Electronic Journal of Differential Equations, Vol. 2017 (2017), No. 137, pp. 1–22.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

WELL-POSEDNESS OF A FULLY COUPLED THERMO-CHEMO-POROELASTIC SYSTEM WITH APPLICATIONS TO PETROLEUM ROCK MECHANICS

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Communicated by Ralph Showalter

Abstract. We consider a system of fully coupled parabolic and elliptic equations constituting the general model of chemical thermo-poroelasticity for a fluid-saturated porous media. The main result of this paper is the developed well-posedness theory for the corresponding initial-boundary problem arising from petroleum rock mechanics applications. Using the proposed pseudo-decoupling method, we establish, subject to some natural assumptions imposed on matrices of diffusion coefficients, the existence, uniqueness, and continuous dependence on initial and boundary data of a weak solution to the problem. Numerical experiments confirm the applicability of the obtained well-posedness results for thermo-chemo-poroelastic models with real-data parameters.

1. Introduction

In this article we investigate a model describing fully coupled processes of quasi-static elastic deformation and thermal, solute, and fluid diffusions in porous media. This work is motivated by petroleum rock mechanics applications dealing with drilling and borehole stability in high-temperature, high-pressure chemically active rock formations. The system of coupled diffusion and deformation equations is taken from Diek, White, and Roegiers [6] and constitutes the general thermo-chemo-poroelasticity theory of porous media saturated by a compressible and thermally expansible fluid.

The underlying equations are formulated in terms of the absolute temperature $T(\mathbf{x},t)$, the solute mass fraction $C(\mathbf{x},t)$, the pore pressure $p(\mathbf{x},t)$, and the vector of solid displacements $\mathbf{u}(\mathbf{x},t)$ and will henceforth be referred to as the TCpu system. In the case of a homogeneous and isotropic medium the equations of the TCpu system take the form:
thermal diffusion

$$\Lambda\dot{T} + \Sigma\dot{C} + \Phi\dot{p} - \frac{k^{\mathrm{T}}}{T_F}\nabla^2 T - C_F D^{\mathrm{T}}\rho_f\tilde{\Omega}\nabla^2 C + K^{\mathrm{T}}\nabla^2 p = -\zeta(\nabla\cdot\dot{\mathbf{u}}), \tag{1.1}$$

2010 *Mathematics Subject Classification.* 35D30, 35E99, 35G16, 35Q74, 35Q86.
Key words and phrases. Parabolic-elliptic system; poroelasticity; thermo-poroelasticity; thermo-chemo-poroelasticity; existence; uniqueness; well-posedness.
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Submitted February 27, 2017. Published May 24, 2017.

solute diffusion

$$\phi\dot{C} - C_F D^{\mathrm{T}}\nabla^2 T - D\nabla^2 C + \frac{k\mathfrak{R}}{\eta}\nabla^2 p = 0, \tag{1.2}$$

fluid diffusion

$$\Gamma\dot{T} + \chi\dot{C} + \Psi\dot{p} + K^{\mathrm{T}}\nabla^2 T + \frac{k\mathfrak{R}}{\eta}\rho_f\tilde{\Omega}\nabla^2 C - \frac{k}{\eta}\nabla^2 p = -\alpha(\nabla\cdot\dot{\mathbf{u}}), \tag{1.3}$$

and the Navier-type elastic equation

$$\Big(K + \frac{G}{3}\Big)\nabla(\nabla\cdot\dot{\mathbf{u}}) + G\nabla^2\dot{\mathbf{u}} = \tilde{\zeta}\nabla\dot{T} - \xi\nabla\dot{C} + \tilde{\alpha}\nabla\dot{p}, \tag{1.4}$$

where the superscript dot ($\dot{}$) denotes a time derivative. The description and values of physical constants are taken from [6] and presented in the appendix of the paper. These equations supplemented by appropriate initial and boundary conditions constitute an initial-boundary value problem (IBVP) defined in an open region $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, exterior to the borehole with a sufficiently smooth boundary Γ. We assume that the outer (far-field) boundary $\Gamma_F \subset \Gamma$ of the region has a nonempty interior relative to Γ and is specified by the following conditions: (i) the absolute temperature, solute mass fraction, pore pressure, and displacements are time-independent, and (ii) displacements and their velocities are negligibly small.

For the sake of convenience, we combine thermal, solute, and fluid diffusion equations into a single vector diffusion equation. To that end, we introduce the vector $\bar{\mathbf{V}} = [T\ C\ p]^T$, with the superscript T meaning transpose, the matrices of diffusion coefficients

$$M = \begin{bmatrix} \Lambda & \Sigma & \Phi \\ 0 & \phi & 0 \\ \Gamma & \chi & \Psi \end{bmatrix}, \quad A = \begin{bmatrix} \frac{k^{\mathrm{T}}}{T_F} & C_F D^{\mathrm{T}}\rho_f\tilde{\Omega} & -K^{\mathrm{T}} \\ C_F D^{\mathrm{T}} & D & -\frac{k\mathfrak{R}}{\eta} \\ -K^{\mathrm{T}} & -\frac{k\mathfrak{R}}{\eta}\rho_f\tilde{\Omega} & \frac{k}{\eta} \end{bmatrix} \tag{1.5}$$

and the coupling vectors

$$\mathbf{b}_0 = \begin{bmatrix} \zeta \\ 0 \\ \alpha \end{bmatrix}, \quad \mathbf{b}_1 = \begin{bmatrix} \tilde{\zeta} \\ -\xi \\ \tilde{\alpha} \end{bmatrix} = \begin{bmatrix} \zeta \\ 0 \\ \alpha \end{bmatrix} + \begin{bmatrix} s_F\tilde{\omega} \\ -\xi \\ -\frac{\tilde{\omega}}{\rho_f} \end{bmatrix} =: \mathbf{b}_0 + \mathbf{b}_d\,. \tag{1.6}$$

With the above notation, the IBVP for the fully coupled TCpu system (1.1)-(1.4) has the form

$$M\dot{\bar{\mathbf{V}}} - A\nabla^2\bar{\mathbf{V}} = -\mathbf{b}_0(\nabla\cdot\dot{\mathbf{u}}), \quad \text{in } \Omega\times(0,t_f), \tag{1.7}$$

$$\Big(K + \frac{G}{3}\Big)\nabla(\nabla\cdot\dot{\mathbf{u}}) + G\nabla^2\dot{\mathbf{u}} = \nabla(\mathbf{b}_1\cdot\dot{\bar{\mathbf{V}}}), \quad \text{in } \Omega\times(0,t_f), \tag{1.8}$$

$$\bar{\mathbf{V}}(\mathbf{x},t) = \mathbf{V}_B(\mathbf{x},t), \quad \text{on } \Gamma\times[0,t_f), \tag{1.9}$$

$$\mathbf{u}(\mathbf{x},t) = \mathbf{0}, \quad \text{on } \Gamma_F\times(0,t_f), \tag{1.10}$$

$$\dot{\mathbf{u}}(\mathbf{x},t) = \mathbf{0}, \quad \text{on } \Gamma_F\times(0,t_f), \tag{1.11}$$

$$\dot{\boldsymbol{\tau}}\mathbf{n} = \big((\mathbf{b}_1\cdot\dot{\mathbf{V}})I + \dot{\boldsymbol{\sigma}}\big)\mathbf{n}, \quad \text{on } \Gamma\setminus\Gamma_F\times(0,t_f), \tag{1.12}$$

$$\bar{\mathbf{V}}(\mathbf{x},0) = \mathbf{V}_I(\mathbf{x}), \quad \text{in } \Omega, \tag{1.13}$$

where $t_f \in (0,\infty)$ stands for a final time, K and G are the bulk and shear moduli, respectively, $\boldsymbol{\tau}$ is the stress tensor, $\mathbf{n}$ is the outward unit normal vector on the boundary, I is the $n\times n$ identity matrix, $n = 2$ or 3 is the dimension of the problem, and $\hat{\boldsymbol{\sigma}}$ is the applied boundary stress tensor. The boundary function $\mathbf{V}_B$ is time-independent on the far-field boundary Γ_F.

The aim of this paper is to establish existence, uniqueness, and continuous dependence on initial and boundary data of a weak solution to the fully coupled IBVP (1.7)-(1.13).

One can find an enormous amount of literature concerning numerical techniques and simulations for models describing coupled thermal, chemical, hydraulic, and quasi-static elastic deformation processes in porous media relevant to petroleum rock mechanics applications. However, there are very few papers dealing with analytical solutions and well-posedness of such coupled problems. Typically, analytical solutions are derived under assumptions that some of the diffusion processes or couplings can be neglected [1, 4, 5, 8, 9, 11, 12, 14], and most well-posedness results are obtained for poroelastic and thermoelastic systems only. The existence, uniqueness, and regularity theory for linear Biot's consolidation models in poroelasticity and a coupled quasi-static problem in thermoelasticity was developed by Showalter [15, 16] from the theory of linear degenerate evolution equations in a Hilbert space. It should be noted that, with slight modifications, these results can be extended to a fully coupled thermo-poroelastic model. In the papers of Barucq, Madaune-Tort, and Saint-Macary [2, 3], the existence and uniqueness of weak solutions to non-linear fully dynamic and quasi-static Biot's consolidation models of poroelasticity for either Newtonian or non-Newtonian fluid were established using Galerkin approximants.

The main difficulty of the problem under consideration is related to the complexity of cross-coupling mechanisms involved in the fully coupled model of thermo-chemo-poroelasticity. In contrast with the fully coupled thermo-poroelastic system, the matrices of diffusion coefficients M and A are non-symmetric and the diffusion equation (1.7) cannot be rescaled to make the coupling vectors $\mathbf{b}_0$ and $\mathbf{b}_1$ equal; concsequently, the techniques presented in [2, 3, 15, 16] are not applicable to the problem (1.7)-(1.13). In our preceding paper [13] we obtained sufficient conditions for Hadamard well-posedness of the system (1.7)-(1.13). However, the question of finding necessary conditions for existence and uniqueness of a solution and its continuous dependence on data remains open and therefore, the well-posedness of the fully coupled TCpu problem (1.7)-(1.13) requires further investigation.

The novel contribution of the present paper consists of the following:

(i) The well-posedness theory is developed for the IBVP (1.7)-(1.13) for the system of fully coupled partial differential equations constituting the general thermo-chemo-poroelasticity theory of porous media saturated by a compressible and thermally expansible fluid. It is shown that, subject to some natural assumptions imposed on matrices of diffusion coefficients, the system (1.7)-(1.13) admits a unique weak solution and this solution depends continuously on initial and boundary data.

(ii) A novel method that allows one to transform the coupled parabolic-elliptic IBVP (1.7)-(1.13) to an IBVP for a single implicit equation is presented. This method will be called the pseudo-decoupling method because it is based on the construction of operators that fully eliminate coupling terms by implicitly incorporating the elliptic IBVP for the elasticity system into the parabolic equation representing diffusion processes. This approach differs from those used before in the literature [15, 16] in the construction of operators that transform a coupled parabolic-elliptic system into an implicit system. In contrast with [15, 16] dealing with anti-symmetric coupling terms, the proposed method makes no assumption on the relation between the coupling vectors $\mathbf{b}_0$ and $\mathbf{b}_1$ and therefore, it is applicable to a wider variety of coupling mechanisms.

The article is organized as follows. In Section 2 we introduce the pseudo-decoupling method to transform the fully coupled parabolic-elliptic IBVP (1.7)-(1.13) to an IBVP for a single implicit equation. The obtained IBVP for the implicit equation is then decomposed into a parabolic IBVP and an implicit IBVP with homogeneous boundary and initial conditions. The well-posedness of the parabolic IBVP is discussed in Section 2. In Section 3 we prove the well-posedness in a weak sense of the implicit IBVP with homogeneous boundary and initial conditions and a generalized source term. Section 4 contains the main results of this paper summarized in Theorem 4.1 that establishes the existence, uniqueness, and continuous dependence on initial and boundary data of a weak solution to the fully coupled IBVP (1.7)-(1.13) for the TCpu system. In Section 5 we provide a numerical example illustrating the applicability of the obtained well-posedness results for the TCpu model with real data.

1.1. **Notation.** Let Ω be a bounded open domain in $\mathbb{R}^n$, $n = 2, 3$, with a sufficiently smooth boundary Γ, $\bar{\Omega} = \Omega \cup \Gamma$, and $\Gamma_F \subset \Gamma$ have a nonempty interior relative to Γ. Throughout this article, $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \bar{\Omega}$, $\mathbf{u} = [u_1\ u_2 \dots u_n]^T$ is the general notation for an $\mathbb{R}^n$-valued function, the superscript T means transpose, and ∂_i stands for the partial derivative with respect to x_i. We introduce the following spaces of vector-valued functions: $\mathbb{H}^n = L^2(\Omega)^n$ and $\mathbb{V}^n = H^1(\Omega)^n$ endowed with their standard norms denoted by $\|\cdot\|_{\mathbb{H}^n}$ and $\|\cdot\|_{\mathbb{V}^n}$, respectively; the space $\mathbb{V}_0^n = H_0^1(\Omega)^n$ with the norm

$$\|\mathbf{u}\|_{0,n} = \Big[\sum_{k=1}^n \int_\Omega |\nabla u_k|^2 \mathrm{d}\Omega\Big]^{1/2}$$

and $\tilde{\mathbb{V}}_0^n = \big\{\boldsymbol{\varphi} \in \mathbb{V}^n : \boldsymbol{\varphi}\big|_{\Gamma_F} = \mathbf{0}\big\}$ with the norm inherited from $\mathbb{V}_0^n$, $n \in \mathbb{N}$. Let $-\infty \le a < b \le \infty$ and X be a Hilbert space. We denote by $L^2(a, b; \mathrm{X})$ the space of L^2-integrable functions from $[a, b]$ into X with the norm

$$\|u\|_{L^2(a,b;\mathrm{X})} = \Big[\int_a^b \|u(t)\|_{\mathrm{X}}^2 \mathrm{d}t\Big]^{1/2}.$$

The space $L^\infty(a, b; \mathrm{X})$ is the space of essentially bounded functions from $[a, b]$ into X equipped with the norm

$$\|u\|_{L^\infty(a,b;\mathrm{X})} = \operatorname{ess\,sup}_{[a,b]} \|u(t)\|_X .$$

We will occasionally use the Einstein summation convention: whenever an index is repeated once in the same term, it implies summation over the specified range of the index. For example, $\sum_{k=1}^n \sum_{l=1}^n a_{ijkl}\varepsilon_{kl} \equiv a_{ijkl}\varepsilon_{kl}$.

2. Pseudo-decoupling method

In this section we present the pseudo-decoupling method to transform the coupled parabolic-elliptic IBVP (1.7)-(1.13) to an IBVP for a single implicit equation and discuss further decomposition of the implicit system.

2.1. **Elastic system.** Elastic deformation of the region Ω is characterized using the linearized strain tensor

$$\boldsymbol{\varepsilon}(\mathbf{u}) = \frac{1}{2}\big(\nabla\mathbf{u} + \nabla\mathbf{u}^T\big), \tag{2.1}$$

where $\mathbf{u}$ is the displacement vector. The material properties of rock are described by the relation between the stress tensor $\boldsymbol{\tau} = [\tau_{ij}]$ and the strain tensor $\boldsymbol{\varepsilon} = [\varepsilon_{ij}]$, the generalized Hooke's law,

$$\tau_{ij} = a_{ijkl}\varepsilon_{kl}(\mathbf{u}), \tag{2.2}$$

where a_{ijkl} are the coefficients of elasticity independent of the strain tensor, with the properties of symmetry

$$a_{ijkl} = a_{jilk} = a_{klij} \tag{2.3}$$

and of ellipticity: there exists a constant $\alpha_0 > 0$ such that

$$a_{ijkl}\epsilon_{ij}\epsilon_{kl} \geq \alpha_0\epsilon_{ij}\epsilon_{ij}$$

for all symmetric $\epsilon \in \mathbb{R}^{n\times n}$. In the case of a homogeneous and isotropic medium, the stress-strain relation (2.2) in terms of the bulk modulus K and the shear modulus G takes the form

$$\tau_{ij} = 2G\varepsilon_{ij} + \Big(K - \frac{2G}{3}\Big)\varepsilon_{kk}\delta_{ij}\,. \tag{2.4}$$

According to the principle of minimum total potential energy, among all admissible displacements satisfying the boundary conditions (1.10) and (1.11), the actual displacement that the region Ω undergoes is the one that minimizes the total potential energy $\mathcal{V}$ of the elastic system (1.8), (1.10)-(1.12):

$$\mathcal{V}(\mathbf{u}) = \mathcal{V}_S(\mathbf{u}) - W_b(\mathbf{u}) - W_S(\mathbf{u}),$$

where

$$\mathcal{V}_S(\mathbf{u}) = \frac{1}{2}\int_\Omega \tau_{ij}(\mathbf{u})\varepsilon_{ij}(\mathbf{u})\mathrm{d}\Omega$$

is the elastic energy of the system;

$$W_b(\mathbf{u}) = \int_\Omega (\mathbf{b}_1\cdot\bar{\mathbf{V}})(\nabla\cdot\mathbf{u})\mathrm{d}\Omega$$

is the work done by body forces due to the absolute temperature, solute mass fraction, and pore pressure; and

$$W_S(\mathbf{u}) = \int_\Gamma (\hat{\boldsymbol{\sigma}}\mathbf{n})\cdot\mathbf{u}\mathrm{d}\Gamma$$

is the work done by applied boundary stress.

Let us define a bilinear form $a_E : \mathbb{V}^n \times \mathbb{V}^n \to \mathbb{R}$ by

$$a_E(\mathbf{u},\boldsymbol{\Phi}) = \int_\Omega \tau_{ij}(\mathbf{u})\varepsilon_{ij}(\boldsymbol{\Phi})\mathrm{d}\Omega\,. \tag{2.5}$$

Then the total potential energy takes the form

$$\mathcal{V}(\mathbf{u}) = \frac{1}{2}a_E(\mathbf{u},\mathbf{u}) - \int_\Omega (\mathbf{b}_1\cdot\bar{\mathbf{V}})(\nabla\cdot\mathbf{u})\mathrm{d}\Omega - \int_\Gamma (\hat{\boldsymbol{\sigma}}\mathbf{n})\cdot\mathbf{u}\mathrm{d}\Gamma\,.$$

Using the principle of minimum total potential energy, it was shown in [13] that the elastic system (1.8), (1.10)-(1.12) is equivalent to

$$\Big(K + \frac{G}{3}\Big)\nabla(\nabla\cdot\mathbf{u}) + G\nabla^2\mathbf{u} = \nabla(\mathbf{b}_1\cdot\bar{\mathbf{V}}), \quad \text{in } \Omega\times(0,t_f),$$

$$\mathbf{u} = \mathbf{0}, \quad \text{on } \Gamma_F \times (0, t_f)$$
$$\boldsymbol{\tau}\mathbf{n} = \big((\mathbf{b}_1 \cdot \bar{\mathbf{V}})I + \hat{\boldsymbol{\sigma}}\big)\mathbf{n}, \quad \text{on } \Gamma \setminus \Gamma_F \times (0, t_f),$$

and the variational form of the above system is

$$a_E(\mathbf{u}, \boldsymbol{\Phi}) - \int_\Omega (\mathbf{b}_1 \cdot \bar{\mathbf{V}})(\nabla \cdot \boldsymbol{\Phi}) \mathrm{d}\Omega - \int_{\Gamma\setminus\Gamma_F} (\hat{\boldsymbol{\sigma}}\mathbf{n}) \cdot \boldsymbol{\Phi} \mathrm{d}\Gamma = 0, \quad \forall \boldsymbol{\Phi} \in \tilde{\mathbb{V}}_0^n . \tag{2.6}$$

We will be working under the following assumption on the applied boundary stress tensor $\hat{\boldsymbol{\sigma}}$.

Assumption 2.1.

$$\hat{\boldsymbol{\sigma}} \in L^2\big(0, t_f; L^2(\Gamma)^{n\times n}\big) \quad \text{and} \quad \dot{\hat{\boldsymbol{\sigma}}} \in L^2\big(0, t_f; L^2(\Gamma)^{n\times n}\big)$$

where $n = 2$ or 3 is the dimension of the problem.

2.2. **Diffusion system.** We observe from (1.5) that the matrix A can be written as a product of a symmetric matrix A_0 and a diagonal matrix R in the form

$$A = A_0 R,$$

where

$$A_0 = \begin{bmatrix} \frac{k^{\mathrm{T}}}{T_F} & C_F D^{\mathrm{T}} & -K^{\mathrm{T}} \\ C_F D^{\mathrm{T}} & \frac{D}{\rho_f \tilde{\Omega}} & -\frac{k\mathfrak{R}}{\eta} \\ -K^{\mathrm{T}} & -\frac{k\mathfrak{R}}{\eta} & \frac{k}{\eta} \end{bmatrix}, \quad R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \rho_f \tilde{\Omega} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \rho_f \tilde{\Omega} > 0 . \tag{2.7}$$

Define the vector

$$\bar{\mathbf{V}}_R = R\bar{\mathbf{V}} . \tag{2.8}$$

This transformation leads to the IBVP equivalent to the diffusion system (1.7), (1.9), and (1.13):

$$M_R \dot{\bar{\mathbf{V}}}_R - A_0 \nabla^2 \bar{\mathbf{V}}_R + \mathbf{b}_0 (\nabla \cdot \dot{\mathbf{u}}) = \mathbf{0}, \quad \text{in } \Omega \times (0, t_f), \tag{2.9}$$
$$\bar{\mathbf{V}}_R(\mathbf{x}, t) = R\mathbf{V}_B(\mathbf{x}, t), \quad \text{on } \Gamma \times [0, t_f), \tag{2.10}$$
$$\bar{\mathbf{V}}_R(\mathbf{x}, 0) = R\mathbf{V}_I(\mathbf{x}), \quad \text{in } \Omega, \tag{2.11}$$

where $M_R = MR^{-1}$ is a non-symmetric matrix.

We shall be using the following assumption imposed on the matrices of diffusion coefficients M_R and A_0.

Assumption 2.2. The matrices $M_R = [m_{ij}]_{i,j=1}^3$ and $A_0 = [a_{ij}]_{i,j=1}^3$ satisfy the following conditions:

(i) $0 < m_{ii} < b_{Ri}^2,\ i = 1, 2, 3$, where $\mathbf{b}_R^T = [b_{R1}\ b_{R2}\ b_{R3}]^T = \mathbf{b}_d^T R^{-1}$.
(ii) $0 < (m_{ij} + m_{ji})^2 < m_{ii} m_{jj},\ i, j = 1, 2, 3$.
(iii) The matrices $\frac{1}{2}\big(M_R + M_R^T\big)$ and A_0 are positive definite.

2.3. **Pseudo-decoupling method.** One may observe from (2.1)-(2.3) and (2.5) that

$$a_E(\mathbf{u}, \mathbf{v}) = a_E(\mathbf{v}, \mathbf{u}), \quad \forall \mathbf{u}, \mathbf{v} \in \mathbb{V}^n$$

and, for every $\mathbf{u}, \mathbf{v} \in \tilde{\mathbb{V}}_0^n$,

$$|a_E(\mathbf{u}, \mathbf{v})| \le n \max_{i,j,k,l} \{a_{ijkl}\} \|\mathbf{u}\|_{0,n} \|\mathbf{v}\|_{0,n} .$$

Also, Korn's inequality and its consequence [7, Chapter III, Theorems 3.1 and 3.3] imply that there exists a constant $\gamma = \gamma(\Omega) > 0$ such that

$$a_E(\mathbf{u}, \mathbf{u}) \geq \gamma \|\mathbf{u}\|_{0,n}^2, \quad \forall \mathbf{u} \in \tilde{\mathbb{V}}_0^n . \tag{2.12}$$

Thus, the bilinear form $a_E(\cdot, \cdot)$ is symmetric, continuous, and coercive on $\tilde{\mathbb{V}}_0^n$.

The above observation allows us to define the linear operator $E_0 : \mathbb{H} \to \tilde{\mathbb{V}}_0^n$ by

$$a_E(E_0\varphi, \mathbf{\Phi}) = \int_\Omega \varphi(\nabla \cdot \mathbf{\Phi}) \mathrm{d}\Omega, \quad \forall \mathbf{\Phi} \in \tilde{\mathbb{V}}_0^n \tag{2.13}$$

and the function $\mathbf{u_B} = \mathbf{u_B}(\hat{\boldsymbol{\sigma}}) \in \tilde{\mathbb{V}}_0^n$ by

$$a_E(\mathbf{u}_B, \mathbf{\Phi}) = \int_\Gamma (\hat{\boldsymbol{\sigma}}\mathbf{n}) \cdot \mathbf{\Phi} \mathrm{d}\Gamma, \quad \forall \mathbf{\Phi} \in \tilde{\mathbb{V}}_0^n. \tag{2.14}$$

The operator E_0 is continuous. Indeed, from (2.12) and (2.13), for every $\varphi \in \mathbb{H}$,

$$\gamma \|E_0\varphi\|_{0,n}^2 \leq a_E(E_0\varphi, E_0\varphi) \leq \|\varphi\|_{\mathbb{H}} \|\nabla \cdot E_0\varphi\|_{\mathbb{H}} \leq \sqrt{n} \|\varphi\|_{\mathbb{H}} \|E_0\varphi\|_{0,n} .$$

Thus,

$$\|E_0\varphi\|_{0,n} \leq \frac{\sqrt{n}}{\gamma} \|\varphi\|_{\mathbb{H}}, \quad \forall \varphi \in \mathbb{H} . \tag{2.15}$$

Applying (2.8), (2.13), and (2.14) to (2.6) gives, for every $\mathbf{\Phi} \in \tilde{\mathbb{V}}_0^n$,

$$\begin{aligned} a_E(\mathbf{u}, \mathbf{\Phi}) &= \int_\Omega (\mathbf{b}_1^T R^{-1} \bar{\mathbf{V}}_R)(\nabla \cdot \mathbf{\Phi}) \mathrm{d}\Omega + \int_{\Gamma \setminus \Gamma_F} (\hat{\boldsymbol{\sigma}}\mathbf{n}) \cdot \mathbf{\Phi} \mathrm{d}\Gamma \\ &= a_E\big(E_0(\mathbf{b}_1^T R^{-1} \bar{\mathbf{V}}_R), \mathbf{\Phi}\big) + a_E(\mathbf{u}_B, \mathbf{\Phi}) . \end{aligned}$$

It follows that

$$\mathbf{u} = E_0(\mathbf{b}_1^T R^{-1} \bar{\mathbf{V}}_R) + \mathbf{u}_B, \tag{2.16}$$

and hence,

$$\nabla \cdot \dot{\mathbf{u}} = \nabla \cdot E_0(\mathbf{b}_1^T R^{-1} \dot{\bar{\mathbf{V}}}_R) + \nabla \cdot \dot{\mathbf{u}}_B \tag{2.17}$$

We now introduce the linear operator $\hat{E} : \mathbb{H}^3 \to \mathbb{H}^3$ by

$$\hat{E}\boldsymbol{\psi} = \begin{bmatrix} \nabla \cdot (E_0\psi_1) \\ \nabla \cdot (E_0\psi_2) \\ \nabla \cdot (E_0\psi_3) \end{bmatrix} . \tag{2.18}$$

The continuity of E_0 implies the continuity of $\hat{E}$, and from (2.15) it is straightforward to show that

$$\|\hat{E}\boldsymbol{\psi}\|_{\mathbb{H}^3} \leq \frac{n}{\gamma} \|\boldsymbol{\psi}\|_{\mathbb{H}^3}, \quad \forall \boldsymbol{\psi} \in \mathbb{H}^3 . \tag{2.19}$$

With the above notation, (2.17) can be rewritten as

$$\nabla \cdot \dot{\mathbf{u}} = \mathbf{b}_1^T R^{-1} \hat{E} \dot{\bar{\mathbf{V}}}_R + \nabla \cdot \dot{\mathbf{u}}_B . \tag{2.20}$$

Substituting (2.20) into (2.9), we obtain the implicit equation

$$\big[M_R + \mathbf{b}_0 \mathbf{b}_1^T R^{-1} \hat{E}\big] \dot{\bar{\mathbf{V}}}_R - A_0 \nabla^2 \bar{\mathbf{V}}_R + \mathbf{b}_0(\nabla \cdot \dot{\mathbf{u}}_B) = \mathbf{0}, \quad \text{in } \Omega \times (0, t_f) . \tag{2.21}$$

The above equation supplemented with the boundary and initial conditions (2.10) and (2.11) yields the IBVP for a single implicit equation,

$$\begin{gathered} \big[M_R + \mathbf{b}_0 \mathbf{b}_1^T R^{-1} \hat{E}\big] \dot{\bar{\mathbf{V}}}_R - A_0 \nabla^2 \bar{\mathbf{V}}_R + \mathbf{b}_0(\nabla \cdot \dot{\mathbf{u}}_B) = \mathbf{0}, \quad \text{in } \Omega \times (0, t_f) , \\ \bar{\mathbf{V}}_R(\mathbf{x}, t) = R\mathbf{V}_B(\mathbf{x}, t), \quad \text{on } \Gamma \times [0, t_f) , \\ \bar{\mathbf{V}}_R(\mathbf{x}, 0) = R\mathbf{V}_I(\mathbf{x}), \quad \text{in } \Omega \end{gathered}$$

that is equivalent to the fully coupled parabolic-elliptic IBVP (1.7)-(1.13). We will denote this IBVP by $\mathcal{P}0$.

By the superposition principle, the problem $\mathcal{P}0$ can be decomposed into the following two subproblems: the (autonomous) parabolic problem, denoted by $\mathcal{P}1$,

$$\begin{gathered} M_R\dot{\bar{\mathbf{W}}} - A_0\nabla^2\bar{\mathbf{W}} = \mathbf{0}, \quad \text{in } \Omega\times(0,t_f), \\ \bar{\mathbf{W}}(\mathbf{x},t) = R\mathbf{V}_B(\mathbf{x},t), \quad \text{on } \Gamma\times[0,t_f), \\ \bar{\mathbf{W}}(\mathbf{x},0) = R\mathbf{V}_I(\mathbf{x}), \quad \text{in } \Omega, \end{gathered} \tag{2.22}$$

and the implicit problem, denoted by $\mathcal{P}2$,

$$\Big[M_R + \mathbf{b}_0\mathbf{b}_1^T R^{-1}\hat{E}\Big]\dot{\mathbf{V}} - A_0\nabla^2\mathbf{V} = \mathbf{F}, \quad \text{in } \Omega\times(0,t_f), \tag{2.23}$$

$$\mathbf{V}(\mathbf{x},t) = \mathbf{0}, \quad \text{on } \Gamma\times[0,t_f),$$

$$\mathbf{V}(\mathbf{x},0) = \mathbf{0}, \quad \text{in } \Omega, \tag{2.24}$$

where

$$\mathbf{F} = -\mathbf{b}_0\mathbf{b}_1^T R^{-1}\hat{E}\dot{\bar{\mathbf{W}}} - \mathbf{b}_0(\nabla\cdot\dot{\mathbf{u}}_B), \tag{2.25}$$

$\dot{\bar{\mathbf{W}}}$ is the time derivative of a solution to the problem $\mathcal{P}1$ and $\mathbf{u}_B$ is defined by (2.14). Then the solution to the problem $\mathcal{P}0$ takes the form

$$\bar{\mathbf{V}}_R = \mathbf{V} + \bar{\mathbf{W}} \tag{2.26}$$

and from (2.8), (2.16), and (2.26), the solution $(\bar{\mathbf{V}},\mathbf{u})$ to the IBVP (1.7)-(1.13) is

$$\bar{\mathbf{V}} = R^{-1}(\mathbf{V}+\bar{\mathbf{W}}), \tag{2.27}$$

$$\mathbf{u} = E_0(\mathbf{b}_1^T R^{-1}\mathbf{V}) + E_0(\mathbf{b}_1^T R^{-1}\bar{\mathbf{W}}) + \mathbf{u}_B\,. \tag{2.28}$$

Remark 2.3. Equations (2.10), (2.22), and (2.26) yield the auxiliary boundary condition for the problem $\mathcal{P}2$:

$$\dot{\mathbf{V}}(\mathbf{x},t) = \mathbf{0}, \quad \text{on } \Gamma\times[0,t_f)\,. \tag{2.29}$$

Remark 2.4. The autonomous IBVP $\mathcal{P}1$ can further be decomposed into the elliptic boundary value problem, denoted by $\mathcal{P}1.1$, with time $t\in[0,t_f]$ as a parameter

$$\begin{gathered} -A_0\nabla^2\mathbf{W}_0 = \mathbf{0}, \quad \text{in } \Omega, \\ \mathbf{W}_0(\mathbf{x},t) = R\mathbf{V}_B(\mathbf{x},t), \quad \text{on } \Gamma, \end{gathered}$$

where $\dot{\mathbf{W}}_0$ satisfies

$$-A_0\nabla^2\dot{\mathbf{W}}_0 = \mathbf{0}, \quad \text{in } \Omega, \tag{2.30}$$

$$\dot{\mathbf{W}}_0(\mathbf{x},t) = R\dot{\mathbf{V}}_B(\mathbf{x},t), \quad \text{on } \Gamma, \tag{2.31}$$

and the parabolic IBVP, denoted by $\mathcal{P}1.2$:

$$\begin{gathered} M_R\dot{\mathbf{W}} - A_0\nabla^2\mathbf{W} = -M_R\dot{\mathbf{W}}_0, \quad \text{in } \Omega\times(0,t_f), \\ \mathbf{W}(\mathbf{x},t) = \mathbf{0}, \quad \text{on } \Gamma\times[0,t_f), \\ \mathbf{W}(\mathbf{x},0) = R\mathbf{V}_I(\mathbf{x}) - \mathbf{W}_0(\mathbf{x},0), \quad \text{in } \Omega, \end{gathered}$$

where $\dot{\mathbf{W}}_0$ is the solution to (2.30)-(2.31). Then the solution to problem $\mathcal{P}1$ is

$$\bar{\mathbf{W}} = \mathbf{W}_0 + \mathbf{W}\,. \tag{2.32}$$

The existence and uniqueness of weak solutions to the subproblems $\mathcal{P}1.1$ and $\mathcal{P}1.2$ and continuous dependence of the solutions on the initial and boundary data follow from the standard elliptic and parabolic theory, yielding the well-posedness in a weak sense of the problem $\mathcal{P}1$. Thus, the question of the well-posedness of the fully coupled IBVP (1.7)-(1.13) amounts to establishing the well-posedness of the problem $\mathcal{P}2$.

2.4. **Problem $\mathcal{P}1$.** Let us now focus our attention on the properties of a weak solution to the problem $\mathcal{P}1$. Henceforth, $\hat{C} > 0$ denotes a generic constant independent of functions to be estimated. We shall start with two lemmas concerning weak solutions to the subproblems $\mathcal{P}1.1$ and $\mathcal{P}1.2$. We omit their proofs because they follow the standard Galerkin methods for elliptic and parabolic problems, respectively.

Lemma 2.5. *Given $\mathbf{V}_B \in L^2\big(0,t_f;H^{1/2}(\Gamma)^3\big)$ with $\dot{\mathbf{V}}_B \in L^2\big(0,t_f;H^{1/2}(\Gamma)^3\big)$, under Assumption 2.2 (iii) on the matrix A_0, there exists a unique weak solution $\mathbf{W}_0 \in L^2\big(0,t_f;\mathbb{V}^3\big)$ to the problem $\mathcal{P}1.1$ such that $\dot{\mathbf{W}}_0 \in L^2\big(0,t_f;\mathbb{V}^3\big)$ and the following estimates hold*

$$
\begin{aligned}
\|\mathbf{W}_0\|_{L^2(0,t_f;\mathbb{V}^3)} &\le \hat{C}\|\mathbf{V}_B\|_{L^2(0,t_f;H^{1/2}(\Gamma)^3)},\\
\|\dot{\mathbf{W}}_0\|_{L^2(0,t_f;\mathbb{V}^3)} &\le \hat{C}\|\dot{\mathbf{V}}_B\|_{L^2(0,t_f;H^{1/2}(\Gamma)^3)}\,.
\end{aligned}
$$

Moreover, for each $t \in [0,t_f]$,

$$
\begin{aligned}
\|\mathbf{W}_0(t)\|_{\mathbb{V}^3} &\le \hat{C}\|\mathbf{V}_B(t)\|_{H^{1/2}(\Gamma)^3},\\
\|\dot{\mathbf{W}}_0(t)\|_{\mathbb{V}^3} &\le \hat{C}\|\dot{\mathbf{V}}_B(t)\|_{H^{1/2}(\Gamma)^3}\,.
\end{aligned}
$$

Lemma 2.6. *Given $\dot{\mathbf{W}}_0 \in L^2(0,t_f;\mathbb{V}^3)$, $\mathbf{V}_I \in \mathbb{V}^3$, and $\mathbf{W}_0(0) \in \mathbb{V}^3$, under Assumption 2.2 (iii), problem $\mathcal{P}1.2$ admits a unique weak solution $\mathbf{W} \in L^\infty\big(0,t_f;\mathbb{V}_0^3\big)$ with $\dot{\mathbf{W}} \in L^2\big(0,t_f;\mathbb{H}^3\big)$ and the solution depends continuously on the data; that is,*

$$
\begin{aligned}
\|\mathbf{W}\|_{L^\infty(0,t_f;\mathbb{V}_0^3)} &\le \hat{C}\big(\|\mathbf{V}_I\|_{\mathbb{V}^3} + \|\mathbf{W}_0(0)\|_{\mathbb{V}^3} + \|\dot{\mathbf{W}}_0\|_{L^2(0,t_f;\mathbb{V}^3)}\big),\\
\|\dot{\mathbf{W}}\|_{L^2(0,t_f;\mathbb{H}^3)} &\le \hat{C}\big(\|\mathbf{V}_I\|_{\mathbb{V}^3} + \|\mathbf{W}_0(0)\|_{\mathbb{V}^3} + \|\dot{\mathbf{W}}_0\|_{L^2(0,t_f;\mathbb{V}^3)}\big)\,.
\end{aligned}
$$

As a direct consequence of Lemma 2.5, Lemma 2.6, and (2.32) we have the following results regarding the well-posedness and properties of a weak solution to the problem $\mathcal{P}1$.

Corollary 2.7. *Given the boundary and initial data $\mathbf{V}_B \in L^2\big(0,t_f;H^{1/2}(\Gamma)^3\big)$ with $\dot{\mathbf{V}}_B \in L^2\big(0,t_f;H^{1/2}(\Gamma)^3\big)$ and $\mathbf{V}_I \in \mathbb{V}^3$, under Assumption 2.2 (iii), problem $\mathcal{P}1$ admits a unique weak solution*

$$
\bar{\mathbf{W}} \in L^2(0,t_f;\mathbb{V}^3) \quad \textit{with} \quad \dot{\bar{\mathbf{W}}} \in L^2(0,t_f;\mathbb{H}^3) \tag{2.33}
$$

and the solution depends continuously on the data in the sense that the following estimates hold

$$
\begin{aligned}
\|\bar{\mathbf{W}}\|_{L^2(0,t_f;\mathbb{V}^3)} \le \hat{C}\big(&\|\mathbf{V}_I\|_{\mathbb{V}^3} + \|\mathbf{V}_B(0)\|_{H^{1/2}(\Gamma)^3}\\
&+ \|\mathbf{V}_B\|_{L^2(0,t_f;H^{1/2}(\Gamma)^3)} + \|\dot{\mathbf{V}}_B\|_{L^2(0,t_f;H^{1/2}(\Gamma)^3)}\big),
\end{aligned} \tag{2.34}
$$

$$
\|\dot{\bar{\mathbf{W}}}\|_{L^2(0,t_f;\mathbb{H}^3)} \le \hat{C}\big(\|\mathbf{V}_I\|_{\mathbb{V}^3} + \|\mathbf{V}_B(0)\|_{H^{1/2}(\Gamma)^3} + \|\dot{\mathbf{V}}_B\|_{L^2(0,t_f;H^{1/2}(\Gamma)^3)}\big)\,. \tag{2.35}
$$

3. Well-posedness of problem $\mathcal{P}2$ with a generalized source term

In this section we address the questions of existence, uniqueness, and continuous dependence on data of a weak solution to the implicit problem $\mathcal{P}2$ with a generalized source term $\mathbf{F}$. Throughout the section we do not impose the relationship (2.25) between $\mathbf{F}$, the solution $\bar{\mathbf{W}}$ to problem $\mathcal{P}1$, and the function $\mathbf{u_B}(\hat{\boldsymbol{\sigma}})$.

3.1. **A priori estimates.** We begin by obtaining formal a priori energy estimates for system $\mathcal{P}2$. An important feature of this system is that the elements of the matrices M_R and A_0 and the coupling vectors $\mathbf{b}_0$ and $\mathbf{b}_1$ differ by more than 20 orders of magnitude. Therefore, a more refined element-wise approach that allows one to determine precise relationships between components of similar magnitude needs to be applied.

We first formally multiply (2.23) by $\dot{\mathbf{V}}^T$, integrate over Ω, and split the first integral on the left-hand side into the sum of two integrals:

$$\int_\Omega \dot{\mathbf{V}}^T M_R \dot{\mathbf{V}} \mathrm{d}\Omega + \int_\Omega \dot{\mathbf{V}}^T \mathbf{b}_0 \mathbf{b}_1^T R^{-1} \hat{E} \dot{\mathbf{V}} \mathrm{d}\Omega - \int_\Omega \dot{\mathbf{V}}^T A_0 \nabla^2 \mathbf{V} \mathrm{d}\Omega = \int_\Omega \dot{\mathbf{V}}^T \mathbf{F} \mathrm{d}\Omega \,. \tag{3.1}$$

We shall estimate each term of (3.1) separately. For the first term on the left-hand side we have

$$\begin{aligned}
&\Big| \int_\Omega \Big[\dot{\mathbf{V}}^T M_R \dot{\mathbf{V}} - \big(m_{11} \dot{V}_1^2 + m_{22} \dot{V}_2^2 + m_{33} \dot{V}_3^2 \big) \Big] \mathrm{d}\Omega \Big| \\
&\le \frac{|m_{12}+m_{21}|}{2} 2 \Big| \int_\Omega \dot{V}_1 \dot{V}_2 \mathrm{d}\Omega \Big| + \frac{|m_{23}+m_{32}|}{2} 2 \Big| \int_\Omega \dot{V}_2 \dot{V}_3 \mathrm{d}\Omega \Big| \\
&\quad + \frac{|m_{13}+m_{31}|}{2} 2 \Big| \int_\Omega \dot{V}_1 \dot{V}_3 \mathrm{d}\Omega \Big| \\
&\le \frac{|m_{12}+m_{21}|}{2} \Big[\frac{1}{\varepsilon_1} \|\dot{V}_1\|_{\mathbb{H}}^2 + \varepsilon_1 \|\dot{V}_2\|_{\mathbb{H}}^2 \Big] + \frac{|m_{23}+m_{32}|}{2} \Big[\frac{1}{\varepsilon_2} \|\dot{V}_2\|_{\mathbb{H}}^2 + \varepsilon_2 \|\dot{V}_3\|_{\mathbb{H}}^2 \Big] \\
&\quad + \frac{|m_{13}+m_{31}|}{2} \Big[\frac{1}{\varepsilon_3} \|\dot{V}_3\|_{\mathbb{H}}^2 + \varepsilon_3 \|\dot{V}_1\|_{\mathbb{H}}^2 \Big] \,,
\end{aligned}$$

where $\varepsilon_i > 0$, $i = 1,2,3$, and therefore,

$$\begin{aligned}
\int_\Omega \dot{\mathbf{V}}^T M_R \dot{\mathbf{V}} \mathrm{d}\Omega \ge \Big[m_{11} - \frac{|m_{12}+m_{21}|}{2} \frac{1}{\varepsilon_1} - \frac{|m_{13}+m_{31}|}{2} \varepsilon_3 \Big] \|\dot{V}_1\|_{\mathbb{H}}^2 \\
+ \Big[m_{22} - \frac{|m_{23}+m_{32}|}{2} \frac{1}{\varepsilon_2} - \frac{|m_{21}+m_{12}|}{2} \varepsilon_1 \Big] \|\dot{V}_2\|_{\mathbb{H}}^2 \\
+ \Big[m_{33} - \frac{|m_{31}+m_{13}|}{2} \frac{1}{\varepsilon_3} - \frac{|m_{32}+m_{23}|}{2} \varepsilon_2 \Big] \|\dot{V}_3\|_{\mathbb{H}}^2 \,.
\end{aligned} \tag{3.2}$$

Using (2.13) and (2.18) together with (1.6) and the definition (2.7) of the matrix R, the second term on the left-hand side of (3.1) can be written as

$$\begin{aligned}
&\int_\Omega \dot{\mathbf{V}}^T \mathbf{b}_0 \mathbf{b}_1^T R^{-1} \hat{E} \dot{\mathbf{V}} \mathrm{d}\Omega \\
&= \int_\Omega (\mathbf{b}_0^T \dot{\mathbf{V}}) \nabla \cdot E_0 (\mathbf{b}_1^T R^{-1} \dot{\mathbf{V}}) \mathrm{d}\Omega \\
&= a_E \big(E_0(\mathbf{b}_0^T \dot{\mathbf{V}}), E_0(\mathbf{b}_1^T R^{-1} \dot{\mathbf{V}}) \big) \\
&= a_E \big(E_0(\mathbf{b}_0^T \dot{\mathbf{V}}), E_0((\mathbf{b}_0 + \mathbf{b}_d)^T R^{-1} \dot{\mathbf{V}}) \big) \\
&= a_E \big(E_0(\mathbf{b}_0^T \dot{\mathbf{V}}), E_0(\mathbf{b}_0^T \dot{\mathbf{V}}) \big) + a_E \big(E_0(\mathbf{b}_0^T \dot{\mathbf{V}}), E_0(\mathbf{b}_R^T \dot{\mathbf{V}}) \big),
\end{aligned} \tag{3.3}$$

where $\mathbf{b}_R^T = [b_{R1}\ b_{R2}\ b_{R3}]^T = \mathbf{b}_d^T R^{-1}$. From (2.12),

$$a_E\big(E_0(\mathbf{b}_0^T\dot{\mathbf{V}}), E_0(\mathbf{b}_0^T\dot{\mathbf{V}})\big) \geq \gamma \|E_0(\mathbf{b}_0^T\dot{\mathbf{V}})\|_{0,n}^2 \tag{3.4}$$

and the last term in (3.3) is estimated as follows:

$$\begin{aligned}
&\big|a_E\big(E_0(\mathbf{b}_0^T\dot{\mathbf{V}}), E_0(\mathbf{b}_R^T\dot{\mathbf{V}})\big)\big| \\
&= \big|a_E\big(E_0(\mathbf{b}_R^T\dot{\mathbf{V}}), E_0(\mathbf{b}_0^T\dot{\mathbf{V}})\big)\big| \\
&= \Big|\int_\Omega (\mathbf{b}_R^T\dot{\mathbf{V}})\nabla\cdot E_0(\mathbf{b}_0^T\dot{\mathbf{V}})\mathrm{d}\Omega\Big| \\
&\leq \|\mathbf{b}_R^T\dot{\mathbf{V}}\|_{\mathbb{H}}\|\nabla\cdot E_0(\mathbf{b}_0^T\dot{\mathbf{V}})\|_{\mathbb{H}} \\
&\leq \frac{\varepsilon}{2}\|\mathbf{b}_R^T\dot{\mathbf{V}}\|_{\mathbb{H}}^2 + \frac{n}{2\varepsilon}\|E_0(\mathbf{b}_0^T\dot{\mathbf{V}})\|_{0,n}^2 \\
&\leq \frac{3\varepsilon}{2}\big(b_{R1}^2\|\dot{V}_1\|_{\mathbb{H}}^2 + b_{R2}^2\|\dot{V}_2\|_{\mathbb{H}}^2 + b_{R3}^2\|\dot{V}_3\|_{\mathbb{H}}^2\big) + \frac{n}{2\varepsilon}\|E_0(\mathbf{b}_0^T\dot{\mathbf{V}})\|_{0,n}^2\,.
\end{aligned}$$

Thus,

$$\begin{aligned}
a_E\big(E_0(\mathbf{b}_0^T\dot{\mathbf{V}}), E_0(\mathbf{b}_R^T\dot{\mathbf{V}})\big) \geq & -\frac{3\varepsilon}{2}\big(b_{R1}^2\|\dot{V}_1\|_{\mathbb{H}}^2 + b_{R2}^2\|\dot{V}_2\|_{\mathbb{H}}^2 + b_{R3}^2\|\dot{V}_3\|_{\mathbb{H}}^2\big) \\
& - \frac{n}{2\varepsilon}\|E_0(\mathbf{b}_0^T\dot{\mathbf{V}})\|_{0,n}^2\,.
\end{aligned} \tag{3.5}$$

To handle the the last term on the left-hand side of (3.1), we use the divergence theorem and the auxiliary boundary condition (2.29) to obtain

$$\begin{aligned}
&-\int_\Omega \dot{\mathbf{V}}^T A_0\nabla^2\mathbf{V}\mathrm{d}\Omega \\
&= \int_\Omega \Big(\nabla\dot{V}_1\cdot(a_{11}\nabla V_1 + a_{12}\nabla V_2 + a_{13}\nabla V_3) + \nabla\dot{V}_2\cdot\big(a_{21}\nabla V_1 + a_{22}\nabla V_2 \\
&\quad + a_{23}\nabla V_3\big) + \nabla\dot{V}_3\cdot(a_{31}\nabla V_1 + a_{32}\nabla V_2 + a_{33}\nabla V_3)\Big)\mathrm{d}\Omega \\
&= \int_\Omega \sum_{i=1}^n \partial_i\dot{\mathbf{V}}^T A_0\partial_i\mathbf{V}\mathrm{d}\Omega\,.
\end{aligned} \tag{3.6}$$

Let us define the bilinear form $\bar{a} : \mathbb{V}_0^3 \times \mathbb{V}_0^3 \to \mathbb{R}$ by

$$\bar{a}(\boldsymbol{\psi}, \mathbf{\Phi}) = \int_\Omega \sum_{i=1}^n \partial_i\boldsymbol{\psi}^T A_0\partial_i\mathbf{\Phi}\mathrm{d}\Omega\,. \tag{3.7}$$

Taking into account the symmetry of A_0, (3.6) and (3.7) yield

$$-\int_\Omega \dot{\mathbf{V}}^T A_0\nabla^2\mathbf{V}\mathrm{d}\Omega = \frac{1}{2}\frac{d}{dt}\bar{a}(\mathbf{V}, \mathbf{V})\,. \tag{3.8}$$

Remark 3.1. The symmetric bilinear form $\bar{a} : \mathbb{V}_0^3 \times \mathbb{V}_0^3 \to \mathbb{R}$ is coercive and continuous. Indeed, Assumption 2.2 (iii) on the matrix A_0 guarantees that there exists $\hat{\alpha} > 0$ such that

$$\bar{a}(\mathbf{\Phi}, \mathbf{\Phi}) \geq \hat{\alpha}\|\mathbf{\Phi}\|_{0,3}^2, \quad \forall \mathbf{\Phi} \in \mathbb{V}_0^3\,. \tag{3.9}$$

On the other hand, for every $\boldsymbol{\psi}, \mathbf{\Phi} \in \mathbb{V}_0^3$,

$$|\bar{a}(\boldsymbol{\psi}, \mathbf{\Phi})| = \Big|\int_\Omega \sum_{i=1}^n \partial_i\boldsymbol{\psi}^T A_0\partial_i\mathbf{\Phi}\mathrm{d}\Omega\Big|$$

$$\begin{aligned}
&\le \max_{1\le j,k\le 3}\{|a_{jk}|\} \sum_{i=1}^{n}\sum_{j,k=1}^{3} \|\partial_i \psi_j\|_{\mathbb{H}} \|\partial_i \Phi_k\|_{\mathbb{H}} \\
&\le \max_{1\le j,k\le 3}\{|a_{jk}|\} \sum_{j=1}^{3}\sum_{i=1}^{n} \|\partial_i \psi_j\|_{\mathbb{H}} \sum_{k=1}^{3}\sum_{i=1}^{n} \|\partial_i \Phi_k\|_{\mathbb{H}} \\
&\le 3n \max_{1\le j,k\le 3}\{|a_{jk}|\} \|\boldsymbol{\psi}\|_{0,3} \|\mathbf{\Phi}\|_{0,3}\,,
\end{aligned}$$

and the result follows.

Applying (3.2), (3.3)-(3.5), and (3.8) to (3.1) we have

$$\begin{aligned}
&2\Big[m_{11} - \frac{|m_{12}+m_{21}|}{2}\frac{1}{\varepsilon_1} - \frac{|m_{13}+m_{31}|}{2}\varepsilon_3 - \frac{3\varepsilon}{2} b_{R1}^2\Big] \|\dot V_1\|_{\mathbb{H}}^2 \\
&+ 2\Big[m_{22} - \frac{|m_{23}+m_{32}|}{2}\frac{1}{\varepsilon_2} - \frac{|m_{21}+m_{12}|}{2}\varepsilon_1 - \frac{3\varepsilon}{2} b_{R2}^2\Big] \|\dot V_2\|_{\mathbb{H}}^2 \\
&+ 2\Big[m_{33} - \frac{|m_{31}+m_{13}|}{2}\frac{1}{\varepsilon_3} - \frac{|m_{32}+m_{23}|}{2}\varepsilon_2 - \frac{3\varepsilon}{2} b_{R3}^2\Big] \|\dot V_3\|_{\mathbb{H}}^2 \\
&+ 2\Big(\gamma - \frac{n}{2\varepsilon}\Big) \|E_0(\mathbf{b}_0^T \dot{\mathbf{V}})\|_{0,n}^2 + \frac{d}{dt}\bar a(\mathbf{V},\mathbf{V}) \\
&\le 2\int_\Omega \dot{\mathbf{V}}^T \mathbf{F} \mathrm{d}\Omega\,.
\end{aligned} \tag{3.10}$$

The right-hand side of (3.10) is majorized by

$$\begin{aligned}
2\int_\Omega \dot{\mathbf{V}}^T \mathbf{F} \mathrm{d}\Omega \le\ & m_{11}\|\dot V_1\|_{\mathbb{H}}^2 + \frac{1}{m_{11}}\|F_1\|_{\mathbb{H}}^2 + m_{22}\|\dot V_2\|_{\mathbb{H}}^2 + \frac{1}{m_{22}}\|F_2\|_{\mathbb{H}}^2 \\
&+ m_{33}\|\dot V_3\|_{\mathbb{H}}^2 + \frac{1}{m_{33}}\|F_3\|_{\mathbb{H}}^2\,.
\end{aligned} \tag{3.11}$$

From (3.10) and (3.11) we obtain

$$\begin{aligned}
&\Big[m_{11} - |m_{12}+m_{21}|\frac{1}{\varepsilon_1} - |m_{13}+m_{31}|\varepsilon_3 - 3\varepsilon b_{R1}^2\Big] \|\dot V_1\|_{\mathbb{H}}^2 \\
&+ \Big[m_{22} - |m_{23}+m_{32}|\frac{1}{\varepsilon_2} - |m_{21}+m_{12}|\varepsilon_1 - 3\varepsilon b_{R2}^2\Big] \|\dot V_2\|_{\mathbb{H}}^2 \\
&+ \Big[m_{33} - |m_{31}+m_{13}|\frac{1}{\varepsilon_3} - |m_{32}+m_{23}|\varepsilon_2 - 3\varepsilon b_{R3}^2\Big] \|\dot V_3\|_{\mathbb{H}}^2 \\
&+ 2\Big(\gamma - \frac{n}{2\varepsilon}\Big) \|E_0(\mathbf{b}_0^T \dot{\mathbf{V}})\|_{0,n}^2 + \frac{d}{dt}\bar a(\mathbf{V},\mathbf{V}) \\
&\le \frac{1}{m_{11}}\|F_1\|_{\mathbb{H}}^2 + \frac{1}{m_{22}}\|F_2\|_{\mathbb{H}}^2 + \frac{1}{m_{33}}\|F_3\|_{\mathbb{H}}^2\,.
\end{aligned}$$

Integrating the last inequality with respect to time and applying (2.24) and (3.9) yield

$$\hat\beta \int_0^t \|\dot{\mathbf{V}}\|_{\mathbb{H}^3}^2 \mathrm{d}s + \hat\gamma \int_0^t \|E_0(\mathbf{b}_0^T \dot{\mathbf{V}})\|_{0,n}^2 \mathrm{d}s + \hat\alpha \|\mathbf{V}(t)\|_{0,3}^2 \le \frac{1}{\underline{m}} \int_0^t \|\mathbf{F}\|_{\mathbb{H}^3}^2 \mathrm{d}s \tag{3.12}$$

for every $t \in [0, t_f]$, where $\hat\beta = \min_{1\le i\le 3} \beta_i$ with

$$\begin{aligned}
&\beta_i = m_{ii} - |m_{ij}+m_{ji}|\frac{1}{\varepsilon_i} - |m_{ik}+m_{ki}|\varepsilon_k - 3\varepsilon b_{Ri}^2\,, \\
&i = 1,2,3,\ j = (i \bmod 3)+1,\ k = (j \bmod 3)+1\,,
\end{aligned} \tag{3.13}$$

$$\hat{\gamma} = 2\Big(\gamma - \frac{n}{2\varepsilon}\Big) \tag{3.14}$$

and $\underline{m} = \min_{1\le i\le 3} m_{ii} > 0$. The values $\varepsilon_i > 0$, $i = 1,2,3$, and $\varepsilon > 0$ must be chosen to satisfy $\hat{\beta} > 0$ and $\hat{\gamma} \ge 0$ and we have the following result.

Lemma 3.2. *Under Assumption 2.2 (i)-(ii), the a priori energy estimate* (3.12) *holds with $\hat{\beta} > 0$ and $\hat{\gamma} \ge 0$ if ε_i, $i = 1,2,3$, and ε satisfy the following conditions.*

$$\frac{p_i|m_{ij}+m_{ji}|}{m_{ii}} \le \varepsilon_i \le \frac{m_{jj}}{q_j|m_{ij}+m_{ji}|}, \quad i = 1,2,3,\ j = (i \bmod 3)+1\,, \tag{3.15}$$

$$\frac{n}{2\gamma} \le \varepsilon \le \min_{1\le i\le 3} \frac{m_{ii}}{3r_i b_{Ri}^2}\,, \tag{3.16}$$

where $p_i > 0$, $q_i > 0$, $r_i > 0$, $\frac{1}{p_i} + \frac{1}{q_i} + \frac{1}{r_i} < 1$, $i = 1,2,3$, $\gamma = \gamma(\Omega)$ is given by (2.12)*, and n is the dimension of the problem.*

Proof. The condition $\hat{\beta} > 0$ is equivalent to $\beta_i > 0$, $i = 1,2,3$. From (3.13) we observe that the latter condition holds true if we assume that there exist $p_i > 0$, $q_i > 0$, and $r_i > 0$, $\frac{1}{p_i} + \frac{1}{q_i} + \frac{1}{r_i} < 1$, $i = 1,2,3$, such that

$$\begin{gathered} |m_{ij}+m_{ji}|\frac{1}{\varepsilon_i} \le \frac{1}{p_i} m_{ii}\,, \\ |m_{ik}+m_{ki}|\varepsilon_k \le \frac{1}{q_i} m_{ii}\,, \\ 3\varepsilon b_{Ri}^2 \le \frac{1}{r_i} m_{ii}, \end{gathered} \tag{3.17}$$

where $i = 1,2,3$, $j = (i \bmod 3)+1$, $k = (j \bmod 3)+1$. The first two inequalities in (3.17) give the following conditions on ε_1:

$$|m_{12}+m_{21}|\frac{1}{\varepsilon_1} \le \frac{1}{p_1} m_{11}, \quad |m_{12}+m_{21}|\varepsilon_1 \le \frac{1}{q_2} m_{22}$$

which implies

$$\frac{p_1|m_{12}+m_{21}|}{m_{11}} \le \varepsilon_1 \le \frac{m_{22}}{q_2|m_{12}+m_{21}|}. \tag{3.18}$$

Similarly, we obtain

$$\frac{p_2|m_{23}+m_{32}|}{m_{22}} \le \varepsilon_2 \le \frac{m_{33}}{q_3|m_{23}+m_{32}|} \tag{3.19}$$

and

$$\frac{p_3|m_{13}+m_{31}|}{m_{33}} \le \varepsilon_3 \le \frac{m_{11}}{q_1|m_{13}+m_{31}|}. \tag{3.20}$$

The existence of positive intervals

$$\Big[\frac{p_i|m_{ij}+m_{ji}|}{m_{ii}}, \frac{m_{jj}}{q_j|m_{ij}+m_{ji}|}\Big], \quad i = 1,2,3,\ j = (i \bmod 3)+1,$$

is guaranteed by Assumption 2.2 (ii); hence, (3.18)-(3.20) yield (3.15).

The condition (3.16) on ε follows immediately from (3.14) and the last inequality in (3.17). Assumption 2.2 (i) together with (2.4), (2.5), (2.12), and the results in Horgan [10] concerning the Korn's constant guarantee the existence of a positive interval for ε. □

Remark 3.3. From formal a priori energy estimate (3.12) we conclude that a weak solution to problem $\mathcal{P}1$ is expected to be

$$\mathbf{V} \in L^\infty(0, t_f; \mathbb{V}_0^3), \quad \dot{\mathbf{V}} \in L^2(0, t_f; \mathbb{H}^3)$$

provided $\mathbf{F} \in L^2(0, t_f; \mathbb{H}^3)$.

3.2. **Abstract formulation.** The preceding remark suggests the weak formulation of the problem $\mathcal{P}2$ as follows: Given $\mathbf{F} \in L^2(0, t_f; \mathbb{H}^3)$, find $\mathbf{V} \in L^\infty(0, t_f; \mathbb{V}_0^3)$ with $\dot{\mathbf{V}} \in L^2(0, t_f; \mathbb{H}^3)$ such that, for all $\boldsymbol{\psi} \in \mathbb{V}_0^3$,

$$\begin{aligned} &\int_\Omega \boldsymbol{\psi}^T M_R \dot{\mathbf{V}} \mathrm{d}\Omega + \int_\Omega \boldsymbol{\psi}^T \mathbf{b}_0 \mathbf{b}_1^T R^{-1} \hat{E} \dot{\mathbf{V}} \mathrm{d}\Omega - \int_\Omega \boldsymbol{\psi}^T A_0 \nabla^2 \mathbf{V} \mathrm{d}\Omega \\ &= \int_\Omega \boldsymbol{\psi}^T \mathbf{F} \mathrm{d}\Omega, \end{aligned} \tag{3.21}$$

$$\mathbf{V}(\mathbf{x}, 0) = \mathbf{0}\,. \tag{3.22}$$

Let us define the continuous bilinear form $m : \mathbb{H}^3 \times \mathbb{H}^3 \to \mathbb{R}$ by

$$m(\boldsymbol{\psi}, \boldsymbol{\Phi}) = \int_\Omega \boldsymbol{\psi}^T M_R \boldsymbol{\Phi} \mathrm{d}\Omega\,.$$

Then the first integral on the left-had side of (3.21) takes the form

$$\int_\Omega \boldsymbol{\psi}^T M_R \dot{\mathbf{V}} \mathrm{d}\Omega = m(\boldsymbol{\psi}, \dot{\mathbf{V}})\,. \tag{3.23}$$

Applying (2.13) and (2.18) to the second term on the left-hand side of (3.21) gives

$$\int_\Omega \boldsymbol{\psi}^T \mathbf{b}_0 \mathbf{b}_1^T R^{-1} \hat{E} \dot{\mathbf{V}} \mathrm{d}\Omega = a_E(E_0(\mathbf{b}_0^T \boldsymbol{\psi}), E_0(\mathbf{b}_1^T R^{-1} \dot{\mathbf{V}}))\,. \tag{3.24}$$

Furthermore, from the definition of the linear operator E_0 (2.13) and the continuity of $a_E(\cdot, \cdot)$ and E_0, we observe that

$$\boldsymbol{\psi}, \boldsymbol{\Phi} \mapsto a_E(E_0(\mathbf{b}_0^T \boldsymbol{\psi}), E_0(\mathbf{b}_1^T R^{-1} \boldsymbol{\Phi}))$$

is a bilinear continuous map from $\mathbb{H}^3 \times \mathbb{H}^3$ to $\mathbb{R}$. Thus, we can define a continuous bilinear form $\bar{l} : \mathbb{H}^3 \times \mathbb{H}^3 \to \mathbb{R}$ by

$$\bar{l}(\boldsymbol{\psi}, \boldsymbol{\Phi}) = m(\boldsymbol{\psi}, \boldsymbol{\Phi}) + a_E(E_0(\mathbf{b}_0^T \boldsymbol{\psi}), E_0(\mathbf{b}_1^T R^{-1} \boldsymbol{\Phi}))\,. \tag{3.25}$$

Combining (3.23)-(3.25) leads to

$$\int_\Omega \boldsymbol{\psi}^T M_R \dot{\mathbf{V}} \mathrm{d}\Omega + \int_\Omega \boldsymbol{\psi}^T \mathbf{b}_0 \mathbf{b}_1^T R^{-1} \hat{E} \dot{\mathbf{V}} \mathrm{d}\Omega = \bar{l}(\boldsymbol{\psi}, \dot{\mathbf{V}})\,. \tag{3.26}$$

Lemma 3.4. *The bilinear form $\bar{l} : \mathbb{H}^3 \times \mathbb{H}^3 \to \mathbb{R}$ is coercive and hence non-degenerate.*

Proof. From (3.2)-(3.5), and (3.26), for every $\dot{\mathbf{V}} \in \mathbb{H}^3$, we have

$$\begin{aligned} \bar{l}(\dot{\mathbf{V}}, \dot{\mathbf{V}}) \geq & \Big[m_{11} - \frac{|m_{12} + m_{21}|}{2} \frac{1}{\varepsilon_1} - \frac{|m_{13} + m_{31}|}{2} \varepsilon_3 - \frac{3\varepsilon}{2} b_{R1}^2\Big] \|\dot{V}_1\|_{\mathbb{H}}^2 \\ &+ \Big[m_{22} - \frac{|m_{23} + m_{32}|}{2} \frac{1}{\varepsilon_2} - \frac{|m_{21} + m_{12}|}{2} \varepsilon_1 - \frac{3\varepsilon}{2} b_{R2}^2\Big] \|\dot{V}_2\|_{\mathbb{H}}^2 \\ &+ \Big[m_{33} - \frac{|m_{31} + m_{13}|}{2} \frac{1}{\varepsilon_3} - \frac{|m_{32} + m_{23}|}{2} \varepsilon_2 - \frac{3\varepsilon}{2} b_{R3}^2\Big] \|\dot{V}_3\|_{\mathbb{H}}^2 \\ &+ \Big(\gamma - \frac{n}{2\varepsilon}\Big) \|E_0(\mathbf{b}_0^T \dot{\mathbf{V}})\|_{0,n}^2 \end{aligned}$$

$$\geq \frac{1}{2}\sum_{i=1}^{3}(m_{ii}+\beta_i)\|\dot{V}_i\|_{\mathbb{H}}^2 + \frac{1}{2}\hat{\gamma}\|E_0(\mathbf{b}_0^T\dot{\mathbf{V}})\|_{0,n}^2 ,$$

where $\beta_i > 0$, $i = 1, 2, 3$, and $\hat{\gamma} \geq 0$ are given by (3.13)-(3.16). It follows that

$$\bar{l}(\dot{\mathbf{V}}, \dot{\mathbf{V}}) \geq \frac{1}{2}\underline{m}\|\dot{\mathbf{V}}\|_{\mathbb{H}^3}^2$$

where $\underline{m} = \min_{1\leq i\leq 3} m_{ii} > 0$, which completes the proof. □

To handle the last term on the left-hand side of (3.21), we use the same argument as in deriving formal a priori estimates (see (3.6) and (3.7) above) and obtain

$$-\int_\Omega \boldsymbol{\psi}^T A_0 \nabla^2 \mathbf{V} \mathrm{d}\Omega = \bar{a}(\boldsymbol{\psi}, \mathbf{V}) . \tag{3.27}$$

Substituting (3.26) and (3.27) into (3.21) yields the following abstract formulation of problem $\mathcal{P}2$ equivalent to (3.21)-(3.22): Given $\mathbf{F} \in L^2(0, t_f; \mathbb{H}^3)$, find $\mathbf{V} \in L^\infty(0, t_f; \mathbb{V}_0^3)$ such that $\dot{\mathbf{V}} \in L^2(0, t_f; \mathbb{H}^3)$ and, for all $\boldsymbol{\psi} \in \mathbb{V}_0^3$,

$$\bar{l}(\boldsymbol{\psi}, \dot{\mathbf{V}}) + \bar{a}(\boldsymbol{\psi}, \mathbf{V}) = (\boldsymbol{\psi}, \mathbf{F})_{\mathbb{H}^3} , \tag{3.28}$$

$$\mathbf{V}(\mathbf{x}, 0) = \mathbf{0} . \tag{3.29}$$

Remark 3.5. The bilinear forms $\bar{l} : \mathbb{H}^3 \times \mathbb{H}^3 \to \mathbb{R}$ and $\bar{a} : \mathbb{V}_0^3 \times \mathbb{V}_0^3 \to \mathbb{R}$ are continuous and coercive and besides, $\bar{a}(\cdot, \cdot)$ is symmetric. Therefore, we can associate with $\bar{l}(\cdot, \cdot)$ and $\bar{a}(\cdot, \cdot)$ the linear bijective operators $\mathcal{L} : \mathbb{H}^3 \to \mathbb{H}^3$ and $\mathcal{A} : \mathbb{V}_0^3 \to H^{-1}(\Omega)^3$, respectively, by setting

$$\bar{l}(\boldsymbol{\psi}, \boldsymbol{\Phi}) = (\boldsymbol{\psi}, \mathcal{L}\boldsymbol{\Phi})_{\mathbb{H}^3}, \quad \boldsymbol{\psi}, \boldsymbol{\Phi} \in \mathbb{H}^3,$$

$$\bar{a}(\boldsymbol{\psi}, \boldsymbol{\Phi}) = \langle \boldsymbol{\psi}, \mathcal{A}\boldsymbol{\Phi} \rangle, \quad \boldsymbol{\psi}, \boldsymbol{\Phi} \in \mathbb{V}_0^3 ,$$

where $\langle \cdot, \cdot \rangle$ is the duality pairing between $\mathbb{V}_0^3$ and $H^{-1}(\Omega)^3$ and $\langle \boldsymbol{\psi}, \boldsymbol{\phi} \rangle = (\boldsymbol{\psi}, \boldsymbol{\phi})_{\mathbb{H}^3}$ for every $\boldsymbol{\psi} \in \mathbb{V}_0^3$ and $\boldsymbol{\phi} \in \mathbb{H}^3$. Then (3.28) can be written as

$$\langle \boldsymbol{\psi}, \mathcal{L}\dot{\mathbf{V}} \rangle + \langle \boldsymbol{\psi}, \mathcal{A}\mathbf{V} \rangle = \langle \boldsymbol{\psi}, \mathbf{F} \rangle, \quad \forall \boldsymbol{\psi} \in \mathbb{V}_0^3$$

and we obtain an alternative formulation of the problem (3.28)-(3.29): Given $\mathbf{F} \in L^2(0, t_f; \mathbb{H}^3)$, find $\mathbf{V} \in L^\infty(0, t_f; \mathbb{V}_0^3)$ such that $\dot{\mathbf{V}} \in L^2(0, t_f; \mathbb{H}^3)$ and

$$\mathcal{L}\dot{\mathbf{V}} + \mathcal{A}\mathbf{V} = \mathbf{F} ,$$

$$\mathbf{V}(\mathbf{x}, 0) = \mathbf{0} .$$

3.3. **Well-posedness in a weak sense.** The main result of this section is formulated in the next theorem.

Theorem 3.6. *Given $\mathbf{F} \in L^2(0, t_f; \mathbb{H}^3)$, under Assumption 2.2, there exists a unique weak solution*

$$\mathbf{V} \in L^\infty(0, t_f; \mathbb{V}_0^3) \quad \textit{with} \quad \dot{\mathbf{V}} \in L^2(0, t_f; \mathbb{H}^3) \tag{3.30}$$

in the sense of (3.28)-(3.29) to the problem $\mathcal{P}2$ and the solution depends continuously on the data $\mathbf{F}$; that is, the mapping

$$\mathbf{F} \mapsto \mathbf{V}, \dot{\mathbf{V}}$$

from $L^2(0, t_f; \mathbb{H}^3)$ to $L^\infty(0, t_f; \mathbb{V}_0^3) \times L^2(0, t_f; \mathbb{H}^3)$ is continuous.

Remark 3.7. The proof of the existence and uniqueness of a weak solution to problem $\mathcal{P}2$ can be completed by the standard Galerkin method. We omit it and focus on continuous dependence results. The key to the proof of existence and uniqueness are energy estimates arising from (3.12), the continuity and coercivity of the bilinear forms $\bar{l}$ and $\bar{a}$, as well as the symmetry of $\bar{a}$.

Proof. The continuous dependence on the data $\mathbf{F}$ follows from the a priori energy estimate (3.12): for every $t \in [0, t_f]$,

$$\hat{\beta}\int_0^t \|\dot{\mathbf{V}}\|^2_{\mathbb{H}^3} \mathrm{d}s + \hat{\gamma}\int_0^t \|E_0(\mathbf{b}_0^T\dot{\mathbf{V}})\|^2_{0,n} \mathrm{d}s + \hat{\alpha}\|\mathbf{V}(t)\|^2_{0,3} \le \frac{1}{\underline{m}}\int_0^t \|\mathbf{F}\|^2_{\mathbb{H}^3} \mathrm{d}s \quad (3.12),$$

where $\hat{\beta}$, $\hat{\alpha}$ and $\underline{m}$ are positive constants and $\hat{\gamma} \ge 0$. The above inequality implies

$$\|\mathbf{V}(t)\|^2_{0,3} \le \frac{1}{\hat{\alpha}\underline{m}}\int_0^{t_f} \|\mathbf{F}\|^2_{\mathbb{H}^3} \mathrm{d}s$$

for every $t \in [0, t_f]$, and thus,

$$\|\mathbf{V}\|_{L^\infty(0,t_f;\mathbb{V}_0^3)} \le \hat{C}\|\mathbf{F}\|_{L^2(0,t_f;\mathbb{H}^3)}\,. \tag{3.31}$$

Taking $t = t_f$ in (3.12) also yields

$$\|\dot{\mathbf{V}}\|_{L^2(0,t_f;\mathbb{H}^3)} \le \hat{C}\|\mathbf{F}\|_{L^2(0,t_f;\mathbb{H}^3)} \tag{3.32}$$

which completes the proof. □

4. Main results

The main results of the paper are summarized in the following theorem that establishes the well-posedness in a weak sense of the fully coupled parabolic-elliptic IBVP (1.7)-(1.13).

Theorem 4.1. *Given the initial data $\mathbf{V}_I \in \mathbb{V}^3$ and the boundary data $\mathbf{V}_B \in L^2\big(0, t_f; H^{1/2}(\Gamma)^3\big)$ with $\dot{\mathbf{V}}_B \in L^2\big(0, t_f; H^{1/2}(\Gamma)^3\big)$ and $\hat{\boldsymbol{\sigma}} \in L^2\big(0, t_f; L^2(\Gamma)^{n\times n}\big)$ with $\dot{\hat{\boldsymbol{\sigma}}} \in L^2\big(0, t_f; L^2(\Gamma)^{n\times n}\big)$, under Assumption 2.2 on the matrices of diffusion coefficients, the IBVP (1.7)-(1.13) for the fully coupled TCpu system (1.1)-(1.4) admits a unique weak solution*

$$(\bar{\mathbf{V}}, \mathbf{u}) \in L^2(0, t_f; \mathbb{V}^3) \times L^2(0, t_f; \tilde{\mathbb{V}}_0^n) \tag{4.1}$$

with

$$(\dot{\bar{\mathbf{V}}}, \dot{\mathbf{u}}) \in L^2(0, t_f; \mathbb{H}^3) \times L^2(0, t_f; \tilde{\mathbb{V}}_0^n) \tag{4.2}$$

and this solution depends continuously on the data $\mathbf{V}_I$, $\mathbf{V}_B(0)$, $\mathbf{V}_B$, $\dot{\mathbf{V}}_B$, $\hat{\boldsymbol{\sigma}}$, and $\dot{\hat{\boldsymbol{\sigma}}}$.

Proof. (i) *Existence and uniqueness.* From the results of Section 2, we know that the solution $(\bar{\mathbf{V}}, \mathbf{u})$ to problem (1.7)-(1.13) is given by (2.27) and (2.28):

$$\bar{\mathbf{V}} = R^{-1}(\mathbf{V} + \bar{\mathbf{W}}),$$
$$\mathbf{u} = E_0(\mathbf{b}_1^T R^{-1}\mathbf{V}) + E_0(\mathbf{b}_1^T R^{-1}\bar{\mathbf{W}}) + \mathbf{u}_B,$$

where $\mathbf{V}$ and $\bar{\mathbf{W}}$ are the solutions to the problems $\mathcal{P}2$ and $\mathcal{P}1$, respectively, and $\mathbf{u}_B$ is defined by (2.14). The well-posedness of a weak solution to the problem $\mathcal{P}1$ is stated in Corollary 2.7, and Theorem 3.6 provides the existence, uniqueness and continuous dependence on data of a weak solution to the problem $\mathcal{P}2$ with a generalized source term $\mathbf{F}$.

The next step toward the proof of Theorem 4.1 is to show that the source term, (2.25),

$$\mathbf{F} = -\mathbf{b}_0\mathbf{b}_1^T R^{-1}\hat{E}\dot{\bar{\mathbf{W}}} - \mathbf{b}_0(\nabla\cdot\dot{\mathbf{u}}_B)$$

corresponding to the IBVP (1.7)-(1.13) satisfies the assumption $\mathbf{F} \in L^2(0,t_f;\mathbb{H}^3)$ of Theorem 3.6. The definition (2.18) of the operator $\hat{E}$ and Corollary 2.7 yield

$$\mathbf{b}_0\mathbf{b}_1^T R^{-1}\hat{E}\dot{\bar{\mathbf{W}}} \in L^2(0,t_f;\mathbb{H}^3)\,. \tag{4.3}$$

Assumption 2.1 and (2.14) imply

$$\mathbf{u}_B \in L^2(0,t_f;\tilde{\mathbb{V}}_0^n)\,, \tag{4.4}$$

$$\dot{\mathbf{u}}_B \in L^2(0,t_f;\tilde{\mathbb{V}}_0^n)\,, \tag{4.5}$$

and thus,

$$\mathbf{b}_0(\nabla\cdot\dot{\mathbf{u}}_B) \in L^2(0,t_f;\mathbb{H}^3)\,. \tag{4.6}$$

From (2.25), (4.3), and (4.6) we conclude that $\mathbf{F} \in L^2(0,t_f;\mathbb{H}^3)$; hence, Theorem 3.6 holds for the problem $\mathcal{P}2$ associated with the IBVP (1.7)-(1.13). The existence and uniqueness of a weak solution $(\bar{\mathbf{V}},\mathbf{u})$ satisfying (4.1) and (4.2) now follow immediately from (2.27), (2.33), (3.30), (4.4), (4.5), and the definition (2.13) of the operator E_0 applied to (2.28) and its time derivative.

(ii) *Continuous dependence on data.* From (2.27), (2.34), (3.31), and (4.1) we have

$$\begin{aligned}\|\bar{\mathbf{V}}\|_{L^2(0,t_f;\mathbb{V}^3)} &\le \hat{C}\big(\|\mathbf{V}\|_{L^\infty(0,t_f;\mathbb{V}_0^3)} + \|\bar{\mathbf{W}}\|_{L^2(0,t_f;\mathbb{V}^3)}\big)\\ &\le \hat{C}\big(\|\mathbf{V}_I\|_{\mathbb{V}^3} + \|\mathbf{V}_B(0)\|_{H^{1/2}(\Gamma)^3} + \|\mathbf{V}_B\|_{L^2(0,t_f;H^{1/2}(\Gamma)^3)}\\ &\quad + \|\dot{\mathbf{V}}_B\|_{L^2(0,t_f;H^{1/2}(\Gamma)^3)} + \|\mathbf{F}\|_{L^2(0,t_f;\mathbb{H}^3)}\big)\,.\end{aligned} \tag{4.7}$$

Similarly, the time derivative of (2.27) together with (2.35), (3.32), and (4.2) give

$$\begin{aligned}\|\dot{\bar{\mathbf{V}}}\|_{L^2(0,t_f;\mathbb{H}^3)} &\le \hat{C}\big(\|\dot{\mathbf{V}}\|_{L^\infty(0,t_f;\mathbb{H}^3)} + \|\dot{\bar{\mathbf{W}}}\|_{L^2(0,t_f;\mathbb{H}^3)}\big)\\ &\le \hat{C}\big(\|\mathbf{V}_I\|_{\mathbb{V}^3} + \|\mathbf{V}_B(0)\|_{H^{1/2}(\Gamma)^3}\\ &\quad + \|\dot{\mathbf{V}}_B\|_{L^2(0,t_f;H^{1/2}(\Gamma)^3)} + \|\mathbf{F}\|_{L^2(0,t_f;\mathbb{H}^3)}\big)\,.\end{aligned} \tag{4.8}$$

Next we estimate the source term on the right-hand side of (4.7) and (4.8). Applying the continuity property (2.19) of $\hat{E}$ and (2.35) to (2.25) gives

$$\begin{aligned}\|\mathbf{F}\|_{L^2(0,t_f;\mathbb{H}^3)} &\le \hat{C}\big(\|\dot{\bar{\mathbf{W}}}\|_{L^2(0,t_f;\mathbb{H}^3)} + \|\nabla\cdot\dot{\mathbf{u}}_B\|_{L^2(0,t_f;L^2(\Omega))}\big)\\ &\le \hat{C}\big(\|\mathbf{V}_I\|_{\mathbb{V}^3} + \|\mathbf{V}_B(0)\|_{H^{1/2}(\Gamma)^3}\\ &\quad + \|\dot{\mathbf{V}}_B\|_{L^2(0,t_f;H^{1/2}(\Gamma)^3)} + \|\nabla\cdot\dot{\mathbf{u}}_B\|_{L^2(0,t_f;\mathbb{H})}\big)\,.\end{aligned} \tag{4.9}$$

Now we evaluate the function $\mathbf{u}_B$ and its time derivative. Taking $\mathbf{\Phi} = \mathbf{u}_B$ in (2.14) and using (2.12) and Assumption 2.1, we have, for every $t \in [0,t_f]$,

$$\begin{aligned}\gamma\|\mathbf{u}_B\|_{0,n}^2 &\le \big|a_E(\mathbf{u}_B,\mathbf{u}_B)\big| = \Big|\int_\Gamma(\hat{\boldsymbol{\sigma}}\mathbf{n})\cdot\mathbf{u}_B\mathrm{d}\Gamma\Big|\\ &\le \|\hat{\boldsymbol{\sigma}}\mathbf{n}\|_{L^2(\Gamma)^n}\|\mathbf{u}_B\|_{L^2(\Gamma)^n} \le \hat{C}\|\hat{\boldsymbol{\sigma}}\|_{L^2(\Gamma)^{n\times n}}\|\mathbf{u}_B\|_{0,n}\,.\end{aligned} \tag{4.10}$$

The above inequality follows from the trace theorem and Poincaré-Friedrichs' inequality. Equation (4.10) implies, for every $t \in [0,t_f]$,

$$\|\mathbf{u}_B(t)\|_{0,n} \le \hat{C}\|\hat{\boldsymbol{\sigma}}(t)\|_{L^2(\Gamma)^{n\times n}}\,. \tag{4.11}$$

The time derivative of (2.14) and Assumption 2.1 yield $\dot{\mathbf{u}}_B \in \tilde{\mathbb{V}}_0^n$. Therefore, differentiating (2.14) with respect to time and taking $\mathbf{\Phi} = \dot{\mathbf{u}}_B$, the above argument also gives

$$\|\dot{\mathbf{u}}_B(t)\|_{0,n} \le \hat{C}\|\dot{\hat{\boldsymbol{\sigma}}}(t)\|_{L^2(\Gamma)^{n\times n}} \tag{4.12}$$

for every $t \in [0, t_f]$, and consequently,

$$\|\nabla \cdot \dot{\mathbf{u}}_B(t)\|_{\mathbb{H}} \le \hat{C}\|\dot{\hat{\boldsymbol{\sigma}}}(t)\|_{L^2(\Gamma)^{n\times n}} . \tag{4.13}$$

Integrating (4.11)-(4.13) over $[0, t_f]$, we get

$$\|\mathbf{u}_B\|_{L^2(0,t_f;\tilde{\mathbb{V}}_0^n)} \le \hat{C}\|\hat{\boldsymbol{\sigma}}\|_{L^2(0,t_f;L^2(\Gamma)^{n\times n})} , \tag{4.14}$$

$$\|\dot{\mathbf{u}}_B\|_{L^2(0,t_f;\tilde{\mathbb{V}}_0^n)} \le \hat{C}\|\dot{\hat{\boldsymbol{\sigma}}}\|_{L^2(0,t_f;L^2(\Gamma)^{n\times n})} , \tag{4.15}$$

$$\|\nabla \cdot \dot{\mathbf{u}}_B\|_{L^2(0,t_f;\mathbb{H})} \le \hat{C}\|\dot{\hat{\boldsymbol{\sigma}}}\|_{L^2(0,t_f;L^2(\Gamma)^{n\times n})} . \tag{4.16}$$

Substituting (4.16) into (4.9) gives

$$\begin{aligned}\|\mathbf{F}\|_{L^2(0,t_f;\mathbb{H}^3)} \le \hat{C}\big(&\|\mathbf{V}_I\|_{\mathbb{V}^3} + \|\mathbf{V}_B(0)\|_{H^{1/2}(\Gamma)^3} \\ &+ \|\dot{\mathbf{V}}_B\|_{L^2(0,t_f;H^{1/2}(\Gamma)^3)} + \|\dot{\hat{\boldsymbol{\sigma}}}\|_{L^2(0,t_f;L^2(\Gamma)^{n\times n})}\big) .\end{aligned} \tag{4.17}$$

Applying (4.17) to (4.7) and (4.8), we obtain respectively

$$\begin{aligned}\|\bar{\mathbf{V}}\|_{L^2(0,t_f;\mathbb{V}^3)} \le \hat{C}\big(&\|\mathbf{V}_I\|_{\mathbb{V}^3} + \|\mathbf{V}_B(0)\|_{H^{1/2}(\Gamma)^3} + \|\mathbf{V}_B\|_{L^2(0,t_f;H^{1/2}(\Gamma)^3)} \\ &+ \|\dot{\mathbf{V}}_B\|_{L^2(0,t_f;H^{1/2}(\Gamma)^3)} + \|\dot{\hat{\boldsymbol{\sigma}}}\|_{L^2(0,t_f;L^2(\Gamma)^{n\times n})}\big)\end{aligned} \tag{4.18}$$

and

$$\begin{aligned}\|\dot{\bar{\mathbf{V}}}\|_{L^2(0,t_f;\mathbb{H}^3)} \le \hat{C}\big(&\|\mathbf{V}_I\|_{\mathbb{V}^3} + \|\mathbf{V}_B(0)\|_{H^{1/2}(\Gamma)^3} \\ &+ \|\dot{\mathbf{V}}_B\|_{L^2(0,t_f;H^{1/2}(\Gamma)^3)} + \|\dot{\hat{\boldsymbol{\sigma}}}\|_{L^2(0,t_f;L^2(\Gamma)^{n\times n})}\big) .\end{aligned} \tag{4.19}$$

Next we show that the **u**-component of the solution depends continuously on data. Applying (2.8) and the continuity of E_0 (2.15) to (2.16) and combining the result with (4.14), and (4.18) yields

$$\begin{aligned}\|\mathbf{u}\|_{L^2(0,t_f;\tilde{\mathbb{V}}_0^n)} \le \hat{C}\big(&\|\mathbf{V}_I\|_{\mathbb{V}^3} + \|\mathbf{V}_B(0)\|_{H^{1/2}(\Gamma)^3} \\ &+ \|\mathbf{V}_B\|_{L^2(0,t_f;H^{1/2}(\Gamma)^3)} + \|\dot{\mathbf{V}}_B\|_{L^2(0,t_f;H^{1/2}(\Gamma)^3)} \\ &+ \|\hat{\boldsymbol{\sigma}}\|_{L^2(0,t_f;L^2(\Gamma)^{n\times n})} + \|\dot{\hat{\boldsymbol{\sigma}}}\|_{L^2(0,t_f;L^2(\Gamma)^{n\times n})}\big) .\end{aligned}$$

Finally, differentiating (2.16) with respect to time and using (2.15), (4.15), and (4.19), we obtain

$$\begin{aligned}\|\dot{\mathbf{u}}\|_{L^2(0,t_f;\tilde{\mathbb{V}}_0^n)} \le \hat{C}\big(&\|\mathbf{V}_I\|_{\mathbb{V}^3} + \|\mathbf{V}_B(0)\|_{H^{1/2}(\Gamma)^3} \\ &+ \|\dot{\mathbf{V}}_B\|_{L^2(0,t_f;H^{1/2}(\Gamma)^3)} + \|\dot{\hat{\boldsymbol{\sigma}}}\|_{L^2(0,t_f;L^2(\Gamma)^{n\times n})}\big)\end{aligned}$$

which completes the proof. □

5. Numerical Example

In this section we provide the results of numerical experiments for the TCpu system with real data illustrating the applicability of the proposed well-posedness theory. The calculations were performed in MATLAB R2015b.

We focus on the two-dimensional case and consider the IBVP (1.7)-(1.13) for the TCpu system defined in an annular region $\Omega \subset \mathbb{R}^2$ with the inner (borehole) radius

R_B and the outer (far-field) radius R_F. The following assumptions are made on the region Ω.

Assumption 5.1. (i) $\delta = \frac{R_B}{R_F} \le e^{-1}$
(ii) The region Ω does not experience pure rotation.

To eliminate pure rotation, we impose the following side condition on the displacement $\mathbf{u} = u\mathbf{i} + v\mathbf{j}$:

$$\int_\Omega \Big(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\Big) \mathrm{d}\Omega = 0\,. \tag{5.1}$$

Using the results obtained by Horgan [10], it was shown in [13] that under Assumption 5.1 and the side condition (5.1) the Korn constant γ appearing in (2.12) is given by

$$\gamma = \frac{G}{2}\Big[1 - \Big(\frac{3}{\delta^{-2} + 1 + \delta^2}\Big)^{1/2}\Big]\,, \tag{5.2}$$

where G is the shear modulus; hence, for $0 < \delta \le e^{-1}$,

$$\frac{G}{2}\Big[1 - \Big(\frac{3}{e^2 + 1 + e^{-2}}\Big)^{1/2}\Big] \le \gamma < \frac{G}{2}\,. \tag{5.3}$$

Example 5.2. In this example, the values of the TCpu model parameters describing the physical properties of a rock/fluid system with various values of the permeability/fluid viscosity ratio, k/η, and the solute reflected fraction, $\mathfrak{R}$, are taken from [6] and represented in Table 3 in the appendix of the paper. With these data, direct calculations show that parts (i) and (ii) of Assumption 2.2 are satisfied as follows regardless of the values of k/η and $\mathfrak{R}$ because the matrix M_R and the vector $\mathbf{b}_R$ are independent of those parameters:

$$\begin{gathered}
0 < m_{11} = 7.1188\text{e+}03 < 2.7196\text{e+}11 = b_{R1}^2 \\
0 < m_{22} = 7.0722\text{e}{-}10 < 2.9166\text{e}{-}03 = b_{R2}^2 \\
0 < m_{33} = 4.9082\text{e}{-}11 < 1.2659\text{e}{-}02 = b_{R3}^2
\end{gathered}$$

and

$$\begin{gathered}
0 < (m_{12} + m_{21})^2 = 1.6800\text{e}{-}12 < 5.0346\text{e}{-}06 = m_{11}m_{22}\,, \\
0 < (m_{23} + m_{32})^2 = 6.9032\text{e}{-}25 < 3.4712\text{e}{-}20 = m_{22}m_{33}\,, \\
0 < (m_{13} + m_{31})^2 = 6.1784\text{e}{-}09 < 3.4941\text{e}{-}07 = m_{11}m_{33}\,.
\end{gathered}$$

From Table 1 we observe that the eigenvalues $\{\mu_i\}$ of the matrix $\frac{1}{2}\big(M_R + M_R^T\big)$ and the eigenvalues $\{\lambda_i\}$ of the matrix A_0 are distinct and positive, which immediately implies the fulfillment of Assumption 2.2 (iii).

Next we determine the values ε_i, $i = 1, 2, 3$, and ε that satisfy conditions (3.15) and (3.16) and consequently, guarantee the energy estimates (3.12) and the well-posedness of the IBVP (1.7)-(1.13). Calculation results for different values of the ratio of radii δ and numbers p_i, q_i, and r_i, $i = 1, 2, 3$, are summarized in Table 2 and are the same for all given values of k/η and $\mathfrak{R}$.

The results of numerical experiments confirm the applicability of the proposed well-posdness theory to the TCpu model with real data parameters.

Table 1. Eigenvalues of the matrices $\frac{1}{2}\big(M_R + M_R^T\big)$ and A_0

Eigenvalue	$k/\eta = 10^{-16}$ $\mathfrak{R} = 0.01$	$k/\eta = 10^{-16}$ $\mathfrak{R} = 0.15$	$k/\eta = 10^{-17}$ $\mathfrak{R} = 0.01$	$k/\eta = 10^{-17}$ $\mathfrak{R} = 0.15$
μ_1	7.1188e+03	7.1188e+03	7.1188e+03	7.1188e+03
μ_2	7.0681e−10	7.0681e−10	7.0681e−10	7.0681e−10
μ_3	4.8865e−11	4.8865e−11	4.8865e−11	4.8865e−11
λ_1	9.8003e−03	9.8003e−03	9.8003e−03	9.8003e−03
λ_2	1.0000e−16	1.0231e−16	9.9910e−18	1.0374e−17
λ_3	4.8252e−18	3.2175e−18	4.6728e−18	4.2899e−18

6. Appendix: Physical parameters of the coupled TCpu system

The elements of the matrices of diffusion coefficients M and A (1.5), the coupling vectors $\mathbf{b}_0$ and $\mathbf{b}_1$ (1.6), the bulk modulus K, and the shear modulus G are defined as follows:

Table 2. Values of ε_i, $i = 1, 2, 3$, and ε

$p_i = \frac{1}{4}$, $q_i = \frac{1}{4}$, $r_i = \frac{1}{4}$		$4.5518\text{e}{-11} \le \varepsilon_1 \le 4.6292\text{e}{+05}$
		$1.3848\text{e}{-12} \le \varepsilon_2 \le 2.3630\text{e}{+02}$
		$4.0036\text{e}{+05} \le \varepsilon_3 \le 3.6227\text{e}{+08}$
	$\delta = 0.01$	$1.1488\text{e}{-10} \le \varepsilon \le 5.1698\text{e}{-09}$
	$\delta = 0.1$	$1.3639\text{e}{-10} \le \varepsilon \le 5.1698\text{e}{-09}$
	$\delta = e^{-1}$	$2.7753\text{e}{-10} \le \varepsilon \le 5.1698\text{e}{-09}$
$p_i = \frac{1}{8}$, $q_i = \frac{1}{3}$, $r_i = \frac{1}{2}$		$2.2759\text{e}{-11} \le \varepsilon_1 \le 3.4719\text{e}{+05}$
		$6.9238\text{e}{-13} \le \varepsilon_2 \le 1.7722\text{e}{+02}$
		$2.0018\text{e}{+05} \le \varepsilon_3 \le 2.7170\text{e}{+08}$
	$\delta = 0.01$	$1.1488\text{e}{-10} \le \varepsilon \le 2.5849\text{e}{-09}$
	$\delta = 0.1$	$1.3639\text{e}{-10} \le \varepsilon \le 2.5849\text{e}{-09}$
	$\delta = e^{-1}$	$2.7753\text{e}{-10} \le \varepsilon \le 2.5849\text{e}{-09}$
$p_i = \frac{1}{10}$, $q_i = \frac{1}{100}$, $r_i = \frac{1}{1000}$		$1.8207\text{e}{-11} \le \varepsilon_1 \le 1.1573\text{e}{+07}$
		$5.5390\text{e}{-13} \le \varepsilon_2 \le 5.9075\text{e}{+03}$
		$1.6014\text{e}{+05} \le \varepsilon_3 \le 9.0567\text{e}{+09}$
	$\delta = 0.01$	$1.1488\text{e}{-10} \le \varepsilon \le 1.2924\text{e}{-06}$
	$\delta = 0.1$	$1.3639\text{e}{-10} \le \varepsilon \le 1.2924\text{e}{-06}$
	$\delta = e^{-1}$	$2.7753\text{e}{-10} \le \varepsilon \le 1.2924\text{e}{-06}$

$$\Lambda = \frac{c}{T_F} - s_F\beta_s\tilde{\omega}, \quad c = (1-\phi)\rho_s c_s + \phi\rho_f c_f, \quad s_F = c_f - \frac{\beta_f p_F}{\rho_f}, \quad \tilde{\omega} = \frac{\omega}{1-C_F},$$
$$\Sigma = \xi\beta_s, \quad \xi = \frac{RT_F\tilde{\omega}(1-2C_F)}{M^s C_F}, \quad \Phi = \Xi - \phi\beta_f + \frac{\beta_s\tilde{\omega}}{\rho_f}, \quad \Xi = (\alpha-\phi)\beta_s,$$
$$\alpha = 1 - \frac{K}{K_s}, \quad K = \frac{E}{3(1-2\nu)}, \quad \Gamma = \Xi - \phi\beta_f - \frac{s_F\tilde{\omega}}{K_s}, \quad \chi = -\frac{\xi}{K_s}, \quad \psi = \frac{\alpha-\phi}{K_s},$$
$$\Psi = \psi + \frac{\phi}{K_f} - \frac{\tilde{\omega}}{\rho_f K_s}, \quad \tilde{\Omega} = \frac{RT_F}{M^s(1-C_F)C_F}, \quad \zeta = K\beta_s, \quad G = \frac{E}{2(1+\nu)}.$$
Other physical properties of the rock/fluid system are given in Table 3.

TABLE 3. Physical properties of the rock/fluid system (data taken from [6])

Property	**Value**
Drained elastic modulus, E	45×10^9 Pa
Solid bulk modulus, K_s	65×10^9 Pa
Drained Poisson's ratio, ν	0.27
Permeability/Fluid viscosity ratio, k/η	10^{-16}, 10^{-17} $m^2/(Pa{\cdot}s)$
Reference porosity, ϕ	0.15
Fluid density, ρ_f	1111 kg/m^3
Fluid specific heat capacity, c_f	4186 J/(Kg°K)
Fluid bulk modulus, K_f	3.3×10^9 Pa
Fluid volumetric thermal expansion coefficient, β_f	3×10^{-4} $°K^{-1}$
Solid density, ρ_s	2.83×10^3 kg/m^3
Solid specific heat capacity, c_s	920 J/(Kg°K)
Solid volumetric thermal expansion coefficient, β_s	2.4×10^{-5} $°K^{-1}$
Rock thermal conductivity coefficient, k^T	4 W/(m°K)
Thermal osmosis coefficient, K^T	10^{-11} $m^2/(s°K)$
Solute molar mass, M^s	0.1111 kg/m^3
Solute reflected fraction, $\mathfrak{R}$	0.01, 0.15
Solute chemical diffusion coefficient, D	10^{-9} m^2/s
Solute thermal diffusion coefficient, D^T	10^{-10} $m^2/(s°K)$
Chemical stress coupling parameter, ω	100 kg/m^3
Universal gas constant, R	8.3144598 JK^{-1}/mol^{-1}
In situ formation pore pressure, p_F	52 MPa
Reference formation solute mass fraction, C_F	0.1
In situ formation temperature, T_F	135°C

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