Electronic Journal of Differential Equations, Vol. 2017 (2017), No. 148, pp. 1–6.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

TRIGONOMETRIC SERIES ADAPTED FOR THE STUDY OF NEUMANN BOUNDARY-VALUE PROBLEMS OF LAMÉ SYSTEMS

BOUBAKEUR MEROUANI, NABIL LIAZIDI

Abstract. In this article, we study the solutions to Neumann boundary-value problems of Lamé system in a sectorial domains. We study directly this problem, by using trigonometric series, without going through the Airy functions. Results using the Airy function are given in [11].

1. Introduction

Let S be the truncated plane sector of angle $\omega \le 2\pi$, and positive radius ρ:

$$S = \{(r\cos\theta, r\sin\theta) \in \mathbb{R}^2, 0 < r < \rho, 0 < \theta < \omega\} \tag{1.1}$$

and Σ the circular boundary part

$$\Sigma = \{(\rho\cos\theta, \rho\sin\theta) \in \mathbb{R}^2, 0 < \theta < \omega\}. \tag{1.2}$$

We are interested in the study of functions u belonging to the Sobolev space $(H^1(S))^2$, and that are solutions to Lamé type system

$$\begin{gathered} Lu = \Delta u + \nu_0 \nabla(\operatorname{div} u) = 0, \quad \text{in } S \\ \sigma(u)\cdot\eta = 0, \quad \text{for } \theta = 0, \omega\,, \end{gathered} \tag{1.3}$$

where

$$\nu_0 = (1-2\nu)^{-1} = \frac{\lambda+\mu}{\mu},$$

λ, μ are Lamé constants, with $\lambda \ge 0$, $\mu > 0$, ν is a real number ($0 < \nu < \frac{1}{2}$) called Poisson coefficient, and

$$\sigma(u) = \begin{pmatrix} \sigma_{11}(u) & \sigma_{12}(u) \\ \sigma_{12}(u) & \sigma_{22}(u) \end{pmatrix}.$$

Here, the components of the stress tensor $\sigma(u)$ are given by Hooke's law

$$\sigma_{ij} = \mu\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big) + \lambda(\nabla . u)\delta_{ij}, \quad i,j = 1,2.$$

We shall analyze the solutions u of this problem which can be written in a series of the form:

$$u(r,\theta) = \sum_{\alpha\in E} c_\alpha r^\alpha v_\alpha(\theta). \tag{1.4}$$

2010 *Mathematics Subject Classification.* 35C09, 35C10, 35J57, 35Q70.
Key words and phrases. Sector; crack; singularity; Lamé; trigonometric series; Airy function.

Submitted March 21, 2017. Published June 23, 2017.

Here E stands for the set of solutions of the equation in a complex variable $\alpha(\nu_0)$

$$\sin^2 \alpha\omega = \alpha^2 \sin^2 \omega, \quad \operatorname{Re} \alpha > 0 \tag{1.5}$$

For further studies of the set E, see, for example, Lozi and Merouani [5, 8].

It is well known that the problem (1.3) is reduced to the problem of Dirichlet for the bilaplacian of the Airy function. In this work, we study directly problem (1.3) and find the results by Tcha-Kondor [11] concerning the Airy function. The relations between the Airy function and the stress tensor $\sigma(u)$ are given in [4] by

$$\sigma_{11}(u) = \frac{\partial^2 H}{\partial^2 x_2}, \quad \sigma_{22}(u) = \frac{\partial^2 H}{\partial^2 x_1}, \quad \sigma_{12}(u) = -\frac{\partial^2 H}{\partial x_1 \partial x_2}, \tag{1.6}$$

We will adapt the technique used in [2, 9, 10, 11], for the bilaplacian and for the Dirichlet's boundary conditions for the Lamé's system. Here, we treat problem of Lamé system in sector for Neumann boundary-value. As in [9], we establish, thanks to the Betti formula instead of Green formula, a relation of orthogonality between the functions v_α and v_β allowing us to compute the coefficients of the singularities which can occur in the solutions, this technique is easier and more direct than the classical one used in [4].

We will focus on the important case of the crack, i.e. $\omega = 2\pi$. The calculations in that case are more explicit and give the known results for the Laplacian as just a particular case.

2. Separation of variables

Replacing u by $r^\alpha(v_{1,\alpha}(\theta), v_{2,\alpha}(\theta))$ in problem (1.3) and using the change of variables

$$\begin{gathered} w_1(\theta) = \cos\theta v_1(\theta) + \sin\theta v_2(\theta), \\ w_2(\theta) = -\sin\theta v_1(\theta) + \cos\theta v_2(\theta) \end{gathered} \tag{2.1}$$

leads us the system

$$\begin{gathered} w_1''(\theta) + (\nu_0 + 1)(\alpha^2 - 1)w_1(\theta) + (\nu_0(\alpha - 1) - 2)w_2'(\theta) = 0, \\ (\nu_0 + 1)w_2''(\theta) + (\alpha^2 - 1)w_2(\theta) + (\nu_0(\alpha + 1) + 2)w_1'(\theta) = 0, \\ w_1'(0) + (\alpha - 1)w_2(0) = 0, \\ (\nu_0(\alpha + 1) + 1)w_1(0) + (\nu_0 + 1)w_2'(0) = 0; \\ ((\nu_0(\alpha + 1) + 1)w_1(\omega) + (\nu_0 + 1)w_2'(\omega)) \sin\omega \\ + (w_1'(\omega) + (\alpha - 1)w_2(\omega)) \cos\omega = 0, \\ ((\nu_0(\alpha + 1) + 1)w_1(\omega) + (\nu_0 + 1)w_2'(\omega)) \cos\omega \\ - (w_1'(\omega) + (\alpha - 1)w_2(\omega)) \sin\omega = 0, \end{gathered} \tag{2.2}$$

By Merouani [7], the solutions of (2.2) are linear combination of the functions

$$\varphi_\alpha(\theta) = \begin{pmatrix} 2v_0\alpha\cos(\alpha - 2)\theta - 2(v_0(\alpha + 2) + 2)\cos\alpha\theta \\ -2v_0\alpha\sin(\alpha - 2)\theta + 2(v_0\alpha - 2)\sin\alpha\theta \end{pmatrix} \tag{2.3}$$

and

$$\psi_\alpha(\theta) = \begin{pmatrix} 2v_0\alpha\sin(\alpha - 2)\theta - 2(v_0\alpha + 2)\sin\alpha\theta \\ 2v_0\alpha\cos(\alpha - 2)\theta - 2(v_0(\alpha - 2) - 2)\cos\alpha\theta \end{pmatrix}, \tag{2.4}$$

A relationship, similar to classical orthogonality, for this system is given by the following theorem.

Theorem 2.1. *Let $w_\alpha = (w_{1,\alpha}, w_{2,\alpha})$ and $w_\beta = (w_{1,\beta}, w_{2,\beta})$ be solutions of* (2.2) *with α and β solutions of* (1.5)*. Then, for $\beta \neq \overline{\alpha}$, we have*

$$\begin{aligned}&[w_\alpha, w_\beta]\\ &= \int_0^\omega \Big[\Big[\frac{1}{(\overline{\beta}-\alpha)}\nu_0(w'_{2,\alpha}, w'_{1,\alpha}) + ((v_0+1)w_{1,\alpha}, w_{2,\alpha})\Big]\begin{pmatrix}\overline{w}_{1,\beta}\\ \overline{w}_{2,\beta}\end{pmatrix}\Big]d\theta = 0\,.\end{aligned} \tag{2.5}$$

Proof. We shall use Betti's formula

$$\int_S (vLu - uLv)dx = \int_\Gamma [v\sigma(u)\cdot\eta - u\sigma(v)\cdot\eta]d\sigma \tag{2.6}$$

where $\eta = \begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}$ is the outward unit vector normal to Σ, and Γ is the boundary of S. For two functions u, v which are solutions of (1.3), using the Betti's formula we obtain

$$\int_\Sigma [v\sigma(u)\cdot\eta - u\sigma(v)\cdot\eta]d\sigma = 0 \tag{2.7}$$

on Σ, for the function $u = r^\alpha\varphi_\alpha$, taking account of the change of variables (2.1), we have

$$\sigma(u)\cdot\eta = \mu r^{\alpha-1}M_{\alpha,v_0}(w_\alpha) \tag{2.8}$$

with $M_{\alpha,v_0}(w_\alpha)$ being the matrix

$$\begin{pmatrix}((v_0-1)w'_{2,\alpha} + (\alpha(v_0+1)+(v_0-1))w_{1,\alpha})\cos\theta - (w'_{1,\alpha} + (\alpha-1)w_{2,\alpha})\sin\theta\\ (w'_{1,\alpha} + (\alpha-1)w_{2,\alpha})\cos\theta + ((v_0-1)w'_{2,\alpha} + (\alpha(v_0+1)+(v_0-1))w_{1,\alpha})\sin\theta\end{pmatrix}$$

The results follow from the application of formula (2.7) to the functions $u = r^\alpha\varphi_\alpha$ and $u = r^\beta\psi_\beta$, and by using relation (2.8). □

Corollary 2.2. *Let w_α and w_β be solutions of* (2.2) *with α and β solutions of* (1.5)*. Suppose in addition that*

$$\int_0^\omega (w'_{2,\alpha}, w'_{1,\alpha})\begin{pmatrix}\overline{w}_{1,\beta}\\ \overline{w}_{2,\beta}\end{pmatrix} = 0 \tag{2.9}$$

and $\alpha \neq \overline{\beta}$. Then

$$[w_\alpha, w_\beta] = \int_0^\omega [((v_0+1)w_{1,\alpha}, w_{2,\alpha})]\begin{pmatrix}\overline{w}_{1,\beta}\\ \overline{w}_{2,\beta}\end{pmatrix}d\theta = 0. \tag{2.10}$$

The above corollary follows by substituting (2.9) in (2.5), to obtain (2.10).

For $\omega_\alpha = r^\alpha\varphi_\alpha$, we define the operator

$$T\omega_\alpha = r^{\alpha-1}\begin{pmatrix}(v_0+1)\omega_{1,\alpha}\\ \omega_{2,\alpha}\end{pmatrix}.$$

Corollary 2.3. *From Corollary 2.2, if $\alpha \neq \overline{\beta}$, we have*

$$\int_\Sigma (T\omega_\alpha\cdot\overline{\omega_\beta} + \omega_\alpha.T\overline{\omega_\beta})d\sigma = 0. \tag{2.11}$$

Proof. From the definition of the operator T and Corollary 2.2 we have

$$\int_\Sigma (T\omega_\alpha.\overline{\omega_\beta} + \omega_\alpha.T\overline{\omega_\beta})d\sigma = 2r^{\alpha+\beta-1}\int((v_0+1)\omega_{1,\alpha}\omega_{1,\overline{\beta}} + \omega_{2,\alpha}\omega_{2,\overline{\beta}})d\theta = 0.$$

□

Corollary 2.4. *Suppose that* $u = \sum_{\alpha\in E} c_\alpha r^\alpha \varphi_\alpha$ *is uniformly convergent in* $\overline{S}$. *If* $[\varphi_{\overline{\beta}}, \varphi_\beta] \neq 0$*, then*

$$C_{\overline{\beta}} = \frac{1}{2}\rho^{-2\overline{\beta}+1}\frac{\int_\Sigma (Tu\cdot \overline{u_\beta} + u.T\overline{u_\beta})d\sigma}{[\varphi_{\overline{\beta}}, \varphi_\beta]}.$$

Proof. For $u = \sum_{\alpha\in E} c_\alpha r^\alpha \varphi_\alpha$ and taking into account the definition of the operator T we have

$$\begin{aligned}
&\int_\Sigma (Tu\cdot\overline{u_\beta} + u\cdot T\overline{u_\beta})d\sigma\\
&= \int_0^\omega \Big(\Big(\sum_{\alpha\in E} c_\alpha r^{\alpha-1}\begin{pmatrix}(v_0+1)\varphi_{1,\alpha}\\ \varphi_{2,\alpha}\end{pmatrix}\Big) r^{\overline{\beta}}\varphi_{\overline{\beta}}\\
&\quad + \Big(\sum_{\alpha\in E} c_\alpha r^\alpha \varphi_\alpha\Big) r^{\overline{\beta}-1}\begin{pmatrix}(v_0+1)\varphi_{1,\overline{\beta}}\\ \varphi_{2,\overline{\beta}}\end{pmatrix}\Big)\,d\theta\\
&= \sum_{\alpha\in E} c_\alpha r^{\overline{\beta}+\alpha-1}\int_0^\omega \Big(\begin{pmatrix}(v_0+1)\varphi_{1,\alpha}\\ \varphi_{2,\alpha}\end{pmatrix}\varphi_{\overline{\beta}} + \varphi_\alpha\begin{pmatrix}(v_0+1)\varphi_{1,\overline{\beta}}\\ \varphi_{2,\overline{\beta}}\end{pmatrix}\Big)d\theta\,.
\end{aligned}$$

From Corollary 2.2 if $\alpha \neq \overline{\beta}$, then

$$\int_\Sigma (Tu\cdot\overline{u_\beta} + u\cdot T\overline{u_\beta})d\sigma = 2C_{\overline{\beta}}[\varphi_{\overline{\beta}}, \varphi_\beta]\rho^{2\overline{\beta}-1}.$$

Expression $c_{\overline{\beta}}$ of Corollary 2.4 results from this last equality. □

The technique we develop for the study of the trigonometric series is based on Theorem 2.1 and Corollary 2.4. To illustrate this, we study the following trigonometric series in the particular case of the crack ($\omega = 2\pi$), which is an important case of singular domains. The explicit knowledge of the roots of (1.5) simplifies computations

3. Complete case study of the crack

To simplify calculations, we decompose every solution u of (1.3) in two parts with respect to θ

$$u = \mathfrak{U}_1 + \mathfrak{U}_2.$$

3.1. **Study of the first part.** The first part is the expression φ_α and is given by (2.3), where

$$E = \{\frac{k}{2},\ k\in\mathbb{N}^*\}$$

because $\omega = 2\pi$. After some calculation, we obtain

$$[\varphi_\alpha, \varphi_\alpha] = 4[v_0^2(v_0+2)\alpha^2 + (v_0(\alpha+2)+2)^2(1+v_0) + (v_0\alpha-2)^2]\pi\rho^{2\alpha-1} \neq 0.$$

We define the sub-sector

$$S_{\rho_0} = S\cap\{(r\cos\theta, r\sin\theta)\in\mathbb{R}^2,\quad r<\rho_0\},\quad \rho_0<\rho.$$

We define the traces on Σ,

$$\mathfrak{U}_1 = \xi_1 \in (\tilde{H}^{1/2}(\Sigma))^2,\quad T\mathfrak{U}_1 = \phi_1 \in (H^{1/2}(\Sigma))^2.$$

Let

$$c_\alpha = A_{\alpha,v_0}\int_0^{2\pi}(\xi_1\begin{pmatrix}(v_0+1)\varphi_{1,\alpha}\\ \varphi_{2,\alpha}\end{pmatrix} + \rho_0\begin{pmatrix}\varphi_{1,\alpha}\\ \varphi_{2,\alpha}\end{pmatrix}\phi_1)(\rho_0,\theta)d\theta, \tag{3.1}$$

with

$$A_{\alpha,v_0} = \frac{\rho_0^{-\alpha}}{8\pi[v_0^2(v_0+2)\alpha^2 + (1+v_0)(v_0(\alpha+2)+2)^2 + (v_0\alpha-2)^2]}.$$

Corollary 3.1. *If* $\mathfrak{U}_1$ *is a solution of* (1.3)*, then*

$$\mathfrak{U}_1 = \sum_{\alpha\in E} c_\alpha r^\alpha \varphi_\alpha \tag{3.2}$$

where c_α *is given by* (3.1)*. The series converges uniformly in* $\overline{S}_{\rho_0}$ *for all* $\rho_0 < \rho$*. Moreover* (3.2) *converges globally in* $(H^1(S_\rho))^2$*, if* $\alpha^{3/2}c_\alpha\rho^\alpha \in l^2$*.*

Proof. (i) if (3.2) occurs, then c_α is expressed by (3.1) under Corollary 2.4.

(ii) if $\mathfrak{U}_1$ is solution of (1.3) and c_α given by (3.1) then $c_\alpha = \circ(\alpha\rho_0^{-\alpha})$. This implies the uniform convergence of the series in $\overline{S}_{\rho_0}$ towards some W_1 satisfying (1.3).

From Grisvard-Geymonat [3], there a exists positive ε, sufficiently small such that the solution of problem (1.3) is written as

$$\mathfrak{U}_1 = \sum_{\alpha\in E} K_\alpha r^\alpha \varphi_\alpha,$$

which converges for $r < \varepsilon$. Then Theorem 2.1 implies that $K_\alpha = c_\alpha$ therefore W_1 and $\mathfrak{U}_1$ coincide in S_ε. They coincide in S_{ρ_0} since they are real analytic. □

Remark 3.2. If ξ_1 belongs to the space $(H^2(]0,2\pi[))^2$ and ϕ_1 to $(H^1(]0,2\pi[))^2$, then $c_\alpha = \circ(\alpha\rho_0^{-\alpha})$ and we have uniform convergence of the series in $\overline{S}_{\rho_0}$ for all $\rho_0 \le \rho$.

3.2. **Study of the second part.** The second part is the expression ψ_α given by (2.4), where

$$E = \{\frac{k}{2},\ k \in \mathbb{N}^*\}$$

because $\omega = 2\pi$. After some calculations, we obtain

$$[\psi_\alpha, \psi_\alpha] = 4[v_0^2(v_0+2)\alpha^2 + (v_0+1)(v_0\alpha+2)^2 + (v_0(\alpha-2)-2)^2]\pi\rho^{2\alpha-1} \neq 0$$

We define the following trace of Σ,

$$\mathfrak{U}_2 = \xi_2 \in (\tilde{H}^{1/2}(\Sigma))^2, \quad T\mathfrak{U}_2 = \phi_2 \in (\tilde{H}^{1/2}(\Sigma))^2.$$

Let

$$d_\alpha = B\alpha, v_0 \int_0^{2\pi} \Big(\xi_1 \begin{pmatrix} (v_0+1)\psi_{1,\alpha} \\ \psi_{2,\alpha} \end{pmatrix} + \rho_0 \begin{pmatrix} \psi_{1,\alpha} \\ \psi_{2,\alpha} \end{pmatrix} \phi_1\Big)(\rho_0,\theta)d\theta. \tag{3.3}$$

with

$$B_{\alpha,v_0} = \frac{\rho_0^{-\alpha}}{8\pi[v_0^2(v_0+2)\alpha^2 + (v_0+1)(v_0\alpha+2)^2 + (v_0(\alpha-2)-2)^2]}.$$

Corollary 3.3. *If* $\mathfrak{U}_2$ *is solution of problem* (1.3) *then*

$$\mathfrak{U}_2 = \sum_{\alpha\in E} d_\alpha r^\alpha \psi_\alpha \tag{3.4}$$

where d_α *is given by* (3.3)*. The series converges uniformly in* $\overline{S}_{\rho_0}$ *for all* $\rho_0 < \rho$*. Moreover* (3.4) *converges globally in* $(H^1(S_\rho))^2$*, if* $\alpha^{3/2}d_\alpha\rho^\alpha \in l^2$*.*

Remark 3.4. For $v_0 = 0$ we obtain the trigonometric series for the Laplace equation in a sector. This is compatible with (1.3) with $v_0 = 0$,

Remark 3.5. Using the formulas (1.6), we find the results concerning the Airy functions.

Equation (2.8) in [10] is false (Typing error). We have to write $\mu r^{\alpha-1}M_{\alpha,v_0}(w_\alpha)$ and not $\frac{1}{\mu}r^{\alpha-1}M_{\alpha,v_0}(w_\alpha)$.

References

[1] Cédric Camier; *Modélisation et étude numé rique de vibrations non-linéaire des plaques circulaires minces imparfaites, Application aux cymbales.* Thèse Présentée et soutenue publiquement le 2 février 2009 pour l'obtention du Docteur de l'Ecole Polytechnique.

[2] W. Chikouche, A. Aibeche; *Coefficients of singularities of the biharmonic problem of Neumann type: case of the crack.* IJMMS 2003: 5, 305-313, Hindawi Publishing Corp.

[3] P. Grisvard-G. Geymonat; *Singularities and constructive methods for treatment*, Proceeding Oberwalfach, Springer-Verlag, 1983, p. 123-126.

[4] P. Grisvard; *Boundary value problems in plane polygons.* Instructions for use. E.D.F. Bulletin de la direction des études et recherches, série C Mathématiques. Informatiques No. 1, 1986. pp.21-59.

[5] R. Lozi; *Résultats numériques de régularit é du problème de Stokes et du Laplatien dans un polygone*, R.A.I.R.O., Aalyse Numér., 12, No. 3, 1978, p. 267-282.

[6] B. Merouani; *Quelques problèmes aux limites pour le système de Lamé dans un secteur plan*, C.R.A.S., t. 304, série I, no. 13, 1987.

[7] B. Merouani; *Solutions singulières du système de l'élasticité dans un polygone pour différentes conditions aux limites*, Maghreb math, Rev, Vol. 5, Nos. 1&2, 1996, pp.95-112.

[8] B. Merouani; *comportements singuliers des solutions du système de l'élasticité dans un polygone*, Thèse de doctorat, U.S.T.H.B., Alger (1990).

[9] B. Merouani, R. Boufenouche; *Coefficients of singularities for a simply supported plate problems in plane sectors*, Electronic Journal of Differential Equations, Vol. 2013(2013), No. 238, p. 1-8.

[10] B. Merouani, R. Boufenouche; *Trigonometric series adapted to the study of boundary value problems of Dirichlet type for the Lamé system (elasticity)*, Electronic Journal of Differential Equations, Vol. 2015(2015), No. 181, p. 1-6.

[11] O. Tcha-Kondor; *Nouvelles séries trigonomé triques adaptées à l'étude de problème aux limite pour l' équation biharmonique.* Etude du cas de la fissure. C. R. Acad. Sci. paris, t. 315, Série I, p. 514-544, 1992.

Boubakeur Merouani

Applied Mathematics Laboratory (LaMA), University of Setif 1, Algeria

E-mail address: mermathsb@hotmail.fr

Nabil Liazidi

Applied Mathematics Laboratory (LaMA), University of Setif 1, Algeria

E-mail address: n.liazidi@outlook.fr