Electronic Journal of Differential Equations, Vol. 2017 (2017), No. 133, pp. 1–10.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

EXISTENCE AND NONEXISTENCE OF SOLUTIONS FOR SUBLINEAR PROBLEMS WITH PRESCRIBED NUMBER OF ZEROS ON EXTERIOR DOMAINS

JANAK JOSHI

Communicated by Jerome A. Goldstein

Abstract. We prove existence of radial solutions of $\Delta u + K(r)f(u) = 0$ on the exterior of the ball, of radius R, centered at the origin in $\mathbb{R}^N$ such that $\lim_{r\to\infty} u(r) = 0$ if $R > 0$ is sufficiently small. We assume $f : \mathbb{R} \to \mathbb{R}$ is odd and there exists a $\beta > 0$ with $f < 0$ on $(0, \beta)$, $f > 0$ on (β, ∞) with f sublinear for large u, and $K(r) \sim r^{-\alpha}$ for large r with $\alpha > 2(N - 1)$. We also prove nonexistence if $R > 0$ is sufficiently large.

1. Introduction

In this article we study radial solutions to

$$\begin{gathered} \Delta u + K(|x|)f(u) = 0 \quad \text{for } R < |x| < \infty, \\ u(x) = 0 \text{ when } |x| = R, \quad \lim_{|x|\to\infty} u(x) = 0, \end{gathered} \tag{1.1}$$

where $u : \mathbb{R}^N \to \mathbb{R}$ with $N \geq 2$, $R > 0$, f is odd and locally Lipschitz. In addition we have the following:

- (H1) $f'(0) < 0$, there exists $\beta > 0$ such that $f(u) < 0$ on $(0, \beta)$, $f(u) > 0$ on (β, ∞).
- (H2) $f(u) = |u|^{p-1}u + g(u)$ with $0 < p < 1$, and $\lim_{u\to\infty} \frac{|g(u)|}{|u|^p} = 0$.
- (H3) Denoting $F(u) \equiv \int_0^u f(t)\,dt$ we assume that there exists γ with $0 < \beta < \gamma$ such that $F < 0$ on $(0, \gamma)$ and $F > 0$ on (γ, ∞).
- (H4) We assume K and K' are continuous on $[R, \infty)$, $K(r) > 0$, and there exists $\alpha > 2(N - 1)$ such that $\lim_{r\to\infty} \frac{rK'}{K} = -\alpha$, $2(N - 1) + \frac{rK'}{K} < 0$.
- (H5) We assume that there exists positive d_1, d_2 such that $d_1 r^{-\alpha} \leq K(r) \leq d_2 r^{-\alpha}$ for $r \geq R$.

Theorem 1.1. *Assume* (H1)–(H5)*, $N > 2$, and $\alpha > 2(N - 1)$. Then for each nonnegative integer n there exists a radial solution, u_n, of* (1.1) *such that u_n has exactly n zeros on (R, ∞) if R is positive and sufficiently small.*

Theorem 1.2. *Let $N \geq 2$ and $\alpha > 2(N-1)$. If R is positive and sufficiently large, then there are no nontrivial radial solutions of* (1.1).

2010 *Mathematics Subject Classification.* 34B40, 35B05.
Key words and phrases. Exterior domain; sublinear; radial solution.

Submitted March 6, 2017. Published May 16, 2017.

The radial solutions of (1.1) on $\mathbb{R}^N$ with f superlinear for large u and $K(r) \equiv 1$ have been well-studied; see for example [1, 2, 9, 11]. Recently there has been interest in studying these problems on $\mathbb{R}^N \backslash B_R(0)$, with various types of non linearities, see [4, 5, 6, 7, 8, 10, 12]. Interest in the topic for this paper comes from the papers [5, 6, 7] by Iaia where he studied the solutions of the differential equations with superlinear and hilltop type of nonlinearity. The type of nonlinearity addressed here has not been studied as extensively as other cases, see [4, 8]. In [8] the same nonlinearity as here was studied but in the case $\alpha < 2$. In this article we use different methods to study the case $\alpha > 2(N-1)$. We use a scaling argument as in [11] to prove existence of solutions for (1.1), (1.2) with large number of zeros.

For the proofs in sections 2 and 3, we will need to temporarily extend K and K' continuously to $(0,\infty)$ so that (H4) and (H5) continue to hold on $(0,\infty)$. We define

$$\tilde{K}(r) = \begin{cases} K(R) - \frac{K'(R)R^{\alpha+1}}{\alpha}[\frac{1}{r^\alpha} - \frac{1}{R^\alpha}] & r \le R \\ K(r) & r \ge R. \end{cases} \tag{1.2}$$

It follows that

$$\lim_{r \to R^-} \tilde{K}(r) = K(R) \quad \text{and} \quad \lim_{r \to R^-} \tilde{K}'(r) = K'(R).$$

It is straightforward to verify that $\tilde{K}$ and $\tilde{K}'$ extend K and K' continuously on $(0,\infty)$ and $\tilde{K}$ satisfies (H4) on $(0,\infty)$. In addition

$$\lim_{r \to 0^+} r^\alpha \tilde{K}(r) = -\frac{K'(R)R^{\alpha+1}}{\alpha} > 0.$$

Therefore with perhaps a smaller positive number d_1 and larger positive number d_2 we may ensure (H5) on $(0,\infty)$.

2. Preliminaries

Since we are interested in radial solutions of (1.1) we denote $r = |x|$ and consider $u(x) = u(|x|)$ where u satisfies

$$u'' + \frac{N-1}{r}u' + K(r)f(u) = 0 \quad \text{for } R < r < \infty, \tag{2.1}$$

$$u(R) = 0, \quad u'(R) = b > 0. \tag{2.2}$$

We will occasionally write $u(r,b)$ to emphasize the dependence of the solution on b. By the standard existence-uniqueness theorem [3] there is a unique solution of (2.1)-(2.2) on $[R, R+\epsilon)$ for some $\epsilon > 0$.

We next consider:

$$E(r) = \frac{1}{2}\frac{u'^2}{K(r)} + F(u). \tag{2.3}$$

It is straightforward using (2.1) and (H4) to show that

$$E'(r) = -\frac{u'^2}{2rK}[2(N-1) + \frac{rK'}{K}] \ge 0. \tag{2.4}$$

Thus E is non-decreasing. Therefore

$$\frac{1}{2}\frac{u'^2}{K(r)} + F(u) = E(r) \ge E(R) = \frac{1}{2}\frac{b^2}{K(R)} \quad \text{for } r \ge R. \tag{2.5}$$

Proof of Theorem 1.2. Suppose there is a nontrivial solution $u(r)$ of (2.1)-(2.2) with $\lim_{r\to\infty} u(r)=0$. Then u has a local maximum at some $M_b > R$ and $u' > 0$ on $[R, M_b)$. Evaluating (2.5) at $r = M_b$ gives

$$F(u(M_b)) \geq \frac{1}{2}\frac{b^2}{K(R)} > 0.$$

So by (H3) we see

$$u(M_b) > \gamma. \tag{2.6}$$

Now by (2.4) we have

$$E(r) \leq E(M_b) \quad \text{for } R \leq r \leq M_b$$

so

$$\frac{1}{2}\frac{u'^2}{K} + F(u) \leq F(u(M_b)) \quad \text{for } R \leq r \leq M_b. \tag{2.7}$$

Rewriting this expression, integrating over $[R, M_b]$ and applying (H5) yields

$$\int_R^{M_b} \frac{|u'|\,dr}{\sqrt{2}\sqrt{F(u(M_b)) - F(u)}} \leq \int_R^{M_b} \sqrt{K} \leq \sqrt{d_2}\int_R^{M_b} r^{-\frac{\alpha}{2}}\,dr. \tag{2.8}$$

Setting $u(r) = t$ we have

$$\int_0^{u(M_b)} \frac{dt}{\sqrt{2}\sqrt{F(u(M_b)) - F(t)}} \leq \frac{2\sqrt{d_2}}{\alpha-2}\big[R^{\frac{2-\alpha}{2}} - M_b^{\frac{2-\alpha}{2}}\big] \leq \frac{2\sqrt{d_2}}{\alpha-2}R^{\frac{2-\alpha}{2}}. \tag{2.9}$$

Next since $f(0)=0$, $f'(0)<0$ and $0<p<1$ it follows that $\lim_{u\to 0}\frac{F(u)}{|u|^{p+1}} = 0$. Also from (H2) we see that $\lim_{u\to\infty}\frac{F(u)}{|u|^{p+1}} = \frac{1}{p+1}$. Then there exists $c_1 > 0$ such that

$$|F(u)| \leq c_1|u|^{p+1} \quad \forall u. \tag{2.10}$$

Moreover, by (H1) and (H3) we have

$$F(u) \geq -F_0 \quad \text{for some } F_0 > 0 \tag{2.11}$$

and so it follows from (2.10) and (2.11) that

$$F(u(M_b)) - F(u) \leq c_1|u(M_b)|^{p+1} + F_0$$

for all u. This along with (2.9) implies

$$\frac{2\sqrt{d_2}}{\alpha-2}R^{\frac{2-\alpha}{2}} \geq \int_0^{u(M_b)} \frac{dt}{\sqrt{F(u(M_b)) - F(t)}} \geq \frac{u(M_b)}{\sqrt{c_1|u(M_b)|^{p+1} + F_0}}. \tag{2.12}$$

Since $u(M_b) > \gamma$ by (2.6) and $0<p<1$, we have

$$\frac{u(M_b)}{\sqrt{c_1|u(M_b)|^{p+1} + F_0}} = \frac{|u(M_b)|^{\frac{1-p}{2}}}{\sqrt{c_1 + \frac{F_0}{|(u(M_b)|^{p+1}}}} \geq \frac{\gamma^{\frac{1-p}{2}}}{\sqrt{c_1 + \frac{F_0}{\gamma^{p+1}}}}. \tag{2.13}$$

Thus by (2.12) and (2.13) it follows that

$$\frac{2\sqrt{d_2}}{\alpha-2}R^{\frac{2-\alpha}{2}} \geq \frac{\gamma^{\frac{1-p}{2}}}{\sqrt{c_1 + \frac{F_0}{\gamma^{p+1}}}}$$

which implies

$$R^{\frac{\alpha-2}{2}} \leq \frac{2\sqrt{d_2}}{\alpha-2}\frac{\sqrt{c_1 + \frac{F_0}{\gamma^{p+1}}}}{\gamma^{\frac{1-p}{2}}}. \tag{2.14}$$

Since $\alpha > 2$ we see that (17) is violated if R is sufficiently large. Hence there are no solutions of (1.1) such that $\lim_{r\to\infty} u(r) = 0$ if R is sufficiently large. This completes the proof. $\square$

3. Proof of Theorem 1.1

To prove Theorem 1.1 we first make the following change of variables. Let

$$u(r) = u_1(r^{2-N}), \tag{3.1}$$

$$R^* = R^{2-N}, \quad b^* = \frac{bR^{N-1}}{N-2}. \tag{3.2}$$

This transforms (2.1)-(2.2) into

$$u_1''(t) + h(t)f(u_1(t)) = 0, \quad \text{with} \quad t = r^{2-N} \text{ for } 0 < t < R^*, \tag{3.3}$$

where

$$u_1(R^*) = 0, \quad u_1'(R^*) = -b^* < 0, \tag{3.4}$$

$$h(t) = \frac{1}{(N-2)^2} t^{\frac{2(N-1)}{2-N}} K(t^{\frac{1}{2-N}}). \tag{3.5}$$

Since $(r^{2(N-1)}K)' < 0$ and $N > 2$ it follows that

$$h'(t) > 0 \quad \text{for } 0 < t < R^*. \tag{3.6}$$

In addition, from (H4)-(H5) we see that

$$\frac{d_1}{(N-2)^2} \le \frac{h(t)}{t^q} \le \frac{d_2}{(N-2)^2} \quad \text{for } 0 < t \le R^*, \tag{3.7}$$

where

$$q = \frac{\alpha - 2(N-1)}{N-2} > 0.$$

Moreover by using (1.2) and (3.5), we can extend h on $(0,\infty)$ such that (3.6)-(3.7) hold on $(0,\infty)$ and that

$$\lim_{t\to\infty} \frac{h(t)}{t^q} = -\frac{K'(R)R^{\alpha+1}}{(N-2)^2\alpha} \equiv L > 0. \tag{3.8}$$

Now instead of considering (3.3)-(3.4) we look at the initial value problem

$$u_1''(t) + h(t)f(u_1(t)) = 0, \quad t > 0 \tag{3.9}$$

with:

$$u_1(0) = 0, \quad u_1'(0) = a > 0. \tag{3.10}$$

From (3.5) and (3.7) it follows that $h(t)$ can be extended to be continuous at $t = 0$ with $h(0) = 0$ and so by the standard existence-uniqueness theorem there is a unique solution of (3.9)-(3.10) on $[0, 2\epsilon]$ for some $\epsilon > 0$. Let

$$E_1(t) = \frac{1}{2}\frac{u_1'^2}{h(t)} + F(u_1).$$

Then using (3.6) and (3.9) we see that

$$E_1'(t) = -\frac{u_1'^2 h'}{2h^2} \le 0.$$

Thus

$$\frac{1}{2}\frac{u_1'^2}{h(t)} + F(u_1) = E_1(t) \le E_1(\epsilon) = \frac{1}{2}\frac{u_1'^2(\epsilon)}{h(\epsilon)} + F(u_1(\epsilon)) \quad \text{for } t > \epsilon. \tag{3.11}$$

It follows from (3.11) that u_1 and u_1' are uniformly bounded on $[\epsilon, t]$ from which it follows that the solution of (3.9)-(3.10) exists on $[0, t]$. Since t is arbitrary, we see the solution of (3.9)-(3.10) exists on $[0, \infty)$.

Lemma 3.1. *Suppose* (H1)–(H5) *hold and* u_1 *solves* (3.9)-(3.10). *Then there exists* $t_a > 0$ *such that* $u(t_a) = \beta$. *Moreover* $\lim_{a\to 0^+} t_a = \infty$.

Proof. From (3.10) we see that u_1 increases initially and $u_1 > 0$ for small $t > 0$. If u_1 has a first local maximum M then $u_1'(M) = 0$ and $u_1''(M) \le 0$. By uniqueness of solutions of initial value problems it follows that $u''(M) < 0$. Then since $h > 0$ it follows from (3.9) that $f(u_1(M)) > 0$ which in turn from (H1) implies $u_1(M) > \beta$. Since $u_1(0) = 0$ it then follows by the Intermediate Value Theorem that the first part of the lemma holds. Otherwise suppose $0 < u_1 < \beta$ and $u_1' > 0$ for $t > 0$. Then by (H1) we have $f(u_1(t)) < 0$ for all $t > 0$. Thus it follows from (3.9) that $u_1''(t) > 0$ for $t > 0$ so $u_1'(t) > a$ and hence $u_1(t) > at$ for $t > 0$. This implies $u_1(t) \to \infty$ as $t \to \infty$ contradicting that $0 < u_1(t) < \beta$ for all $t > 0$. Hence there exists $t_a > 0$ such that $u_1(t_a) = \beta$ and $0 < u_1 < \beta$ on $(0, t_a)$. This proves the first part of Lemma 3.1.

Now we let

$$E_2(t) = \frac{1}{2}u_1'^2 + h(t)F(u_1).$$

By (H3) and (3.6) we see

$$E_2'(t) = \big[\frac{1}{2}u_1'^2 + h(t)F(u_1)\big]' = h'(t)F(u_1) < 0 \quad \text{if } u_1 < \gamma$$

and since $0 \le u_1 \le \beta < \gamma$ on $[0, t_a]$ this implies

$$\frac{1}{2}u_1'^2 + h(t)F(u_1) \le \frac{1}{2}a^2 \quad \text{on } [0, t_a]. \tag{3.12}$$

Also by (H2)-(H3) there exists $c_2 > 0$ such that

$$F(u_1) \ge -c_2 u_1^2 \quad \text{on } [0, \gamma]. \tag{3.13}$$

It then follows from (3.12) and (3.13) that

$$\frac{1}{2}u_1'^2 - c_2\, h(t)\, u_1^2 \le \frac{1}{2}u_1'^2 + h(t)F(u_1) \le \frac{1}{2}a^2 \quad \text{on } [0, t_a]$$

which on rewriting and using (3.7) gives

$$u_1'^2 \le a^2 + 2c_2 h(t)u_1^2 \le a^2 + \frac{2c_2 d_2}{(N-2)^2}t^q u_1^2 \quad \text{on } [0, t_a]$$

which implies

$$u_1' \le a + c_3 t^{q/2} u_1 \quad \text{on } [0, t_a] \tag{3.14}$$

where $c_3 = \frac{\sqrt{2c_2 d_2}}{N-2}$. After rewriting (3.14), multiplying by $e^{-y(s)}$ where $y(t) = \frac{c_3\, t^{\frac{q}{2}+1}}{\frac{q}{2}+1}$, and integrating on $(0, t)$ we see

$$u_1 e^{-y(t)} = \int_0^t \Big(u_1 e^{-y(s)}\, ds\Big)' \le \int_0^t a\, e^{-y(s)}\, ds \text{ on } [0, t_a].$$

This implies

$$u_1 \le a\, e^{y(t)} \int_0^t e^{-y(s)}\, ds \text{ on } [0, t_a]$$

and since $e^{-y(t)} \le 1$ it then follows that

$$\beta = u_1(t_a) \le a\, e^{y(t_a)} \int_0^{t_a} e^{-y(s)}\, ds \le a\, t_a\, e^{y(t_a)}. \tag{3.15}$$

Now suppose $|t_a| \le S$ for some constant S. Since $y(t)$ is continuous on $[0, M]$ then $y(t_a)$ is bounded and thus the right side of (3.15) goes to 0 as $a \to 0^+$. This is a contradiction as the left side is $\beta > 0$. Hence $t_a \to \infty$ as $a \to 0^+$. This completes the proof □

We now use a rescaling argument to show u_1 has a large number of zeros if $a > 0$ is sufficiently large. For this we let

$$v_\lambda(t) = \lambda^{-\frac{2+q}{1-p}} u_1(\lambda t), \tag{3.16}$$

where $q = \frac{\alpha - 2(N-1)}{N-2} > 0$ (from (3.7)). Also from (3.3) and (H2) we have

$$u_1''(\lambda\, t) + h(\lambda t)[u_1^p(\lambda t) + g(u_1(\lambda t))] = 0.$$

It now follows using (3.16) that

$$v_\lambda'' + \frac{h(\lambda t)}{(\lambda t)^q} t^q \Big[v_\lambda^p + \lambda^{-\frac{p(q+2)}{1-p}} g(\lambda^{\frac{2+q}{1-p}} v_\lambda) \Big] = 0 \tag{3.17}$$

with

$$v_\lambda(0) = 0, \quad v_\lambda'(0) = \lambda^{-\frac{p+q+1}{1-p}} a.$$

On setting $a = \lambda^{\frac{1+p+q}{1-p}}$ we have

$$v_\lambda(0) = 0, \quad v_\lambda'(0) = 1. \tag{3.18}$$

Integrating (3.17) on $(0, t)$ and using (3.18) gives

$$v_\lambda'(t) = 1 - \int_0^t \frac{h(\lambda s)}{(\lambda s)^q} s^q \Big[v_\lambda^p(s) + \lambda^{-\frac{p(q+2)}{1-p}} g\Big(\lambda^{\frac{2+q}{1-p}} v_\lambda(s)\Big)\Big]\, ds. \tag{3.19}$$

Integrating on $(0, t)$ by using (3.18) gives

$$v_\lambda(t) = t - \int_0^t \int_0^s \frac{h(\lambda x)}{(\lambda x)^q} x^q \Big[v_\lambda^p(x) + \lambda^{-\frac{p(q+2)}{1-p}} g\Big(\lambda^{\frac{2+q}{1-p}} v_\lambda(x)\Big)\Big]\, dx\, ds. \tag{3.20}$$

Therefore

$$|v_\lambda(t)| \le t + \int_0^t \int_0^s \Big|\frac{h(\lambda x)}{(\lambda x)^q}\Big| x^q \Big| v_\lambda^p(x) + \lambda^{-\frac{p(q+2)}{1-p}} g\big(\lambda^{\frac{2+q}{1-p}} v_\lambda(x)\big)\Big|\, dx\, ds.$$

Using (3.7) yields

$$|v_\lambda(t)| \le t + \frac{d_2}{(N-2)^2} \int_0^t s^q \int_0^s \Big| v_\lambda^p(x) + \lambda^{-\frac{p(q+2)}{1-p}} g\big(\lambda^{\frac{2+q}{1-p}} v_\lambda(x)\big)\Big|\, dx\, ds. \tag{3.21}$$

Since from (H2) we have $|\frac{g(u)}{u^p}| \to 0$ as $u \to \infty$, it follows that given $\epsilon > 0$ there exists u_0 such that $|g(u)| \le \epsilon |u|^p$ for $|u| \ge u_0$ and also the continuity of g implies $|g(u)| \le c_4$ for $u \le u_0$ where c_4 is some positive constant. Thus

$$|g(u)| \le c_4 + \epsilon\, |u|^p \quad \text{for all } u. \tag{3.22}$$

Now rewriting (3.21) and using (3.22) we see that

$$|v_\lambda(t)| \le t + \frac{d_2}{(N-2)^2}\int_0^t s^q \int_0^t \big[(1+\epsilon)|v_\lambda^p(x)| + \frac{c_4}{\lambda^{\frac{(2+q)p}{1-p}}}\big]\,dx\,ds. \tag{3.23}$$

Now choose $\lambda > 0$ large enough so that $\frac{c_4}{\lambda^{\frac{(2+q)p}{1-p}}} \le 1$. Since $0 < p < 1$ it follows that $|v_\lambda|^p \le 1 + |v_\lambda|$. Hence it follows from (3.23) that for large λ,

$$|v_\lambda(t)| \le t + \Big[\frac{d_2}{(q+1)(N-2)^2}t^{q+2}\Big] + \Big[\frac{d_2(1+\epsilon)}{(q+1)(N-2)^2}t^{q+1}\int_0^t |v_\lambda(x)|\,dx\Big]$$

so for large λ,

$$|v_\lambda(t)| \le t + Bt^{q+2} + \Big[At^{q+1}\int_0^t |v_\lambda(x)|\,dx\Big] \tag{3.24}$$

where

$$A = \frac{d_2(1+\epsilon)}{(q+1)(N-2)^2}, \quad B = \frac{d_2}{(q+1)(N-2)^2}.$$

Now let $w(t) = \int_0^t |v_\lambda(x)|\,dx$. Then $w'(t) = |v_\lambda(t)|$ and hence from (41) we have

$$|v_\lambda(t)| = w'(t) \le t + B\,t^{q+2} + A\,t^{q+1}w(t). \tag{3.25}$$

Thus

$$w'(t) - A\,t^{q+1}w(t) \le t + B\,t^{q+2}$$

and therefore

$$\int_0^t \big(w\,e^{-(\frac{A}{q+2})t^{q+2}}\big)' \le [t + B\,t^{q+2}]e^{-(\frac{A}{q+2})t^{q+2}}.$$

which implies

$$w(t) \le e^{(\frac{A}{q+2})t^{q+2}}\int_0^t [s + 2A\,s^{q+2}]e^{-(\frac{A}{q+2})t^{q+2}}\,ds.$$

Since $e^{-(\frac{A}{q+2})t^{q+2}} \le 1$ it follows that

$$w(t) \le e^{(\frac{A}{q+2})t^{q+2}}\big[\frac{t^2}{2} + \frac{B\,t^{q+3}}{q+3}\big].$$

Thus for any fixed $0 < T < \infty$ we have

$$w(t) \le e^{(\frac{A}{q+2})T^{q+2}}\big[\frac{T^2}{2} + \frac{B\,T^{q+3}}{q+3}\big] = C_T \quad \text{on } [0,T]. \tag{3.26}$$

Now from (3.25) and (3.26) we have

$$|v_\lambda(t)| \le T + B\,T^{q+2} + A\,T^{q+1}C_T = D_T \quad \text{on } [0,T] \tag{3.27}$$

and using (3.7), (3.22) and (3.27) in (3.19) it follows that

$$|v_\lambda'(t)| \le 1 + \frac{d_2(1+D_T)(2+\epsilon)}{(q+1)(N-2)^2}T^{q+1} = Q_T \quad \text{on } [0,T]. \tag{3.28}$$

Using this inequality along with (3.9), (3.22) and (3.27) in (3.17) it follows, for sufficiently large λ, that

$$|v_\lambda''(t)| \le \frac{d_2(2+\epsilon)(1+D_T)}{(N-2)^2}T^q = J_T \quad \text{on } [0,T].$$

where C_T, D_T, Q_T, J_T are constants for the fixed $T < \infty$. Hence by the Arzela-Ascoli theorem $v_\lambda \to v$ and $v_\lambda' \to v'$ uniformly on $[0,T]$ as $\lambda \to \infty$ for some subsequence still denoted by v_λ. Since T is arbitrary we see that v and v' are continuous on $[0,\infty)$.

Now by (3.19) we have

$$\lim_{\lambda\to\infty} v'_\lambda(t) = 1 - \lim_{\lambda\to\infty}\Big(\int_0^t \frac{h(\lambda s)}{(\lambda s)^q} s^q\big[v_\lambda^p(s) + \lambda^{-\frac{(2+q)p}{1-p}} g\big(\lambda^{\frac{2+q}{1-p}} v_\lambda(s)\big)\big]\,ds\Big).$$

But we know that $\lim_{\lambda\to\infty} v'_\lambda(t) = v'(t)$ so

$$v'(t) = 1 - \lim_{\lambda\to\infty}\Big(\int_0^t \frac{h(\lambda s)}{(\lambda s)^q} s^q\big[v_\lambda^p(s) + \lambda^{-\frac{(2+q)p}{1-p}} g\big(\lambda^{\frac{2+q}{1-p}} v_\lambda(s)\big)\big]\,ds\Big). \tag{3.29}$$

Next we show that

$$\lim_{\lambda\to\infty}\int_0^t \Big(\frac{h(\lambda\, s)}{(\lambda s)^q}\,\lambda^{-\frac{(2+q)p}{1-p}} g(\lambda^{\frac{2+q}{1-p}} v_\lambda)\Big)\,ds = 0.$$

From (3.7) and (3.22) it follows that

$$\begin{aligned}
&\Big|\int_0^t \frac{h(\lambda\, s)}{(\lambda s)^q} s^q \lambda^{-\frac{(2+q)p}{1-p}} g\big(\lambda^{\frac{2+q}{1-p}} v_\lambda(s)\big)\,ds\Big|\\
&\le \frac{d_2}{(N-2)^2}\int_0^t s^q\Big(c_4\lambda^{-\frac{(2+q)p}{1-p}} + \epsilon\, v_\lambda^p(s)\Big)\,ds\\
&\le \frac{d_2\, c_4}{(q+1)(N-2)^2}\frac{T^{q+1}}{\lambda^{\frac{(2+q)p}{1-p}}} + \Big(\frac{d_2\, D_T^p\, T^{q+1}}{(q+1)(N-2)^2}\Big)\epsilon \quad \text{on } [0,T].
\end{aligned}$$

Since $\frac{T^{q+1}}{\lambda^{\frac{(2+q)p}{1-p}}} \to \infty$ as $\lambda\to\infty$ and $\epsilon>0$ is arbitrary it follows that

$$\lim_{\lambda\to\infty}\int_0^t \Big(\frac{h(\lambda\, s)}{(\lambda s)^q}\,\lambda^{-\frac{(2+q)p}{1-p}} g(\lambda^{\frac{2+q}{1-p}} v_\lambda)\Big)\,ds = 0.$$

Therefore from (3.27) it follows that

$$v'(t) = 1 - \lim_{\lambda\to\infty}\int_0^t \Big(\frac{h(\lambda\, s)}{(\lambda s)^q} s^q v_\lambda^p(s)\Big)\,ds \tag{3.30}$$

and since from (3.8) we have $0 < \lim_{t\to\infty}\frac{h(t)}{t^q} = L$ then it follows from (3.27) that

$$v'(t) = 1 - L\int_0^t s^q v^p(s)\,ds$$

therefore $v''(t) = -L\, t^q\, v^p(t)$. Hence v satisfies:

$$v''(t) + Lt^q v^p(t) = 0, \tag{3.31}$$
$$v(0) = 0, \quad v'(0) = 1. \tag{3.32}$$

Lemma 3.2. *Suppose v satisfies (3.30)-(3.31). Then $v(t)$ has an infinite number of zeros on $(0,\infty)$.*

Proof. We first show v has a local maximum. If not $v'>0$ and $v(t)\ge c_0$ for some $c_0>0$ on (t_0,∞) for some $t_0>0$. Then from (3.31) we see

$$-v'' \ge L\, c_0\, t^q \tag{3.33}$$

Integrating on $[t_0,t]$ gives

$$-v'(t) \ge -v'(t_0) + \frac{Lc_0}{q+1}[t^{q+1} - t_0^{q+1}].$$

This implies $v'(t) \to -\infty$ as $t \to \infty$ which contradicts that $v' > 0$. Thus there exists an $M > 0$ such that $v'(M) = 0$ and $v(M) > 0$. Thus by (3.31) $v''(M) < 0$. So v has a local maximum at M. Integrating (3.31) on $[M, t]$ gives

$$-v'(t) = L \int_M^t s^q v^p \, ds. \tag{3.34}$$

Now let us assume that $v > 0$ for $t > M$ then (3.34) implies that $v' < 0$ for $t > M$ hence estimating (3.34) on $[M, t]$ gives

$$-v'(t) = L \int_M^t s^q v^p \, ds \geq L v^p(t) \big[\frac{t^{q+1} - M^{q+1}}{q+1} \big]$$

which on rewriting it yields

$$-v^{-p} \, v' \geq L \big[\frac{t^{q+1} - M^{q+1}}{q+1} \big].$$

Integrating this on $[M, t]$ gives

$$\frac{v^{1-p}(M)}{1-p} - \frac{v^{1-p}(t)}{1-p} \geq L \int_M^t \frac{s^{q+1} - M^{q+1}}{q+1} \, ds$$

Thus

$$\frac{v^{1-p}(M)}{1-p} - \frac{L}{(q+1)(q+2)} \left[t^{q+2} - (q+2) M^{q+1} t + (q+1) M^{q+2} \right] \geq \frac{v^{1-p}(t)}{1-p}$$

the left side of which goes to $-\infty$ as $t \to \infty$ since $L > 0$ and $q > 0$ hence

$$\frac{v^{1-p}(t)}{1-p} \to -\infty \quad \text{as } t \to \infty.$$

This is a contradiction since we assumed $v > 0$ for $t > M$. Hence there exists $z_1 > 0$ such that $v(z_1) = 0$ and $v > 0$ on $[M, z_1]$ so $v'(z_1) \leq 0$. In addition by uniqueness of solution of initial value problems it follows that $v'(z_1) < 0$. Similarly we can then show that v has a local minimum, $m > z_1$ and that v has a second zero $z_2 > m$. Proceeding similarly we can show that v has infinitely many zeros on $[0, \infty)$. This completes the proof. □

Since $v_\lambda \to v$ uniformly on compact sets and since v has an infinite number of zeros, so v_λ has large number of zeros for large values of λ and hence u_1 satisfying (3.9)-(3.10) has a large number of zeros for large a.

Proof of Theorem 1.1. From Lemma 3.2 we see that u_1 has a first zero, $z_1(a)$, if a is sufficiently large. Note by continuous dependence on initial conditions $z_1(a)$ is a continuous function of a. Now choose $R > 0$ small enough so that $z_1(a) < R^{2-N}$. Then by Lemma 3.1 if a is sufficiently small $t_a > R^{2-N}$ and so $u_1(t, a) > 0$ on $(0, R^{2-N})$ if a is sufficiently small. Thus $\{a > 0 : z_1(a) < R^{2-N}\}$ is nonempty and bounded from below. Then let:

$$a_0 = \inf\{a : z_1(a) < R^{2-N}\}.$$

Since $z_1(a)$ is continuous it follows that $z_1(a_0) \leq R^{2-N}$.

Now suppose $z_1(a_0) < R^{2-N}$. Then by the continuous dependency of the solutions on initial conditions we have $z_1(a) < R^{2-N}$ for $a < a_0$ with a sufficiently close to a_0 which contradicts the definition of a_0. So $z_1(a_0) = R^{2-N}$ and hence $u_1(z_1(a_0)) = u_1(R^{2-N}) = 0$. Now let $U_0(r) = u_1(r^{2-N}, a_0)$. Then U_0 satisfies (2.1) and (2.2) with $b = u_1'(R^{2-N})(2-N)R^{1-N} > 0$, $\lim_{r\to\infty} U_0(r) = 0$ and $U_0 > 0$ on (R, ∞).

Next using Lemma 3.2 we see that if a is sufficiently large then u_1 has two zeros $z_1(a)$ and $z_2(a)$ with $u_1 > 0$ on $(0, z_1(a))$ and $u_1 < 0$ on $(z_1(a), z_2(a))$. Choose $R > 0$ small enough so that $z_2(a) < R^{2-N}$. Let

$$a_1 = \inf\{a|\ z_2(a) < R^{2-N}\}.$$

As above we can show that $z_2(a_1) = R^{2-N}$. Let $U_1(r) = -u_1(r^{2-N}, a_1)$. Then $U_1'(R) = (N-2)R^{1-N}u_1(R^{2-N}) > 0$, U_1 has one zero on (R, ∞), satisfies (2.1) and (2.2) and $\lim_{r\to\infty} U_1(r) = 0$. Proceeding inductively, we obtain $a_2, a_3, a_4, \dots, a_n$ for any non negative integer n such that $u_1(R^{2-N}, a_n) = 0$ by choosing R sufficiently small. Define $U_n(r) = (-1)^n u_1(r^{2-N}, a_n)$. Then $U_n'(R) = (-1)^n(2-N)R^{1-N}u_1'(R^{2-N}, a_n) > 0$ and as above we can show that U_n has n zeros on (R, ∞), satisfies (2.1) and (2.2) and $\lim_{r\to\infty} U_n(r) = 0$. This completes the proof. $\square$

References

[1] H. Berestycki, P. L. Lions; Non-linear scalar field equations I, *Arch. Rational Mech. Anal.*, Volume 82, 313-347, 1983.

[2] H. Berestycki P. L. Lions; Non-linear scalar field equations II, *Arch. Rational Mech. Anal.*, Volume 82, 347-375, 1983.

[3] G. Birkhoff, G. C. Rota; *Ordinary Differential Equations*, Ginn and Company, 1962.

[4] A. Castro, L. Sankar, R. Shivaji; Uniqueness of nonnegative solutions for semipositone problems on exterior domains, *Journal of Mathematical Analysis and Applications*, Volume 394, Issue 1, 432-437, 2012.

[5] J. Iaia; Existence of solutions for semilinear problems with prescribed number of zeros on exterior domains, *Journal of Mathematical Analysis and Applications*, Volume 446, Issue 1, 591-604, 2017.

[6] J. Iaia, "Loitering at the hilltop on exterior domains," *Electronic Journal of the Qualitative Theory of Differential Equations*, No. 82, 1-11, 2015.

[7] J. Iaia; Existence and nonexistence for semilinear equations on exterior domains, *Electronic Journal of Differential equations*, Vol. 2016 (2016), No. 227, 1-9.

[8] J. Iaia; Existence of solutions for sublinear equations on exterior domains, preprint.

[9] C. K. R. T. Jones, T. Kupper; On the infinitely many solutions of a semi-linear equation, *SIAM J. Math. Anal.*, Volume 17, 803-835, 1986.

[10] E. Lee, L. Sankar, R. Shivaji; Positive solutions for infinite semipositone problems on exterior domains, *Differential and Integral Equations*, Volume 24, Number 9/10, 861-875, 2011.

[11] K. McLeod, W. C. Troy, F. B. Weissler; Radial solutions of $\Delta u + f(u) = 0$ with prescribed numbers of zeros, *Journal of Differential Equations*, Volume 83, Issue 2, 368-373, 1990.

[12] L. Sankar, S. Sasi, R. Shivaji; Semipositone problems with falling zeros on exterior domains, *Journal of Mathematical Analysis and Applications*, Volume 401, Issue 1, 146-153, 2013.

Janak Joshi

Department of Mathematics, University of North Texas, Denton, TX 76203, USA

E-mail address: janakrajjoshi@my.unt.edu