Electronic Journal of Differential Equations, Vol. 2017 (2017), No. 144, pp. 1–29.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

GLOBAL SUBSONIC FLOW IN A 3-D INFINITELY LONG CURVED NOZZLE

WENXIA CHEN, GANG XU, QIN XU

Communicated by Goong Chen

Abstract. In this article, we focus on the existence and stability of a subsonic global solution in an infinitely long curved nozzle for the three-dimensional steady potential flow equation. By introducing some suitably weighted Hölder spaces and establishing a series of a priori estimates on the solution to second order linear elliptic equation in an unbounded strip domain with two Neumann boundary conditions and one periodic boundary condition with respect to some variable, we show that the global subsonic solution of potential flow equation in a 3-D nozzle exists uniquely when the state of subsonic flow at negative infinity is given. Meanwhile, the asymptotic state of the subsonic solution at positive infinity as well as the asymptotic behavior at minus infinity are also studied.

1. Introduction and statement of main results

The existence of global subsonic flows in infinitely long nozzles or past obstacles is a fundamental problem in fluid dynamics. Such a problem has been extensively studied by many authors (see [1, 2, 4, 5, 7, 8, 9, 10, 14, 15, 16] and the references therein). For examples, for 2-D or 3-D potential flow equations, if the speed of the gas is assumed to be suitably low and the gas passes an obstacle, then it is shown in [4] and [9, 10] respectively that the whole subsonic flows outside 2-D or 3-D obstacles exist uniquely. Very recently, such a 2-D result on potential flow equation has been extended into the 2-D full Euler system case in [7] when it is supposed that the low velocity gas hits a symmetric obstacle. With respect to the 2-D subsonic potential flows in the infinitely long nozzles, those authors in [14, 15, 16] have shown the global existence and stability, in particular, those authors in [16] established the monotonicity of the maximum of the flow speed with respect to the incoming mass flux. In this paper, our focus is on the 3-D subsonic potential flow equation in a 3-D infinitely long nozzle.

Now we use the potential flow equation to describe the motion of the subsonic gas in a 3-D nozzle. Let $\varphi(x)$ be the potential of velocity $u = (u_1, u_2, u_3)$, i.e.,

2010 *Mathematics Subject Classification.* 35L70, 35L65, 35L67, 76N15.
Key words and phrases. Subsonic flow; potential flow equation; weighted Hölder space; global solution.

Submitted April 27, 2017. Published June 17, 2017.

$u_i = \partial_i \varphi$, then it follows from the Bernoulli's law that

$$\frac{1}{2}|\nabla\varphi|^2 + h(\rho) = C_0, \tag{1.1}$$

here $\nabla = (\partial_1, \partial_2, \partial_3)$, $h(\rho) = c^2(\rho)/(\gamma - 1)$ is the specific enthalpy for the polytropic gas with the state equation $P = A\rho^\gamma$ $(1 < \gamma < 3)$ and the sonic speed $c(\rho) = \sqrt{P'(\rho)}$, $C_0 = \frac{1}{2}q_0^2 + h(\rho_0)$ stands for the Bernoulli's constant, where the far velocity field $(q_0, 0, 0)$ at minus infinity of the nozzle is subsonic, i.e., $q_0 < c(\rho_0)$ holds true.

By use of (1.1) and the implicit function theorem, the density function $\rho(x)$ of gas can be expressed as

$$\rho = h^{-1}(C_0 - \frac{1}{2}|\nabla\varphi|^2) \equiv H(\nabla\varphi). \tag{1.2}$$

Substituting (1.2) into the mass conservation equation $\sum_{j=1}^{3} \partial_j(\rho u_j) = 0$ of gas yields

$$\begin{aligned} &\big((\partial_1\varphi)^2 - c^2\big)\partial_1^2\varphi + \big((\partial_2\varphi)^2 - c^2\big)\partial_2^2\varphi + \big((\partial_3\varphi)^2 - c^2\big)\partial_3^2\varphi \\ &+ 2\partial_1\varphi\partial_2\varphi\partial_{12}^2\varphi + 2\partial_1\varphi\partial_3\varphi\partial_{13}^2\varphi + 2\partial_2\varphi\partial_3\varphi\partial_{23}^2\varphi = 0, \end{aligned} \tag{1.3}$$

here $c = c(H(\nabla\varphi))$.

We assume that the 3-D infinitely nozzle Ω_0(Figure 1) is bounded by the walls: $\gamma_1 = \{x : x_2 = \varepsilon\tilde{f}_1(x_1, x_3), x_1 \in \mathbb{R}, x_3 \in (0, \frac{1}{2})\}$, $\gamma_2 = \{x : x_2 = 1 + \varepsilon\tilde{f}_2(x_1, x_3), x_1 \in \mathbb{R}, x_3 \in (0, \frac{1}{2})\}$, $\gamma_3 = \{x : x_3 = 0\}$ and $\gamma_4 = \{x : x_3 = \frac{1}{2}\}$, here $\tilde{f}_i(x_1, x_3) \in C_0^\infty((-X_0, X_0) \times (0, \frac{1}{2}))$ for some fixed positive constant X_0, and $\varepsilon > 0$ is a suitably small constant, for example, we can choose

$$\tilde{f}_1(x_1, x_3) = \begin{cases} -\exp\big(\frac{1}{x_1^2 + (x_3 - \frac{1}{4})^2 - \frac{1}{25}}\big) & \text{for } \sqrt{x_1^2 + (x_3 - \frac{1}{4})^2} \le 1/5 \\ 0 & \text{for } \sqrt{x_1^2 + (x_3 - \frac{1}{4})^2} > 1/5 \end{cases}$$

and $\tilde{f}_2(x_1, x_3) = -\tilde{f}_1(x_1, x_3)$. As illustrated in [6, (3.8), (3.9)], by the anti-symmetric extension in the x_3-direction with respect to the function $\tilde{f}_i(x_1, x_3)(i = 1, 2)$ and the potential function $\varphi(x)$, then the domain Ω_0 can be changed to

$$\Omega_1 = \{x : \varepsilon f_1^0(x_1, x_3) \le x_2 \le 1 + \varepsilon f_2^0(x_1, x_3), x_1 \in \mathbb{R}, x_3 \in [0, 1]\}$$

(Figure 2), where

$$f_i^0(x_1, x_3) = \begin{cases} \tilde{f}_i(x_1, x_3), & 0 \le x_3 \le 1/2 \\ \tilde{f}_i(x_1, 1 - x_3), & 1/2 \le x_3 \le 1. \end{cases}$$

Then by using periodic extension in the x_3-direction, we can use the following unbounded strip domain Ω instead of Ω_1 to consider our problem (1.1) with $\varphi(x_1, x_2, x_3) = \varphi(x_1, x_2, x_3 + 1)$, where Ω is bounded by $\Gamma_1 = \{x : x_2 = \varepsilon f_1(x_1, x_3), -\infty < x_1, x_3 < +\infty\}$ and $\Gamma_2 = \{x : x_2 = 1 + \varepsilon f_2(x_1, x_3), -\infty < x_1, x_3 < +\infty\}$, here $f_i(x_1, x_3 + 1) = f_i(x_1, x_3)$ and $f_i(x_1, x_3) \in C_0^\infty((-X_0, X_0) \times (-\infty, +\infty))$. More concretely, $f_i(x_1, x_3) = f_i^0(x_1, x_3 - l)$ for $l < x_3 \le l + 1$ with $l \in \mathbb{Z}$.

Since the flow is tangent to the nozzle walls, then one has

$$(\partial_1\varphi, \partial_2\varphi, \partial_3\varphi) \cdot (\varepsilon\partial_1 f_i, -1, \varepsilon\partial_3 f_i) = 0 \quad \text{on } \Gamma_i, \quad i = 1, 2. \tag{1.4}$$

In addition, we suppose that the the state of subsonic flow at minus infinity satisfies

$$\lim_{x_1 \to -\infty} (\varphi(x) - q_0 x_1) = 0. \tag{1.5}$$

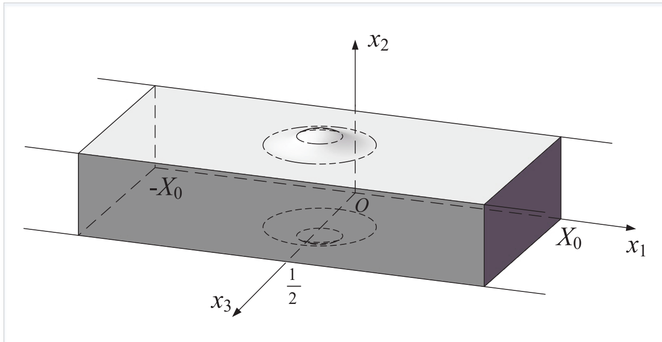

FIGURE 1. Domain Ω_0

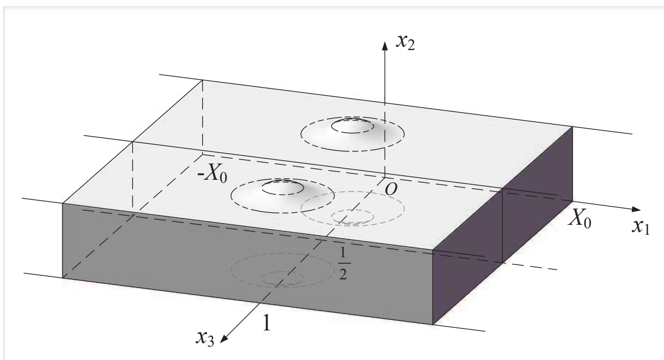

FIGURE 2. Anti-symmetric extension Ω_0 to Ω_1 with respect to $x_3 = \frac{1}{2}$

On the other hand, from the physical point of view (see [4, 5, 7, 8, 9, 10] and the references therein), when a subsonic flow in an unbounded domain is called to be stable, it should admit a determined state at infinity. Namely,

$$\lim_{x_1 \to +\infty} \nabla\varphi(x) \quad \text{exists for } x \in \Omega. \tag{1.6}$$

Our result read as follows.

Theorem 1.1. *If the 3-D unbounded strip domain Ω is defined by $\Gamma_1 = \{x : x_2 = \varepsilon f_1(x_1, x_3), -\infty < x_1, x_3 < +\infty\}$ and $\Gamma_2 = \{x : x_2 = 1 + \varepsilon f_2(x_1, x_3), -\infty < x_1, x_3 < +\infty\}$, here $f_i(x_1, x_3 + 1) = f_i(x_1, x_3)$ and $f_i(x_1, x_3) \in C_0^\infty((-X_0, X_0) \times (-\infty, +\infty))$ for some fixed constant $X_0 > 0$, then there exists a small constant $\varepsilon_0 > 0$ such that the problem (1.3)-(1.6) has a global smooth solution $\varphi(x)$ as $\varepsilon < \varepsilon_0$, which admits*

(i) $\varphi(x_1, x_2, x_3) = \varphi(x_1, x_2, x_3 + 1)$.
(ii) $|\nabla\varphi| < c(H(\nabla\varphi))$. *Namely, the flow is globally subsonic in the whole domain Ω.*

(iii) *For* $x_1 < 0$ *and* $x \in \Omega$, *there exist a suitable constant* $\delta_0 > 0$ *and a constant* $C_0 > 0$ *such that*
$$|\varphi(x) - q_0 x_1| + |\nabla(\varphi(x) - q_0 x_1)| \le C_0 \varepsilon e^{-\delta_0 |x_1|}.$$
(iv) *For* $x_1 > 0$ *and* $x \in \Omega$, *there exists a constant* $C_0 > 0$ *such that*
$$|\varphi(x) - q_0 x_1| \le C_0 \varepsilon (1 + x_1).$$
(v) $\lim_{x_1 \to +\infty, x \in \Omega} \nabla\varphi(x) = (q_0, 0, 0)$ *holds. Moreover, for* $x_1 > 0$ *and* $x \in \Omega$, *there exists a constant* $C_0 > 0$ *such that*
$$|\nabla_{x_2,x_3} \varphi(x)| \le C_0 \varepsilon e^{-\delta_0 x_1},$$
here $\delta_0 > 0$ *is given in (iii).*

Besides the estimates described by Theorem 1.1, we can give more detailed asymptotic properties on the subsonic solution $\varphi(x)$ and its derivatives in Ω when $x_1 \to \pm\infty$. This will be stated more precisely in Theorem 2.4.

Although there have been many results on the weighted $W^{2,p}(\Omega)$ $(1 < p < \infty)$ estimates of solution to the second order linear elliptic equation in an unbounded strip domain Ω or half-space Ω (see [3, 12, 13] and the references therein), it is difficult for us to use these results to treat the existence of solution to the quasilinear elliptic equation (1.3) as well as the asymptotic state and asymptotic behavior at minus or positive infinity of solution since the related weighted Sobolev spaces in [3, 12, 13] can not be imbedded into the suitable Hölder space $C^{\delta}(\bar{\Omega})$ with some positive constant $\delta > 0$.

Now we mention some works which are related to this paper. In [4, 9, 10], for the case of the gas past an obstacle, by using the Kelvin transformation, those authors have reduced the exterior domain problem on the 2-D or 3-D potential flow equation into a boundary value problem in a bounded domain. From this, together with the maximum principle, some a priori estimates on the solutions to second order linear elliptic equations and Schauder fixed point theorem, those authors have shown that the global subsonic flow field exists uniquely outside the obstacle. Although it seems that the subsonic nozzle flow problem is perhaps similar to or even simpler than the one for the subsonic flow past a profile, which is also roughly described in [5, page 75] as "the problem of finding a subsonic flow in a given channel is mathematically simpler than that of finding the flow past an airfoil", we find that these two problems have actually some differences:

(i) One is that the Kelvin transformation used in [4] and [9, 10] can not be applied directly to our nozzle problem due to the different geometric properties between the exterior domain and the infinitely long nozzle.

(ii) Another one is that the asymptotic properties of subsonic flow at minus infinity and positive infinity are very different for the nozzle problem (however, the far fields of subsonic flow at infinity are uniform for the subsonic flow past an obstacle, one can see [4, 9, 10]).

There also have some essential differences between 2-D and 3-D subsonic nozzle flow. With respect to the 2-D subsonic nozzle flow, those authors in [16] use the stream function ψ to reduce the 2-D potential flow equation into a second order quasilinear elliptic equation on ψ, meanwhile, the fixed nozzle wall conditions are correspondingly changed into the Dirichlet boundary value conditions on ψ. By use of this kind of crucial reduction in 2-D case (at this time, the maximum principle and comparison principle can be directly applied due to the appearance of Dirichlet boundary value), together with some a priori uniform estimates on the solutions to the suitably modified nonlinear equations in some well-chosen bounded domains with the suitable Dirichlet boundary values, those authors established the global existence, stability and the monotonicity of the maximum of the flow speed with respect to the incoming mass flux. However, for the 3-D subsonic nozzle flows, the streamline function method does not work (this is also illustrated in Chapter VI of [8]), we have to directly treat the 3-D potential flow equation with the fixed nozzle wall condition, which is described by the Neumann boundary value condition. In this case, the crucial comparison principle on second order elliptic equations can not be used and further the L^∞ norm estimate of $\varphi - q_0x_1$ can not be obtained directly. Therefore, we have to use some new ingredients to overcome this essential difficulty.

Next we comment on the proof of the main result in this paper. By introducing some suitable coordinate transformation and linearizing the nonlinear equation (1.3), we can actually get a Laplacian equation $\Delta u = f$ in an unbounded strip domain $\tilde{\Omega} = \{(z_1, z_2, z_3) : -\infty < z_1 < \infty, 0 < z_2 < 1, -\infty < z_3 < \infty\}$ with two Neumann boundary conditions on $z_2 = 0$ and $z_2 = 1$, one periodic boundary condition on the variable z_3, one Dirichlet boundary value condition at minus infinity (i.e., $z_1 \to -\infty$) and one restriction condition on the existence of $\lim_{z_1\to+\infty} \nabla_z u(z)$. In order to solve such a Laplacian equation in $\tilde{\Omega}$, our ingredient is to use the separation variable method to write out the formal expression of $u(z)$. From this, together with some delicate analysis, we can show that this formal expression is actually a solution of $\Delta u = f$ and its derivatives will decay at the rate of $e^{-\delta_0|z_1|}$ ($\delta_0 > 0$ is a suitable constant) for $z_1 < 0$; on the other hand, for $z_1 > 0$, the solution $u(z)$ increases at the rate of $(1+z_1)$ meanwhile its partial derivative $\partial_{z_1} u$ is bounded and the partial derivatives $(\partial_{z_2} u, \partial_{z_3} u)$ decay at the rate of $e^{-\delta_0 z_1}$. In terms of these properties, some inhomogeneous weighted Hölder spaces will be introduced by us and further be used to treat the regularity and existence of solution to the second order nonlinear elliptic problem in an unbounded strip domain. In this procedure, some detailed analysis on the expression of solution will be required, moreover, a priori estimates with different weighted norms are required to be established. Subsequently, by using the continuity method, we can complete the proof of Theorem 1.1.

This article is organized as follows. In §2, we reformulate the problem (1.3) with (1.4)-(1.6), and then give a more precise descriptions on Theorem 1.1 in some suitably weighted Hörder spaces. In §3, we will linearize the nonlinear problem (1.3) with (1.4)-(1.6). By such a linearization, we essentially obtain the Laplacian equation $\Delta u = \tilde{f}(z)$ in the strip domain $\tilde{\Omega} = \{z = (z_1, z_2, z_3) : -\infty < z_1 < \infty, 0 < z_2 < 1, -\infty < z_3 < \infty\}$ with two Neumann boundary conditions on $z_2 = 0$ and $z_2 = 1$, one periodic condition on z_3 together with $\lim_{z_1\to-\infty} u(z) = 0$ and the requirement on the existence of $\lim_{z_1\to\infty} \nabla_z u(z)$. By use of Sturm-Liouville theorem and the separation variable method, we can derive the formal expression of $u(z)$ in $\tilde{\Omega}$. Subsequently, it follows from some detailed estimates that we can obtain the existence and regularity of $u(z)$ in $\tilde{\Omega}$. In §4, based on the crucial estimates and properties given in §3, by using the suitable iteration scheme, we can complete the proof of Theorem 1.1 and further obtain the asymptotic behavior of $\nabla_x\varphi$ at negative and positive infinity in the strip domain Ω respectively.

2. Reformulation on (1.3)-(1.6) and more precise descriptions on Theorem 1.1

In this section, we first introduce some notation and weighted Hölder norms so that Theorem 1.1 can be given a more precise description.

Let $\Omega \subset \mathbb{R}^3$ be an open set including the origin $O = (0,0,0)$, if $u \in C^{m,\alpha}(\Omega)$ with $0 \le \alpha < 1$, then we define the following weighted Hölder norms for $x, y \in \Omega$, some positive constant $\delta > 0$ and $m \in \mathbb{N} \cup \{0\}$:

$$[u]^{(\delta)}_{m,0;\Omega} \equiv \sum_{|\beta|=m} \sup_{x\in\Omega} e^{\delta|x_1|}|D^\beta u(x)|;$$

$$[u]^{(\delta)}_{m,\alpha;\Omega} \equiv \sum_{|\beta|=m} \sup_{x,y\in\Omega} e^{\delta d_{x,y}} \frac{|D^\beta u(x) - D^\beta u(y)|}{|x-y|^\alpha}, \quad \text{here } d_{x,y} = \min(|x_1|, |y_1|);$$

$$|u|^{(\delta)}_{m,\alpha;\Omega} \equiv \sum_{0\le k\le m} [u]^{(\delta)}_{k,0;\Omega} + [u]^{(\delta)}_{m,\alpha;\Omega};$$

$$\begin{aligned}\|u\|^{(\delta)}_{m,\alpha;\Omega} \equiv &\sup_{x\in\Omega;x_1<0} e^{\delta|x_1|}|u(x)| + \sup_{x\in\Omega;x_1>0} (1+x_1)^{-1}|u(x)| \\ &+ \sup_{x\in\Omega;x_1<0} e^{\delta|x_1|}|\partial_{x_1} u(x)| + \sup_{x\in\Omega;x_1>0} |\partial_{x_1} u(x)| \\ &+ \sup_{x\in\Omega} e^{\delta|x_1|}(|\partial_{x_2} u(x)| + |\partial_{x_3} u(x)|) + \sum_{2\le k\le m} [u]^{(\delta)}_{k,0;\Omega} + [u]^{(\delta)}_{m,\alpha;\Omega},\end{aligned}$$

and the corresponding function spaces are defined as

$$H^{(\delta)}_{m,\alpha}(\Omega) = \{u(x) \in C^{m,\alpha}(\Omega) : |u|^{(\delta)}_{m,\alpha} < +\infty\},$$
$$\mathbb{H}^{(\delta)}_{m,\alpha}(\Omega) = \{u(x) \in C^{m,\alpha}(\Omega) : \|u\|^{(\delta)}_{m,\alpha} < +\infty\}.$$

Lemma 2.1. *For $u(x) \in C^{m,\alpha}(\bar\Omega)$, one has*

(i) $H^{(\delta)}_{m,\alpha}(\Omega) \subset \mathbb{H}^{(\delta)}_{m,\alpha}(\Omega)$.
(ii) $|\partial_{x_i} u|^{(\delta)}_{m-1,\alpha;\Omega} \le \|u\|^{(\delta)}_{m,\alpha;\Omega}$ *for $i = 2, 3$ and $m \ge 1$.*
(iii) $|D^2 u|^{(\delta)}_{m-2,\alpha;\Omega} \le \|u\|^{(\delta)}_{m,\alpha;\Omega}$ *for $m \ge 2$.*

Since these properties can be directly verified by using the definitions of the norms $|\cdot|^{(\delta)}_{m,\alpha}$ and $\|\cdot\|^{(\delta)}_{m,\alpha}$, then we omit their proof.

By using of the weighted Hölder norms introduced above, Theorem 1.1 can be stated more precisely as follows.

Theorem 2.2. *Under the assumptions of Theorem 1.1, in the domain $\Omega = \{x : -\infty < x_1 < +\infty, \varepsilon f_1(x_1,x_3) < x_2 < 1 + \varepsilon f_2(x_1,x_3), -\infty < x_3 < +\infty\}$, problem (1.3)-(1.6) has a unique solution $\varphi(x) \in C^{6,\alpha}(\Omega)$ (any fixed constant $0 < \alpha < 1$), which satisfies*

(i) $\|\varphi(x) - q_0 x_1\|^{(\delta_0)}_{6,\alpha;\Omega} \le \tilde C\varepsilon$, *here $\delta_0 > 0$ is some suitable constant.*
(ii) $\lim_{x\in\Omega;x_1\to+\infty} \nabla\varphi(x) = (q_0, 0, 0)$.

Remark 2.3. From the results on the interior regularities and boundary regularities of solutions to second order elliptic equations (see [11, Chapter 6]), we know that $\varphi(x) \in C^\infty(\bar\Omega)$ holds in Theorem 2.2.

For the requirements to show Theorem 2.2, we intend to introduce the following transformation so that the domain Ω can be changed into a standard strip domain

$\tilde{\Omega} \equiv \{z = (z_1, z_2, z_3) : -\infty < z_1 < \infty, 0 < z_2 < 1, -\infty < z_3 < \infty\}$:

$$
\begin{gathered}
z_1 = x_1, \\
z_2 = \frac{x_2 - \varepsilon f_1(x_1, x_3)}{1 + \varepsilon f_2(x_1, x_3) - \varepsilon f_1(x_1, x_3)}, \\
z_3 = x_3.
\end{gathered} \tag{2.1}
$$

In this case, for notational convenience, we still denote by $\varphi(z)$ as the solution instead of $\varphi(x)$ under the transformation (2.1). It follows from a direct computation that the problem (1.3)-(1.6) can be changed into

$$
\begin{gathered}
\sum_{i,j=1}^{3} A_{ij}(z, \nabla_z \varphi)\partial^2_{z_i z_j}\varphi + B(z, \nabla_z\varphi)\partial_{z_2}\varphi = 0 \quad \text{in } \tilde{\Omega} \\
b_{11}(z)\partial_{z_1}\varphi + \partial_{z_2}\varphi + b_{13}(z)\partial_{z_3}\varphi = 0 \quad \text{on } z_2 = 0, \\
b_{21}(z)\partial_{z_1}\varphi + \partial_{z_2}\varphi + b_{23}(z)\partial_{z_3}\varphi = 0 \quad \text{on } z_2 = 1, \\
\varphi(z_1, z_2, z_3 + 1) = \varphi(z_1, z_2, z_3), \\
\lim_{z_1 \to -\infty} (\varphi(z) - q_0 z_1) = 0, \\
\lim_{z \in \tilde{\Omega}; z_1 \to +\infty} \nabla_z \varphi(z) \text{ exists},
\end{gathered} \tag{2.2}
$$

where

$$
A_{11}(z, \nabla_z\varphi) = c^2(H(\nabla_x\varphi)) - (\partial_{x_1}\varphi)^2,
$$

$$
A_{22}(z, \nabla_z\varphi) = \sum_{i=1}^{3}\big(c^2(H(\nabla_x\varphi)) - (\partial_{x_i}\varphi)^2\big)(\frac{\partial z_2}{\partial x_i})^2 - 2\sum_{1\le i<j\le 3}\partial_{x_i}\varphi\partial_{x_j}\varphi\frac{\partial z_2}{\partial x_i}\frac{\partial z_2}{\partial x_j},
$$

$$
A_{33}(z, \nabla_z\varphi) = c^2(H(\nabla_x\varphi)) - (\partial_{x_3}\varphi)^2,
$$

$$
\begin{aligned}
A_{12}(z, \nabla_z\varphi) &= A_{21}(z, \nabla_z\varphi) \\
&= (c^2(H(\nabla_x\varphi)) - (\partial_{x_1}\varphi)^2)\frac{\partial z_2}{\partial x_1} - \partial_{x_1}\varphi\partial_{x_2}\varphi\frac{\partial z_2}{\partial x_2} - \partial_{x_1}\varphi\partial_{x_3}\varphi\frac{\partial z_2}{\partial x_3},
\end{aligned}
$$

$$
A_{13}(z, \nabla_z\varphi) = A_{31}(z, \nabla_z\varphi) = -\partial_{x_1}\varphi\partial_{x_3}\varphi,
$$

$$
\begin{aligned}
A_{23}(z, \nabla_z\varphi) &= A_{32}(z, \nabla_z\varphi) \\
&= (c^2(H(\nabla_x\varphi)) - (\partial_{x_3}\varphi)^2)\frac{\partial z_2}{\partial x_3} - \partial_{x_1}\varphi\partial_{x_3}\varphi\frac{\partial z_2}{\partial x_1} - \partial_{x_2}\varphi\partial_{x_3}\varphi\frac{\partial z_2}{\partial x_2},
\end{aligned}
$$

$$
B(z, \nabla_z\varphi) = \sum_{i=1}^{3}\big(c^2(H(\nabla_x\varphi)) - (\partial_{x_i}\varphi)^2\big)\frac{\partial^2 z_2}{\partial x_i^2} - 2\sum_{1\le i<j\le 3}\partial_{x_i}\varphi\partial_{x_j}\varphi\frac{\partial^2 z_2}{\partial x_i\partial x_j},
$$

$$
b_{ij}(z) = \frac{\varepsilon\partial_{x_j} f_i}{\varepsilon\partial_{x_1} f_i\frac{\partial z_2}{\partial x_1} - \frac{\partial z_2}{\partial x_2} + \varepsilon\partial_{x_3} f_i\frac{\partial z_2}{\partial x_3}}, \quad i = 1, 2;\ j = 1, 3
$$

with

$$
\partial_{x_1}\varphi = \partial_{z_1}\varphi + \partial_{z_2}\varphi\frac{\partial z_2}{\partial x_1}, \quad \partial_{x_2}\varphi = \partial_{z_2}\varphi\frac{\partial z_2}{\partial x_2}, \quad \partial_{x_3}\varphi = \partial_{z_3}\varphi + \partial_{z_2}\varphi\frac{\partial z_2}{\partial x_3}.
$$

By the transformation (2.1), together with the properties of $f_i(x_1, x_3)$ $(i = 1, 2)$ and the definition of the norm $\|\cdot\|^{(\delta)}_{m,\alpha}$, to show Theorem 2.2, we only need to establish the following theorem.

Theorem 2.4. *Under the assumptions in Theorem 1.1, problem* (2.2) *has a unique solution* $\varphi(z) \in C^{6,\alpha}(\tilde{\Omega})$ *which satisfies*

(i) $\|\varphi(z) - q_0 z_1\|^{(\delta_0)}_{6,\alpha;\tilde{\Omega}} \le \tilde{C}\varepsilon$.
(ii) $\lim_{z\in\tilde{\Omega}; z_1\to+\infty} \nabla_z \varphi(z) = (q_0, 0, 0)$.

In next sections, we will focus on the proof of Theorem 2.4.

3. Solvability and a priori estimates for the linearized problem of (2.2)

To solve the nonlinear problem (2.2), we first consider its linearized case, which corresponds to a mixed boundary value problem of a second order linear elliptic equation in an infinitely long strip domain $\tilde{\Omega}$. In terms of the smallness of perturbed nozzle walls and by using direct computations, the linearized problem of (2.2) can be essentially expressed as:

$$
\begin{gathered}
\begin{aligned}
\bar{L}(v)\dot{u} &\equiv \sum_{i,j=1}^{3} a_{ij}(z, \nabla_z v)\partial^2_{z_i z_j}\dot{u} \\
&\equiv \sum_{i=1}^{3} \big(c^2(H(\nabla_z v)) - (\partial_{z_i} v)^2\big)\partial^2_{z_i}\dot{u} - 2\sum_{1\le i<j\le 3} \partial_{z_i} v \partial_{z_j} v \partial^2_{z_i z_j}\dot{u} \\
&= \dot{f} \quad \text{in } \tilde{\Omega},
\end{aligned} \\
\partial_{z_2}\dot{u} = \dot{g}_1 \quad \text{on } z_2 = 0, \\
\partial_{z_2}\dot{u} = \dot{g}_2 \quad \text{on } z_2 = 1, \\
\dot{u}(z_1, z_2, z_3 + 1) = \dot{u}(z_1, z_2, z_3), \\
\lim_{z_1\to-\infty,\, z\in\tilde{\Omega}} \dot{u}(z) = 0, \\
\lim_{z_1\to+\infty,\, z\in\tilde{\Omega}} \nabla_z \dot{u}(z) \text{ exists},
\end{gathered} \tag{3.1}
$$

where $v(z_1, z_2, z_3), \dot{f}(z_1, z_2, z_3)$ and $\dot{g}_i(z_1, z_3)$ are all 1-periodic functions with respect to the variable z_3, and $v \in \mathbb{H}^{(\delta_0)}_{6,\alpha}(\tilde{\Omega})$ with $\|v - q_0 z_1\|^{(\delta_0)}_{6,\alpha;\tilde{\Omega}} < \varepsilon$ and $\delta_0 > 0$ a suitably fixed constant.

It is easy to verify that the coefficients of problem (3.1) satisfy the following uniformly elliptic condition in $\tilde{\Omega}$:

$$
\lambda|\xi|^2 \le \sum_{i,j=1}^{3} a_{ij}(z, \nabla_z v)\xi_i\xi_j \le \Lambda|\xi|^2, \tag{3.2}
$$

for all $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$ and $z \in \tilde{\Omega}$, here λ and Λ are two appropriate constants.

Next, we study the solvability of problem (3.1) as well as the regularity and a priori estimates of solution $\dot{u}(z)$ to (3.1). To this end, we first study the Laplacian equation in $\mathbb{R}^3$ with the following boundary conditions:

$$
\begin{gathered}
L_0 u \equiv \Delta u = \tilde{f} \quad \tilde{\Omega}, \\
\partial_{z_2} u = \tilde{g}_1 \quad \text{on } z_2 = 0, \\
\partial_{z_2} u = \tilde{g}_2 \quad \text{on } z_2 = 1, \\
u(z_1, z_2, z_3 + 1) = u(z_1, z_2, z_3), \\
\lim_{z_1 \to -\infty} u(z) = 0, \\
\lim_{z_1 \to +\infty} \nabla_z u(z) \text{ exists.}
\end{gathered}
\tag{3.3}
$$

where $\tilde{f}(z) \in H^{(\delta_0)}_{4,\alpha}(\tilde{\Omega})$ and $\tilde{g}_i(z_1, z_3) \in H^{(\delta_0)}_{5,\alpha}(\tilde{\Omega})$ $(i = 0, 1)$ are all 1-periodic functions with respect to the variable z_3.

For the later uses, we now give a lemma on the function $\tilde{f}(z)$.

Lemma 3.1. *For $\tilde{f} \in H^{(\delta_0)}_{4,\alpha}(\tilde{\Omega})$, if we set*

$$
\begin{gathered}
f_{m0}(z_1) = 2 \int_0^1 \int_0^1 \tilde{f}(z) \cos(m\pi z_2) dz_2 dz_3, \\
f^1_{0n}(z_1) = 2 \int_0^1 \int_0^1 \tilde{f}(z) \cos(2n\pi z_3) dz_2 dz_3, \\
f^2_{0n}(z_1) = 2 \int_0^1 \int_0^1 \tilde{f}(z) \sin(2n\pi z_3) dz_2 dz_3, \\
f^1_{mn}(z_1) = 4 \int_0^1 \int_0^1 \tilde{f}(z) \cos(m\pi z_2) \cos(2n\pi z_3) dz_2 dz_3, \\
f^2_{mn}(z_1) = 4 \int_0^1 \int_0^1 \tilde{f}(z) \cos(m\pi z_2) \sin(2n\pi z_3) dz_2 dz_3
\end{gathered}
$$

for $m, n \in \mathbb{N}$, then

$$
\begin{gathered}
|f_{m0}(z_1)| \leq \frac{C}{m^2} |\tilde{f}|^{(\delta_0)}_{4,\alpha;\tilde{\Omega}} e^{-\delta_0 |z_1|}, \\
|f^i_{0n}(z_1)| \leq \frac{C}{n^2} |\tilde{f}|^{(\delta_0)}_{4,\alpha;\tilde{\Omega}} e^{-\delta_0 |z_1|}, \quad i = 1, 2, \\
|f^i_{mn}(z_1)| \leq \frac{C}{m^2 n^2} |\tilde{f}|^{(\delta_0)}_{4,\alpha;\tilde{\Omega}} e^{-\delta_0 |z_1|}, \quad i = 1, 2.
\end{gathered}
$$

Proof. Integrating by parts, we arrived at

$$
\begin{aligned}
f_{m0}(z_1) &= 2 \int_0^1 \Big(\frac{1}{m\pi} \tilde{f} \sin(m\pi z_2)|^{z_2=1}_{z_2=0} - \frac{1}{m\pi} \int_0^1 \partial_{z_2} \tilde{f} \sin(m\pi z_2) dz_2 \Big) dz_3 \\
&= 2 \int_0^1 \Big(\frac{1}{m^2\pi^2} \partial_{z_2} \tilde{f} \cos(m\pi z_2)|^{z_2=1}_{z_2=0} - \frac{1}{m^2\pi^2} \int_0^1 \partial^2_{z_2} \tilde{f} \cos(m\pi z_2) dz_2 \Big) dz_3.
\end{aligned}
$$

Because $\tilde{f} \in H^{(\delta_0)}_{4,\alpha}(\tilde{\Omega})$, we have $|D^\beta \tilde{f}(z)| \leq |\tilde{f}|^{(\delta_0)}_{4,\alpha;\tilde{\Omega}} e^{-\delta_0 |z_1|}$ for $|\beta| \leq 4$. From this, we derive that

$$
|f_{m0}(z_1)| \leq \frac{C}{m^2} |\tilde{f}|^{(\delta_0)}_{4,\alpha;\tilde{\Omega}} e^{-\delta_0 |z_1|}.
$$

Analogously, by using the periodic property of $\tilde f$ with respect to z_3 and integration by parts, we have

$$\int_0^1 \tilde f \cos(2n\pi z_3)dz_3 = -\frac{1}{4n^2\pi^2}\int_0^1 \partial_{z_3}^2 \tilde f \cos(2n\pi z_3)dz_3. \tag{3.4}$$

This yields

$$|f^1_{0n}(z_1)| \le \frac{C}{n^2}|f|^{(\delta_0)}_{4,\alpha;\tilde\Omega} e^{-\delta_0|z_1|}.$$

Analogously, $|f^2_{0n}(z_1)| \le \frac{C}{n^2}|f|^{(\delta_0)}_{4,\alpha;\tilde\Omega} e^{-\delta_0|z_1|}$ also holds.

Next, we estimate $f^i_{mn}(z_1)$ for $i = 1, 2$. It follows from (3.4) and integration by parts with respect to z_2 that

$$\begin{aligned} f^1_{mn}(z_1) &= -\frac{1}{n^2\pi^2}\int_0^1\int_0^1 \partial^2_{z_3}\tilde f \cos(m\pi z_2)\cos(2n\pi z_3)dz_2dz_3 \\ &= \frac{1}{m^2n^2\pi^4}\Big(\int_0^1 \big(\cos(m\pi z_2)\partial_{z_2}\partial^2_{z_3}\tilde f\big)|^{z_2=1}_{z_2=0}\cos(2n\pi z_3)dz_3 \\ &\quad - \int_0^1\int_0^1 \partial^2_{z_2}\partial^2_{z_3}\tilde f\cos(m\pi z_2)\cos(2n\pi z_3)dz_2dz_3\Big). \end{aligned}$$

This yields

$$|f^1_{mn}(z_1)| \le \frac{C}{m^2n^2}|f|^{(\delta_0)}_{4,\alpha;\tilde\Omega} e^{-\delta_0|z_1|}.$$

Analogously, $|f^2_{mn}(z_1)| \le \frac{C}{m^2n^2}|f|^{(\delta_0)}_{4,\alpha;\tilde\Omega} e^{-\delta_0|z_1|}$ holds. □

Lemma 3.2. *If $\tilde f \in H^{(\delta_0)}_{4,\alpha}(\tilde\Omega)$ and $\tilde g_i \in H^{(\delta_0)}_{5,\alpha}(\tilde\Omega)$ with $0 < \delta_0 < \pi$, then the equation (3.3) has a solution $u \in C^2(\overline{\tilde\Omega})$, which satisfies the estimate*

$$\|u\|^{(\delta_0)}_{2,0;\tilde\Omega} \le C(|\tilde f|^{(\delta_0)}_{4,\alpha;\tilde\Omega} + \sum_{i=1}^2 |\tilde g_i|^{(\delta_0)}_{5,\alpha;\tilde\Omega}). \tag{3.5}$$

Proof. We intend to use the method of separation variables to study the solvability and regularities of solution u to (3.3). To this end, we firstly focus on its corresponding homogeneous problem.

Let us consider the nontrivial solutions of the problem

$$\begin{gathered} \Delta u = 0 \quad \text{in } \tilde\Omega, \\ \partial_{z_2}u = 0 \quad \text{on } z_2 = 0, \\ \partial_{z_2}u = 0 \quad \text{on } z_2 = 1, \\ u(z_1, z_2, z_3 + 1) = u(z_1, z_2, z_3). \end{gathered} \tag{3.6}$$

Set $u(z) = X(z_1)Y(z_2)Z(z_3)$, then from (3.6) it follows that

$$\begin{gathered} Y''(z_2) + \lambda Y(z_2) = 0, \\ Y'(0) = 0, \\ Y'(1) = 0, \end{gathered} \tag{3.7}$$

and

$$\begin{gathered} Z''(z_3) + \mu Z(z_3) = 0, \\ Z(z_3 + 1) = Z(z_3), \end{gathered} \tag{3.8}$$

and

$$X''(z_1) - (\lambda + \mu)X(z_1) = 0, \tag{3.9}$$

here $\lambda, \mu \in \mathbb{R}$.

By a simple computation, we can show that the eigenvalues of (3.7) are $\lambda_m = (m\pi)^2$ $(m = 0, 1, 2, \dots)$, and the corresponding eigenfunctions are $\cos(m\pi z_2)$. In addition, we can compute that the eigenvalues of (3.8) are $\mu_n = (2n\pi)^2$ $(n = 0, 1, 2, \dots)$, and the corresponding eigenfunctions are $\cos(2n\pi z_3)$ and $\sin(2n\pi z_3)$ respectively.

We now solve equation (3.6) by using the eigenfunction expansion method in terms of the complete orthogonal basis $\{\cos m\pi z_2 \cos 2n\pi z_3, \cos m\pi z_2 \sin 2n\pi z_3\}_{m,n=0}^{+\infty}$.

Set $h(z) = \frac{1}{2}(\tilde{g}_2(z_1, z_3) - \tilde{g}_1(z_1, z_3))z_2^2 + \tilde{g}_1(z_1, z_3)z_2$ and $v(z) = u(z) - h(z)$, then it follows from (3.3) that $v(z)$ satisfies

$$\begin{gathered}
\Delta v = \tilde{f} - \Delta h \equiv f \qquad in \quad \tilde{\Omega}, \\
\partial_{z_2} v = 0 \quad \text{on } z_2 = 0, \\
\partial_{z_2} v = 0 \quad \text{on } z_2 = 1, \\
v(z_1, z_2, z_3 + 1) = v(z_1, z_2, z_3), \\
\lim_{z_1 \to -\infty} v(z) = 0, \\
\lim_{z_1 \to +\infty} \nabla_z v(z) \text{ exists.}
\end{gathered} \tag{3.10}$$

Let

$$\begin{aligned}
v(z) = X_{00}(z_1) &+ \sum_{m=1}^{\infty} X_{m0}(z_1)\cos(m\pi z_2) \\
&+ \sum_{n=1}^{\infty} \big(X_{0n}^1(z_1)\cos(2n\pi z_3) + X_{0n}^2(z_1)\sin(2n\pi z_3)\big) \\
&+ \sum_{m,n=1}^{\infty} \big(X_{mn}^1(z_1)\cos(m\pi z_2)\cos(2n\pi z_3) \\
&+ X_{mn}^2(z_1)\cos(m\pi z_2)\sin(2n\pi z_3)\big)
\end{aligned} \tag{3.11}$$

and

$$\begin{aligned}
f(z) = f_{00}(z_1) &+ \sum_{m=1}^{\infty} f_{m0}(z_1)\cos(m\pi z_2) \\
&+ \sum_{n=1}^{\infty} \big(f_{0n}^1(z_1)\cos(2n\pi z_3) + f_{0n}^2(z_1)\sin(2n\pi z_3)\big) \\
&+ \sum_{m,n=1}^{\infty} \big(f_{mn}^1(z_1)\cos(m\pi z_2)\cos(2n\pi z_3) + f_{mn}^2(z_1)\cos(m\pi z_2)\sin(2n\pi z_3)\big),
\end{aligned}$$

where

$$\begin{gathered}
f_{00}(z_1) = \int_0^1 \int_0^1 f(z)\,dz_2 dz_3 \\
f_{m0}(z_1) = 2\int_0^1 \int_0^1 f(z)\cos(m\pi z_2)\,dz_2 dz_3,
\end{gathered}$$

$$
\begin{aligned}
f_{0n}^1(z_1) &= 2\int_0^1\int_0^1 f(z)\cos(2n\pi z_3)dz_2dz_3,\\
f_{0n}^2(z_1) &= 2\int_0^1\int_0^1 f(z)\sin(2n\pi z_3)dz_2dz_3,\\
f_{mn}^1(z_1) &= 4\int_0^1\int_0^1 f(z)\cos(m\pi z_2)\cos(2n\pi z_3)dz_2dz_3,\\
f_{mn}^1(z_1) &= 4\int_0^1\int_0^1 f(z)\cos(m\pi z_2)\sin(2n\pi z_3)dz_2dz_3.
\end{aligned}
$$

Next, we determine the terms $X_{00}(z_1)$, $X_{m0}(z_1)$, $X_{0n}^i(z_1)$ and $X_{mn}^i(z_1)$ $(i = 1, 2)$ in (3.11). It follows from (3.10) and (3.11) that we can formally obtain

$$
\begin{gathered}
X_{00}''(z_1) = f_{00}(z_1),\\
\lim_{z_1\to-\infty} X_{00}(z_1) = 0, \quad \lim_{z_1\to+\infty} X_{00}'(z_1) \text{ exists},
\end{gathered} \tag{3.12}
$$

$$
\begin{gathered}
X_{m0}''(z_1) - m^2\pi^2 X_{m0}(z_1) = f_{m0}(z_1),\\
\lim_{z_1\to-\infty} X_{m0}(z_1) = 0, \quad \lim_{z_1\to+\infty} X_{m0}'(z_1) \text{ exists},
\end{gathered} \tag{3.13}
$$

$$
\begin{gathered}
(X_{0n}^i)''(z_1) - 4n^2\pi^2 X_{m0}^i(z_1) = f_{0n}^i(z_1),\\
\lim_{z_1\to-\infty} X_{0n}^i(z_1) = 0, \quad \lim_{z_1\to+\infty} (X_{0n}^i)'(z_1) \text{ exists},
\end{gathered} \tag{3.14}
$$

$$
\begin{gathered}
(X_{mn}^i)''(z_1) - (m^2+4n^2)\pi^2 X_{mn}^i(z_1) = f_{mn}^i(z_1),\\
\lim_{z_1\to-\infty} X_{mn}^i(z_1) = 0, \quad \lim_{z_1\to+\infty} (X_{mn}^i)'(z_1) \text{ exists}.
\end{gathered} \tag{3.15}
$$

Solving these ordinary differential equations directly yield

$$
X_{00}(z_1) = \int_{-\infty}^{z_1}\int_{-\infty}^{t} f_{00}(\xi)d\xi dt, \tag{3.16}
$$

$$
X_{m0}(z_1) = e^{m\pi z_1}\int_{+\infty}^{z_1} e^{-2m\pi t}\int_{-\infty}^{t} e^{m\pi\xi} f_{m0}(\xi)d\xi dt, \quad m \geq 1, \tag{3.17}
$$

$$
X_{0n}^i(z_1) = e^{2n\pi z_1}\int_{+\infty}^{z_1} e^{-2n\pi t}\int_{-\infty}^{t} e^{2n\pi\xi} f_{0n}^i(\xi)d\xi dt, \quad n \geq 1, \tag{3.18}
$$

$$
\begin{aligned}
X_{mn}^i(z_1) &= e^{\sqrt{m^2+4n^2}\pi z_1}\int_{+\infty}^{z_1} e^{-2\sqrt{m^2+4n^2}\pi t}\int_{-\infty}^{t} e^{\sqrt{m^2+4n^2}\pi\xi} f_{mn}^i(\xi)d\xi dt,\\
&\quad m, n \geq 1.
\end{aligned} \tag{3.19}
$$

We now analyze the expressions in (3.16)-(3.19). This will be divided into three parts.

Part 1. Estimate of $X_{00}(z_1)$. By using the expression of $X_{00}(z_1)$ in (3.16) and integrating by parts, one has

$$
X_{00}(z_1) = t\int_{-\infty}^{t} f_{00}(\xi)d\xi\Big|_{-\infty}^{z_1} - \int_{-\infty}^{z_1} t f_{00}(t)dt. \tag{3.20}
$$

By $f(z) \in H_{3,\alpha}^{(\delta_0)}(\tilde\Omega)$, we have

$$
|f_{00}(z_1)| \leq |f|_{3,\alpha;\tilde\Omega}^{(\delta_0)} e^{-\delta_0|z_1|}. \tag{3.21}
$$

Thus

$$\lim_{t\to-\infty} t\int_{-\infty}^{t} f_{00}(\xi)dt = 0.$$

This and (3.20), yield

$$X_{00}(z_1) = z_1\int_{-\infty}^{z_1} f_{00}(t)dt - \int_{-\infty}^{z_1} tf_{00}(t)dt. \tag{3.22}$$

For $z_1 < 0$, it follows from (3.16) and (3.21) that

$$|X_{00}(z_1)| \le \int_{-\infty}^{z_1}\int_{-\infty}^{t} |f_{00}(\xi)|d\xi dt \le |f|^{(\delta_0)}_{3,\alpha;\tilde\Omega}\int_{-\infty}^{z_1}\int_{-\infty}^{t} e^{\delta_0\xi}d\xi dt \le \frac{1}{\delta_0^2}|f|^{(\delta_0)}_{3,\alpha;\tilde\Omega}e^{\delta_0 z_1}.$$

For $z_1 > 0$, by using (3.21)-(3.22), we have

$$\begin{aligned}|X_{00}(z_1)| &\le z_1\int_{-\infty}^{z_1}|f_{00}(t)|dt + \int_{-\infty}^{z_1}|tf_{00}(t)|dt\\ &\le |f|^{(\delta_0)}_{3,\alpha;\tilde\Omega}\Big(z_1\int_{-\infty}^{0}e^{\delta_0 t}dt + z_1\int_0^{z_1}e^{-\delta_0 t}dt - \int_{-\infty}^{0}te^{\delta_0 t}dt + \int_0^{z_1}te^{-\delta_0 t}dt\Big)\\ &\le C(1+z_1)|f|^{(\delta_0)}_{3,\alpha;\tilde\Omega}.\end{aligned}$$

This means

$$\|X_{00}(z_1)\|^{(\delta_0)}_{0,0} \le C|f|^{(\delta_0)}_{3,\alpha;\tilde\Omega}. \tag{3.23}$$

Next we estimate $X_{00}'(z_1)$. Note that

$$X_{00}'(z_1) = \int_{-\infty}^{z_1} f_{00}(t)dt.$$

If $z_1 < 0$, then one has

$$|X_{00}'(z_1)| \le |f|^{(\delta_0)}_{3,\alpha;\tilde\Omega}\int_{-\infty}^{z_1} e^{\delta_0 t}dt \le \frac{1}{\delta_0}|f|^{(\delta_0)}_{3,\alpha;\tilde\Omega}e^{\delta_0 z_1}.$$

If $z_1 > 0$, then

$$|X_{00}'(z_1)| \le |f|^{(\delta_0)}_{3,\alpha;\tilde\Omega}\Big(\int_{-\infty}^{0} e^{\delta_0 t}dt + \int_0^{z_1} e^{-\delta_0 t}dt\Big) \le \frac{2}{\delta_0}|f|^{(\delta_0)}_{3,\alpha;\tilde\Omega}. \tag{3.24}$$

Thus, we arrive at

$$\|X_{00}'(z_1)\|^{(\delta_0)}_{0,0} \le C|f|^{(\delta_0)}_{3,\alpha;\tilde\Omega}$$

and

$$\lim_{z_1\to+\infty} X_{00}'(z_1) = \int_{-\infty}^{+\infty}\int_0^1\int_0^1 f(t,z_2,z_3)dz_2dz_3dt. \tag{3.25}$$

Part 2. Estimate of $X_{m0}(z_1)$ with $m \ge 1$. By (3.17) and integration by parts, we have

$$\begin{aligned}X_{m0}(z_1) &= -\frac{1}{2m\pi}e^{m\pi z_1}\Big(e^{-2m\pi t}\int_{-\infty}^{t} e^{m\pi\xi}f_{m0}(\xi)d\xi\Big|_{+\infty}^{z_1} - \int_{+\infty}^{z_1} e^{-m\pi t}f_{m0}(t)dt\Big)\\ &= -\frac{1}{2m\pi}\Big(e^{-m\pi z_1}\int_{-\infty}^{z_1} e^{m\pi t}f_{m0}(t)dt + e^{m\pi z_1}\int_{z_1}^{+\infty} e^{-m\pi t}f_{m0}(t)dt\Big),\end{aligned} \tag{3.26}$$

where we have used that

$$\lim_{t\to+\infty}\frac{\int_{-\infty}^{t}e^{m\pi\xi}f_{m0}(\xi)d\xi}{e^{2m\pi t}}=\lim_{t\to+\infty}\frac{f_{m0}(t)}{2m\pi e^{m\pi t}}=0.$$

It is noted that by Lemma 3.1 and the proof of Lemma 3.1, we have

(i) If $z_1<0$, then

$$\begin{aligned}\Big|e^{-m\pi z_1}\int_{-\infty}^{z_1}e^{m\pi t}f_{m0}(t)dt\Big| &\le \frac{C}{m^2}|f|_{3,\alpha;\tilde\Omega}^{(\delta_0)}e^{-m\pi z_1}\int_{-\infty}^{z_1}e^{(m\pi+\delta_0)t}dt\\ &=\frac{C}{m^2(m\pi+\delta_0)}|f|_{3,\alpha;\tilde\Omega}^{(\delta_0)}e^{\delta_0 z_1}\end{aligned}\tag{3.27}$$

and

$$\begin{aligned}&\Big|e^{m\pi z_1}\int_{z_1}^{+\infty}e^{-m\pi t}f_{m0}(t)dt\Big|\\ &\le \frac{C}{m^2}|f|_{3,\alpha;\tilde\Omega}^{(\delta_0)}e^{m\pi z_1}\Big(\int_{z_1}^{0}e^{(\delta_0-m\pi)t}dt+\int_0^{+\infty}e^{-(m\pi+\delta_0)t}dt\Big)\\ &\le \frac{C}{m^2(m\pi-\delta_0)}|f|_{3,\alpha;\tilde\Omega}^{(\delta_0)}e^{\delta_0 z_1}.\end{aligned}\tag{3.28}$$

(ii) If $z_1>0$, then

$$\begin{aligned}&\Big|e^{-m\pi z_1}\int_{-\infty}^{z_1}e^{m\pi t}f_{m0}(t)dt\Big|\\ &\le \frac{C}{m^2}|f|_{3,\alpha;\tilde\Omega}^{(\delta_0)}e^{-m\pi z_1}\Big(\int_{-\infty}^{0}e^{(m\pi+\delta_0)t}dt+\int_0^{z_1}e^{(m\pi-\delta_0)t}dt\Big)\\ &\le \frac{C}{m^2(m\pi-\delta_0)}|f|_{3,\alpha;\tilde\Omega}^{(\delta_0)}e^{-\delta_0 z_1}\end{aligned}\tag{3.29}$$

and

$$\begin{aligned}\Big|e^{m\pi z_1}\int_{z_1}^{+\infty}e^{-m\pi t}f_{m0}(t)dt\Big| &\le \frac{C}{m^2}|f|_{3,\alpha;\tilde\Omega}^{(\delta_0)}e^{m\pi z_1}\int_{z_1}^{+\infty}e^{-(m\pi+\delta_0)t}dt\\ &\le\frac{C}{m^2(m\pi+\delta_0)}|f|_{3,\alpha;\tilde\Omega}^{(\delta_0)}e^{-\delta_0 z_1}.\end{aligned}\tag{3.30}$$

Substituting (3.27)-(3.28) and (3.29)-(3.30) in (3.26) yields

$$|X_{m0}(z_1)|\le\frac{C}{m^3(m\pi-\delta_0)}|f|_{3,\alpha;\tilde\Omega}^{(\delta_0)}e^{-\delta_0|z_1|}.$$

Namely,

$$|X_{m0}(z_1)|_{0,0}^{(\delta_0)}\le\frac{C}{m^3(m\pi-\delta_0)}|f|_{3,\alpha;\tilde\Omega}^{(\delta_0)}.\tag{3.31}$$

Next, we estimate $X'_{m0}(z_1)$. Since

$$X'_{m0}(z_1)=\frac12\Big(e^{-m\pi z_1}\int_{-\infty}^{z_1}e^{m\pi t}f_{m0}(t)dt-e^{m\pi z_1}\int_{z_1}^{+\infty}e^{-m\pi t}f_{m0}(t)dt\Big),$$

by using (3.27)-(3.28) and (3.29)-(3.30), we have

$$|X'_{m0}(z_1)|\le\frac{C}{m^2(m\pi-\delta_0)}|f|_{3,\alpha;\tilde\Omega}^{\delta_0}e^{-\delta_0|z_1|}.$$

This means

$$|X'_{m0}(z_1)|_{0,0}^{(\delta_0)} \leq \frac{C}{m^2(m\pi-\delta_0)}|f|_{3,\alpha;\tilde{\Omega}}^{\delta_0}, \tag{3.32}$$

$$\lim_{z_1\to\infty} X'_{m0}(z_1) = 0. \tag{3.33}$$

Part 3. Estimates of $X_{0n}^i(z_1)$ and $X_{mn}^i(z_1)$ with $i=1,2$ and $m,n\geq 1$. As in Step 2, it follows from a direct computation that

$$\begin{aligned} X_{0n}^i(z_1) &= -\frac{1}{2n\pi}\Big(\int_{-\infty}^{z_1} e^{2n\pi(t-z_1)} f_{0n}^i(t)dt + \int_{z_1}^{+\infty} e^{2n\pi(z_1-t)} f_{0n}^i(t)dt\Big),\\ X_{mn}^i(z_1) &= -\frac{1}{2\pi\sqrt{m^2+4n^2}}\Big(\int_{-\infty}^{z_1} e^{\pi\sqrt{m^2+4n^2}(t-z_1)} f_{mn}^i(t)dt \\ &\quad + \int_{z_1}^{+\infty} e^{\pi\sqrt{m^2+4n^2}(z_1-t)} f_{mn}^i(t)dt\Big). \end{aligned}$$

Similar to the estimates on $X_{m0}^i(z_1)$, we can arrive at

$$\begin{aligned} |X_{0n}^i(z_1)| &\leq \frac{C}{n^3(2n\pi-\delta_0)}|f|_{3,\alpha;\tilde{\Omega}}^{(\delta_0)} e^{-\delta_0|z_1|},\\ |X_{mn}^i(z_1)| &\leq \frac{C}{m^2n^2\sqrt{m^2+4n^2}(\pi\sqrt{m^2+4n^2}-\delta_0)} \\ &\quad \times \Big(|\tilde{f}|_{4,\alpha;\tilde{\Omega}}^{(\delta_0)} + \sum_{i=1}^{2}|\tilde{g}_i|_{5,\alpha;\tilde{\Omega}}^{(\delta_0)}\Big) e^{-\delta_0|z_1|}. \end{aligned} \tag{3.34}$$

Namely,

$$|X_{0n}^i(z_1)|_{0,0}^{(\delta_0)} \leq \frac{C}{n^3(2n\pi-\delta_0)}|f|_{3,\alpha;\tilde{\Omega}}^{(\delta_0)}, \tag{3.35}$$

$$\begin{aligned} |X_{mn}^i(z_1)|_{0,0}^{(\delta_0)} &\leq \frac{C}{m^2n^2\sqrt{m^2+4n^2}(\pi\sqrt{m^2+4n^2}-\delta_0)} \\ &\quad \times \Big(|\tilde{f}|_{4,\alpha;\tilde{\Omega}}^{(\delta_0)} + \sum_{i=1}^{2}|\tilde{g}_i|_{5,\alpha;\tilde{\Omega}}^{(\delta_0)}\Big). \end{aligned} \tag{3.36}$$

Analogously,

$$\begin{aligned} |(X_{0n}^i)'(z_1)|_{0,0}^{(\delta_0)} &\leq \frac{C}{n^2(2n\pi-\delta_0)}|f|_{3,\alpha;\tilde{\Omega}}^{(\delta_0)},\\ |(X_{mn}^i)'(z_1)|_{0,0}^{(\delta_0)} &\leq \frac{C}{m^2n^2(\pi\sqrt{m^2+4n^2}-\delta_0)}\Big(|\tilde{f}|_{4,\alpha;\tilde{\Omega}}^{(\delta_0)} + \sum_{i=1}^{2}|\tilde{g}_i|_{5,\alpha;\tilde{\Omega}}^{(\delta_0)}\Big) \end{aligned} \tag{3.37}$$

and

$$\lim_{z_1\to\infty}(X_{0n}^i)'(z_1) = 0, \quad \lim_{z_1\to\infty}(X_{mn}^i)'(z_1) = 0. \tag{3.38}$$

Based on Parts 1–3, we now show that the formal solution (3.11) is actually a classical solution of (3.10). For convenience, we set

$$v(z) = X_{00}(z_1) + I(z), \tag{3.39}$$

where $I(z)=\sum_{k=1}^5 I_k(z)$ with

$$\begin{aligned}
I_1(z)\equiv I_1(z_1,z_2)&=\sum_{m=1}^{\infty} X_{m0}(z_1)\cos(m\pi z_2),\\
I_2(z)\equiv I_2(z_1,z_3)&=\sum_{m=1}^{\infty} X_{0n}^1(z_1)\cos(2n\pi z_3),\\
I_3(z)\equiv I_3(z_1,z_3)&=\sum_{m=1}^{\infty} X_{0n}^2(z_1)\sin(2n\pi z_3),\\
I_4(z)&=\sum_{m,n=1}^{\infty} X_{mn}^1(z_1)\cos(m\pi z_2)\cos(2n\pi z_3),\\
I_5(z)&=\sum_{m,n=1}^{\infty} X_{mn}^2(z_1)\cos(m\pi z_2)\sin(2n\pi z_3).
\end{aligned}$$

Next, we show that $I_k(z)$ $(1\le k\le 5)$ is convergent for $(z_1,z_2,z_3)\in(-\infty,+\infty)\times[0,1]\times(-\infty,+\infty)$. Indeed, by using (3.31), we have

$$|I_1(z)|\le\sum_{m=1}^{+\infty}\frac{C}{m^3(m\pi-\delta_0)}|f|_{3,\alpha}^{(\delta_0)}e^{-\delta_0 z_1}\le C|f|_{3,\alpha;\tilde\Omega}^{(\delta_0)}e^{-\delta_0 z_1}, \tag{3.40}$$

$$|I_1(z)|_{0,0}^{(\delta_0)}\le C|f|_{3,\alpha;\tilde\Omega}^{(\delta_0)}. \tag{3.41}$$

In terms of (3.35)-(3.36), we have

$$|I_2(z)|+|I_3(z)|\le|\le\sum_{n=1}^{+\infty}\frac{C}{n^3(2n\pi-\delta_0)}|f|_{3,\alpha;\tilde\Omega}^{(\delta_0)}e^{-\delta_0|z_1|}\le C|f|_{3,\alpha;\tilde\Omega}^{(\delta_0)}e^{-\delta_0|z_1|}, \tag{3.42}$$

$$\begin{aligned}
|I_4(z)|+|I_5(z)|&\le\sum_{m,n=1}^{+\infty}\frac{C}{m^2n^2\sqrt{m^2+4n^2}(\pi\sqrt{m^2+4n^2}-\delta_0)}\\
&\quad\times\Big(|\tilde f|_{4,\alpha;\tilde\Omega}^{(\delta_0)}+\sum_{i=1}^2|\tilde g_i|_{5,\alpha;\tilde\Omega}^{(\delta_0)}\Big)e^{-\delta_0|z_1|} \qquad (3.43)\\
&\le C(|\tilde f|_{4,\alpha;\tilde\Omega}^{(\delta_0)}+\sum_{i=1}^2|\tilde g_i|_{5,\alpha;\tilde\Omega}^{(\delta_0)})e^{-\delta_0|z_1|}.
\end{aligned}$$

This means

$$|I_k(z)|_{0,0}^{(\delta_0)}\le C(|\tilde f|_{4,\alpha;\tilde\Omega}^{(\delta_0)}+\sum_{i=1}^2|\tilde g_i|_{5,\alpha;\tilde\Omega}^{(\delta_0)})\quad for\quad k=2,3,4,5. \tag{3.44}$$

Thus, the series $I(z)$ and further $v(z)$ are continuous because of the uniform convergence of $I_k(z)$ in any compact subset of $\tilde\Omega=(-\infty,+\infty)\times[0,1]\times(-\infty,+\infty)$.

Next, we show $I(z)\in C^1(\tilde\Omega)$ and further $v(z)\in C^1(\tilde\Omega)$. It is noted that

$$\begin{aligned}
|\partial_{z_1}I_1(z)|&\le\sum_{m=1}^{+\infty}|X_{m0}'(z_1)|\\
&\le\sum_{m=1}^{+\infty}\frac{C}{m^2(m\pi-\delta_0)}|f|_{3,\alpha;\tilde\Omega}^{(\delta_0)}e^{-\delta_0|z_1|}
\end{aligned}$$

$$\le C|f|^{(\delta_0)}_{3,\alpha;\tilde\Omega}e^{-\delta_0|z_1|}$$

and

$$\begin{aligned}|\partial_{z_2}I_1(z)| &\le \sum_{m=1}^{+\infty} m\pi|X_{m0}(z_1)|\\ &\le \sum_{m=1}^{+\infty}\frac{C}{m^2(m\pi-\delta_0)}|f|^{(\delta_0)}_{3,\alpha;\tilde\Omega}e^{-\delta_0|z_1|} \le C|f|^{(\delta_0)}_{3,\alpha;\tilde\Omega}e^{-\delta_0|z_1|}.\end{aligned}$$

Therefore, $I_1(z)\in C^1(\tilde\Omega)$ holds, and satisfies the estimate

$$|\nabla_z I_1(z)|^{(\delta_0)}_{0,0} \le C|f|^{(\delta_0)}_{3,\alpha;\tilde\Omega}.$$

Analogously, $I_k(z)\in C^1(\tilde\Omega)$ $(k=2,3,4)$ holds. Moreover, we have

$$|\nabla_z I(z)| \le C(|\tilde f|^{(\delta_0)}_{4,\alpha;\tilde\Omega} + \sum_{i=1}^{2}|\tilde g_i|^{(\delta_0)}_{5,\alpha;\tilde\Omega})e^{-\delta_0|z_1|}$$

and

$$|\nabla_z I(z)|^{(\delta_0)}_{0,0} \le C(|\tilde f|^{(\delta_0)}_{4,\alpha;\tilde\Omega} + \sum_{i=1}^{2}|\tilde g_i|^{(\delta_0)}_{5,\alpha;\tilde\Omega}). \tag{3.45}$$

Finally, we show $I(z)\in C^2(\tilde\Omega)$ and further $v(z)\in C^2(\tilde\Omega)$. By using the expression of $I_1(z)$ and (3.13), we have

$$\partial^2_{z_1}I_1(z) = \sum_{m=1}^{+\infty}X''_{m0}(z_1)\cos(m\pi z_2) = \sum_{m=1}^{+\infty}(m^2\pi^2X_{m0}(z_1)+f_{m0}(z_1))\cos(m\pi z_2).$$

It follows from Lemma 3.1 and (3.31) that

$$\begin{aligned}|\partial^2_{z_1}I_1(z)| &\le \sum_{m=1}^{+\infty}\big(\frac{C}{m(m\pi-\delta_0)}+\frac{C}{m^2}\big)(|\tilde f|^{(\delta_0)}_{4,\alpha;\tilde\Omega} + \sum_{i=1}^{2}|\tilde g_i|^{(\delta_0)}_{5,\alpha;\tilde\Omega})e^{-\delta_0|z_1|}\\ &\le C(|\tilde f|^{(\delta_0)}_{4,\alpha;\tilde\Omega} + \sum_{i=1}^{2}|\tilde g_i|^{(\delta_0)}_{5,\alpha;\tilde\Omega})e^{-\delta_0|z_1|}.\end{aligned}$$

Analogously, one has

$$\begin{aligned}|\partial^2_{z_1}I_2(z)|+|\partial^2_{z_1}I_3(z)| &\le \sum_{n=1}^{+\infty}\big(\frac{C}{n(2n\pi-\delta_0)}+\frac{C}{n^2}\big)(|\tilde f|^{(\delta_0)}_{4,\alpha;\tilde\Omega} + \sum_{i=1}^{2}|\tilde g_i|^{(\delta_0)}_{5,\alpha;\tilde\Omega})e^{-\delta_0|z_1|}\\ &\le C(|\tilde f|^{(\delta_0)}_{4,\alpha;\tilde\Omega} + \sum_{i=1}^{2}|\tilde g_i|^{(\delta_0)}_{5,\alpha;\tilde\Omega})e^{-\delta_0|z_1|}\end{aligned}$$

and

$$\begin{aligned}&|\partial^2_{z_1}I_4(z)|+|\partial^2_{z_1}I_5(z)|\\ &\le \sum_{m,n=1}^{+\infty}\big(\frac{C\sqrt{m^2+4n^2}}{m^2n^2(\pi\sqrt{m^2+4n^2}-\delta_0)}+\frac{C}{m^2n^2}\big)(|\tilde f|^{(\delta_0)}_{4,\alpha;\tilde\Omega} + \sum_{i=1}^{2}|\tilde g_i|^{(\delta_0)}_{5,\alpha;\tilde\Omega})e^{-\delta_0|z_1|}\\ &\le C(|\tilde f|^{(\delta_0)}_{4,\alpha;\tilde\Omega} + \sum_{i=1}^{2}|\tilde g_i|^{(\delta_0)}_{5,\alpha;\tilde\Omega})e^{-\delta_0|z_1|}.\end{aligned}$$

This implies

$$|\partial_{z_1}^2 I(z)|_{0,0}^{(\delta_0)} \le C(|\tilde f|_{4,\alpha;\tilde\Omega}^{(\delta_0)} + \sum_{i=1}^2 |\tilde g_i|_{5,\alpha;\tilde\Omega}^{(\delta_0)}). \tag{3.46}$$

Similarly, we can arrive at

$$|\partial_{z_iz_j}^2 I(z)|_{0,0}^{(\delta_0)} \le C(|\tilde f|_{4,\alpha;\tilde\Omega}^{(\delta_0)} + \sum_{i=1}^2 |\tilde g_i|_{5,\alpha;\tilde\Omega}^{(\delta_0)}) \quad \text{for } 1\le i\le 3,\ 1<j\le 3. \tag{3.47}$$

Combining (3.23)-(3.25), (3.41) and (3.44)-(3.47) yield (3.5).

On the other hand, it follows from (3.25), (3.33), (3.38) and the uniform convergence of $\partial_{z_1} I(z)$ with respect to z_1 that

$$\begin{aligned} \lim_{z_1\to+\infty} \partial_{z_1} u(z) = \int_{-\infty}^{+\infty}\int_0^1\int_0^1 \tilde f(t,z_2,z_3)dz_2dz_3dt \\ - \int_{-\infty}^{+\infty}\int_0^1 (\tilde g_2(z_1,z_3)-\tilde g_1(z_1,z_3))dz_1dz_3, \end{aligned} \tag{3.48}$$

$$\lim_{z_1\to+\infty} \partial_{z_k} u(z) = 0 \quad \text{for } k=2,3. \tag{3.49}$$

The proof is complete. □

To obtain the higher regularities and higher order norm (i.e., $\|u\|_{6,\alpha}^{(\delta_0)}$) estimates of $u(z)$ to (3.3) and further treat the nonlinear problem (3.1) in the unbounded strip domain $\tilde\Omega$, we have to overcome the difficulty induced by the exponent weight $e^{\delta_0|z_1|}$ in the spaces $H_{6,\alpha}^{(\delta_0)}$ or $\mathbb{H}_{6,\alpha}^{(\delta_0)}$ (it is noted that in the general case, the weighted Hölder space with the weight $|d_x|^\nu (\nu\in\mathbb{R})$ is only used to obtain a priori estimates of solutions to second order elliptic equations, here d_x stands for the distance of the point x to the boundary or some parts of boundary. One can be referred to [11, Chapter 6]). For this end, first we take a suitable transformation (see (3.53) below) to change the unbounded domain $\tilde\Omega$ into an unbounded domain Q which is bounded by two cones $\{y : y_3 = \mu_1\sqrt{y_1^2+y_2^2}\}$ and $\{y : y_3 = \mu_2\sqrt{y_1^2+y_2^2}\}$ with two suitable fixed constants $\mu_1>\mu_2>0$. In this case, the exponent weight $e^{\delta_0|z_1|}$ in the z-coordinates is equivalent to the weight $|y|^{\delta_0}$ in the y-coordinates. From this, as in [17, 18], the estimate of solution in the weighted Hölder space with the weight $|y|^{\delta_0}$ can be obtained. On the other hand, due to the different properties of $u(z)$ as $z_1\to-\infty$ or $z_1\to\infty$, we have to introduce another transformation (see (3.72) below) such that the estimate of solution in the weighted Hölder space with the weight $|\tilde y|^{-\delta_0}$ can be also obtained. Combining these two cases, together with some delicate analysis, we can finally obtain the estimates of $\|u\|_{6,\alpha}^{(\delta_0)}$. One can see the details below.

For notational convenience, we use a weighted Hölder norm which is introduced in [11, Chapter 6] and the references therein as follows:

Let $D\subset\mathbb{R}^3$ be an open set, for $x,y\in D$, we define $r_{x,y}=\min(|x|,|y|)$. For $m\in\mathbb{N}\cup\{0\}$, $\alpha\in\mathbb{R}^+$, $\mu\in\mathbb{R}^+$, $\mu_1,\mu_2\in\mathbb{R}$ and $v\in C^{m,\alpha}(D)$, we define

$$[v]]_{m,0;D}^{(\mu)} \equiv \sum_{|\beta|=m}\sup_{x\in D}|x|^{m+\mu}|D^\beta v(x)|,$$

$$[v]]_{m,\alpha;D}^{(\mu)} \equiv \sum_{|\beta|=m}\sup_{x,y\in D;x\ne y} r_{x,y}^{m+\alpha+\mu}\frac{|D^\beta v(x)-D^\beta v(y)|}{|x-y|^\alpha},$$

$$|v\|^{(\mu)}_{m,\alpha;D} \equiv \sum_{0\le i\le m} [v]]^{(\mu)}_{i,0;D} + [v]]^{(\mu)}_{m,\alpha;D},$$

$$[[v]]^{(\mu_1,\mu_2)}_{m,0;D} \equiv \max\Big\{ \sup_{|x|<1} \sum_{|\beta|=m} |x|^{m+\mu_1}|D^\beta v(x)|, \sup_{|x|>1} \sum_{|\beta|=m} |x|^{m+\mu_2}|D^\beta v(x)|\Big\},$$

$$]^{(\mu_1,\mu_2)}_{m,\alpha;D} \equiv \max\Big\{ \sup_{0<r_{x,y}<1} \sum_{|\beta|=m} r^{\mu_1+m+\alpha}_{x,y} \frac{|D^\beta v(x) - D^\beta v(y)|}{|x-y|^\alpha},$$

$$\sup_{r_{x,y}>1} \sum_{|\beta|=m} r^{\mu_2+m+\alpha}_{x,y} \frac{|D^\beta v(x) - D^\beta v(y)|}{|x-y|^\alpha}\Big\},$$

$$|||v|||^{(\mu_1,\mu_2)}_{m,\alpha;D} \equiv \sum_{0\le i\le m} [[v]]^{(\mu_1,\mu_2)}_{i,0;D} + [[v]]^{(\mu_1,\mu_2)}_{m,\alpha;D}.$$

Now let's consider the equation

$$\begin{gathered} \Delta w = \hat f \quad \text{in } \tilde\Omega \\ w(z_1,0,z_3) = w(z_1,1,z_3) = 0, \\ w(z_1,z_2,z_3+1) = w(z_1,z_2,z_3), \\ \lim_{z_1\to-\infty} w(z) = \lim_{z_1\to+\infty} w(z) = 0, \end{gathered} \tag{3.50}$$

where $\hat f \in H^{(\delta_0)}_{3,\alpha}(\tilde\Omega)$ with $\hat f(z_1,z_2,z_3+1) = \hat f(z_1,z_2,z_3)$.

Lemma 3.3. *If $w \in H^{(\delta_0)}_{5,\alpha}(\tilde\Omega)$ is a solution of (3.50), which satisfies*

$$\sup_{z\in\tilde\Omega}(e^{\delta_0|z_1|}|w(z)|) \le C|\hat f|^{(\delta_0)}_{3,\alpha;\tilde\Omega}, \tag{3.51}$$

then we have

$$|w|^{(\delta_0)}_{5,\alpha;\tilde\Omega} \le C|\hat f|^{(\delta_0)}_{3,\alpha;\tilde\Omega}.$$

Proof. First, we introduce a coordinate transformation:

$$\begin{gathered} y_1 = e^{z_1}\cos(2\pi z_3)\sin(\frac{\pi z_2}{4} + \frac{\pi}{8}), \quad y_2 = e^{z_1}\sin(2\pi z_3)\sin(\frac{\pi z_2}{4} + \frac{\pi}{8}), \\ y_3 = e^{z_1}\cos(\frac{\pi z_2}{4} + \frac{\pi}{8}). \end{gathered} \tag{3.52}$$

In this case, the strip domain $\tilde\Omega$ is changed into an unbounded domain D which is bounded by two infinitely long cones $\{y : y_3 = \cot\frac{\pi}{8}\sqrt{y_1^2+y_2^2}\}$ and $\{y : y_3 = \cot\frac{3\pi}{8}\sqrt{y_1^2+y_2^2}\}$.

It follows from the transformation (3.52) that (3.50) can be changed into the problem

$$\begin{gathered} \sum_{i,j=1}^3 \tilde a_{ij}(y)\partial_{ij}w + \sum_{i=1}^3 \tilde b_i(y)\partial_i w = F(y) \\ \text{in } D \equiv \{y : \cot\frac{3\pi}{8}\sqrt{y_1^2+y_2^2} < y_3 < \cot\frac{\pi}{8}\sqrt{y_1^2+y_2^2}\}, \\ w(y_1,y_2,y_3) = 0 \quad \text{on } cot\frac{3\pi}{8}\sqrt{y_1^2+y_2^2} = y_3, \\ w(y_1,y_2,y_3) = 0 \quad \text{on } cot\frac{\pi}{8}\sqrt{y_1^2+y_2^2} = y_3, \\ w(0,0,0) = \lim_{r\to+\infty} w(z_1,z_2,z_3) = 0, \end{gathered} \tag{3.53}$$

where

$$\tilde{a}_{11}=\frac{1}{|y|^2}\Big(y_1^2+(2\pi)^2y_2^2+(\frac{\pi}{4})^2\frac{y_1^2y_3^2}{y_1^2+y_2^2}\Big),$$
$$\tilde{a}_{22}=\frac{1}{|y|^2}\Big(y_2^2+(2\pi)^2y_1^2+(\frac{\pi}{4})^2\frac{y_2^2y_3^2}{y_1^2+y_2^2}\Big),$$
$$\tilde{a}_{33}=\frac{1}{|y|^2}\Big(y_3^2+(\frac{\pi}{4})^2(y_1^2+y_2^2)\Big),\quad \tilde{a}_{12}=\frac{y_1y_2}{|y|^2}\Big(1+(\frac{\pi}{4})^2\frac{y_3^2}{y_1^2+y_2^2}-(2\pi)^2\Big),$$
$$\tilde{a}_{13}=\frac{y_1y_3}{|y|^2}\big(1-(\frac{\pi}{4})^2\big),\quad \tilde{a}_{23}=\frac{y_2y_3}{|y|^2}\big(1-(\frac{\pi}{4})^2\big),$$
$$\tilde{b}_1=\frac{y_1}{|y|^2}\big(1-(\frac{\pi}{4})^2-(2\pi)^2\big),\quad \tilde{b}_2=\frac{y_2}{|y|^2}\big(1-(\frac{\pi}{4})^2-(2\pi)^2\big),$$
$$\tilde{b}_3=\frac{y_3}{|y|^2}\big(1-(\frac{\pi}{4})^2\big),\quad F=\frac{1}{|y|^2}\hat{f}.$$

In addition, from (3.51) and the transformation (3.52) it follows that

$$\sup|y|^{\delta_0}|w|\le C|\hat{f}|^{(\delta_0)}_{3,\alpha;\tilde{\Omega}}. \tag{3.54}$$

Next, we show the estimate

$$|w\|^{(\delta_0)}_{5,\alpha;D}\le C|\hat{f}|^{(\delta_0)}_{3,\alpha;\tilde{\Omega}}. \tag{3.55}$$

To this end, for any fixed point $y_0=(y_1^0,y_2^0,y_3^0)\in D$, we set $d_0=\mu|y_0|$ with $0<\mu<<1$ and $B_{d_0}(y_0)\equiv B(y_0,d_0)$, and define the map $T:B_{d_0}(y_0)\to B_1(0)$ by $T(y)=\frac{y-y_0}{d_0}$ for $y\in B_{d_0}(y_0)$. In order to estimate $w(y)$ in D, we distinguish two cases:

(i) $B_{d_0}(y_0)\subset\subset D$, and
(ii) $B_{d_0}(y_0)\bigcap\partial D\neq\emptyset$.

In case (i), set $\tilde{w}(x)=\frac{1}{d_0}w(y_0+d_0x)$ for $x\in B_1(0)$, then it follows from a direct computation that $\tilde{w}(x)$ satisfies

$$\begin{aligned}&\sum_{i,j=1}^{2}\tilde{a}_{ij}(y_0+d_0x)\partial_{ij}\tilde{w}(x)+\sum_{i=1}^{2}d_0\tilde{b}_i(y_0+d_0x)\partial_i\tilde{w}(x)\\&=d_0F(y_0+d_0x).\end{aligned} \tag{3.56}$$

By the Schauder interior estimate (for example, see [11, Chapter 6]), one has

$$\|\tilde{w}\|_{5,\alpha;B_{\frac12}(0)}\le C(\|\tilde{w}\|_{0;B_1(0)}+\|d_0F\|_{3,\alpha;B_1(0)}). \tag{3.57}$$

where the positive constant C depends only on α.

For $y\in B_{\frac{d_0}{2}}(y_0)$, then $(\frac{1}{\mu}-\frac12)d_0\le|y|\le(\frac12+\frac{1}{\mu})d_0$, and (3.57) implies

$$\begin{aligned}&\sum_{m=1}^{5}|y|^{m+\delta_0}|D_y^mw|\\&\le C(\frac12+\frac1\mu)^{1+\delta_0}\big(d_0^{\delta_0}|w|_{0;B_{d_0}(y_0)}+\sum_{m=1}^{3}d_0^{2+m+\delta_0}|D_y^mF(y)|_{0;B_{d_0}(y_0)}\\&\quad+d_0^{5+\delta_0}[D_y^3F(y)]_{0,\alpha;B_{d_0}(y_0)}\big).\end{aligned} \tag{3.58}$$

Combining (3.54) with (3.58) and noting

$$|||F(y)|||^{(2-\delta_0,2+\delta_0)}_{3,\alpha;D} \le |||\hat{f}(y)|||^{(-\delta_0,\delta_0)}_{3,\alpha;D}$$

yield

$$\begin{aligned} |w\|^{(\delta_0)}_{5,0;B_{\frac{d_0}{2}}(y_0)} &\le C(|\hat{f}|^{(\delta_0)}_{3,\alpha;\tilde{\Omega}} + |||F(y)|||^{(2-\delta_0,2+\delta_0)}_{3,\alpha;D}) \\ &\le C(|\hat{f}|^{(\delta_0)}_{3,\alpha;\tilde{\Omega}} + |||\hat{f}(y)|||^{(-\delta_0,\delta_0)}_{3,\alpha;D}). \end{aligned} \tag{3.59}$$

In case (ii), set $\tilde{w}(x) = \frac{1}{d_0} w(y_0 + d_0 x)$ for $x \in M \equiv T(B_{d_0}(y_0) \bigcap D)$. As in Case (i), but it follows from the Schauder boundary estimate that

$$\|\tilde{w}\|_{5,\alpha;B_{\frac{1}{2}(0)} \bigcap M} \le C(\|\tilde{w}\|_{0;M} + \|d_0 F\|_{3,\alpha;M}). \tag{3.60}$$

Similar to (3.57) and (3.58), we can arrive at

$$|w\|^{(\delta_0)}_{5,0;B_{\frac{d_0}{2}}(y_0) \bigcap D} \le C(|\hat{f}|^{(\delta_0)}_{3,\alpha;\tilde{\Omega}} + |||\hat{f}(y)|||^{(-\delta_0,\delta_0)}_{3,\alpha;D}). \tag{3.61}$$

Therefore, by (3.59) and (3.61), we have

$$|w\|_{5,0;D} \le C(|\hat{f}|^{(\delta_0)}_{3,\alpha;\tilde{\Omega}} + |||\hat{f}(y)|||^{(-\delta_0,\delta_0)}_{3,\alpha;D}). \tag{3.62}$$

Next, we estimate $[D^5 w]]^{(\delta_0)}_{0,\alpha;D}$. Let y, y' be distinct points in D with $|y| \le |y'|$. We now consider the following two cases:

(a) $\text{dist}(y, y') \le \frac{d}{2}$;

(b) $\text{dist}(y, y') > \frac{d}{2}$, here $d = \mu|y|$.

In case (a), (3.54), (3.57) and (3.60) imply

$$\begin{aligned} |y|^{5+\delta_0+\alpha} \frac{|D^5 w(y) - D^5 w(y')|}{|y-y'|^\alpha} &\le C(d^{\delta_0}|w|_{0;B_d(y)} + d^{2+\delta_0}\|F\|_{3,\alpha;B_d(y)}) \\ &\le C(|\hat{f}|^{(\delta_0)}_{3,\alpha;\tilde{\Omega}} + |||\hat{f}(y)|||^{(-\delta_0,\delta_0)}_{3,\alpha;D}). \end{aligned} \tag{3.63}$$

In case (b), (3.62) implies

$$\begin{aligned} &|y|^{5+\delta_0+\alpha} \frac{|D^5 w(y) - D^5 w(y')|}{|y-y'|^\alpha} \\ &\le C d^{5+\delta_0}(|D^5 w(y)| + |D^5 w(y')|) \\ &\le C(\frac{1}{2} + \frac{1}{\mu})^{-1}(|y|^{5+\delta_0}|D^5 w(y)| + |y'|^{5+\delta_0}|D^5 w(y')|) \\ &\le C(|\hat{f}|^{(\delta_0)}_{3,\alpha;\tilde{\Omega}} + |||\hat{f}(y)|||^{(-\delta_0,\delta_0)}_{3,\alpha;D}). \end{aligned} \tag{3.64}$$

Taking the supremum with respect to y and y' in (3.63) and (3.64) respectively, we obtain

$$[D^5 w]]^{(\delta_0)}_{0,\alpha;D} \le C(|\hat{f}|^{(\delta_0)}_{3,\alpha;\tilde{\Omega}} + |||\hat{f}(y)|||^{(-\delta_0,\delta_0)}_{3,\alpha;D}). \tag{3.65}$$

Next we show that

$$|||\hat{f}(y)|||^{(-\delta_0,\delta_0)}_{3,\alpha;D} \le C|\hat{f}|^{(\delta_0)}_{3,\alpha;\tilde{\Omega}}. \tag{3.66}$$

In fact, by using a direct computation, we can arrive at If $z_1 > 0$, namely, $|y| > 1$,

$$\begin{gathered} e^{\delta_0|z_1|}|\hat{f}(z)| = |y|^{\delta_0}|\hat{f}(y)|, \\ e^{\delta_0|z_1|}\sum_{k=1}^{3}|D_z^k \hat{f}(z)| \sim \sum_{k=1}^{3}|y|^{k+\delta_0}|D_y^k \hat{f}(y)|. \end{gathered} \tag{3.67}$$

If $z_1 < 0$, namely, $|y| < 1$,

$$\begin{gathered} e^{\delta_0 |z_1|} |\hat{f}(z)| = |y|^{-\delta_0} |\hat{f}(y)|, \\ e^{\delta_0 |z_1|} \sum_{k=1}^{3} |D_z^k \hat{f}(z)| \sim \sum_{k=1}^{3} |y|^{k-\delta_0} |D_y^k \hat{f}(y)|. \end{gathered} \tag{3.68}$$

In addition, for $z, \tilde{z} \in \tilde{\Omega}$, we have

$$\begin{aligned} & e^{\delta_0 \min\{|z_1|,|\tilde{z}_1|\}} \frac{|D_z^3 \hat{f}(z) - D_z^3 \hat{f}(\tilde{z})|}{|z-\tilde{z}|^\alpha} \\ & \sim \sup_{y \in D; |y|<1} \sum_{k=1}^{3} |y|^{k-\delta_0} |D_y^k \hat{f}(y)| + \sup_{y \in D; |y|>1} \sum_{k=1}^{3} |y|^{k+\delta_0} |D_y^k \hat{f}(y)| \\ & \quad + \sup_{0<d_{y,\tilde{y}}<1} d_{y,\tilde{y}}^{3+\alpha-\delta_0} \frac{|D_y^3 \hat{f}(y) - D_y^3 \hat{f}(\tilde{y})|}{|y-\tilde{y}|^\alpha} \\ & \quad + \sup_{d_{y,\tilde{y}}>1} d_{y,\tilde{y}}^{3+\alpha+\delta_0} \frac{|D_y^3 \hat{f}(y) - D_y^3 \hat{f}(\tilde{y})|}{|y-\tilde{y}|^\alpha}, \end{aligned} \tag{3.69}$$

here $|y| = e^z$, $|\tilde{y}| = e^{\tilde{z}}$ and

$$d_{y,\tilde{y}} = \begin{cases} \max(|y|,|\tilde{y}|) & \text{if } \min(|y|,|\tilde{y}|) < \min(|y|^{-1},|\tilde{y}|^{-1}) \\ \min(|y|,|\tilde{y}|) & \text{if } \min(|y|,|\tilde{y}|) \geq \min(|y|^{-1},|\tilde{y}|^{-1}) \end{cases}$$

Therefore, combining (3.67)-(3.68) with (3.69) and noting $d_{y,\tilde{y}} \geq r_{y,\tilde{y}}$ yield (3.66).

Substituting (3.66) into (3.65), we obtain

$$[D^5 w]]_{0,\alpha;D}^{(\delta_0)} \leq C|\hat{f}|_{3,\alpha;\tilde{\Omega}}^{(\delta_0)}. \tag{3.70}$$

Returning to the coordinate $z = (z_1, z_2, z_3)$ for $[D^5 w]]_{0,\alpha;D}^{(\delta_0)}$, we can derive

$$\begin{aligned} & \sum_{|\beta|\leq 5} \sup_{z\in\tilde{\Omega}} e^{\delta_0 z_1} |D^\beta w(z)| + \sum_{|\beta|=5} \sup_{z,\tilde{z}\in\tilde{\Omega}; z\neq\tilde{z}} e^{\delta_0 \min(z_1,\tilde{z}_1)} \frac{|D_z^\beta w(z) - D_z^\beta w(\tilde{z})|}{|z-\tilde{z}|^\alpha} \\ & \leq C|\hat{f}|_{3,\alpha;\tilde{\Omega}}^{(\delta_0)}. \end{aligned} \tag{3.71}$$

On the other hand, if we introduce the coordinate transformation:

$$\begin{gathered} \tilde{y}_1 = e^{-z_1} \cos(2\pi z_3) \sin(\frac{\pi z_2}{4} + \theta_0), \quad \tilde{y}_2 = e^{-z_1} \sin(2\pi z_3) \sin(\frac{\pi z_2}{4} + \theta_0), \\ \tilde{y}_3 = e^{-z_1} \cos(\frac{\pi z_2}{4} + \theta_0), \end{gathered} \tag{3.72}$$

then by using the same method to deduce (3.71), we can arrive at

$$\begin{aligned} & \sum_{|\beta|\leq 5} \sup_{z\in\tilde{\Omega}} e^{\delta_0 z_1} |D^\beta w(z)| + \sum_{|\beta|=5} \sup_{z,\tilde{z}\in\tilde{\Omega}; z\neq\tilde{z}} e^{\delta_0 \max(z_1,\tilde{z}_1)} \frac{|D_z^\beta w(z) - D_z^\beta w(\tilde{z})|}{|z-\tilde{z}|^\alpha} \\ & \leq C|\hat{f}|_{3,\alpha;\tilde{\Omega}}^{(\delta_0)}. \end{aligned} \tag{3.73}$$

Combining (3.71) with (3.73) yields

$$[w]_{i,0;\tilde{\Omega}}^{(\delta_0)} \leq C|\hat{f}|_{3,\alpha;\tilde{\Omega}}^{(\delta_0)}, \quad i = 1, 2, \dots, 5; \tag{3.74}$$

$$\sup_{z_1\tilde{z}_1>0} e^{\delta_0 \min\{|z_1|,|\tilde{z}_1|\}} \frac{|D^5 w(z) - D^5 w(\tilde{z})|}{|z-\tilde{z}|^\alpha} \leq C|\hat{f}|_{3,\alpha;\tilde{\Omega}}^{(\delta_0)}. \tag{3.75}$$

Next we consider the case $z_1\tilde z_1<0$ in (3.75). Without loss of generality, we assume $z_1<0<\tilde z_1$. Moreover, we consider the following two cases:

(I) $\tilde z_1-z_1\ge 1$,
(II) $\tilde z_1-z_1<1$.

For case (I), we have

$$e^{\delta_0\min\{-z_1,\tilde z_1\}}\frac{|D^5w(z)-D^5w(\tilde z)|}{|z-\tilde z|^\alpha}\le e^{-\delta_0 z_1}|D^5w(z)|+e^{\delta_0\tilde z_1}|D^5w(\tilde z)|\le C|\hat f|^{(\delta_0)}_{3,\alpha;\tilde\Omega}.$$

For case (II), it follows from (3.71) and (3.73) that

$$e^{\delta_0\min\{-z_1,\tilde z_1\}}\frac{|D^5w(z)-D^5w(\tilde z)|}{|z-\tilde z|^\alpha}\le e^{2\delta_0}\cdot e^{-\delta_0 z_1}\frac{|D^5w(z)-D^5w(\tilde z)|}{|z-\tilde z|^\alpha}\le C|\hat f|^{(\delta_0)}_{3,\alpha;\tilde\Omega}.$$

Thus, we have proved that

$$[w]^{(\delta_0)}_{5,\alpha;\tilde\Omega}\le C|\hat f|^{(\delta_0)}_{3,\alpha;\tilde\Omega}. \tag{3.76}$$

Namely, by using (3.74) and (3.76), the proof is complete. □

Based on Lemmas 3.2 and 3.3. we now give the estimate of $\|u\|^{(\delta_0)}_{6,\alpha;\tilde\Omega}$, for the solution of (3.3).

Lemma 3.4. *Under the assumptions of Lemma 3.2, the solution $u(z)$ of (3.3) satisfies the estimate*

$$\|u\|^{(\delta_0)}_{6,\alpha;\tilde\Omega}\le C(|\tilde f|^{(\delta_0)}_{4,\alpha;\tilde\Omega}+\sum_{i=1}^2|\tilde g_i|^{(\delta_0)}_{5,\alpha;\tilde\Omega}). \tag{3.77}$$

Proof. To prove (3.77), by using Lemma 3.2, it only suffices to prove

$$\begin{gathered}|\partial_{z_k}u|^{(\delta_0)}_{5,\alpha;\tilde\Omega}\le C(|\tilde f|^{(\delta_0)}_{4,\alpha;\tilde\Omega}+\sum_{i=1}^2|\tilde g_i|^{(\delta_0)}_{5,\alpha;\tilde\Omega}),\quad k=2,3,\\ |\partial^2_{z_1}u|^{(\delta_0)}_{4,\alpha;\tilde\Omega}\le C(|\tilde f|^{(\delta_0)}_{4,\alpha;\tilde\Omega}+\sum_{i=1}^2|\tilde g_i|^{(\delta_0)}_{5,\alpha;\tilde\Omega}).\end{gathered} \tag{3.78}$$

Set $w(z)=\partial_{z_2}v$, where $v(z)$ is a solution of (3.10). Then it is easy to know that $w(z)$ satisfies the equation (3.50) with $\hat f(z)=\partial_{z_2}f(z)$. Therefore, by Lemma 3.3, we have

$$|\partial_{z_2}v|^{(\delta_0)}_{5,\alpha;\tilde\Omega}\le C|\hat f|^{(\delta_0)}_{3,\alpha;\tilde\Omega}\le C|f|^{(\delta_0)}_{4,\alpha;\tilde\Omega}.$$

Due to $u(z)=v(z)+h(z)$ with $h(z)=\frac12(\tilde g_2(z_1,z_3)-\tilde g_1(z_1,z_3))z_2^2+\tilde g_1(z_1,z_3)z_2$, then

$$|\partial_{z_2}u|^{(\delta_0)}_{5,\alpha;\tilde\Omega}\le C(|\tilde f|^{(\delta_0)}_{4,\alpha;\tilde\Omega}+\sum_{i=1}^2|\tilde g_i|^{(\delta_0)}_{5,\alpha;\tilde\Omega}). \tag{3.79}$$

By using similar method to the one in Lemma 3.3 (compared with the problem (3.50), $\partial_{z_3}v$ will satisfy the same equation which admits two Neumann boundary conditions on $y_2=0$ and y_2=1 instead of the Dirichlet boundary conditions of (3.50) and the same restrictions in (3.50) as $z_1\to\pm\infty$), we can arrive at

$$|\partial_{z_3}v|^{(\delta_0)}_{5,\alpha;\tilde\Omega}\le C|\partial_{z_3}f|^{(\delta_0)}_{3,\alpha;\tilde\Omega}\le C|f|^{(\delta_0)}_{4,\alpha;\tilde\Omega}$$

and further

$$|\partial_{z_3} u|^{(\delta_0)}_{5,\alpha;\tilde\Omega} \le C(|\tilde f|^{(\delta_0)}_{4,\alpha;\tilde\Omega} + \sum_{i=1}^{2} |\tilde g_i|^{(\delta_0)}_{5,\alpha;\tilde\Omega}). \tag{3.80}$$

On the other hand, substituting (3.79)-(3.80) into the equation (3.3) yields

$$|\partial^2_{z_1} u|^{(\delta_0)}_{4,\alpha;\tilde\Omega} \le C(|\tilde f|^{(\delta_0)}_{4,\alpha;\tilde\Omega} + \sum_{i=1}^{2} |\tilde g_i|^{(\delta_0)}_{5,\alpha;\tilde\Omega}).$$

The proof is complete. □

Based on Lemma 3.4, we now derive uniform estimates on the solution $\dot u(z)$ to problem (3.1).

Lemma 3.5. *Suppose that the assumption* (3.2) *holds, and $\dot u \in C^2(\overline{\tilde\Omega})$ is a solution of* (3.1)*. Then there exists a positive constant δ_0 such that for any $\dot f \in H^{(\delta_0)}_{4,\alpha}(\tilde\Omega)$, $\dot g_i \in H^{(\delta_0)}_{5,\alpha}(\tilde\Omega)(i=1,2)$, we have $\dot u \in \mathbb{H}^{(\delta_0)}_{6,\alpha}(\tilde\Omega)$ with*

$$\|\dot u\|^{(\delta_0)}_{6,\alpha;\tilde\Omega} \le C(|\dot f|^{(\delta_0)}_{4,\alpha;\tilde\Omega} + \sum_{i=1}^{2} |\dot g_i|^{(\delta_0)}_{5,\alpha;\tilde\Omega}), \tag{3.81}$$

where $C > 0$ depends only on the constants Λ and λ in (3.2).

Proof. Firstly, we introduce the coordinate transformation

$$\tilde z_1 = k_1 z_1, \quad \tilde z_2 = k_2 z_2, \quad \tilde z_3 = k_3 z_3 \tag{3.82}$$

with $k_1 = \frac{1}{\sqrt{c^2(\rho_0)-q_0^2}}$ and $k_2 = k_3 = \frac{1}{c(\rho_0)}$. Under this transformation, the domain $\tilde\Omega$ is changed into the domain $Q \equiv (-\infty,+\infty)\times[0,\frac{1}{c(\rho_0)}]\times(-\infty,+\infty)$, and the equation (3.1) can be rewritten as

$$\begin{gathered}
\Delta \dot u = \bar f \quad \text{in } Q,\\
\partial_{\tilde z_2}\dot u = \tilde g_1 \quad \text{on } \tilde z_2 = 0,\\
\partial_{\tilde z_2}\dot u = \tilde g_2 \quad \text{on } \tilde z_2 = l,\\
\dot u(\tilde z_1,\tilde z_2,\tilde z_3 + l) = \dot u(\tilde z_1,\tilde z_2,\tilde z_3),\\
\lim_{\tilde z_1\to-\infty} \dot u = 0,\\
\lim_{\tilde z_1\to+\infty} \nabla_{\tilde z}\dot u \text{ exists}.
\end{gathered} \tag{3.83}$$

where $l = 1/c(\rho_0)$, and

$$\begin{gathered}
\bar f = \dot f + \sum_{i=1}^{3}(1 - k_i^2(c^2(\nabla v) - \partial^2_{z_1} v))\partial^2_{\tilde z_i}\dot u - 2\sum_{1\le i<j\le 3} k_i k_j \partial_{z_i} v \partial_{z_j} v \partial_{\tilde z_i \tilde z_j}\dot u,\\
\tilde g_i = c(\rho_0)\dot g, \quad i = 1,2.
\end{gathered} \tag{3.84}$$

For simplicity and without loss of generality, we assume $l = 1$ in (3.83). By the assumption $\|v - q_0 z_1\|^{(\delta_0)}_{6,\alpha;\tilde\Omega} < \varepsilon$ and Lemma 2.1, we have

$$|\bar f|^{(\delta_0)}_{4,\alpha;\tilde\Omega} \le O(\varepsilon)\|\dot u\|^{(\delta_0)}_{6,\alpha;\tilde\Omega} + |\dot f|^{(\delta_0)}_{4,\alpha;\tilde\Omega}. \tag{3.85}$$

On the other hand, by using Lemma 3.4, one has

$$\|\dot{u}\|^{(\delta_0)}_{6,\alpha;\tilde{\Omega}} \le C(|\bar{f}|^{(\delta_0)}_{4,\alpha;\tilde{\Omega}} + \sum_{i=1}^{2} |\dot{g}_i|^{(\delta_0)}_{5,\alpha;\tilde{\Omega}}). \tag{3.86}$$

Substituting (3.84) into (3.85) yields (3.81).

Moreover, from (3.48) and (3.49) it follows that

$$\begin{gathered} \lim_{z_1\to+\infty} |\partial_{z_1}\dot{u}| \le C(|\dot{f}|^{(\delta_0)}_{4,\alpha;\tilde{\Omega}} + \sum_{i=1}^{2} |\dot{g}_i|^{(\delta_0)}_{5,\alpha;\tilde{\Omega}}), \\ \lim_{z_1\to+\infty} \partial_{z_i}\dot{u} = 0, \quad i = 2,3. \end{gathered} \tag{3.87}$$

Therefore, the proof is complete. □

Based on Lemmas 3.2 and 3.5, from the standard continuity method (see [11, Theorem 5.2]) we have the following result.

Theorem 3.6. *There exists a unique solution $\dot{u} \in \mathbb{H}^{(\delta_0)}_{6,\alpha}(\tilde{\Omega})$ to problem (3.1) for some $\delta_0 > 0$, which admits the following estimate*

$$\|\dot{u}\|^{(\delta_0)}_{6,\alpha;\tilde{\Omega}} \le C\big(|\dot{f}|^{(\delta_0)}_{4,\alpha;\tilde{\Omega}} + \sum_{i=1}^{2} |\dot{g}_i|^{(\delta_0)}_{5,\alpha;\tilde{\Omega}}\big). \tag{3.88}$$

4. Proofs of Theorems 1.1, 2.2, and 2.4

In this section, first we use the contraction mapping principle to show Theorem 2.4. To this end, we define the space $K = \{\psi(z) : \psi(z) - \varphi_0(z) \in \mathbb{H}^{(\delta_0)}_{6,\alpha}(\tilde{\Omega}), \psi(z_1, z_2, z_3 + 1) = \psi(z_1, z_2, z_3), \|\psi(z) - \varphi_0(z)\|^{(\delta_0)}_{6,\alpha;\tilde{\Omega}} \le \varepsilon\}$ with $\varphi_0(z) = q_0 z_1$.

Set $\varphi = \dot{\varphi} + \varphi_0$, then $\dot{\varphi}$ satisfies

$$\begin{gathered} L(\psi)\dot{\varphi} = \sum_{i,j=1}^{3} a_{ij}(z, D\psi)\partial^2_{z_i z_j}\dot{\varphi} = \dot{f}(z, D\psi, D^2\psi) \quad \text{in } \tilde{\Omega}, \\ G_1(\psi)\dot{\varphi} = \partial_{z_2}\dot{\varphi} = \dot{g}_1(z, D\psi) \quad \text{on } z_2 = 0, \\ G_2(\psi)\dot{\varphi} = \partial_{z_2}\dot{\varphi} = \dot{g}_2(z, D\psi) \quad z_2 = 1, \\ \dot{\varphi}(z_1, z_2, z_3 + 1) = \dot{\varphi}(z_1, z_2, z_3), \\ \lim_{z_1\to-\infty} \dot{\varphi} = 0, \\ \lim_{z_1\to+\infty} \nabla_z\dot{\varphi} \text{ exists}, \end{gathered} \tag{4.1}$$

where

$$\begin{aligned} \dot{f}(z, D\psi, D^2\psi) &= (L(\varphi_0)\varphi_0 - L(\psi)\varphi_0) \\ &\quad + \sum_{i,j=1}^{3} (a_{ij}(z, \nabla\psi) - A_{ij}(z, \nabla\psi))\partial_{z_i z_j}\dot{\psi} - B(z, \nabla\psi)\partial_{z_2}\dot{\psi}, \\ \dot{g}_i(D\psi) &= -b_{i1}(z)\partial_{z_1}\dot{\psi} - b_{i3}(z)\partial_{z_3}\dot{\psi}, \quad i = 1, 2 \end{aligned}$$

with $\dot{\psi} = \psi - \varphi_0$.

Define the nonlinear mapping J by $J(\psi) = \varphi$.

Lemma 4.1. *Suppose that α and δ_0 are the positive constants given in Lemma 3.5. Then there exists an $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$, J is a mapping from K to itself.*

Proof. By the definitions of $A_{ij}(z, \nabla\psi)$ and $B(z, \nabla\psi)$ in (2.2), we arrived at

$$\begin{aligned}
&|(a_{ij}(z, \nabla\psi) - A_{ij}(z, \nabla\psi))\partial_{z_i z_j}\dot{\psi}|^{(\delta_0)}_{4,\alpha;\tilde{\Omega}} \\
&\le |a_{ij}(z, \nabla\psi) - A_{ij}(z, \nabla\psi)|^{(0)}_{4,\alpha;\tilde{\Omega}} \|\dot{\psi}\|^{(\delta_0)}_{6,\alpha;\tilde{\Omega}} \le C\varepsilon^2, \\
&|B(z, \nabla\psi)\partial_{z_2}\dot{\psi}|^{(\delta_0)}_{4,\alpha;\tilde{\Omega}} \le |B(z, \nabla\psi)|^{(0)}_{4,\alpha;\tilde{\Omega}} \|\dot{\psi}\|^{(\delta_0)}_{6,\alpha;\tilde{\Omega}} \le C\varepsilon^2.
\end{aligned}$$

In addition, by using $\varphi_0 = q_0 z_1$, we have

$$L(\varphi_0)\varphi_0 - L(\psi)\varphi_0 = 0.$$

Thus, we arrive at

$$|\dot{f}|^{(\delta_0)}_{4,\alpha;\tilde{\Omega}} \le C\varepsilon^2. \tag{4.2}$$

Analogously, one has

$$|\dot{g}_i|^{(\delta_0)}_{5,\alpha;\tilde{\Omega}} \le C\varepsilon^2, \quad i = 1, 2. \tag{4.3}$$

It follows from Theorem 3.6 that

$$\|\dot{\varphi}\|^{(\delta_0)}_{6,\alpha;\tilde{\Omega}} \le C(|\dot{f}|^{(\delta_0)}_{4,\alpha;\tilde{\Omega}} + \sum_{i=1}^{2} |\dot{g}_i|^{(\delta_0)}_{5,\alpha;\tilde{\Omega}}) \le C\varepsilon^2, \tag{4.4}$$

where $C > 0$ depends only on Λ, λ.

Choose $\varepsilon_0 = \frac{1}{2C}$, then for any $0 < \sigma < \varepsilon < \varepsilon_0$, by (4.4) we obtain

$$\|\dot{\varphi}\|^{(\delta_0)}_{6,\alpha;\tilde{\Omega}} < \varepsilon. \tag{4.5}$$

This means that the mapping J is from K into itself. □

Next we show that the mapping J defined above is contractible.

Lemma 4.2. *Under the assumptions of Lemma 4.1, the mapping J is a contractible mapping from K to itself.*

Proof. Take $\psi_1, \psi_2 \in K$. Let $\varphi_i = J\psi_i$ and $\dot{\varphi}_i = \varphi_i - \varphi_0$, then we have

$$\begin{gathered}
L(\psi_2)(\varphi_2 - \varphi_1) = \dot{f}(z, D\psi_2, D^2\psi_2) - \dot{f}(z, D\psi_1, D^2\psi_1) - (L(\psi_2) - L(\psi_1))\dot{\varphi}_1 \quad \text{in } \tilde{\Omega}, \\
\partial_{z_2}(\varphi_2 - \varphi_1) = \dot{g}_1(z, \psi_2) - \dot{g}_1(z, \psi_1) \quad \text{on } z_2 = 0, \\
\partial_{z_2}(\varphi_2 - \varphi_1) = \dot{g}_2(z, \psi_2) - \dot{g}_2(z, \psi_1) \quad \text{on } \ z_2 = 1, \\
(\varphi_2 - \varphi_1)(z_1, z_2, z_3 + 1) = (\varphi_2 - \varphi_1)(z_1, z_2, z_3), \\
\lim_{z_1 \to -\infty} (\varphi_2 - \varphi_1) = 0, \\
\lim_{z_1 \to +\infty} \nabla_z(\varphi_2 - \varphi_1) \text{ exists.}
\end{gathered} \tag{4.6}$$

As in Lemma 4.1, a direct computation yields

$$\begin{gathered}
|\dot{f}(z, D\psi_2, D^2\psi_2) - \dot{f}(z, D\psi_1, D^2\psi_1)|^{(\delta_0)}_{4,\alpha;\tilde{\Omega}} \le C\varepsilon\|\psi_2 - \psi_1\|^{(\delta_0)}_{6,\alpha;\tilde{\Omega}}, \\
|\dot{g}_i(z, \psi_2) - \dot{g}_i(z, \psi_1)|^{(\delta_0)}_{5,\alpha;\tilde{\Omega}} \le C\varepsilon\|\psi_2 - \psi_1\|^{(\delta_0)}_{6,\alpha;\tilde{\Omega}}, \quad i = 1, 2; \\
|(L(\psi_2) - L(\psi_1))\dot{\varphi}_1|^{(\delta_0)}_{4,\alpha;\tilde{\Omega}} \le C\varepsilon\|\psi_2 - \psi_1\|^{(\delta_0)}_{6,\alpha;\tilde{\Omega}}.
\end{gathered}$$

It follows from Theorem 3.6 that
$$\|\varphi_2-\varphi_1\|^{(\delta_0)}_{6,\alpha;\tilde\Omega}\le C\varepsilon\|\psi_2-\psi_1\|^{(\delta_0)}_{6,\alpha;\tilde\Omega}.$$
Choosing appropriately small ε_0 and letting $0<\varepsilon<\varepsilon_0$ yields
$$\|J(\psi_2)-J(\psi_1)\|^{(\delta_0)}_{6,\alpha;\tilde\Omega}\le \frac{1}{2}\|\psi_2-\psi_1\|^{(\delta_0)}_{6,\alpha;\tilde\Omega}.$$
This means that J is a contractible mapping. □

Based on Lemma 4.1 and Lemma 4.2, we now show Theorem 2.2.

Proof of Theorem 2.4. By Lemmas 4.1 and 4.2, we know that the mapping $J\psi=\varphi$ has a unique fixed point in $\mathbb{H}^{(\delta_0)}_{6,\alpha}(\tilde\Omega)$.

Next, we show $\lim_{z_1\to+\infty}\nabla_z\varphi(z)$ exists. Since for $Z_1>Z_2>0$, we have
$$\begin{aligned}
&|\partial_{z_1}\varphi(Z_1,z_2,z_3)-\partial_{z_1}\varphi(Z_2,z_2,z_3)|\\
&=(Z_1-Z_2)|\int_0^1\partial^2_{z_1}\varphi(\theta Z_1+(1-\theta)Z_2,z_2,z_3)d\theta|\\
&\le C(Z_1-Z_2)|\int_0^1 e^{-\delta_0(\theta Z_1+(1-\theta)Z_2)}d\theta|\le Ce^{-\delta_0 Z_2}.
\end{aligned}$$
This means that there exists a function $q(z_2,z_3)$ such that $\partial_{z_1}\varphi(z_1,z_2,z_3)$ converges to $q(z_2,z_3)$ uniformly as $z_1\to+\infty$. On the other hand, $|\partial^2_{z_1z_k}\varphi(z_1,z_2,z_3)|\le Ce^{-\delta_0 z_1}$ for $k=2,3$, this implies that $\partial^2_{z_1z_k}\varphi(z_1,z_2,z_3)$ converges to 0 uniformly as $z_1\to+\infty$. Therefore, we can arrive at $\partial_{z_2}q(z_2,z_3)=\partial_{z_3}q(z_2,z_3)\equiv 0$, namely, $q(z_2,z_3)\equiv q$, here q is a constant which will be determined later on. In addition, $|\partial_{z_k}\varphi(z)|\le Ce^{-\delta_0|z_1|}$ $(k=2,3)$, then $\lim_{z_1\to\pm\infty}\partial_{z_k}\varphi(z)=0$. From the analysis above, we can also obtain under the x-coordinates,
$$\lim_{x_1\to\infty}\partial_{x_1}\varphi=q\quad\text{and}\quad\lim_{x_1\to\pm\infty}\partial_{x_i}\varphi=0\quad\text{for } i=2,3. \tag{4.7}$$
We now show that $q=q_0$ holds. Integrating the mass conservation equation $\sum_{j=1}^3\partial_{x_j}(\rho(|\nabla\varphi|)\partial_{x_j}\varphi)=0$ in $\Omega_R=\Omega\cap\{x:-R\le x_1\le R,0<x_3<1\}$ yields
$$0=-\int_{x_1=-R}\rho(\nabla\varphi)\partial_{x_1}\varphi d\sigma+\int_{x_1=R}\rho(\nabla\varphi)\partial_{x_1}\varphi d\sigma. \tag{4.8}$$
Using (4.7) and letting $R\to+\infty$ in (4.8), we arrive at
$$\rho(q)q=\rho(q_0)q_0. \tag{4.9}$$
On the other hand, it follows from (1.2) that
$$\rho(q)q=(\frac{\gamma-1}{A\gamma})^{\frac{1}{\gamma-1}}(2C_0-q^2)^{\frac{1}{\gamma-1}}q.$$
A direct computation yields
$$(\rho(q)q)'=(\frac{\gamma-1}{A\gamma})^{\frac{1}{\gamma-1}}(2C_0-q^2)^{-\frac{\gamma+1}{\gamma-1}}(2C_0-\frac{\gamma+1}{\gamma-1}q^2). \tag{4.10}$$
In addition, by using $q<c(q_0)$ and (1.2), we have
$$C_0=\frac{1}{2}q^2+\frac{c^2(q)}{\gamma-1}>\frac{1}{2}q^2+\frac{1}{\gamma-1}q^2=\frac{\gamma+1}{2(\gamma-1)}q^2. \tag{4.11}$$
Thus, substituting (4.11) into (4.10) yields
$$(\rho(q)q)'>0. \tag{4.12}$$

Combining (4.9) with (4.12) implies $q = q_0$. Thus, we complete the proof. □

Since the proofs of Theorem 1.1 and 2.2 come directly from Theorem 2.4, then we omit them.

Acknowledgments. Wenxia Chen is supported by the National Natural Science Foundation of China (No.11501253). Gang Xu and Qin Xu are supported by the National Natural Science Foundations of China (No.11571141).

References

[1] H. W. Alt, L. A. Caffarelli, A. Friedman; *Compressible flows of jets and cavities,* J. of differential Equations, 56, no. 1 (1985), 82–141.
[2] H. W. Alt, L. A. Caffarelli, A. Friedman; *Axially symmetric jet flows,* Arch. R. Mech. Anal. 81, no. 2 (1983), 97–149.
[3] C. Amrouche, F. Bonzon; *Exterior problems in the half-space for the Laplace operator in weighted Sobolev spaces,* J. of differential Equations, 246, no. 5 (2009), 1894-1920.
[4] L. Bers; *Existence and uniqueness of a subsonic flow past a given profile,* Comm. Pure Appl. Math. 7 (1954), 441-504.
[5] L. Bers; *Mathematical aspects of subsonic and transonic gas dynamics*, Surveys in Applied Mathematics, Vol.3, John Wiley & Sons, Inc., New York: Chapman & Hall, Ltd., London, 1958.
[6] G. Chen, M. Feldman; *Multidimensional transonic shocks and free boundary problems for nonlinear equations of mixed type*, J. Amer. Math. Soc. 16, no. 3 (2003), 461–494.
[7] J. Chen; *Subsonic flows for the full Euler equations in half plane*, J. Hyperbolic Differ. Equ. 6 , no. 2 (2009), 207–228.
[8] R. Courrant, K. O. Friedrichs; *Supersonic flow and shock waves*, Interscience Publishers Inc., New York, 1948.
[9] G. Dong, B. Ou; *Subsonic flows around a body in space*, Comm. P. D. E. 18 (1993), no. 1-2 (1993), 355–379.
[10] R. Finn, D. Gilbarg; *Three-dimensional subsonic flows and asymptotic estimates for elliptic partial differential equations*, Acta Math. 98 (1957), 265-296.
[11] D. Gilbarg, N. S. Tudinger; *Elliptic partial differential equations of second order*, Second edition. Grundlehren der Mathematischen Wissenschaften, 224, Springer, Berlin-New York, 1998.
[12] P. Grisvard; *Elliptic problems in nonsmooth domains*, Monographs and Studies in Mathematics, 24. Pitman (Advanced Publishing Program), Boston, MA, 1985.
[13] V. A. Kozlov, V. G. Maz'ya, J. Rossman; *Elliptic boundary value problems in domains with point singularities*, Mathematical Survey and Monographs, Vol. 52, Amer. Math. Soc., Providence, RI, 1997.
[14] L. C. Woods; *Compressible subsonic flow in two-dimensional channels I. Basic mathematical theory*, Aero. Quart. 6 (1955), 205–220.
[15] L. C. Woods; *Compressible subsonic flow in two-dimensional channels II. The application of the theory to problems of channel flow*, Aero. Quart. 6 (1955), 254–276.
[16] C. Xie, Z. Xin; *Global subsonic and subsonic-sonic flows through infinitely long nozzles.* Indiana Univ. Math. J. 56, no. 6 (2007), 2991–3023.
[17] G. Xu, H. Yin; *Global transonic conic shock wave for the symmetrically perturbed supersonic flow past a cone*, J. Differetnail Equations, 245, no. 11 (2008), 3389-3432.
[18] G. Xu, H. Yin; *Global multidimensional transonic conic shock wave for the perturbed supersonic flow past a cone*, SIAM J. Math. Anal., 41 , no. 1 (2009), 178-218.

Wenxia Chen
Faculty of Science, Jiangsu University, Zhenjiang, Jiangsu 212013, China
E-mail address: chenwx@ujs.edu.cn

Gang Xu (corresponding author)
Faculty of Science, Jiangsu University, Zhenjiang, Jiangsu 212013, China
E-mail address: gxu@ujs.edu.cn

Qin Xu
Faculty of Science, Jiangsu University, Zhenjiang, Jiangsu 212013, China
E-mail address: qinxu_math@163.com