Electronic Journal of Differential Equations, Vol. 2017 (2017), No. 141, pp. 1–13.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

EXISTENCE AND ASYMPTOTIC BEHAVIOR OF POSITIVE SOLUTIONS FOR SEMILINEAR FRACTIONAL NAVIER BOUNDARY-VALUE PROBLEMS

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Communicated by Vicentiu Radulescu

Abstract. We study the existence, uniqueness, and asymptotic behavior of positive continuous solutions to the fractional Navier boundary-value problem

$$D^{\beta}(D^{\alpha}u)(x) = -p(x)u^{\sigma}, \quad \in (0,1),$$
$$\lim_{x\to 0} x^{1-\beta}D^{\alpha}u(x) = 0, \quad u(1) = 0,$$

where $\alpha, \beta \in (0,1]$ such that $\alpha+\beta>1$, D^{β} and D^{α} stand for the standard Riemann-Liouville fractional derivatives, $\sigma \in (-1,1)$ and p being a nonnegative continuous function in $(0,1)$ that may be singular at $x=0$ and satisfies some conditions related to the Karamata regular variation theory. Our approach is based on the Schäuder fixed point theorem.

1. Introduction

The existence, uniqueness and asymptotic behavior of positive continuous solutions related to fractional differential equations have been studied by many researchers. Many fractional differential equations subject to various boundary conditions have been addressed; see, for instance, [1, 2, 4, 5, 7, 8, 14, 16, 18, 19, 21, 22, 23, 25, 28, 29, 30] and the reference therein. It is known that fractional differential equations have extensive applications in various fields of science and engineering. Many phenomena in viscoelasticity, electrochemistry, control theory, porous media, electromagnetism and other fields, can be modeled by fractional differential equations. Also it provides an excellent tool to describe the hereditary properties of various materials and processes. Concerning the development of theory methods and applications of fractional calculus, we refer to [6, 9, 10, 11, 12, 13, 15, 17, 23, 24, 26, 28] and the references therein for discussions of various applications.

In [18], Mâagli et al considered the following fractional initial value problem

$$\begin{gathered} D^{\beta}u(x) = p(x)u^{\sigma}, \quad x \in (0,1), \\ \lim_{x\to 0^+} x^{1-\beta}u(x) = 0, \end{gathered} \tag{1.1}$$

2010 *Mathematics Subject Classification.* 34A08, 34B15, 34B18, 34B27.
Key words and phrases. Fractional Navier differential equations; Dirichlet problem; positive solution; asymptotic behavior; Schauder fixed point theorem.

Submitted February 11, 2017. Published May 25, 2017.

where $\beta \in (0,1)$, $\sigma < 1$ and p is a nonnegative measurable function on $(0,1)$. By a potential theory approach associated to D^β and some technical tools relying to Karamata regular variation theory, the authors proved the existence, uniqueness and asymptotic behavior of a positive solution to problem (1.1).

Bachar et al [1] studied the following fractional Navier boundary value problem

$$\begin{gathered} D^\beta(D^\alpha u)(x) + u(x)f(x,u(x)) = 0, \quad x \in (0,1), \\ \lim_{x\to 0^+} D^{\beta-1}u(x) = 0, \quad \lim_{x\to 0^+} D^{\alpha-1}(D^\beta u)(x) = \xi, \\ u(1) = 0, \quad D^\beta u(1) = -\varsigma, \end{gathered} \tag{1.2}$$

where $\alpha, \beta \in (1,2]$ and $\xi, \varsigma \geq 0$ are such that $\xi + \varsigma > 0$ and $f(x,s)$ is a nonnegative continuous function on $(0,1) \times [0,\infty)$. Under some appropriate condition on the function f and by a perturbation argument method, the authors proved the existence of a unique positive solution to problem (1.2).

Inspired by the above-mentioned papers, we aim at studying similar problem in the case of fractional Navier boundary value problem. More precisely, we are concerned with the following semilinear fractional Navier boundary-value problem

$$\begin{gathered} D^\beta(D^\alpha)u(x) = -p(x)u^\sigma, \quad x \in (0,1), \\ \lim_{x\to 0^+} x^{1-\beta}D^\alpha u(x) = 0, \quad u(1) = 0, \end{gathered} \tag{1.3}$$

where $\alpha, \beta \in (0,1]$ such that $\alpha + \beta > 1$, $\sigma \in (-1,1)$ and p is a nonnegative continuous function on $(0,1)$ and satisfies some appropriate assumptions related to the Karamata class $\mathcal{K}$ (see Definition 1.1 below). Using the Schäuder fixed point theorem, we prove the existence of a unique positive continuous solution to problem (1.3). Further, by applying the Karamata regular variation theory, we establish sharp estimates on such a solution. To state our existence result, we need some notations. We first introduce the Karamata class $\mathcal{K}$.

Definition 1.1. The class $\mathcal{K}$ is the set of Karamata functions L defined on $(0,\eta]$ by

$$L(t) := c\exp\Big(\int_t^\eta \frac{z(s)}{s}ds\Big)$$

for some $\eta > 1$, where $c > 1$ and $z \in C([0,\eta])$ such that $z(0) = 0$.

Remark 1.2. It is clear that a function L is in $\mathcal{K}$ if and only if L is a positive function in $C^1((0,\eta])$ for some $\eta > 1$, such that $\lim_{t\to 0^+} \frac{tL'(t)}{L(t)} = 0$.

As a typical example of function belonging to the class $\mathcal{K}$, we quote

$$L(t) = \prod_{j=1}^m \big(\log(\frac{w}{t})\big)^{\xi_j}$$

where ξ_j are real numbers, $\log_j x = \log\circ\log\dots\log x$ (j times) and w is a sufficiently large positive real number such that L is defined and positive on $(0,\eta]$ for some $\eta > 1$. For two nonnegative functions f and g defined on a set S, the notation $f(x) \approx g(x)$, $x \in S$, means that there exists $c > 0$ such that $\frac{1}{c}f(x) \leq g(x) \leq cf(x)$ for all $x \in S$. We denote $x^+ = \max(x,0)$ for $x \in \mathbb{R}$ and by $\mathcal{B}^+((0,1))$ the set of all nonnegative measurable functions on $(0,1)$. $C((0,1))$ (resp. $C([0,1])$) dentes the set of all continuous functions in $(0,1)$ (resp. $[0,1]$). Also, for $r>0$, we denote the weighted space of continuous functions on $[0,1]$ by

$$C_r([0,1]) = \{f \in C((0,1]) : t^r f \in C([0,1])\}.$$

For $\alpha \in (0,1)$, we put ω_α the function defined in $(0,1]$ by $\omega_\alpha(x) = x^{\alpha-1}$.

In problem (1.3), we assume that p is a nonnegative function on $(0,1)$ satisfying the following condition:

(H1) $p \in C((0,1))$ such that

$$p(x) \approx x^{-\lambda} L_1(x)(1-x)^{-\mu} L_2(1-x), \quad x \in (0,1), \tag{1.4}$$

where $\lambda + (1-\alpha)\sigma \le 1$, $\mu \le \alpha+\beta$ and $L_1, L_2 \in \mathcal{K}$ satisfying

$$\int_0^\eta t^{(\alpha-1)\sigma-\lambda} L_1(t)dt < \infty, \quad \int_0^\eta t^{\alpha+\beta-1-\mu} L_2(t)dt < \infty. \tag{1.5}$$

We define the function θ on $[0,1]$ by

$$\theta(x) := (1-x)^{\min(\frac{\alpha+\beta-\mu}{1-\sigma},1)} \big(\tilde{L_2}(1-x)\big)^{\frac{1}{1-\sigma}}, \tag{1.6}$$

where

$$\tilde{L_2}(x) := \begin{cases} \int_0^x \frac{L_2(t)}{t} dt, & \text{if } \mu = \alpha+\beta, \\ L_2(x), & \text{if } \alpha+\beta-1+\sigma < \mu < \alpha+\beta, \\ \int_x^\eta \frac{L_2(t)}{t} dt, & \text{if } \mu = \alpha+\beta-1+\sigma, \\ 1, & \text{if } \mu < \alpha+\beta-1+\sigma. \end{cases} \tag{1.7}$$

Our existence result is the following.

Theorem 1.3. *Let $\sigma \in (-1,1)$ and assume that p satisfies* (H1). *Then problem* (1.3) *has a unique positive solution $u \in C_{1-\alpha}([0,1])$ satisfying for $x \in (0,1)$*

$$u(x) \approx \omega_\alpha(x)\theta(x). \tag{1.8}$$

The rest of this article is organized as follows. In Section 2, we prove some sharp estimates on the Green's function $H(x,t)$ of the operator $u \to -D^\beta(D^\alpha u)$, with boundary conditions $\lim_{x\to 0^+} x^{1-\beta} D^\alpha u(x) = u(1) = 0$. In Section 3, we present some known results on functions belong to the class $\mathcal{K}$ and we establish sharp estimates on some potential functions. Exploiting theses results, we prove Theorem 1.3 by means of the Schäuder fixed point theorem. Finally, we give an example to illustrate our existence result.

2. Fractional calculus and estimates on the Green's Function

2.1. **Fractional calculus.** For the convenience of the reader, we recall in this section some basic definitions of fractional calculus (see [10, 25, 29]).

Definition 2.1. The Riemann-Liouville fractional integral of order $\gamma > 0$ for a measurable function $f : (0,\infty) \to \mathbb{R}$ is defined as

$$I^\gamma f(x) = \frac{1}{\Gamma(\gamma)} \int_0^x (x-t)^{\gamma-1} f(t)dt, \quad x > 0,$$

provided that the right-hand side is pointwise defined on $(0,\infty)$. Here Γ is the Euler Gamma function.

Definition 2.2. The Riemann-Liouville fractional derivative of order $\gamma > 0$ of a measurable function $f : (0, \infty) \to \mathbb{R}$ is defined as

$$D^\gamma f(x) = \frac{1}{\Gamma(n-\gamma)} \Big(\frac{d}{dx}\Big)^n \int_0^x (x-t)^{n-\gamma-1} f(t)dt = \Big(\frac{d}{dx}\Big)^n I^{n-\gamma} f(x),$$

provided that the right-hand side is pointwise defined on $(0, \infty)$. Here $n = [\gamma] + 1$, where $[\gamma]$ denotes the integer part of the number γ.

Lemma 2.3 ([10, 25])**.** *Let $\gamma > 0$ and $u \in C((0,1)) \cap L^1((0,1))$. Then we have the following assertions:*

(i) *For $\beta > 0$, $I^\beta I^\gamma u = I^{\alpha+\gamma} u$ for $\beta + \gamma \geq 1$ and $D^\gamma I^\gamma u = u$.*
(ii) *$D^\gamma u(x) = 0$ if and only if $u(x) = c_1 x^{\gamma-1} + c_2 x^{\gamma-2} + \cdots + c_m x^{\gamma-m}$, $c_i \in \mathbb{R}$, $i = 1, \dots, m$, where m is the smallest integer greater than or equal to γ.*
(iii) *Assume that $D^\gamma u \in C((0,1)) \cap L^1((0,1))$; then*
$$I^\gamma D^\gamma u(x) = u(x) + c_1 x^{\gamma-1} + c_2 x^{\gamma-2} + \cdots + c_m x^{\gamma-m},$$
$c_i \in \mathbb{R}$, $i = 1, \dots, m$, where m is the smallest integer greater than or equal to γ.

2.2. **Estimates on the Green's function.** In this section, we derive the corresponding Green's function for the homogeneous boundary value problem (1.3) and we prove some estimates on this function. To this end we need the following lemma.

Lemma 2.4 ([3])**.** *For $\lambda, \mu \in (0, \infty)$ and $a, t \in [0,1]$, we have*

$$\min(1, \frac{\mu}{\lambda})(1 - at^\lambda) \leq 1 - at^\mu \leq \max(1, \frac{\mu}{\lambda})(1 - at^\lambda).$$

Lemma 2.5. *Let $\alpha, \beta \in (0,1]$ such that $\alpha + \beta > 1$. Let $f \in C([0,1])$, then the following boundary-value problem*

$$\begin{gathered} D^\beta(D^\alpha u)(x) = -f(x), \quad x \in (0,1), \\ \lim_{x\to 0^+} x^{1-\beta} D^\alpha u(x) = u(1) = 0 \end{gathered} \tag{2.1}$$

has a unique solution given by

$$u(x) = \int_0^1 H(x,t) f(t) dt, \tag{2.2}$$

where for $x, t \in (0,1)$,

$$H(x,t) = \frac{1}{\Gamma(\alpha+\beta)} (x^{\alpha-1}(1-t)^{\alpha+\beta-1} - ((x-t)^+)^{\alpha+\beta-1}) \tag{2.3}$$

is the Green's function of the operator $u \to -D^\beta(D^\alpha u)$, with boundary conditions $\lim_{x\to 0^+} x^{1-\beta} D^\alpha u(x) = u(1) = 0$.

Proof. By Lemma 2.3, we can reduce equation $D^\beta(D^\alpha u)(x) = -f(x)$ to an equivalent equation

$$D^\alpha u(x) = -I^\beta f(x) + c_1 x^{\beta-1}.$$

The boundary condition $\lim_{x\to 0^+} x^{1-\beta} D^\alpha u(x) = 0$ implies that $c_1 = 0$ and we have

$$D^\alpha u(x) = -I^\beta f(x). \tag{2.4}$$

Using again Lemma 2.3, we can reduce the equation (2.4) to an equivalent integral equation

$$u(x) = -I^\alpha I^\beta f(x) + c_2 x^{\alpha-1} = -I^{\alpha+\beta} f(x) + c_2 x^{\alpha-1}.$$

The boundary condition $u(1) = 0$ gives

$$c_2 = I^{\alpha+\beta} f(1) = \frac{1}{\Gamma(\alpha+\beta)} \int_0^1 (1-t)^{\alpha+\beta-1} f(t) dt.$$

Therefore, the unique solution of problem (2.1) is

$$\begin{aligned} u(x) &= \frac{1}{\Gamma(\alpha+\beta)} \Big(x^{\alpha-1} \int_0^1 (1-t)^{\alpha+\beta-1} f(t) dt - \int_0^x (x-t)^{\alpha+\beta-1} f(t) dt \Big) \\ &= \int_0^1 H(x,t) f(t) dt. \end{aligned}$$

□

Proposition 2.6. *Let $\alpha, \beta \in (0,1]$ such that $\alpha+\beta > 1$. Then we have*

(i) *For $(x,t) \in (0,1) \times (0,1)$, the Green's function $H(x,t)$ satisfies*

$$\frac{\alpha+\beta-1}{\beta\Gamma(\alpha+\beta)} K(x,t) \le H(x,t) \le \frac{1}{\Gamma(\alpha+\beta)} K(x,t), \tag{2.5}$$

where $K(x,t) := x^{\alpha-1}(1-t)^{\alpha+\beta-2}(1-\max(x,t))$.

(ii)

$$\begin{aligned} &\frac{(\alpha+\beta-1) x^{\alpha-1}(1-x)(1-t)^{\alpha+\beta-1}}{\beta\Gamma(\alpha+\beta)} \\ &\le H(x,t) \\ &\le \frac{x^{\alpha-1}(1-t)^{\alpha+\beta-2} \min(1-t, 1-x)}{\Gamma(\alpha+\beta)}. \end{aligned} \tag{2.6}$$

Proof. (i) From the explicit expression of the Green's function given by (2.3), for $x, t \in (0,1)$ we have

$$H(x,t) = \frac{x^{\alpha-1}(1-t)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \Big(1 - x^\beta \Big(\frac{(x-t)^+}{x(1-t)} \Big)^{\alpha+\beta-1} \Big).$$

Since $\frac{(x-t)^+}{x(1-t)} \in (0,1]$ for $t \in [0,1)$, then by applying Lemma 2.4 with $a = x^\beta$, $\mu = \alpha+\beta-1$ and $\lambda = \beta$, we obtain

$$\begin{aligned} &\frac{(\alpha+\beta-1) x^{\alpha-1}(1-t)^{\alpha+\beta-1}}{\beta\Gamma(\alpha+\beta)} \Big(1 - \Big(\frac{(x-t)^+}{(1-t)} \Big)^\beta \Big) \\ &\le H(x,t) \\ &\le \frac{x^{\alpha-1}(1-t)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \Big(1 - \Big(\frac{(x-t)^+}{(1-t)} \Big)^\beta \Big). \end{aligned}$$

Since $\frac{(x-t)^+}{(1-t)} \in (0,1]$ for $t \in (0,1)$, then again by Lemma 2.4 with $a = \lambda = 1$, $\mu = \beta$ and using the fact that $(1-t) - (x-t)^+ = 1 - \max(x,t)$, we deduce (2.5).

(ii) Inequality (2.6) follows from the fact that for $x, t \in [0,1]$,

$$(1-t)(1-x) \le 1 - \max(x,t) = \min(1-t, 1-x).$$

□

In the sequel, we denote the kernel V defined on $\mathcal{B}^+((0,1))$ by

$$Vf(x) := \int_0^1 H(x,t)f(t)dt, \quad x \in (0,1).$$

As an immediately consequence of the assertion (ii) of Proposition 2.6, we obtain the following result.

Corollary 2.7. *Let $f \in \mathcal{B}^+((0,1))$, then the function $x \to Vf(x)$ is in $C_{1-\alpha}([0,1])$ if and only if $\int_0^1 (1-t)^{\alpha+\beta-1} f(t)dt < \infty$.*

Lemma 2.8. *Let $\alpha, \beta \in (0,1]$. Let $f \in C((0,1))$ such that the map $t \to (1-t)^{\alpha+\beta-1}f(t)$ is integrable and $|f(t)| \le t^{-\delta}L(t)$ for t near 0, with $\delta \le 1$ and $L \in \mathcal{K}$ satisfying $\int_0^\eta t^{-\delta}L(t)dt < \infty$. Then the function $x \to I^\beta f(x) \in C((0,1)) \cap L^1((0,1))$ and $\lim_{x\to 0} x^{1-\beta} I^\beta f(x) = 0$.*

Proof. Put $h(t) = t^{-\delta}L(t)$ and let $0 < a < 1$. Since $f \in C((0,1))$, there exists $c > 0$ such that $|f(t)| \le ch(t)$ for $t \in (0,a]$.

Now, as in [18, Theorem 2], we show that the function $x \to I^\beta f(x)$ is continuous on $(0,a]$ and $\lim_{x\to 0} x^{1-\beta} I^\beta f(x) = 0$.

Thus the mapping $x \to I^\beta f(x)$ is continuous on $(0,1)$ and $\lim_{x\to 0} x^{1-\beta} I^\beta f(x) = 0$. Moreover, we have

$$\begin{aligned}
\int_0^1 | I^\beta f(x) | \, dx &\le \frac{1}{\Gamma(\beta)} \int_0^1 \Big(\int_0^x (x-t)^{\beta-1} |f(t)| dt \Big) dx \\
&= \frac{1}{\Gamma(\beta)} \int_0^1 |f(t)| \Big(\int_t^1 (x-t)^{\beta-1} dx \Big) dt \\
&= \frac{1}{\Gamma(\beta+1)} \int_0^1 (1-t)^\beta |f(t)| dt \\
&\le \frac{1}{\Gamma(\beta+1)} \int_0^1 (1-t)^{\alpha+\beta-1} |f(t)| dt < +\infty.
\end{aligned}$$

This shows that $I^\beta f \in L^1((0,1))$. □

Proposition 2.9. *Let $\alpha, \beta \in (0,1]$ such that $\alpha + \beta > 1$. Let $f \in C((0,1))$ such that the map $t \to (1-t)^{\alpha+\beta-1} f(t)$ is integrable and $|f(t)| \le t^{-\delta}L(t)$ near 0, with $\delta \le 1$ and $L \in \mathcal{K}$ satisfying $\int_0^\eta t^{-\delta}L(t)dt < \infty$. Then Vf is the unique solution in $C_{1-\alpha}([0,1])$ of the boundary value problem*

$$\begin{gathered}
D^\beta(D^\alpha u)(x) = -f, \quad x \in (0,1), \\
\lim_{x\to 0^+} x^{1-\beta} D^\alpha u(x) = u(1) = 0.
\end{gathered} \tag{2.7}$$

Proof. From Corollary 2.7, the function Vf is in $C_{1-\alpha}([0,1])$ and we have for $x \in (0,1)$,

$$Vf(x) = \frac{x^{\alpha-1}}{\Gamma(\alpha+\beta)} \int_0^1 (1-t)^{\alpha+\beta-1} f(t)dt - \frac{1}{\Gamma(\alpha+\beta)} \int_0^x (x-t)^{\alpha+\beta-1} f(t)dt.$$

That is

$$Vf(x) = \frac{x^{\alpha-1}}{\Gamma(\alpha+\beta)} \int_0^1 (1-t)^{\alpha+\beta-1} f(t)dt - I^{\alpha+\beta} f(x).$$

So, by Lemma 2.3, we obtain

$$D^\alpha(Vf)(x) = -I^\beta f(x). \tag{2.8}$$

Applying the operator D^β on both sides of (2.8) and using Lemma 2.3, we have

$$D^\beta(D^\alpha Vf)(x) = -f(x) \quad \text{for } x \in (0,1).$$

Next, we need to verify that Vf satisfies the boundary conditions. By Proposition 2.6 (ii), there exists a nonnegative constant c such that

$$|Vf(x)| \leq cx^{\alpha-1} \int_0^1 (1-t)^{\alpha+\beta-2} \min(1-t, 1-x)|f(t)|dt.$$

By Lebesgue's theorem, we deduce that $\lim_{x\to 1} Vf(x) = 0$. On the other hand, from (2.8) and Lemma 2.8, we conclude that $\lim_{x\to 0^+} x^{1-\beta} D^\alpha Vf(x) = 0$.

Finally, we prove the uniqueness. Let $u, v \in C_{1-\alpha}([0,1])$ be two solution of (2.7) and put $w = u-v$. Then $w \in C_{1-\alpha}([0,1])$ and $D^\beta(D^\alpha w) = 0$. Hence, it follows from Lemma 2.3 (ii) that $D^\alpha w(x) = c_1 x^{\beta-1}$. Using the fact that $\lim_{x\to 0^+} x^{1-\beta} D^\alpha w(x) = 0$, we deduce that $c_1 = 0$ and then $D^\alpha w(x) = 0$. Using again Lemma 2.3 (ii), we conclude that $w(x) = c_2 x^{\alpha-1}$. Since $w(1) = 0$, then $c_2 = 0$, this implies that $w(x) = 0$ and therefore $u = v$. □

3. Existence result

In this section, we aim at proving Theorem 1.3.

3.1. **Karamata class and sharp estimates on some potential functions.**

In this subsection, we recall some fundamental properties of functions belonging to the class $\mathcal{K}$ and we establish estimates on some potential functions.

Lemma 3.1 ([19, 30]). *Let $\gamma \in \mathbb{R}$ and L be a function in $\mathcal{K}$ defined on $(0,\eta]$. Then we have that*

(i) *if $\gamma > -1$, then $\int_0^\eta s^\gamma L(s)ds$ converges and $\int_0^t s^\gamma L(s)ds \underset{t\to 0^+}{\sim} \frac{t^{1+\gamma}L(t)}{\gamma+1}$;*

(ii) *if $\gamma < -1$, then $\int_0^\eta s^\gamma L(s)ds$ diverges and $\int_t^\eta s^\gamma L(s)ds \underset{t\to 0^+}{\sim} -\frac{t^{1+\gamma}L(t)}{\gamma+1}$.*

Lemma 3.2 ([3, 30]). (i) *Let $L \in \mathcal{K}$ and $\epsilon > 0$. So then we have*

$$\lim_{t\to 0^+} t^\epsilon L(t) = 0.$$

(ii) *Let L_1 and $L_2 \in \mathcal{K}$ defined on $(0,\eta]$ and $p \in \mathbb{R}$. Then functions*

$$L_1 + L_2, L_1 L_2, L_1^p \text{ belong to the class } \mathcal{K}.$$

(iii) *Let $L \in \mathcal{K}$ defined on $(0,\eta]$. So then we have*

$$\lim_{t\to 0^+} \frac{L(t)}{\int_t^\eta \frac{L(s)}{s}ds} = 0.$$

In particular the function

$$t \to \int_t^\eta \frac{L(s)}{s}ds \in \mathcal{K}.$$

If further $\int_0^\eta \frac{L(s)}{s}ds$ converges, then we have

$$\lim_{t\to 0^+} \frac{L(t)}{\int_0^t \frac{L(s)}{s}ds} = 0.$$

In particular the function

$$t \to \int_0^t \frac{L(s)}{s}ds \in \mathcal{K}.$$

Next, we shall prove sharp estimates on the potential function $V(p(\omega_\alpha\theta)^\sigma)$, where p is a function satisfying (H1) and θ is the function given in (1.6). To this end, we need the following proposition.

Proposition 3.3. *Let $\alpha,\beta\in(0,1]$ such that $\alpha+\beta>1$ and let $\gamma\le 1$, $\nu\le\alpha+\beta$ and $L_3,L_4\in\mathcal{K}$ with*

$$\int_0^\eta t^{-\gamma}L_3(t)dt<\infty,\quad \int_0^\eta t^{\alpha+\beta-1-\nu}L_4(t)dt<\infty. \tag{3.1}$$

Put

$$b(x)=x^{-\gamma}L_3(x)(1-x)^{-\nu}L_4(1-x)\quad \text{for } x\in(0,1).$$

Then, for $x\in(0,1)$, we have

$$Vb(x)\approx x^{\alpha-1}(1-x)^{\min(\alpha+\beta-\nu,1)}\widetilde{L}_4(1-x),$$

where

$$\widetilde{L}_4(x):=\begin{cases}\int_0^x\frac{L_4(t)}{t}dt, & \text{if } \nu=\alpha+\beta,\\ L_4(x), & \text{if } \alpha+\beta-1<\nu<\alpha+\beta,\\ \int_x^\eta\frac{L_4(t)}{t}dt, & \text{if } \nu=\alpha+\beta-1,\\ 1, & \text{if } \nu<\alpha+\beta-1.\end{cases}$$

Proof. For $x\in(0,1]$, we have

$$Vb(x)=\int_0^1 H(x,t)b(t)dt.$$

Using Proposition 2.6 (i), we obtain that

$$\begin{aligned}Vb(x)\approx{}& x^{\alpha-1}(1-x)\int_0^x t^{-\gamma}L_3(t)(1-t)^{\alpha+\beta-2-\nu}L_4(1-t)dt\\ &+x^{\alpha-1}\int_x^1 t^{-\gamma}L_3(t)(1-t)^{\alpha+\beta-1-\nu}L_4(1-t)dt.\end{aligned}$$

In what follows, we distinguish two cases.

Case 1. $0<x\le\frac12$. In this case $1-x\approx 1$. So, we obtain

$$\begin{aligned}Vb(x)\approx{}& x^{\alpha-1}(1-x)\int_0^x t^{-\gamma}L_3(t)(1-t)^{\alpha+\beta-2-\nu}L_4(1-t)dt\\ &+x^{\alpha-1}\Big(\int_x^{1/2} t^{-\gamma}L_3(t)(1-t)^{\alpha+\beta-1-\nu}L_4(1-t)dt\\ &+\int_{\frac12}^1 t^{-\gamma}L_3(t)(1-t)^{\alpha+\beta-1-\nu}L_4(1-t)dt\Big).\\ \approx{}& x^{\alpha-1}\Big(\int_0^x t^{-\gamma}L_3(t)dt+\int_x^{\frac12}t^{-\gamma}L_3(t)dt\\ &+\int_0^{1/2}t^{\alpha+\beta-1-\nu}L_4(t)dt\Big)\\ \approx{}& x^{\alpha-1}\Big(\int_0^{1/2}t^{-\gamma}L_3(t)dt+\int_0^{1/2}t^{\alpha+\beta-1-\nu}L_4(t)dt\Big).\end{aligned}$$

Using hypothesis (3.1), we deduce that for $0<x\le\frac12$

$$Vb(x)\approx x^{\alpha-1}. \tag{3.2}$$

Case 2. $\frac{1}{2} \le x \le 1$. In this case, we have $x \approx 1$. Therefore, we obtain

$$\begin{aligned} Vb(x) &\approx x^{\alpha-1}(1-x)\Big(\int_0^{1/2} t^{-\gamma}L_3(t)(1-t)^{\alpha+\beta-2-\nu}L_4(1-t)dt \\ &\quad + \int_{\frac{1}{2}}^{x} t^{-\gamma}L_3(t)(1-t)^{\alpha+\beta-2-\nu}L_4(1-t)dt\Big) \\ &\quad + x^{\alpha-1}\int_x^1 t^{-\gamma}L_3(t)(1-t)^{\alpha+\beta-1-\nu}L_4(1-t)dt \\ &\approx (1-x)\Big(\int_0^{1/2} t^{-\gamma}L_3(t)dt + \int_{\frac{1}{2}}^{x}(1-t)^{\alpha+\beta-2-\nu}L_4(1-t)dt\Big) \\ &\quad + \int_x^1 (1-t)^{\alpha+\beta-1-\nu}L_4(1-t)dt. \end{aligned}$$

Since $\int_0^\eta t^{-\gamma}L_3(t)dt < \infty$, we deduce that

$$Vb(x) \approx (1-x)\Big(1 + \int_{1-x}^{1/2} t^{\alpha+\beta-2-\nu}L_4(t)dt\Big) + \int_0^{1-x} t^{\alpha+\beta-1-\nu}L_4(t)dt.$$

Using Lemma 3.1 and hypothesis (3.1), we deduce that

$$\int_0^{1-x} t^{\alpha+\beta-1-\nu}L_4(t)dt \approx \begin{cases} \int_0^{1-x}\frac{L_4(t)}{t}dt, & \text{if } \nu = \alpha+\beta, \\ (1-x)^{\alpha+\beta-\nu}L_4(x), & \text{if } \nu < \alpha+\beta \end{cases}$$

and

$$1 + \int_{1-x}^{1/2} t^{\alpha+\beta-2-\nu}L_4(t)dt \approx \begin{cases} (1-x)^{\alpha+\beta-1-\nu}L_4(x), & \text{if } \alpha+\beta-1 < \nu \le \alpha+\beta, \\ \int_{1-x}^{\eta}\frac{L_4(t)}{t}dt, & \text{if } \nu = \alpha+\beta-1, \\ 1, & \text{if } \nu < \alpha+\beta-1. \end{cases}$$

Hence, it follows by Lemma 3.2 and hypothesis (3.1) that for $\frac{1}{2} \le x \le 1$,

$$Vb(x) \approx \begin{cases} \int_0^{1-x}\frac{L_4(t)}{t}dt, & \text{if } \nu = \alpha+\beta, \\ (1-x)^{\alpha+\beta-\nu}L_4(x), & \text{if } \alpha+\beta-1 < \nu < \alpha+\beta, \\ (1-x)\int_{1-x}^{\eta}\frac{L_4(t)}{t}dt, & \text{if } \nu = \alpha+\beta-1, \\ 1-x, & \text{if } \nu < \alpha+\beta-1. \end{cases}$$

That is,

$$Vb(x) \approx (1-x)^{\min(\alpha+\beta-\nu,1)}\widetilde{L}_4(1-x). \tag{3.3}$$

This and (3.2) imply that for $x \in (0,1)$, we have

$$Vb(x) \approx x^{\alpha-1}(1-x)^{\min(\alpha+\beta-\nu,1)}\widetilde{L}_4(1-x).$$

This ends the proof. □

The following proposition plays a crucial role in the proof of Theorem 1.3

Proposition 3.4. *Let p be a function satisfying* (H1). *Then, for $x \in (0,1)$, we have*

$$V(p(\omega_\alpha\theta)^\sigma)(x) \approx \omega_\alpha(x)\theta(x).$$

Proof. Let p be a function satisfying (H1). Let $\gamma = \lambda + (1-\alpha)\sigma$ and $\nu = \mu - \sigma \min(\frac{\alpha+\beta-\mu}{1-\sigma}, 1)$, where the constants λ and μ are given in (H1).

Since $\lambda \leq 1 + (\alpha-1)\sigma$ and $\mu \leq \alpha+\beta$, we verify that $\gamma \leq 1$ and $\nu \leq \alpha+\beta$. On the other hand, by using (1.4) and (1.6), we have

$$p(x)(\omega_\alpha \theta)^\sigma(x) \approx x^{-\gamma}(1-x)^{-\nu} L_1(x) L_2(1-x)\big(\tilde{L_2}(1-x)\big)^{\frac{\sigma}{1-\sigma}}.$$

So, using Lemma 3.2 and Proposition 3.3 with $L_4 = L_2\big(\tilde{L_2}\big)^{\frac{\sigma}{1-\sigma}}$, we deduce that for $x \in (0,1)$,

$$V(p(\omega_\alpha \theta)^\sigma)(x) \approx \omega_\alpha(x)(1-x)^{\min(\alpha+\beta-\nu,1)}\tilde{L_4}(1-x).$$

Since $\min(\alpha+\beta-\nu, 1) = \min(\frac{\alpha+\beta-\mu}{1-\sigma}, 1)$, we conclude by elementary calculus that for $x \in (0,1)$,

$$V(p(\omega_\alpha \theta)^\sigma)(x) \approx \omega_\alpha(x)(1-x)^{\min(\frac{\alpha+\beta-\mu}{1-\sigma},1)}\tilde{L_4}(1-x) \approx \omega_\alpha(x)\theta(x).$$

This completes the proof. □

Proof of Theorem 1.3. Let p be a function satisfying (H1) and let θ be the function given in (1.6). By Proposition 3.4, there exists $M \geq 1$ such that for each $x \in [0,1]$

$$\frac{1}{M}\theta(x) \leq x^{1-\alpha} V(p(\omega_\alpha \theta)^\sigma)(x) \leq M\theta(x).$$

We shall use a fixed point argument to construct a solution to problem (1.3). For this end, put $c = M^{\frac{1}{1-|\sigma|}}$ and consider the closed convex set

$$\Lambda := \{v \in C([0,1]) : \frac{1}{c}\theta(x) \leq v(x) \leq c\theta(x)\}.$$

Obviously, the function θ belongs to $C([0,1])$ and so Λ is not empty. We define the operator T on Λ by

$$Tv(x) = x^{1-\alpha} V(p(\omega_\alpha v)^\sigma)(x), \quad x \in [0,1].$$

For this choice of c, we can easily get that for $v \in \Lambda$ and $x \in [0,1]$, we have

$$\frac{1}{c}\theta(x) \leq Tv(x) \leq c\theta(x).$$

Now, since the function $(x,t) \to x^{1-\alpha} H(x,t)$ is continuous on $[0,1] \times [0,1]$ and the function $t \to (1-t)^{\alpha+\beta-1} p(t) t^{(\alpha-1)\sigma}\theta^\sigma(t)$ is integrable on $(0,1)$, we deduce that the operator T is compact from Λ to itself. So, by the Schäuder fixed point theorem, there exists a function $v \in \Lambda$ such that

$$Tv(x) = v(x), \quad x \in [0,1].$$

Put $u(x) = \omega_\alpha(x) v(x)$. Then $u \in C_{1-\alpha}([0,1])$ and satisfies the integral equation

$$u(x) = V(pu^\sigma)(x) \quad x \in (0,1)$$

and

$$u(x) \approx \omega_\alpha(x)\theta(x).$$

It remains to prove that u is a positive solution of problem (1.3). Indeed, we obviously that the function pu^σ is continuous in $(0,1)$ and the map $t \to (1-t)^{\alpha+\beta-1} p(t) u^\sigma(t)$ is integrable. Moreover, by hypothesis (H1) there exists a positive constant c such that

$$p(t)u^\sigma(t) \leq ct^{-\lambda+(\alpha-1)\sigma} L_1(t) \quad \text{near } 0,$$

with $\lambda+(1-\alpha)\sigma\le 1$ and $L_1\in\mathcal{K}$ satisfying $\int_0^1 t^{-\lambda+(\alpha-1)\sigma}L_1(t)dt<\infty$. Hence, it follows from Proposition 2.9 that the function u is a continuous solution of problem (1.3). Finally, let us show that problem (1.3) has a unique positive solution in the cone

$$\Gamma:=\{u\in C_{1-\alpha}([0,1]):u\ \approx\omega_\alpha\theta\}.$$

So, we assume that u and v are arbitrary solutions of problem (1.3) in Γ. Since $u,v\in\Gamma$, then there exists a constant $m\ge 1$ such that

$$\frac{1}{m}\le\frac{u}{v}\le m \text{ in } (0,1).$$

This implies that the set $J:=\{m\ge 1:\frac{1}{m}\le\frac{u}{v}\le m\}$ is not empty. Now let $m_0:=\inf J$. It is easy to see that $m_0\ge 1$. This gives that $u^\sigma\le m_0^{|\sigma|}v^\sigma$.

On the other hand, putting $z:=m_0^{|\sigma|}v-u$, we have

$$\begin{gathered}D^\beta(D^\alpha z)=-p(x)(m_0^{|\sigma|}v^\sigma-u^\sigma)\le 0,\quad (0,1),\\ \lim_{x\to 0^+}x^{1-\beta}D^\alpha z(x)=z(1)=0.\end{gathered}$$

This implies by Proposition 2.9 that $m_0^{|\sigma|}v-u=V(p(m_0^{|\sigma|}v^\sigma-u^\sigma))\ge 0$. By symmetry, we obtain that $m_0^{|\sigma|}u\ge v$. Hence, $m_0^{|\sigma|}\in J$. Using the fact that $m_0:=\inf J$ and $|\sigma|<1$, we get $m_0=1$. Then, we conclude that $u=v$. $\square$

To illustrate the result in Theorem 1.3, we give the following example.

Example 3.5. Let $\sigma\in(-1,1)$ and p be a nonnegative continuous function on $(0,1)$ such that

$$p(x)\approx x^{-\lambda}\big(1-x\big)^{-\mu}\big(\log\big(\frac{3}{x}\big)\big)^{-s}\big(\log\big(\frac{3}{1-x}\big)\big)\big)^{-r},$$

where $\lambda+(1-\alpha)\sigma\le 1$, $\mu\le\alpha+\beta$ and $r,s\in\mathbb{R}$. If one of the following conditions holds:

- $\lambda+(1-\alpha)\sigma\le 1$ and $s>1$;
- $\lambda+(1-\alpha)\sigma<1$ and $s\in\mathbb{R}$.

Then by Theorem 1.3, problem (1.3) has a unique positive solution $u\in C_{1-\alpha}([0,1])$ satisfying the following estimates:

(i) If $\mu=\alpha+\beta$ and $r>1$, then for $x\in(0,1)$,

$$u(x)\approx x^{\alpha-1}\big(\log\big(\frac{3}{1-x}\big)\big)^{\frac{1-r}{1-\sigma}}.$$

(ii) If $\alpha+\beta-1+\sigma<\mu<\alpha+\beta$, then for $x\in(0,1)$,

$$u(x)\approx x^{\alpha-1}(1-x)^{\frac{\alpha+\beta-\mu}{1-\sigma}}\big(\log\big(\frac{3}{1-x}\big)\big)^{\frac{-r}{1-\sigma}}.$$

(iii) If $\mu=\alpha+\beta-1+\sigma$ and $r=1$, then for $x\in(0,1)$,

$$u(x)\approx x^{\alpha-1}(1-x)\big(\log\big(\log\big(\frac{3}{1-x}\big)\big)\big)^{\frac{1}{1-\sigma}}.$$

(iv) If $\mu=\alpha+\beta-1+\sigma$ and $r<1$, then for $x\in(0,1)$,

$$u(x)\approx x^{\alpha-1}(1-x)\big(\log\big(\frac{3}{1-x}\big)\big)^{\frac{1-r}{1-\sigma}}.$$

(v) If $\mu < \alpha + \beta - 1 + \sigma$ or $\mu = \alpha + \beta - 1 + \sigma$ and $r > 1$, then for $x \in (0, 1)$,

$$u(x) \approx x^{\alpha-1}(1 - x).$$

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