h^2 e^{2s\varphi}\,dt\,da\,dx\Big).$$

Thus, the proof is complete. □

Remark 4.2. Lemma 4.1 remains valid for any function $\pi \in C([0,1],(-\infty,0)) \cap C^1([0,1]\backslash\{x_0\},(-\infty,0))$ satisfying

$$|\pi_x| \le \frac{c}{\sqrt{k}}, \quad \text{for } x\in[0,1]\backslash\{x_0\},$$

where $c>0$. see [22, Proposition 4.2] for more details.

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