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NONTRIVIAL SOLUTIONS OF INCLUSIONS INVOLVING PERTURBED MAXIMAL MONOTONE OPERATORS

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Abstract. Let X be a real reflexive Banach space and X^* its dual space. Let $L : X \supset D(L) \to X^*$ be a densely defined linear maximal monotone operator, and $T : X \supset D(T) \to 2^{X^*}$, $0 \in D(T)$ and $0 \in T(0)$, be strongly quasibounded maximal monotone and positively homogeneous of degree 1. Also, let $C : X \supset D(C) \to X^*$ be bounded, demicontinuous and of type (S_+) w.r.t. to $D(L)$. The existence of nonzero solutions of $Lx + Tx + Cx \ni 0$ is established in the set $G_1 \setminus G_2$, where $G_2 \subset G_1$ with $\overline{G}_2 \subset G_1$, G_1, G_2 are open sets in X, $0 \in G_2$, and G_1 is bounded. In the special case when $L = 0$, a mapping $G : \overline{G}_1 \to X^*$ of class (P) introduced by Hu and Papageorgiou is also incorporated and the existence of nonzero solutions of $Tx+Cx+Gx \ni 0$, where T is only maximal monotone and positively homogeneous of degree $\alpha \in (0, 1]$, is obtained. Applications to elliptic partial differential equations involving p-Laplacian with $p \in (1, 2]$ and time-dependent parabolic partial differential equations on cylindrical domains are presented.

1. Introduction and preliminaries

Let X be a real reflexive Banach space with its dual space X^*. The norms of X, X^* will be denoted by $\|\cdot\|_X$ and $\|\cdot\|_{X^*}$, respectively. We denote by $\langle x^*, x\rangle$ the value of the functional $x^* \in X^*$ at $x \in X$. The symbols $\partial D, \mathring{D}, \overline{D}$, denote the strong boundary, interior and closure of the set D, respectively. The symbol $B_Y(0, R)$ denotes the open ball of radius R with center at 0 in a Banach space Y.

If $\{x_n\}$ is a sequence in X, we denote its strong convergence to x_0 in X by $x_n \to x_0$ and its weak convergence to x_0 in X by $x_n \rightharpoonup x_0$. An operator $T : X \supset D(T) \to Y$ is said to be "bounded" if it maps bounded subsets of the domain $D(T)$ onto bounded subsets of Y. The operator T is said to be "compact" if it maps bounded subsets of $D(T)$ onto relatively compact subsets in Y. It is said to be "demicontinuous" if it is strong-weak continuous on $D(T)$. The symbols $\mathbb{R}$ and $\mathbb{R}_+$ denote $(-\infty, \infty)$ and $[0, \infty)$, respectively. The normalized duality mapping $J : X \supset D(J) \to 2^{X^*}$ is defined by

$$Jx = \{x^* \in X^* : \langle x^*, x\rangle = \|x\|^2,\ \|x^*\| = \|x\|\}, \quad x \in X.$$

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The Hahn-Banach theorem ensures that $D(J) = X$, and therefore $J : X \to 2^{X^*}$ is a multivalued mapping defined on the whole space X.

By a well-known renorming theorem due to Trojanski [27], one can always renorm the reflexive Banach space X with an equivalent norm with respect to which both X and X^* become locally uniformly convex (therefore strictly convex). Henceforth, we assume that X is a locally uniformly convex reflexive Banach space. With this setting, the normalized duality mapping J is single-valued homeomorphism from X onto X^* and satisfies

$$J(\alpha x) = \alpha J(x), \quad (\alpha, x) \in \mathbb{R}_+ \times X.$$

For a multivalued operator T from X to X^*, we write $T : X \supset D(T) \to 2^{X^*}$, where $D(T) = \{x \in X : Tx \neq \emptyset\}$ is the effective domain of T. We denote by $Gr(T)$ the graph of T, i.e., $Gr(T) = \{(x, y) : x \in D(T), y \in Tx\}$.

An operator $T : X \supset D(T) \to 2^{X^*}$ is said to be "monotone" if for every $x, y \in D(T)$ and every $u \in Tx, v \in Ty$ we have

$$\langle u - v, x - y\rangle \geq 0.$$

A monotone operator T is said to be "maximal monotone" if $Gr(T)$ is maximal in $X \times X^*$, when $X \times X^*$ is partially ordered by the set inclusion. In our setting, a monotone operator T is maximal if and only if $R(T + \lambda J) = X^*$ for all $\lambda \in (0, \infty)$. If T is maximal monotone, then the operator $T_t \equiv (T^{-1} + tJ^{-1})^{-1} : X \to X^*$ called the Yosida approximant is bounded, demicontinuous, maximal monotone and such that $T_t x \rightharpoonup T^{\{0\}}x$ as $t \to 0^+$ for every $x \in D(T)$, where $T^{\{0\}}x$ denotes the element $y^* \in Tx$ of minimum norm, i.e., $\|T^{\{0\}}x\| = \inf\{\|y^*\| : y^* \in Tx\}$. In our setting, this infimum is always attained and $D(T^{\{0\}}) = D(T)$. Also, $T_t x \in TJ_t x$, where $J_t \equiv I - tJ^{-1}T_t : X \to X$ and satisfies $\lim_{t\to 0} J_t x = x$ for all $x \in \overline{\operatorname{co} D(T)}$, where $\operatorname{co} A$ denotes the convex hull of the set A. In addition, $x \in D(T)$ and $t_0 > 0$ imply $\lim_{t\to t_0} T_t x = T_{t_0} x$. The operators T_t, J_t were introduced by Brézis, Crandall and Pazy in [9]. For their basic properties, we refer the reader to [9] as well as Pascali and Sburlan [23, pp. 128-130].

We need the following lemmas about maximal monotone operators.

Lemma 1.1 ([28, p. 915]). *Let $T : X \supset D(T) \to 2^{X^*}$ be maximal monotone. Then the following are true:*

(i) *$\{x_n\} \subset D(T)$, $x_n \to x_0$ and $Tx_n \ni y_n \rightharpoonup y_0$ imply $x_0 \in D(T)$ and $y_0 \in Tx_0$.*

(ii) *$\{x_n\} \subset D(T)$, $x_n \rightharpoonup x_0$ and $Tx_n \ni y_n \to y_0$ imply $x_0 \in D(T)$ and $y_0 \in Tx_0$.*

The next lemma is essentially due to Brézis, Crandall and Pazy [9], and its proof can be found in [3].

Lemma 1.2. *Assume that the operators $T : X \supset D(T) \to 2^{X^*}$ and $S : X \supset D(S) \to 2^{X^*}$ are maximal monotone, with $0 \in D(T) \cap D(S)$ and $0 \in S(0) \cap T(0)$. Assume, further, that $T + S$ is maximal monotone and that there is a sequence $\{t_n\} \subset (0, \infty)$ such that $t_n \downarrow 0$, and a sequence $\{x_n\} \subset D(S)$ such that $x_n \rightharpoonup x_0 \in X$ and $T_{t_n} x_n + w_n^* \rightharpoonup y_0^* \in X^*$, where $w_n^* \in Sx_n$. Then the following are true.*

(i) *The inequality*

$$\lim_{n\to\infty} \langle T_{t_n} x_n + w_n^*, x_n - x_0\rangle < 0 \tag{1.1}$$

is impossible.

(ii) *If*
$$\lim_{n\to\infty}\langle T_{t_n}x_n + w_n^*, x_n - x_0\rangle = 0, \tag{1.2}$$
then $x_0 \in D(T+S)$ *and* $y_0^* \in (T+S)x_0$.

Definition 1.3. An operator $T : X \supset D(T) \to 2^{X^*}$ is said to be "strongly quasibounded" if for every $S > 0$ there exists $K(S) > 0$ such that
$$\|x\| \le S, \quad \text{and} \quad \langle x^*, x\rangle \le S, \quad \text{for some } x^* \in Tx,$$
imply $\|x^*\| \le K(S)$.

Browder and Hess have shown in [13] that a monotone operator T with $0 \in \mathring{D}(T)$ is strongly quasibounded. The proof of the following lemma, which is due to Browder and Hess [13], can also be found in [17, Lemma D].

Lemma 1.4. *Let* $T : X \supset D(T) \to 2^{X^*}$ *be a strongly quasibounded maximal monotone operator such that* $0 \in T(0)$. *Let* $\{t_n\} \subset (0,\infty)$ *and* $\{u_n\} \subset X$ *be such that*
$$\|u_n\| \le S, \quad \langle T_{t_n}u_n, u_n\rangle \le S, \quad \text{for all } n,$$
where S *is a positive constant. Then there exists a number* $K = K(S) > 0$ *such that* $\|T_{t_n}u_n\| \le K$ *for all* n.

Definition 1.5. An operator $G : X \supset D(G) \to 2^{X^*}$ is said to belong to class (P) if it maps bounded sets to relatively compact sets, for every $x \in D(G)$, $G(x)$ is closed and convex subsets of X^* and $G(\cdot)$ is upper-semicontinuous (usc), i.e., for every closed set $F \subset X^*$, the set $G^-(F) = \{x \in D(G) : G(x) \cap F \ne \emptyset\}$ is closed in X.

An important fact about a compact-set valued upper-semicontinuous operator G is that it is closed. Furthermore, for every sequence $\{[x_n, y_n]\} \subset Gr(G)$ such that $x_n \to x \in D(G)$, the sequence $\{y_n\}$ has a cluster point in $G(x)$.

Definition 1.6. Let $L : X \supset D(L) \to X^*$ be a densely defined linear maximal monotone operator and $C : X \supset D(C) \to X^*$ be bounded and demicontinuous. We say that $C : X \supset D(C) \to X^*$ is of type (S_+) w.r.t. to $D(L)$ if for every sequence $\{x_n\} \subset D(L) \cap D(C)$ with $x_n \rightharpoonup x_0$ in X, $Lx_n \rightharpoonup Lx_0$ in X^* and
$$\limsup_{n\to\infty}\langle Cx_n, x_n - x_0\rangle \le 0,$$
we have $x_n \to x_0$ in X. In this case, if $L = 0$, then C is of class (S_+).

Definition 1.7. The family $C(t) : X \supset D \to X^*, t \in [0,1]$, of operators is said to be a "homotopy of type (S_+) w.r.t. $D(L)$" if for any sequences $\{x_n\} \subset D(L) \cap D$ with $x_n \rightharpoonup x_0$ in X and $Lx_n \rightharpoonup Lx_0$ in X^*, $\{t_n\} \subset [0,1]$ with $t_n \to t_0$ and
$$\limsup_{n\to\infty}\langle C(t_n)x_n, x_n - x_0\rangle \le 0,$$
we have $x_n \to x_0$ in $X, x_0 \in D$ and $C(t_n)x_n \rightharpoonup C(t_0)x_0$ in X^*. In this case, if $L = 0$, then $C(t)$ is a homotopy of type (S_+). A homotopy of type (S_+) w.r.t. $D(L)$ is "bounded" if the set
$$\{C(t)x : t \in [0,1], x \in D\}$$
is bounded.

Let G be an open and bounded subset of X. Let $L : X \supset D(L) \to X^*$ be densely defined linear maximal monotone, $T : X \supset D(T) \to 2^{X^*}$ maximal monotone, and $C(s) : X \supset \overline{G} \to X^*$, $s \in [0,1]$, a bounded homotopy of type (S_+) w.r.t. $D(L)$. Since the graph $Gr(L)$ of L is closed in $X \times X^*$, the space $Y = D(L)$ associated with the graph norm

$$\|x\|_Y = \|x\|_X + \|Lx\|_{X^*}, \quad x \in Y,$$

becomes a real reflexive Banach space. We may now assume that Y and its dual Y^* are locally uniformly convex.

Let $j : Y \to X$ be the natural embedding and $j^* : X^* \to Y^*$ its adjoint. Note that since $j : Y \to X$ is continuous, we have $D(j^*) = X^*$, which implies that j^* is also continuous. Since j^{-1} is not necessarily bounded, we have, in general, $j^*(X^*) \neq Y^*$. Moreover, $j^{-1}(\overline{G}) = \overline{G} \cap D(L)$ is closed and $j^{-1}(G) = G \cap D(L)$ is open, and

$$\overline{j^{-1}(G)} \subset j^{-1}(\overline{G}), \quad \partial(j^{-1}(G)) \subset j^{-1}(\partial G).$$

We define $M : Y \to Y^*$ by

$$(Mx, y) = \langle Ly, J^{-1}(Lx)\rangle, \quad x, y \in D(L).$$

Here, the duality pair $(\cdot,\cdot)$ is in $Y^* \times Y$ and J^{-1} is the inverse of the duality map $J : X \to X^*$ and is identified with the duality map from X^* to $X^{**} = X$. Also, for every $x \in Y$ such that $Mx \in j^*(X^*)$, we have $J^{-1}(Lx) \in D(L^*)$ and

$$Mx = j^* \circ L^* \circ J^{-1}(Lx), \tag{1.3}$$

$$(Mx - My, x - y) = \langle Lx - Ly, J^{-1}(Lx) - J^{-1}(Ly)\rangle \geq 0 \tag{1.4}$$

for all $y \in Y$ such that $My \in j^*(X^*)$.

We now define $\hat{L} : Y \to Y^*$ and $\hat{C}(s) : j^{-1}(\overline{G}) \to Y^*$ by

$$\hat{L} = j^* \circ L \circ j \quad \text{and} \quad \hat{C}(s) = j^* \circ C(s) \circ j$$

respectively, and for every $t > 0$, we also define $\hat{T}_t : Y \to Y^*$ by

$$\hat{T}_t = j^* \circ T_t \circ j,$$

where T_t is the Yosida approximant of T.

Kartsatos and the author developed a new degree theory in [2] for the triplet $L+T+C$, where L is densely defined linear maximal monotone, T is (possibly nonlinear) maximal monotone and strongly quasibounded, and C is bounded, demicontinuous and of type (S_+) w.r.t. the set $D(L)$. This degree theory extends the degree theory of Berkovits and Mustonen [8] who considered the case $T = 0$. As in [8], the construction of the degree mapping in [2] uses the graph norm topology of the space $Y = D(L)$ and is based on the Skrypnik degree and its invariance under homotopies of type (S_+). In fact, it is shown that the mapping

$$H(t, x) := \hat{L} + \hat{T}_t + \hat{C} + tMx, \quad (t, x) \in (0, \infty) \times j^{-1}(\overline{G}), \tag{1.5}$$

has the Skrypnik degree, $\mathrm{d_S}(H(t,\cdot), \widetilde{G}, 0)$, under the usual boundary condition on the boundary of an open and bounded set $\widetilde{G} \subset Y$, which remains fixed for all sufficiently small $t \in (0,\infty)$. Then the degree is defined by

$$d(L + T + C, G, 0) = \lim_{t \downarrow 0} \mathrm{d_S}(\hat{L} + \hat{T}_t + \hat{C} + tM, \widetilde{G}, 0), \tag{1.6}$$

where G is an open bounded subset of X related to $\widetilde{G}$. The operator C above satisfies the (S_+)-condition w.r.t. $Y = D(L)$ and T is strongly quasibounded and maximal monotone with $0 \in T(0)$. In order to show that the degree $\mathrm{d_S}$ is fixed as above, it can be shown, in addition, that the family of mappings $f^t := H(t,\cdot)$ is a homotopy of class (S_+) in the sense of Browder [10, Definition 3, p. 69] on every interval $[t_1,t_2] \subset (0,t_0]$, where t_0 is an appropriate fixed positive number. The approach discussed here is that of Berkovits and Mustonen in [8] and Addou and Memri in [1].

In Section 2, we establish the existence of nonzero solutions of the inclusion $Lx + Tx + Cx \ni 0$, where L, C are as above and T is a strongly quasibounded maximal monotone operator and positively homogeneous of degree 1. This result is in the spirit of similar results in [3] for operators of the form $T + C$, where T is single-valued maximal monotone, $0 = T(0)$, and C bounded demicontinuous and of type (S_+). Mild and natural boundary conditions are considered in order to establish the result by utilizing the graph norm topology on $D(L)$ and relevant topological degree theory. The theory is applicable to parabolic partial differential equations in divergence form on cylindrical domains.

In Section 3, the existence of nonzero solutions of $Tx+Cx+Gx \ni 0$ is established by utilizing the topological degree theories developed by Browder [13] and Skrypnik [26]. In this case, T is only maximal monotone with $0 \in T(0)$ and positively homogeneous of degree $\alpha \in (0,1]$, and C is bounded demicontinuous of type of (S_+). This result extends and generalizes a similar result in [3] for $\alpha = 1$ and $G = 0$ and has applications to elliptic boundary value problems involving p-Laplacian.

For additional facts and various topological degree theories related to the subject of this paper, the reader is referred to Kartsatos and the author [4], Kartsatos and Lin [16], and Kartsatos and Skrypnik [20, 18]. For information on various concepts and ideas of Nonlinear Analysis used herein, the reader is referred to Barbu [7], Browder [11], Pascali and Sburlan [23], Simons [24], Skrypnik [25, 26], and Zeidler [28].

The following lemma from [5] about the boundedness of the solutions of a homotopy equation will be needed in the sequel.

Lemma 1.8. *Let $G \subset X$ be open and bounded. Assume the following:*

- (A1) $L : X \supset D(L) \to X^*$ *is linear, maximal monotone with $D(L)$ dense in X;*
- (A2) $T : X \supset D(T) \to 2^{X^*}$ *is strongly quasibounded, maximal monotone with $0 \in T(0)$;*
- (A3) $C(t) : X \supset \overline{G} \to X^*$ *is a bounded homotopy of type (S_+) w.r.t. $D(L)$.*

Then, for a continuous curve $f(s), 0 \le s \le 1$, in X^*, the set

$$K = \left\{x \in j^{-1}(\overline{G}) : \hat{L} + \hat{T}_t + \hat{C}(s) + tMx = j^* f(s), \text{ for some } t > 0,\ s \in [0,1]\right\}$$

is bounded in Y. Thus, there exists $R > 0$ such that $K \subset B_Y(R)$, where $B_Y(R))$ is the open ball of Y of radius R.

Lemma 1.9 below taken from Kartsatos and Skrypnik [19] will be used in the proof of Theorem 2.2.

Lemma 1.9. *Let $T : X \supset D(T) \to 2^{X^*}$ be maximal monotone and such that $0 \in D(T)$ and $0 \in T(0)$. Then the mapping $(t,x) \to T_t x$ is continuous on the set $(0,\infty) \times X$.*

Definition 1.10. An operator $T : X \supset D(T) \to 2^{X^*}$ is said to be positively homogeneous of degree $\alpha > 0$ if, for a fixed $\alpha > 0$, $x \in D(T)$ implies $tx \in D(T)$ for all $t \in \mathbb{R}_+$ and $T(tx) = t^\alpha Tx$.

The following lemma, which plays an important role in the existence theorems of Section 2 and Section 3, shows in particular that the Yosida approximants of a positively homogeneous maximal monotone operator of degree α are also positively homogeneous only when $\alpha = 1$.

Lemma 1.11. *Let $T : X \supset D(T) \to 2^{X^*}$ is maximal monotone and positively homogeneous of degree $\alpha > 0$. Then, for each $t > 0$, the Yosida approximant T_t satisfies*

$$T_t(sx) = s^\alpha T_{ts^{\alpha-1}}(x) \quad \text{for all } (s, x) \in (0, +\infty) \times X. \tag{1.7}$$

Proof. Let

$$y = T_t(sx) = (T^{-1} + tJ^{-1})^{-1}(sx),$$

for $t, s > 0$, $x \in X$. The homogeneity of the duality mapping J implies

$$\begin{aligned} y \in T(-tJ^{-1}y + sx) &= T\Big(s\Big(-\frac{t}{s}J^{-1}y + x\Big)\Big) \\ &= s^\alpha T\Big(-\frac{t}{s}J^{-1}y + x\Big) \\ &= s^\alpha T\Big(-\frac{t}{s^{1-\alpha}}J^{-1}\big(\frac{y}{s^\alpha}\big) + x\Big). \end{aligned}$$

This is equivalent to

$$x \in T^{-1}\Big(\frac{y}{s^\alpha}\Big) + ts^{\alpha-1}J^{-1}\Big(\frac{y}{s^\alpha}\Big)$$

and

$$y = s^\alpha (T^{-1} + ts^{\alpha-1}J^{-1})^{-1}x = s^\alpha T_{ts^{\alpha-1}}(x).$$

□

2. Nonzero solutions of $Lx + Tx + Cx \ni 0$

Guo and Lakshmikantham have shown in [14] the following result for compact operators defined on a cone in a Banach space. The operator T satisfies non-contractive and non-expansive type of conditions only on the boundary of the subsets G_1, G_2 of X for the existence of a nonzero fixed of T.

Theorem 2.1. *Let X be a Banach space and K a positive cone in X which induces a partial ordering "$\le$" in X. Let G_1, $G_2 \subset X$ be open, $0 \in G_2$, $\overline{G_2} \subset G_1$, G_1 bounded, and $T : K \cap \overline{G}_1 \to K$ compact with $T(0) = 0$. Suppose that one of the following two conditions holds.*

(1) *$Tx \ngeq x$ for $x \in K \cap \partial G_1$, and $Tx \nleq x$ for $x \in K \cap \partial G_2$;*
(2) *$Tx \nleq x$ for $x \in K \cap \partial G_1$, and $Tx \ngeq x$ for $x \in K \cap \partial G_2$.*

Then there exists a fixed point of T in $K \cap (G_1 \setminus G_2)$.

By imposing certain conditions only on the boundary of sets G_1, G_2, the author and Kartsatos [3] established the existence of nonzero solutions of $Tx + Cx = 0$, where T is positively homogeneous of degree 1 and single-valued maximal monotone, and C is a bounded demicontinuous of type (S_+). The following result is obtained in the spirit of [3, Theorem 6, p.1246] in the context of the Berkovits-Mustonen theory in [8].

Theorem 2.2. *Assume that $G_1, G_2 \subset X$ are open, bounded with $0 \in G_2$ and $\overline{G_2} \subset G_1$. Let $L : X \supset D(L) \to X^*$ be linear maximal monotone with $\overline{D(L)} = X$, and $T : X \supset D(T) \to 2^{X^*}$ strongly quasibounded, maximal monotone and positively homogeneous of degree 1. Also, let $C : \overline{G_1} \to X^*$ be bounded, demicontinuous and of type (S_+) w.r.t. to $D(L)$. Moreover, assume the following:*

- (H1) *there exists $v^* \in X^* \setminus \{0\}$ such that $Lx + Tx + Cx \not\ni \lambda v^*$ for all $(\lambda, x) \in \mathbb{R}_+ \times (D(L) \cap D(T) \cap \partial G_1)$, and*
- (H2) *$Lx + Tx + Cx + \lambda Jx \not\ni 0$ for all $(\lambda, x) \in \mathbb{R}_+ \times (D(L) \cap D(T) \cap \partial G_2)$.*

Then the inclusion $Lx + Tx + Cx \ni 0$ has a solution $x \in D(L) \cap D(T) \cap (G_1 \setminus G_2)$.

Proof. To solve the inclusion
$$Lx + Tx + Cx \ni 0, \quad x \in \overline{G_1}, \tag{2.1}$$
let us consider the associated equation
$$\hat{L}x + \hat{T}_t x + \hat{C}x + tMx = 0, \quad t \in (0, +\infty),\ x \in j^{-1}(\overline{G_1}). \tag{2.2}$$
One can show as in [2] that there exists $R > 0$ such that the open ball $B_Y(0, R) = \{y \in Y : \|y\|_Y < R\}$ contains all solutions of (2.2). We shall prove that (2.2) has a solution $x_t \in j^{-1}(G_1 \setminus G_2)$ for all sufficiently small t. We first claim that there exist $\tau_0 > 0$, $t_0 > 0$ such that
$$\hat{L}x + \hat{T}_t x + \hat{C}x + tMx = \tau j^* v^* \tag{2.3}$$
has no solution in $G^1_R(Y) := j^{-1}(G_1) \cap B_Y(0, R)$ for all $t \in (0, t_0]$ and all $\tau \in [\tau_0, \infty)$. Assume the contrary and let $\{\tau_n\} \subset (0, \infty)$, $\{t_n\} \subset (0, 1)$ and $\{x_n\} \subset G^1_R(Y)$ such that $\tau_n \to \infty$, $t_n \downarrow 0$ and
$$\hat{L}x_n + \hat{T}_{t_n} x_n + \hat{C}x_n + t_n Mx_n = \tau_n j^* v^*. \tag{2.4}$$
We note that j^* is one-to-one because $j(Y) = Y$ which is dense in X. This implies that $j^* v^*$ is nonzero, and therefore $\|\tau_n j^* v^*\|_{Y^*} \to +\infty$. Also, the sequence $\{x_n\}$ is bounded in Y and so we may assume that $x_n \rightharpoonup x_0$ in X and $Lx_n \rightharpoonup Lx_0$ in X^*. In particular, $\{Lx_n\}$ is bounded in X^*. Since $Mx_n \in j^*(X^*)$, we have $J^{-1}(Lu) \in D(L^*)$ and
$$Mx_n = j^* L^* J^{-1}(Lx_n).$$
Since j^*, L^*, J^{-1} are bounded, we obtain the boundedness of $\{M(x_n)\}$. It is clear that $\hat{C}x_n$ is bounded in Y^*, and therefore (2.4) implies that $\|\hat{L}x_n + \hat{T}_{t_n} x_n\|_{Y^*} \to \infty$. Define
$$\alpha_n = \frac{1}{\|\hat{L}x_n + \hat{T}_{t_n} x_n\|_{Y^*}} \quad \text{and} \quad u_n = \alpha_n x_n.$$
It is obvious that $u_n \to 0$ in Y.

Since T is positively homogeneous of degree 1, T_t is also positively homogeneous of degree 1 by Lemma 1.11. From (2.4), we obtain
$$(\hat{L} + \hat{T}_{t_n})(\alpha_n x_n) + \alpha_n \hat{C}x_n + t_n \alpha_n Mx_n = \tau_n \alpha_n j^* v^*. \tag{2.5}$$
Since $\|(\hat{L} + \hat{T}_{t_n})(\alpha_n x_n)\|_{Y^*} = 1$, (2.5) implies
$$\tau_n \alpha_n \to \frac{1}{\|j^* v^*\|_{Y^*}},$$
and therefore
$$(\hat{L} + \hat{T}_{t_n})(u_n) = (\hat{L} + \hat{T}_{t_n})(\alpha_n x_n) \to y_0,$$

where

$$y_0 = \frac{j^* v_*}{\|j^* v^*\|_{Y^*}}.$$

Since $u_n \to 0$, we have

$$\lim_{n\to\infty} \langle (\hat{L} + \hat{T}_{t_n}) u_n, u_n \rangle = \langle y_0, 0 \rangle = 0.$$

Since $\hat{L}, \hat{T}_{t_n}$, and $\hat{L} + \hat{T}_{t_n}$ are maximal monotone, by Lemma 1.2, (ii), we have

$$y_0 = (\hat{L} + \hat{T})(0) = 0,$$

which is a contradiction to $\|y_0\|_{Y^*} = 1$.

We now consider the homotopy $H : [0,1] \times Y \to Y^*$ defined by

$$H(s,x) = \hat{L}x + \hat{T}_t x + \hat{C}x + tMx - s\tau_0 j^* v^*, \quad s \in [0,1],\ x \in j^{-1}(\overline{G_1}), \tag{2.6}$$

where $t \in (0, t_0]$ is fixed. It can be easily seen that $C - s\tau_0 v^*$ is bounded demicontinuous on $\overline{G_1}$ and of type (S_+) w.r.t. $D(L)$.

We now show that the equation $H(s,x) = 0$ has no solution on the boundary $\partial G_R^1(Y)$. Here, the number $R > 0$ is increased if necessary so that the ball $B_Y(0,R)$ now also contains all solutions x of $H(s,x) = 0$. To this end, assume the contrary so that there exist $\{t_n\} \subset (0, t_0]$, $\{s_n\} \subset [0,1]$, and $\{x_n\} \subset \partial G_R^1(Y)$ such that $t_n \to 0$, $s_n \to s_0$, $x_n \rightharpoonup x_0$ in Y, $T_{t_n} x_n \rightharpoonup w^*$ in X^* and $Cx_n \rightharpoonup c^*$ and

$$\hat{L}x_n + \hat{T}_{t_n} x_n + \hat{C}x_n + t_n M x_n = s_n \tau_0 j^* v^*. \tag{2.7}$$

Here, the boundedness of $\{T_{t_n}\}$ follows as in Step I of [5, Prop. 1]. Since $x_n \rightharpoonup x_0$ in Y, we have $x_n \rightharpoonup x_0$ in X and $Lx_n \rightharpoonup Lx_0$ in X^*. Also, since $x_n \in B_Y(0,R)$ and

$$\partial(j^{-1}(G_1) \cap B_Y(0,R)) \subset \partial(j^{-1}(G_1)) \cup \partial B_Y(0,R) \subset j^{-1}(\partial G_1) \cup \partial B_Y(0,R),$$

we have $x_n \in j^{-1}(\partial G_1) = \partial G_1 \cap Y \subset \partial G_1$. From (2.7) we obtain

$$\langle Lx_n + T_{t_n} x_n + Cx_n + t_n L^* J^{-1}(Lx_n), x_n - x_0 \rangle = s_n \tau_0 \langle v^*, x_n - x_0 \rangle. \tag{2.8}$$

If we assume

$$\limsup_{n\to\infty} \langle Cx_n, x_n - x_0 \rangle > 0, \tag{2.9}$$

we easily get a contradiction using a standard argument in relation to Lemma 1.2, (i). This is because $L+T$ is maximal monotone because T is strongly quasibounded (cf. Pascali and Sburlan [23, Proposition, p. 142]). Consequently,

$$\limsup_{n\to\infty} \langle Cx_n, x_n - x_0 \rangle \le 0. \tag{2.10}$$

Since C is demicontinuous and of type (S_+) w.r.t. $D(L)$, we obtain $x_n \to x_0$ and $Cx_n \rightharpoonup c^* = Cx_0$. From (2.8), we obtain

$$\lim_{n\to\infty} \langle Lx_n + T_{t_n} x_n, x_n - x_0 \rangle = 0.$$

Using Lemma 1.2, (ii), we obtain $x_0 \in D(T)$ and $w^* \in Tx_0$. Then, in view of (2.8), it follows that

$$\langle Lx_0 + w^* + Cx_0 - s_0 \tau_0 v^*, u \rangle = 0$$

for all $u \in Y$. Since Y is dense in X, we have

$$Lx_0 + Tx_0 + Cx_0 \ni s_0 \tau_0 v^*,$$

which contradicts the hypothesis (H1) because $x_0 \in D(L) \cap D(T) \cap \partial G_1$.

We shrink t_0 if necessary so that

$$H(s,x)=0, \quad s\in[0,1],\ x\in\overline{G_R^1(Y)}$$

has no solution on the boundary $\partial G_R^1(Y)$ for all $t\in(0,t_0]$ and all $s\in[0,1]$. The mapping $H(s,x)$ is an admissible homotopy for the Skrypnik's degree. The Skyrpnik's degree, $\mathrm{d_S}(H(s,\cdot),G_R^1(Y),0)$, is well-defined and remains constant for all $s\in[0,1]$. Also, the degree, $\mathrm{d}(L+T+C,G_1,0)$, developed in [2] is defined as

$$\mathrm{d}(L+T+C-\tau_0 v^*,G_1,0)=\lim_{t\to 0+}\mathrm{d_S}(H(1,\cdot),G_R^1(Y),0).$$

By shrinking t_0 further if necessary, we have

$$\mathrm{d}(L+T+C-\tau_0 v^*,G_1,0)=\mathrm{d_S}(H(1,\cdot),G_R^1(Y),0), \quad \text{for all } t\in(0,t_0].$$

Suppose, if possible, that

$$\mathrm{d_S}(H(1,\cdot),G_R^1(Y),0)\neq 0$$

for some $t_1\in(0,t_0]$. Then there exists $x_0\in G_R^1(Y)$ such that

$$\hat{L}x+\hat{T}_{t_1}x+\hat{C}x+t_1Mx=\tau_0 j^*v^*.$$

This contradicts the choice of τ_0 as stated in (2.3). Since

$$\mathrm{d_S}(H(0,\cdot),G_R^1(Y),0)=\mathrm{d}_S(H(1,\cdot),G_R^1(Y),0),$$

we have

$$\mathrm{d}_S(\hat{L}+\hat{T}_t+\hat{C}+tM,G_R^1(Y),0)=\mathrm{d}_S(H(0,\cdot),G_R^1(Y),0)=0 \tag{2.11}$$

for all $t\in(0,t_0]$.

Next, we consider the homotopy $\widetilde{H}:[0,1]\times Y\to Y^*$ defined by

$$\widetilde{H}(s,x)=s(\hat{L}x+\hat{T}_tx+\hat{C}x)+tMx+(1-s)\hat{J}x, \quad s\in[0,1],\ x\in j^{-1}(\overline{G_2}).$$

As in [5, Step III, p.29], it can be shown that there exists $t_0>0$ (choose it even smaller than the one used previously if necessary) such that all the solutions

$$\widetilde{H}(s,x)=0,\ t\in(0,t_0],\ s\in[0,1]$$

are bounded in Y. We enlarge the previous number $R>0$ if necessary so that all solutions of $\widetilde{H}(s,x)=0$ as above are contained in $B_Y(0,R)$ in Y.

We first show that there exists $t_1\in(0,t_0]$ such that the equation $\widetilde{H}(s,x)=0$ has no solutions on $\partial G_R^2(Y)$ for any $t\in(0,t_1]$ and any $s\in[0,1]$. Here, $G_R^2(Y):=j^{-1}(G_2)\cap B_Y(0,R)$. Suppose that the contrary is true. Then there must exist sequences $\{t_n\}\subset(0,t_0]$, $\{s_n\}\subset[0,1]$, $\{x_n\}\subset\partial G_R^2(Y)$ such that

$$s_n(\hat{L}x_n+\hat{T}_{t_n}x_n+\hat{C}x_n)+t_nMx_n+(1-s_n)\hat{J}x_n=0. \tag{2.12}$$

We may assume that $t_n\downarrow 0$, $s_n\to s_0$, $x_n\rightharpoonup x_0$ in X and $Lx_n\rightharpoonup Lx_0$ in X^*. As in the previous part, we can show that $x_n\in\partial G_2\cap Y\subset\partial G_2$. If $s_n=0$ for some n, then we obtain $t_nMx_n+\hat{J}x_n=0$. Since M is monotone for such x_n's by (1.3), (1.4), and $\hat{J}$ is strictly monotone, we obtain $x_n=0$ which is a contradiction to $0\in G_2$. We may now assume that $s_n\in(0,1]$. Suppose $s_0=0$. Dividing both sides of (2.12), we obtain

$$\hat{L}x_n+\hat{T}_{t_n}x_n+\hat{C}x_n+\frac{t_n}{s_n}Mx_n=-\frac{1-s_n}{s_n}\hat{J}x_n, \tag{2.13}$$

which implies

$$\langle Cx_n, x_n\rangle \le -\frac{(1-s_n)}{s_n}\|x_n\|_X^2.$$

Since $x_n \in \partial G_2$, the sequence $\{\|x_n\|_X\}$ is bounded away from zero. This leads to a contradiction to the boundedness of $\{\langle Cx_n, x_n\rangle\}$ because $(1-s_n)/s_n \to \infty$.

Assume that $s_0 = 1$. Now, by Lemma 1.4, the strong quasiboundedness of T implies that the sequence $\{T_{t_n}x_n\}$ is bounded, and so we may assume that $T_{t_n}x_n \rightharpoonup w^*$ for some $w^* \in X^*$. From (2.12), we obtain

$$\lim_{n\to\infty}\langle Lx_n + T_{t_n}x_n + Cx_n, x_n - x_0\rangle = 0. \tag{2.14}$$

If (2.9) is true, we obtain a contradiction to (i) of Lemma 1.2. Therefore (2.10) must hold true. With (2.14), this implies $x_n \to x_0 \in \partial G_2$, and therefore $x_0 \in D(T)$ and $Lx_0 + Tx_0 + Cx_0 \ni 0$. This is a contradiction to hypothesis (H2) for $\lambda = 0$. For the remaining case $s_0 \in (0,1)$, one can see that (2.13) is replaced with

$$\limsup_{n\to\infty}\langle Lx_n + T_{t_n}x_n + Cx_n, x_n - x_0\rangle \le 0. \tag{2.15}$$

We may assume that $T_{t_n}x_n \rightharpoonup w^*(\text{some}) \in X^*$. By using the monotonicity of L, T_{t_n}, the continuity of T_t from Lemma 1.9 and a standard argument, we obtain $x_n \to x_0 \in \partial G_2$, and hence (2.13) implies

$$\langle Lx_0 + w^* + Cx_0 + \frac{1-s_0}{s_0}Jx_0, u\rangle = 0$$

for all $u \in Y$. By the density of Y in X, we obtain

$$Lx_0 + Tx_0 + Cx_0 + \frac{1-s_0}{s_0}Jx_0 \ni 0,$$

which contradicts hypothesis (H2).

At this time, we replace the number t_0 chosen previously with t_1 and call it t_0 again. Let us fix $t \in (0, t_0]$ and consider the homotopy equation

$$\widetilde{H}(s,x) = s(\hat{L}x+\hat{T}_tx+\hat{C}x)+tMx+(1-s)\hat{J}x = 0, \quad s \in [0,1],\ x \in \overline{G_R^2(Y)}. \tag{2.16}$$

It is already shown that (2.16) has no solution on $\partial G_R^2(Y)$. We note that $\widetilde{H}$ is an affine homotopy of bounded demicontinuous operators of type (S_+) on $\overline{G_R^2(Y)}$; namely, $\hat{L} + \hat{T}_t + \hat{C} + tM$ and $tM + \hat{J}$. We also note here that $tM + \hat{J}$ is strictly monotone. Therefore $\widetilde{H}(s,x)$ is an admissible homotopy for the Skrypnik's degree, d_S, which satisfies

$$\mathrm{d}_S(\widetilde{H}(1,\cdot), G_R^2(Y), 0) = \mathrm{d}_S(\widetilde{H}(0,\cdot), G_R^2(Y), 0). \tag{2.17}$$

This implies

$$\mathrm{d}_S(\hat{L} + \hat{T}_t + \hat{C} + tM, G_R^2(Y), 0) = \mathrm{d}_S(tM + \hat{J}, G_R^2(Y), 0) = 1 \tag{2.18}$$

for all $t \in (0, t_0]$. The last equality follows from [10, Theorem 3, (iv)]. From (2.11) and (2.18), we obtain

$$\mathrm{d}_S(\hat{L} + \hat{T}_t + \hat{C} + tM, G_R^1(Y), 0) \ne \mathrm{d}_S(\hat{L} + \hat{T}_t + \hat{C} + tM, G_R^2(Y), 0)$$

for all $t \in (0, t_0]$. By the excision property of the Skrypnik's degree, for each $t \in (0, t_0]$, there exists a solution $x_t \in G_R^1(Y) \setminus G_R^2(Y)$ of the equation

$$\hat{L}x + \hat{T}_tx + \hat{C}x + tMx = 0.$$

We now pick a sequence $\{t_n\} \subset (0, t_0]$ such that $t_n \downarrow 0$, and denote the corresponding solution x_t by x_n, i.e.

$$\hat{L}x_n + \hat{T}_{t_n}x_n + \hat{C}x_n + t_nMx_n = 0.$$

Since Y is reflexive, we have $x_n \rightharpoonup x_0 \in Y$ by passing to a subsequence. This implies $x_n \to x_0$ in X and $Lx_n \rightharpoonup Lx_0$ in X^*. By the strong quasiboundedness of T, we may assume that $T_{t_n}x_n \rightharpoonup w^* \in X^*$. If (2.9) holds, then we obtain a contradiction by Lemma 1.2, (i). Then (2.10) must be valid. Since C is of type (S_+) w.r.t. $D(L)$, we obtain $x_n \to x_0 \in \overline{G_R^1(Y) \setminus G_R^2(Y)}$, and by Lemma 1.1, we have $x_0 \in D(T)$ and $Lx_0 + w^* + Cx_0 = 0$, and therefore $Lx_0 + Tx_0 + Cx_0 \ni 0$.

It remains to show that $x_0 \in G_1 \setminus G_2$. Since

$$G_R^1(Y) \setminus G_R^2(Y) = (G_1 \setminus G_2) \cap Y \cap B_Y(0, R) \subset G_1 \setminus G_2,$$

we have $x_n \in G_1 \setminus G_2$ for all n, and so

$$x_0 \in \overline{G_1 \setminus G_2} \subset (G_1 \setminus G_2) \cup \partial(G_1 \setminus G_2) \subset (G_1 \setminus G_2) \cup \partial G_1 \cup \partial G_2$$

By hypotheses (H1) and (H2), $x_0 \notin \partial G_1 \cup \partial G_2$. Thus, $x_0 \in D(L) \cap D(T) \cap (G_1 \setminus G_2)$. □

3. Nonzero solutions of $Tx + Cx + Gx \ni 0$

Hu and Papageorgiou [15] generalized the degree theory of Browder [12] to the mappings of the form $T + C + G$, where T is maximal monotone with $0 \in T(0)$, C bounded demicontinuous of type (S_+) and G belongs to class (P). In this section, with an application of Browder and Skrypnik degree theories, the existence of nonzero solutions of the inclusion $Tx+Cx+Gx \ni 0$ is established with an additional condition of positive homogeneity of degree $\alpha \in (0, 1]$ on T. The result extends and generalizes a similar result by Kartsatos and the author in [3, Theorem 6, p.1246, for $\alpha = 1$ and $G = 0$] to a multivalued T with $\alpha \in (0, 1]$ and $G \neq 0$. This result is new for $\alpha \in (0, 1)$ and applies to partial differential equations involving p-Laplacian with $p \in (1, 2]$.

In what follows, the norms in X and X^* are both denoted by $\|\cdot\|$ and will be understood from the context of their use.

Theorem 3.1. *Assume that $G_1, G_2 \subset X$ are open, bounded with $0 \in G_2$ and $\overline{G_2} \subset G_1$. Let $T : X \supset D(T) \to 2^{X^*}$ be maximal monotone, and positively homogeneous of degree $\alpha \in (0, 1]$, $C : \overline{G}_1 \to X^*$ bounded, demicontinuous and of type (S_+), and $G : \overline{G}_1 \to 2^{X^*}$ of class (P). Moreover, assume the following:*

- (H3) *There exists $v_0^* \in X^* \setminus \{0\}$ such that $Tx + Cx + Gx \not\ni \lambda v_0^*$ for every $(\lambda, x) \in \mathbb{R}_+ \times (D(T) \cap \partial G_1)$;*
- (H4) *$Tx + Cx + Gx + \lambda Jx \not\ni 0$ for every $(\lambda, x) \in \mathbb{R}_+ \times (D(T) \cap \partial G_2)$.*

Then the inclusion $Tx+Cx+Gx \ni 0$ has a nonzero solution $x \in D(T) \cap (G_1 \setminus G_2)$.

Proof. We consider the inclusion

$$Tx + Cx + Gx \ni 0$$

and then the associated approximate equation

$$T_tx + Cx + g_\epsilon x = 0. \tag{3.1}$$

Here, $\epsilon > 0$ and $g_\epsilon : \overline{G_1} \to X^*$ is an approximate continuous Cellina-selection (cf. [15], [6, Lemma 6, p. 236]) satisfying

$$g_\epsilon x \in G(B_\epsilon(x) \cap \overline{G_1}) + B_\epsilon(0)$$

for all $x \in \overline{G_1}$ and $g_\epsilon(\overline{G_1}) \subset \overline{\mathrm{conv}} G(\overline{G_1})$.

We show that equation (3.1) has a solution $x_{t,\epsilon}$ in $G_1 \setminus G_2$ for all sufficiently small t and ϵ. To this end, we first show that there exist $\tau_0 > 0$, $t_0 > 0$ and $\epsilon_0 > 0$ such that the equation

$$T_t x + Cx + g_\epsilon x = \tau v_0^* \tag{3.2}$$

has no solution in G_1 for every $\tau \geq \tau_0$, $t \in (0, t_0]$ and $\epsilon \in (0, \epsilon_0]$.

Assuming the contrary, let $\{\tau_n\} \subset (0, \infty)$, $\{t_n\} \subset (0, \infty)$, $\{\epsilon_n\} \subset (0, \infty)$ and $\{x_n\} \subset G_1$ be such that $\tau_n \to \infty$, $t_n \downarrow 0$, $\epsilon_n \downarrow 0$ and

$$T_{t_n} x_n + Cx_n + g_{\epsilon_n} x_n = \tau_n v_0^*. \tag{3.3}$$

We may assume that $g_{\epsilon_n} x_n \to g^* \in X^*$ in view of the properties of G. Then $\|T_{t_n} x_n\| \to \infty$ as $\|\tau_n v_0^*\| \to \infty$ and $\{Cx_n\}$ is bounded.

Thus, from (3.3), we obtain

$$\frac{T_{t_n} x_n}{\|T_{t_n} x_n\|} + \frac{Cx_n}{\|T_{t_n} x_n\|} + \frac{g_{\epsilon_n} x_n}{\|T_{t_n} x_n\|} = \frac{\tau_n}{\|T_{t_n} x_n\|} v_0^*, \tag{3.4}$$

In view of (1.7), we obtain

$$\frac{T_{t_n} x_n}{\|T_{t_n} x_n\|} = T_{t_n \lambda_n}\Big(\frac{x_n}{\|T_{t_n} x_n\|^{1/\alpha}}\Big), \tag{3.5}$$

where

$$\lambda_n = \|T_{t_n} x_n\|^{(\alpha-1)/\alpha}.$$

It clear that $\lambda_n \to 0$ for $\alpha \in (0,1)$ and $\lambda_n = 1$ for $\alpha = 1$. Then (3.4) implies

$$1 - \Big\|\frac{Cx_n}{\|T_{t_n} x_n\|} + \frac{g_{\epsilon_n} x_n}{\|T_{t_n} x_n\|}\Big\| \leq \frac{\tau_n \|v_0^*\|}{\|T_{t_n} x_n\|} \leq 1 + \Big\|\frac{Cx_n}{\|T_{t_n} x_n\|} + \frac{g_{\epsilon_n} x_n}{\|T_{t_n} x_n\|}\Big\|.$$

Thus,

$$\frac{\tau_n \|v_0^*\|}{\|T_{t_n} x_n\|} \to 1 \quad \text{and} \quad \frac{\tau_n}{\|T_{t_n} x_n\|} \to \frac{1}{\|v_0^*\|} \quad \text{as } n \to \infty. \tag{3.6}$$

Let

$$u_n = \frac{x_n}{\|T_{t_n} x_n\|^{1/\alpha}}.$$

We have $u_n \to 0$. By (3.4), (3.5) and (3.6), we obtain $T_{t_n \lambda_n} u_n \to h$ with

$$h = \frac{v_0^*}{\|v_0^*\|}.$$

Therefore

$$\lim_{n\to\infty} \langle T_{t_n \lambda_n} u_n, u_n \rangle = \langle h, 0 \rangle = 0.$$

Since $t_n \lambda_n \to 0$, by (ii) of Lemma 1.2 with $S = 0$ we obtain, $0 \in D(T)$ and $h = T(0)$. Since $T(0) = 0$, this is a contradiction to $\|h\| = 1$.

We now consider the homotopy mapping

$$H_1(s, x, t, \epsilon) = T_t x + Cx + g_\epsilon x - s\tau_0 v_0^*, \quad s \in [0,1],\ x \in \overline{G_1}, \tag{3.7}$$

where $t \in (0, t_0]$ and $\epsilon \in (0, \epsilon_0]$ are fixed. For every $s \in [0,1]$ the operator $x \mapsto Cx - s\tau_0 v_0^*$ is demicontinuous and bounded on $\overline{G_1}$. In order to see that it is of type (S_+), assume that $\{x_n\} \subset \overline{G_1}$ satisfies $x_n \rightharpoonup x_0 \in X$ and

$$\limsup_{n\to\infty} \langle Cx_n - s\tau_0 v_0^*, x_n - x_0\rangle \le 0.$$

Then

$$\limsup_{n\to\infty} \langle Cx_n, x_n - x_0\rangle \le 0,$$

which by the (S_+)-property of C, implies $x_n \to x_0 \in \overline{G_1}$. Before we consider the Skrypnik degree of this homotopy on the set G_1, we show that the equation $H_1(s, x, t, \epsilon) = 0$ has no solution on the boundary of G_1 for all sufficiently small $t \in (0, t_0]$, $\epsilon \in (0, \epsilon_0]$ and all $s \in [0, 1]$. To this end, assume the contrary and let $\{x_n\} \subset \partial G_1$, $\{t_n\} \subset (0, t_0]$, $\{s_n\} \subset [0,1]$ and $\{\epsilon_n\} \subset (0, \epsilon_0]$ such that $t_n \downarrow 0$, $s_n \to s_0$ for some $s_0 \in [0,1]$, $\epsilon_n \downarrow 0$ and

$$T_{t_n} x_n + Cx_n + g_{\epsilon_n} x_n = s_n \tau_0 v_0^*.$$

We may assume that $x_n \rightharpoonup x_0 \in X$. Since $\{Cx_n\}$ is bounded, we may assume that $Cx_n \rightharpoonup y_0^* \in X^*$ and $g_{\epsilon_n} x_n \to g^*$. Then we have $T_{t_n} x_n \rightharpoonup -y_0^* - g^* + s_0\tau_0 v_0^*$. From

$$\langle T_{t_n} x_n, x_n - x_0\rangle + \langle Cx_n, x_n - x_0\rangle = \langle g_{\epsilon_n} x_n + s_n \tau_0 v_0^*, x_n - x_0\rangle,$$

we obtain

$$\lim_{n\to\infty} [\langle T_{t_n} x_n, x_n - x_0\rangle + \langle Cx_n, x_n - x_0\rangle] = 0. \tag{3.8}$$

Let us assume that

$$\limsup_{n\to\infty} \langle Cx_n, x_n - x_0\rangle > 0. \tag{3.9}$$

Then there exists a subsequence of $\{x_n\}$, which we still denote by $\{x_n\}$, such that

$$\lim_{n\to\infty} \langle Cx_n, x_n - x_0\rangle = q, \tag{3.10}$$

for some constant $q > 0$. By (3.8) and (3.10), we obtain

$$\lim_{n\to\infty} \langle T_{t_n} x_n, x_n - x_0\rangle = -q < 0.$$

Applying (i) of Lemma 1.2 with $S = 0$, we obtain a contradiction. Therefore (3.9) is false and we now only have

$$\limsup_{n\to\infty} \langle Cx_n, x_n - x_0\rangle \le 0.$$

Since C is of type (S_+), we have $x_n \to x_0 \in \partial G_1$. Since C is also demicontinuous, $Cx_n \rightharpoonup Cx_0$. This implies

$$T_{t_n} x_n \rightharpoonup -Cx_0 - g^* + s_0 \tau_0 v_0^*.$$

Applying (ii) of Lemma 1.2 with $S = 0$, we obtain $x_0 \in D(T) \cap \partial G_1$ and

$$Tx_0 + Cx_0 + Gx_0 \ni s_0 \tau_0 v_0^*,$$

which is a contradiction to our hypothesis (H3). Thus, we may now choose t_0 and ϵ_0 further so that we also have that $H_1(s, x, t, \epsilon) = 0$ has no solution $x \in \partial G_1$ for all $t \in (0, t_0]$, $\epsilon \in (0, \epsilon_0]$ and all $s \in [0, 1]$. It is clear that the mapping $H_1(s, x, t, \epsilon)$ is an admissible homotopy for Skrypnik's degree and the Skrypnik degree $d_S(H_1(s, \cdot, t, \epsilon), G_1, 0)$ is well-defined and is constant for all $s \in [0, 1]$ and for all $t \in (0, t_0]$, $\epsilon \in (0, \epsilon_0]$. Consequently, the Browder's degree generalized by Hu and Papageorgiou [15], $\mathrm{d_{HP}}$, is well-defined and satisfies

$$\mathrm{d_{HP}}(T + C + G - \tau_0 v_0^*, G_1, 0) = \mathrm{d_S}(T_t + C + g_\epsilon - \tau_0 v_0^*, G_1, 0) \tag{3.11}$$

for $t \in (0, t_0], \epsilon \in (0, \epsilon_0]$.

Assume that

$$\mathrm{d_S}(H_1(1, \cdot, t_1, \epsilon_1), G_1, 0) \neq 0,$$

for some sufficiently small $t_1 \in (0, t_0]$ and $\epsilon_1 \in (0, \epsilon_0]$. Then, the equation

$$T_{t_1} x + Cx + g_{\epsilon_1} x = \tau_0 v_0^*$$

has a solution in the set G_1. However, this contradicts our choice of the number τ_0 in (3.2). Consequently,

$$\mathrm{d_S}(T_t + C + g_\epsilon, G_1, 0) = \mathrm{d_S}(H_1(0, \cdot, t_1, \epsilon_1), G_1, 0) = 0, \quad t \in (0, t_0],\ \epsilon \in (0, \epsilon_0].$$

We next consider the homotopy mapping

$$H_2(s, x, t, \epsilon) = s(T_t x + Cx + g_\epsilon x) + (1 - s)Jx, \quad (s, x) \in [0, 1] \times \overline{G_2}. \tag{3.12}$$

We first show that there exist $t_1 \in (0, t_0]$, $\epsilon_1 \in (0, \epsilon_0]$ such that the equation $H_2(s, x, t, \epsilon) = 0$ has no solution on ∂G_2 for any $s \in [0, 1]$, any $t \in (0, t_1]$ and any $\epsilon \in (0, \epsilon_1]$.

Let us assume the contrary. Then there exist sequences $t_n \in (0, t_0]$, $\epsilon_n \in (0, \epsilon_1]$, $s_n \in [0, 1]$, and $x_n \in \partial G_2$ such that $t_n \downarrow 0$, $\epsilon_n \downarrow 0$, $s_n \to s_0 \in [0, 1]$, $x_n \rightharpoonup x_0 \in X$, $Cx_n \rightharpoonup y_0^* \in X^*$, $g_{\epsilon_n} x_n \to g^* \in X^*$, $Jx_n \rightharpoonup z_0^* \in X^*$, and

$$s_n(T_{t_n} x_n + Cx_n + g_{\epsilon_n} x_n) + (1 - s_n)Jx_n = 0. \tag{3.13}$$

$s_n = 0$ is impossible because $J(0) = 0$ and J is injective, we may assume that $s_n > 0$, for all n. If $s_n \to 0$,

$$\langle T_{t_n} x_n + Cx_n, x_n \rangle = -\Big(\frac{1}{s_n} - 1\Big)\langle Jx_n, x_n \rangle - \langle g_{\epsilon_n} x_n, x_n \rangle \to -\infty \tag{3.14}$$

because $\{\|x_n\|\}$ is bounded below away from zero. Since $\langle T_{t_n} x_n, x_n \rangle \geq 0$ and $\{\langle Cx_n, x_n \rangle\}$ is bounded, we see that (3.14) is impossible. Thus $s_0 \in (0, 1]$ and (3.13) implies that

$$T_{t_n} x_n \rightharpoonup -y_0^* - g^* - \Big(\frac{1}{s_0} - 1\Big) z_0^*.$$

Also, from (3.13),

$$\begin{aligned} &\langle T_{t_n} x_n + Cx_n, x_n - x_0 \rangle \\ &= -\Big(\frac{1}{s_n} - 1\Big)\langle g_{\epsilon_n} x_n + Jx_n, x_n - x_0 \rangle \\ &= -\Big(\frac{1}{s_n} - 1\Big)\big[\langle Jx_n - Jx_0, x_n - x_0 \rangle + \langle g_{\epsilon_n} x_n + Jx_0, x_n - x_0 \rangle\big] \\ &\leq -\Big(\frac{1}{s_n} - 1\Big)\langle g_{\epsilon_n} x_n + Jx_0, x_n - x_0 \rangle, \end{aligned} \tag{3.15}$$

by the monotonicity of the duality mapping J. Since $s_0 \in (0, 1]$ and $x_n \rightharpoonup x_0$, we see from (3.15) that

$$\limsup_{n\to\infty} \{q_n := \langle T_{t_n} x_n + Cx_n, x_n - x_0 \rangle\} \leq 0.$$

Let

$$\limsup_{n\to\infty} \langle Cx_n, x_n - x_0 \rangle > 0. \tag{3.16}$$

Then, for some subsequence of $\{n\}$ denoted by $\{n\}$ again, we have

$$\lim_{n\to\infty}\langle Cx_n, x_n - x_0\rangle = q > 0. \tag{3.17}$$

From

$$\langle T_{t_n}x_n, x_n - x_0\rangle = q_n - \langle Cx_n, x_n - x_0\rangle,$$

we see that

$$\limsup_{n\to\infty}\langle T_{t_n}x_n, x_n - x_0\rangle \le \limsup_{n\to\infty} q_n + \lim_{n\to\infty}[-\langle Cx_n, x_n - x_0\rangle] \le -q < 0.$$

This implies

$$\limsup_{n\to\infty}\langle T_{t_n}x_n, x_n - x_0\rangle < 0.$$

Using (i) of Lemma 1.2, we conclude that (3.16) is impossible, and therefore (3.16) holds with "$\le$" in place of "$>$". Since C is of type (S_+), we have $x_n \to x_0 \in \partial G_2$. This implies $Cx_n \rightharpoonup Cx_0$, $Jx_n \to Jx_0$ and

$$T_{t_n}x_n \rightharpoonup -Cx_0 - g^* - \Big(\frac{1}{s_0} - 1\Big)Jx_0.$$

Since $x_n \to x_0$, we have

$$\lim_{n\to\infty}\langle T_{t_n}x_n, x_n - x_0\rangle = 0.$$

Using *ii* of Lemma 1.2, we have $x_0 \in D(T)$ and

$$-Cx_0 - g^* - \Big(\frac{1}{s_0} - 1\Big)Jx_0 \in Tx_0.$$

By a property of the selection $g_{\epsilon_n}x_n$ (cf. [15, p. 238]), we have $g^* \in G(x_0)$. This implies

$$Tx_0 + Cx_0 + Gx_0 + \Big(\frac{1}{s_0} - 1\Big)Jx_0 \ni 0.$$

We arrived at a contradiction to our hypothesis (H4) because $x_0 \in D(T)\cap\partial G_2$. For the sake of convenience, we assume that t_0 and ϵ_0 are sufficiently small so that we may take $t_1 = t_0$ and $\epsilon_1 = \epsilon_0$.

It is now clear that the mapping $H_2(s, x, t, \epsilon)$ is an admissible homotopy for Skrypnik's degree and so the Skrypnik degree $d_S(H_2(s, \cdot, t, \epsilon), G_2, 0)$ is well-defined and constant for all $s \in [0, 1]$, all $t \in (0, t_0]$ and all $\epsilon \in (0, \epsilon_0]$. By the invariance of the Skrypnik degree, for all $t \in (0, t_0]$, $\epsilon \in (0, \epsilon_0]$, we have

$$\begin{aligned} d_S(H_2(1, \cdot, t, \epsilon), G_2, 0) &= d_S(T_t + C + g_\epsilon, G_2, 0)\\ &= d_S(H_2(0, \cdot, t, \epsilon), G_2, 0)\\ &= d_S(J, G_2, 0) = 1. \end{aligned}$$

Thus, for all $t \in (0, t_0]$, $\epsilon \in (0, \epsilon_0]$, we have

$$d_S(T_t + C + g_\epsilon, G_1, 0) \ne d_S(T_t + C + g_\epsilon, G_2, 0).$$

From the excision property of the Skrypnik degree, which is an easy consequence of its finite-dimensional approximations, we obtain a solution $x_{t,\epsilon} \in G_1 \setminus G_2$ of $T_tx + Cx + g_\epsilon x = 0$ for every $t \in (0, t_0]$ and every $\epsilon \in (0, \epsilon_0]$. We let $t_n \in (0, t_0]$ and $\epsilon_n \in (0, \epsilon_0]$ be such that $t_n \downarrow 0$, $\epsilon_n \downarrow 0$ and let $x_n \in G_1 \setminus G_2$ be the corresponding solutions of $T_tx + Cx + g_\epsilon x = 0$. We have

$$T_{t_n}x_n + Cx_n + g_{\epsilon_n}x_n = 0.$$

We may assume that $x_n \rightharpoonup x_0$ and $g_{\epsilon_n} x_n \to g^* \in X^*$. We have

$$\langle T_{t_n} x_n, x_n - x_0\rangle = -\langle Cx_n + g_{\epsilon_n} x_n, x_n - x_0\rangle.$$

If

$$\limsup_{n\to\infty} \langle Cx_n + g_{\epsilon_n} x_n, x_n - x_0\rangle > 0,$$

then we obtain a contradiction from (i) of Lemma 1.2. Consequently,

$$\limsup_{n\to\infty} \langle Cx_n + g_{\epsilon_n} x_n, x_n - x_0\rangle \le 0,$$

and hence

$$\limsup_{n\to\infty} \langle Cx_n, x_n - x_0\rangle \le 0.$$

By the (S_+)-property of C, we obtain $x_n \to x_0 \in \overline{G_1 \setminus G_2}$. Then $Cx_n \rightharpoonup Cx_0$ and $T_{t_n} x_n \rightharpoonup -Cx_0 - g^*$. Using this in (ii) of Lemma 1.1, we obtain $x_0 \in D(T)$ and $-Cx_0 - g^* \in Tx_0$. By a property of the selection $g_{\epsilon_n} x_n$ (cf. [15, p. 238]), we have $g^* \in G(x_0)$ and therefore $Tx_0 + Cx_0 + Gx_0 \ni 0$ by Lemma 1.1. We also have

$$x_0 \in \overline{G_1 \setminus G_2} = (G_1 \setminus G_2) \cup \partial(G_1 \setminus G_2) \subset (G_1 \setminus G_2) \cup \partial G_1 \cup \partial G_2.$$

By conditions (H3) and (H4), we have $x_0 \notin \partial G_1 \cup \partial G_2$. Thus, $x_0 \in D(T) \cap (G_1 \setminus G_2)$ and the proof is complete. $\square$

4. Applications

Application 1. We consider the space $X = W_0^{m,p}(\Omega)$ with the integer $m \ge 1$, the number $p \in (1,\infty)$, and the domain $\Omega \subset \mathbb{R}^N$ with smooth boundary. We let N_0 denote the number of all multi-indices $\alpha = (\alpha_1, \dots, \alpha_N)$ such that $|\alpha| = \alpha_1 + \dots + \alpha_N \le m$. For $\xi = (\xi_\alpha)_{|\alpha|\le m} \in \mathbb{R}^{N_0}$, we have a representation $\xi = (\eta, \zeta)$, where $\eta = (\eta_\alpha)_{|\alpha|\le m-1} \in \mathbb{R}^{N_1}$, $\zeta = (\zeta_\alpha)_{|\alpha|=m} \in \mathbb{R}^{N_2}$ and $N_0 = N_1 + N_2$. We let

$$\xi(u) = (D^\alpha u)_{|\alpha|\le m}, \quad \eta(u) = (D^\alpha u)_{|\alpha|\le m-1}, \quad \zeta(u) = (D^\alpha u)_{|\alpha|=m},$$

where

$$D^\alpha u = \prod_{i=1}^N \Big(\frac{\partial}{\partial x_i}\Big)^{\alpha_i}.$$

Also, let $q = p/(p-1)$.

We now consider the partial differential operator in divergence form

$$(Au)(x) = \sum_{|\alpha|\le m} (-1)^{|\alpha|} D^\alpha A_\alpha(x, u(x), \dots, D^m u(x)), \quad x \in \Omega.$$

The coefficients $A_\alpha : \Omega \times \mathbb{R}^{N_0} \to \mathbb{R}$ are assumed to be Carathéodory functions, i.e., each $A_\alpha(x,\xi)$ is measurable in x for fixed $\xi \in \mathbb{R}^{N_0}$ and continuous in ξ for almost all $x \in \Omega$. We consider the following conditions:

(H5) There exist $p \in (1,\infty)$, $c_1 > 0$ and $\kappa_1 \in L^q(\Omega)$ such that

$$|A_\alpha(x,\xi)| \le c_1 |\xi|^{p-1} + \kappa_1(x), \quad x \in \Omega, \ \ \xi \in \mathbb{R}^{N_0}, \ \ |\alpha| \le m.$$

(H6) The Leray-Lions Condition

$$\sum_{|\alpha|=m} [A_\alpha(x,\eta,\zeta_1) - A_\alpha(x,\eta,\zeta_2)](\zeta_{1_\alpha} - \zeta_{2_\alpha}) > 0$$

is satisfied for every $x \in \Omega$, $\eta \in \mathbb{R}^{N_1}$, $\zeta_1, \zeta_2 \in \mathbb{R}^{N_2}$ with $\zeta_1 \ne \zeta_2$.

(H7)
$$\sum_{|\alpha|\le m}[A_\alpha(x,\xi_1)-A_\alpha(x,\xi_2)](\xi_{1_\alpha}-\xi_{2_\alpha}))\ge 0$$
is satisfied for every $x\in\Omega$, $\xi_1,\xi_2\in\mathbb{R}^{N_0}$.

(H8) There exist $c_2>0$, $\kappa_2\in L^1(\Omega)$ such that
$$\sum_{|\alpha|\le m}A_\alpha(x,\xi)\xi_\alpha\ge c_2|\xi|^p-\kappa_2(x),\quad x\in\Omega,\ \xi\in\mathbb{R}^{N_0}.$$

If an operator $T:W_0^{m,p}(\Omega)\to W^{-m,q}(\Omega)$ is given by
$$\langle Tu,v\rangle=\int_\Omega\sum_{|\alpha|\le m}A_\alpha(x,\xi(u))D^\alpha v,\quad u,\,v\in W_0^{m,p}(\Omega), \tag{4.1}$$
then conditions (H5), (H7) imply that it is bounded, continuous and monotone (cf. e.g. Kittila [22, pp. 25-26], Pascali and Sburlan [23, pp. 274-275]). Since T is continuous, it is maximal monotone. Similarly, condition (H5), with A replaced by B, implies that the operator
$$\langle Cu,v\rangle=\int_\Omega\sum_{|\alpha|\le m}B_\alpha(x,\xi(u))D^\alpha v,\qquad u,\,v\in W_0^{m,p}(\Omega), \tag{4.2}$$
is a bounded continuous mapping. We also know that conditions (H5), (H6) and (H8), with B in place of A everywhere, imply that the operator C is of type (S_+) (cf. Kittila [22, p. 27]).

We also consider a multifunction $H:\Omega\times\mathbb{R}^{N_1}\to 2^{\mathbf{R}}$ such that

(H9) $H(x,r)=[\varphi(x,r),\psi(x,r)]$ is measurable in x and u.s.c. in r, where $\varphi,\psi:\Omega\times\mathbb{R}^{N_1}\to\mathbf{R}$ are measurable functions;

(H10) $|H(x,r)|=\max[|\varphi(x,r)|,|\psi(x,r)|]\le a(x)+c_2|r|$ a.e. on $\Omega\times\mathbb{R}^{N_1}$ and $a(\cdot)\in L^q(\Omega)$, $c_2>0$.

Define $G:W_0^{m,p}\to 2^{W^{-m,q}(\Omega)}$ by
$$\begin{gathered}Gu=\Big\{h\in W^{-m,q}(\Omega):\exists w\in L^q(\Omega)\text{ such that }w(x)\in H(x,u(x))\\ \text{and }\langle h,v\rangle=\int_\Omega w(x)v(x)\text{ for all }v\in W_0^{m,p}(\Omega)\Big\}.\end{gathered}$$
It is well-known that G is u.s.c and compact with closed and convex values (cf. [15, p. 254]), and therefore is of class (P).

We now state the following theorem as an application of Theorem 3.1.

Theorem 4.1. *Assume that the operators T, C and G defined as above with $T(0)=0$, $C(0)=0$. Assume, further, that the rest of the conditions of Theorem 3.1 are satisfied for two balls $G_1=B_r(0)$ and $G_2=B_q(0)$, where $0<q<r$. Then the Dirichlet boundary value problem*
$$\begin{gathered}(Au)(x)+(Bu)(x)+(Hu)(x)\ni 0,\quad x\in\Omega,\\ (D^\alpha u)(x)=0,\quad x\in\partial\Omega,\quad |\alpha|\le m-1,\end{gathered}$$
has a "weak" nonzero solution $u\in B_r(0)\setminus B_q(0)\subset W_0^{m,p}(\Omega)$, which satisfies the equation $Tu+Cu+Gu\ni 0$.

In light of recent degree theories for more general combinations of operators, such as the ones in [4], the results of this paper may be generalized. For the triplet $T+C+G$ in Theorem 2.2, the existence of nonzero solutions for the homogeneity condition for degree $\alpha > 1$ ($p > 2$ for p-Laplacian operator A in Theorem 4.1) needs further work.

Application 2. Let Ω be a bounded open set in $\mathbb{R}^N$ with smooth boundary, $m \geq 1$ an integer, and $T > 0$. Set $Q = \Omega \times [0, a]$. We consider the differential operator

$$\begin{aligned} &\frac{\partial u(x,t)}{\partial t} + \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha A_\alpha(x, t, u(x,t), Du(x,t), \dots, D^m u(x,t)) \\ &+ \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha B_\alpha(x, t, u(x,t), Du(x,t), \dots, D^m u(x,t)) \end{aligned} \tag{4.3}$$

in Q. The coefficients $A_\alpha = A_\alpha(x,t,\xi)$, are defined for $(x,t) \in Q$, $\xi = \{\xi_\gamma, |\gamma| \leq m\} = (\eta, \zeta) \in \mathbb{R}^{N_0}$ with $\eta = \{\eta_\gamma, |\gamma| \leq m-1\} \in \mathbb{R}^{N_1}$, $\zeta = \{\zeta_\gamma, |\gamma| = m\} \in \mathbb{R}^{N_2}$, and $N_1 + N_2 = N_0$. We assume that each coefficient $A_\alpha(x,t,\xi)$ satisfies the usual Carathéodory conditions. We consider the following conditions.

(H11) (Continuity) For some $p \geq 2$, $c_1 > 0$, $g \in L^q(Q)$ with $q = p/(p-1)$, we have
$$|A_\alpha(x,t,\eta,\zeta)| \leq c_1(|\zeta|^{p-1} + |\eta|^{p-1} + g(x,t)),$$
for $(x,t) \in Q$, $\xi = (\eta, \zeta) \in \mathbb{R}^{N_0}$, $|a| \leq m$.

(H12) (Monotonicity)
$$\sum_{|\alpha| \leq m} (A_\alpha(x,t,\xi_1) - A_\alpha(x,t,\xi_2))(\xi_{1_\gamma} - \xi_{2_\gamma}) \geq 0, \quad (x,t) \in Q,\ \xi_1, \xi_2 \in \mathbb{R}^{N_0}.$$

(H13) (Leray-Lions)
$$\sum_{|\alpha| = m} (A_\alpha(x,t,\eta,\zeta) - A_\alpha(x,t,\eta,\zeta^*))(\zeta_\gamma - \zeta^*_\gamma) > 0,$$
for $(x,t) \in Q$, $\eta \in \mathbb{R}^{N_1}$, $\zeta, \zeta^* \in \mathbb{R}^{N_2}$.

(H14) (Coercivity) There exist $c_0 > 0$ and $h \in L^1(Q)$ such that
$$\sum_{|a| \leq m} A_\alpha(x,t,\xi) \geq c_0 |\xi|^p - h(x,t), \quad (x,t) \in Q,\ \xi \in \mathbb{R}^{N_0}.$$

Under the condition (H11), the second term of (4.3) generates a continuous bounded operator $T : X \to X^*$, where $X = L^p(0,a;V), X^* = L^q(0,a;V^*)$, and $V = W_0^{m,p}(\Omega)$. It is defined by
$$\langle Tu, v \rangle = \sum_{|\alpha| \leq m} \int_Q A_\alpha(x,t,u,Du,\dots,D^m u) D^\alpha v, \quad u, v \in X.$$

This operator is also maximal monotone under the condition (H12). Under (H11), (H13) and (H14) (with "A" replaced by "B" and the other necessary changes) the third term of (4.3) generates a continuous, bounded operator C which satisfies the condition (S_+) w.r.t. $D(L)$, where the operator L is defined below. The operator C is defined by
$$\langle Cu, v \rangle = \sum_{|\alpha| \leq m} \int_Q B_\alpha(x,t,u,Du,\dots,D^m u) D^\alpha v, \quad u, v \in X.$$

The operator $\partial/\partial t$ generates an operator $L : X \supset D(L) \to X^*$, where

$$D(L) = \{v \in X : v' \in X^*,\ v(0) = 0\},$$

via the relation

$$\langle Lu, v\rangle = \int_0^a \langle u'(t), v(t)\rangle dt, \quad u \in D(L),\ v \in X.$$

The symbol $u'(t)$ above is the generalized derivative of $u(t)$, i.e.

$$\int_0^a \langle u'(t), \varphi(t)\rangle\, dt = -\int_0^a \langle \varphi'(t), u(t)\rangle\, dt, \quad \varphi \in C_0^\infty(0, a; X).$$

One can verify, as in Zeidler [28], that L is a linear densely defined maximal monotone operator.

Let K be an unbounded closed convex proper subset of X with $0 \in \mathring{K}$. Let $\varphi_K : X \to \mathbb{R}_+ \cup \{\infty\}$ be defined by

$$\varphi_K(x) = \begin{cases} 0 & \text{if } x \in K, \\ \infty & \text{otherwise.} \end{cases}$$

The function φ_K is proper convex and lower semicontinuous on X, and $x^* \in \partial\varphi_K(x)$, for $x \in K$, if and only if

$$\langle x^*, y - x\rangle \le 0, \quad \text{for all } y \in K.$$

Also,

$$D(\partial\varphi_K) = K \quad \text{and} \quad 0 \in \partial\varphi_K(x), \quad x \in K,$$
$$\partial\varphi_K(x) = \{0\}, \quad x \in \mathring{K}.$$

The operator $\partial\varphi_K : X \supset K \to 2^{X^*}$ is maximal monotone with $0 \in \mathring{D}(\partial\varphi_K)$ and $0 \in \partial\varphi_K(0)$. It is thus strongly quasibounded. For these facts see, e.g., Kenmochi [21]. In addition, the sum $\partial\varphi_K + T$ is a multivalued strongly quasibounded maximal monotone operator from K to 2^{X^*}.

As an application of Theorem 2.2, we state the following theorem.

Theorem 4.2. *Assume that the operators L, T, C are as above with A_α satisfying (H11), (H12) and $T(0) = 0$, $C(0) = 0$, and B_α satisfying (H11), (H13) and (H14) with the necessary notational changes. Assume, further, that the rest of the conditions of Theorem 2.2 are satisfied for two balls $G_1 = B_r(0)$ and $G_2 = B_q(0)$, in $X = L^p(0, a, V)$, where $0 < q < r$ and $V = W_0^m(\Omega)$. Then the inclusion*

$$Lu + \partial\varphi_K(u) + Tu + Cu \ni 0$$

has a nonzero solution $u \in B_r(0) \setminus B_q(0)$.

The mapping $\partial\varphi_K$ above is essential because the operator $T + C$ is demicontinuous, bounded and of type (S_+) w.r.t. $D(L)$, and therefore it reduces to another operator exactly like C (cf. [5, p.41]).

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