Electronic Journal of Differential Equations, Vol. 2017 (2017), No. 153, pp. 1–21.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

TWO-DIMENSIONAL PRODUCT-TYPE SYSTEMS OF DIFFERENCE EQUATIONS OF DELAY-TYPE (2,2,1,2)

STEVO STEVIĆ

Communicated by Vicentiu Radulescu

Abstract. We prove that the following class of systems of difference equations is solvable in closed form:

$$z_{n+1} = \alpha z_{n-1}^a w_n^b, \quad w_{n+1} = \beta w_{n-1}^c z_{n-1}^d, \quad n \in \mathbb{N}_0,$$

where $a, b, c, d \in \mathbb{Z}$, $\alpha, \beta, z_{-1}, z_0, w_{-1}, w_0 \in \mathbb{C} \setminus \{0\}$. We present formulas for its solutions in all the cases. The most complex formulas are presented in terms of the zeros of three different associated polynomials to the systems corresponding to the cases $a = 0$, $c = 0$ and $abcd \neq 0$, respectively, which on the other hand depend on some of parameters a, b, c, d.

1. Introduction

There has been a considerable interest in difference equations and systems of difference equations (see, for example, [1, 2, 3, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35]). Books [1, 7, 9, 10, 12], among others, present some classical methods for solving some classes of the equations and systems. The topic has re-attracted some attention in the last decade (see, for example, [2, 3, 20, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35] and numerous related references therein). Many nonlinear equations and systems which have appeared in the literature recently were solved by transforming them to the classical solvable ones, by using some suitable changes of variables (see, for example, [3, 20, 29], as well as the related references therein). On the other hand, almost two decades ago Papaschinopoulos and Schinas have started investigating some concrete systems of difference equations ([14, 15, 16]), which motivated other experts to investigate some related ones ([3, 5, 13, 17, 18, 19, 24, 25, 26, 27, 28, 30, 31, 32, 33, 33, 34, 35]). Majority of the systems investigated therein belong to the class of symmetric or close to symmetric systems. Namely, special cases of the following symmetric two-dimensional system of difference equations

$$x_n = f(x_{n-k}, y_{n-l}), \quad y_n = f(y_{n-k}, x_{n-l}),$$

$n \in \mathbb{N}_0$, have been studied a lot.

2010 *Mathematics Subject Classification.* 39A20, 39A45.
Key words and phrases. System of difference equations; product-type system; solvable in closed form.

Submitted April 21, 2017. Published June 27, 2017.

Many recent papers on difference equations and systems study only their positive solutions. One of the reasons for this is the fact that numerous difference equations and systems present some population models. Some systems of difference equations include product-type ones, which are obtained for some special values of its coefficients and/or parameters. It is well-known how product-type systems with positive initial values and coefficients are solved. Studying non-positive solutions is a more interesting problem. We started investigating complex-valued product-type systems in [31]. Another system was investigated in [32], where some further basic steps concerning solvability of such systems were presented. Having published [29] we came up with an idea of studying product-type systems which, unlike the ones in [31] and [32], have some multipliers. The system in [26] was the first complex-valued product-type system with multipliers that we have studied. Papers [26, 31, 32], suggested us to study the solvability of the following two-dimensional product-type system

$$z_n = \alpha z_{n-k}^a w_{n-l}^b, \quad w_n = \beta w_{n-m}^c z_{n-s}^d, \quad n \in \mathbb{N}, \tag{1.1}$$

where $a, b, c, d, k, l, m, s \in \mathbb{N}$ and $\alpha, \beta \in \mathbb{C}$. We call system (1.1) two-dimensional product-type system of *delay-type* (k, m, l, s).

The main problem, which is a relatively big project, is to present closed-form formulas for the solvable systems of the type (this was not done for the systems in [32] and [34]). The systems in [28] and [35] were more complex than the one in [26], and non-trivial analyses of the form of their solutions in terms of the initial values and especially parameters were needed. The structures of the solutions for the system in [33] is simpler, so the corresponding analysis was also simpler. Another approach in dealing with the solvability problem for product-type systems can be found in [27]. To each system a few polynomials can be associated for determining the form of its solutions. A fourth order one helped in presenting formulas for solutions to the system in [30].

The solvability of system (1.1) of delay-type $(2, 2, 1, 2)$, that is, of

$$z_{n+1} = \alpha z_{n-1}^a w_n^b, \quad w_{n+1} = \beta w_{n-1}^c z_{n-1}^d, \quad n \in \mathbb{N}_0, \tag{1.2}$$

where $a, b, c, d \in \mathbb{Z}$, $\alpha, \beta, z_{-1}, z_0, w_{-1}, w_0 \in \mathbb{C}$, is investigated in this paper, continuing the project in [26, 27, 28, 30, 31, 32, 33, 34, 35].

Note that if some of the quantities $\alpha, \beta, w_{-1}, w_0, z_{-1}, z_0$ are zero, then the solutions are either not defined or trivial, so not interesting. Hence, the case is excluded from further considerations. If $m, n \in \mathbb{Z}$, then $\overline{m,n} = \{j \in \mathbb{Z},\ m \le j \le n\}$, while $\sum_{j=n}^{n-1} f_j = 0$ for each $n \in \mathbb{Z}$.

2. Auxiliary results

The following set of lemmas presents standard tools for dealing with the problem of solvability of product-type systems with small delays k, l, m, s. For the first lemma, which is classical one, see, for example, [4, 9, 32].

Lemma 2.1. *Let*

$$p_k(t) = c_k \prod_{j=1}^{k} (t - t_j),$$

$c_k \neq 0$ and $t_i \neq t_j$, $i \neq j$. Then

$$\sum_{j=1}^{k} \frac{t_j^s}{p_k'(t_j)} = 0, \quad 0 \le s \le k-2,$$

$$\sum_{j=1}^{k} \frac{t_j^{k-1}}{p_k'(t_j)} = \frac{1}{c_k}.$$

For the second lemma see, for example, [9, 12] (or [28] for a general method for calculating the sums in the lemma).

Lemma 2.2. *Let*

$$s_n^{(i)}(z) = \sum_{j=1}^{n} j^i z^{j-1}, \quad n \in \mathbb{N},$$

where $i \in \mathbb{N}_0$ and $z \in \mathbb{C}$. Then

$$s_n^{(0)}(z) = \frac{1-z^n}{1-z},$$

$$s_n^{(1)}(z) = \frac{1-(n+1)z^n + nz^{n+1}}{(1-z)^2},$$

$$s_n^{(2)}(z) = \frac{1+z-(n+1)^2 z^n + (2n^2+2n-1)z^{n+1} - n^2 z^{n+2}}{(1-z)^3},$$

$$s_n^{(3)}(z) = \frac{n^3 z^n (z-1)^3 - 3n^2 z^n (z-1)^2 + 3nz^n(z^2-1) - (z^n-1)(z^2+4z+1)}{(1-z)^4},$$

for every $z \in \mathbb{C} \setminus \{1\}$ and $n \in \mathbb{N}$.

The results in [22] are suitably reformulated in the following lemma.

Lemma 2.3. *Let*

$$P_4(t) = t^4 + bt^3 + ct^2 + dt + e, \quad \Delta_0 = c^2 - 3bd + 12e,$$

$$\Delta_1 = 2c^3 - 9bcd + 27b^2 e + 27d^2 - 72ce, \quad \Delta = \frac{1}{27}(4\Delta_0^3 - \Delta_1^2),$$

$$P = 8c - 3b^2, \quad Q = b^3 + 8d - 4bc, \quad D = 64e - 16c^2 + 16b^2 c - 16bd - 3b^4.$$

(a) *If $\Delta < 0$, then two zeros of P_4 are real and different, and two are complex conjugate;*

(b) *If $\Delta > 0$, then all the zeros of P_4 are real or none is. More precisely,*
  1 *if $P < 0$ and $D < 0$, then all four zeros of P_4 are real and different;*
  2 *if $P > 0$ or $D > 0$, then there are two pairs of complex conjugate zeros of P_4.*

(c) *If $\Delta = 0$, then and only then P_4 has a multiple zero. The following cases can occur:*
  1 *if $P < 0$, $D < 0$ and $\Delta_0 \neq 0$, then two zeros of P_4 are real and equal and two are real and simple;*
  2 *if $D > 0$ or ($P > 0$ and ($D \neq 0$ or $Q \neq 0$)), then two zeros of P_4 are real and equal and two are complex conjugate;*
  3 *if $\Delta_0 = 0$ and $D \neq 0$, there is a triple zero of P_4 and one simple, all real;*
  4 *if $D = 0$, then*

4.1 *if* $P < 0$ *there are two double real zeros of* P_4*;*
4.2 *if* $P > 0$ *and* $Q = 0$ *there are two double complex conjugate zeros of* P_4*;*
4.3 *if* $\Delta_0 = 0$*, then all four zeros of* P_4 *are real and equal to* $-b/4$*.*

3. Main results

This section contains our main results. Before formulating them we give a list of a few members of sequences z_n and w_n that we use in proofs of the results

$$\begin{gathered} z_1 = \alpha z_{-1}^a w_0^b, \quad z_2 = \alpha\beta^b w_{-1}^{bc} z_{-1}^{bd} z_0^a \\ w_1 = \beta w_{-1}^c z_{-1}^d, \quad w_2 = \beta w_0^c z_0^d. \end{gathered} \tag{3.1}$$

Theorem 3.1. *Assume that* $a, c, d \in \mathbb{Z}$*,* $b = 0$*,* $\alpha, \beta, z_{-1}, z_0, w_{-1}, w_0 \in \mathbb{C} \setminus \{0\}$*. Then system* (1.2) *is solvable in closed form.*

Proof. From the condition $b = 0$, we have

$$z_{n+1} = \alpha z_{n-1}^a, \quad w_{n+1} = \beta w_{n-1}^c z_{n-1}^d, \quad n \in \mathbb{N}_0. \tag{3.2}$$

The first equation in (3.2) yields

$$z_{2n+i} = \alpha^{\sum_{j=0}^{n-1} a^j} z_i^{a^n}, \tag{3.3}$$

for $n \in \mathbb{N}$ and $i = -1, 0$, from which it follows that

$$z_{2n+i} = \alpha^{\frac{1-a^n}{1-a}} z_i^{a^n}, \tag{3.4}$$

for $n \in \mathbb{N}$ and $i = -1, 0$, when $a \neq 1$, and

$$z_{2n+i} = \alpha^n z_i, \tag{3.5}$$

for $n \in \mathbb{N}$ and $i = -1, 0$, when $a = 1$.

Using (3.3) in the second equation in (3.2) we obtain

$$w_{2n+i} = \beta\alpha^{d\sum_{j=0}^{n-2} a^j} z_i^{da^{n-1}} w_{2(n-1)+i}^c, \tag{3.6}$$

for $n \in \mathbb{N}$ and $i = -1, 0$.

From (3.6) and by induction it is proved that

$$w_{2n+i} = \beta^{\sum_{l=0}^{n-1} c^l} \alpha^{d\sum_{l=0}^{n-1} c^l \sum_{j=0}^{n-l-2} a^j} z_i^{d\sum_{l=0}^{n-1} c^l a^{n-l-1}} w_i^{c^n}, \tag{3.7}$$

for $n \in \mathbb{N}$ and $i = -1, 0$.

Case $a \neq 1 \neq c \neq a$**.** From (3.7) and by Lemma 2.2, in this case, we have

$$\begin{aligned} w_{2n+i} &= \beta^{\sum_{l=0}^{n-1} c^l} \alpha^{d\sum_{l=0}^{n-1} c^l \sum_{j=0}^{n-l-2} a^j} z_i^{d\sum_{l=0}^{n-1} c^l a^{n-l-1}} w_i^{c^n} \\ &= \beta^{\frac{1-c^n}{1-c}} \alpha^{d\sum_{l=0}^{n-1} c^l \frac{1-a^{n-l-1}}{1-a}} z_i^{d\frac{a^n-c^n}{a-c}} w_i^{c^n} \\ &= \beta^{\frac{1-c^n}{1-c}} \alpha^{\frac{d}{1-a}\left(\frac{1-c^n}{1-c} - \frac{a^n-c^n}{a-c}\right)} z_i^{d\frac{a^n-c^n}{a-c}} w_i^{c^n} \\ &= \beta^{\frac{1-c^n}{1-c}} \alpha^{\frac{d(a-c+(1-a)c^n+(c-1)a^n)}{(1-a)(1-c)(a-c)}} z_i^{d\frac{a^n-c^n}{a-c}} w_i^{c^n}, \end{aligned} \tag{3.8}$$

for $n \in \mathbb{N}$ and $i = -1, 0$.

Case $a = c \neq 1$**.** From (3.7) and by Lemma 2.2, in this case, we have

$$\begin{aligned} w_{2n+i} &= \beta^{\sum_{l=0}^{n-1} a^l} \alpha^{d \sum_{l=0}^{n-1} a^l \sum_{j=0}^{n-l-2} a^j} z_i^{d \sum_{l=0}^{n-1} a^l a^{n-l-1}} w_i^{a^n} \\ &= \beta^{\frac{1-a^n}{1-a}} \alpha^{d \sum_{l=0}^{n-1} a^l \frac{1-a^{n-l-1}}{1-a}} z_i^{dna^{n-1}} w_i^{a^n} \\ &= \beta^{\frac{1-a^n}{1-a}} \alpha^{\frac{d}{1-a}(\frac{1-a^n}{1-a} - na^{n-1})} z_i^{dna^{n-1}} w_i^{a^n} \\ &= \beta^{\frac{1-a^n}{1-a}} \alpha^{\frac{d((n-1)a^n - na^{n-1}+1)}{(1-a)^2}} z_i^{dna^{n-1}} w_i^{a^n}, \end{aligned} \tag{3.9}$$

for $n \in \mathbb{N}$ and $i = -1, 0$.

Case $a \neq 1 = c$**.** From (3.7) and by Lemma 2.2, in this case, we have

$$\begin{aligned} w_{2n+i} &= \beta^{\sum_{l=0}^{n-1} 1} \alpha^{d \sum_{l=0}^{n-1} \sum_{j=0}^{n-l-2} a^j} z_i^{d \sum_{l=0}^{n-1} a^{n-l-1}} w_i \\ &= \beta^n \alpha^{d \sum_{l=0}^{n-1} \frac{1-a^{n-l-1}}{1-a}} z_i^{d \frac{a^n-1}{a-1}} w_i \\ &= \beta^n \alpha^{\frac{d}{1-a}(n - \frac{a^n-1}{a-1})} z_i^{d \frac{a^n-1}{a-1}} w_i \\ &= \beta^n \alpha^{d \frac{a^n - an + n - 1}{(1-a)^2}} z_i^{d \frac{a^n-1}{a-1}} w_i, \end{aligned} \tag{3.10}$$

for $n \in \mathbb{N}$ and $i = -1, 0$.

Case $a = 1 \neq c$**.** From (3.7) and by Lemma 2.2, in this case, we have

$$\begin{aligned} w_{2n+i} &= \beta^{\sum_{l=0}^{n-1} c^l} \alpha^{d \sum_{l=0}^{n-1} c^l (n-l-1)} z_i^{d \sum_{l=0}^{n-1} c^l} w_i^{c^n} \\ &= \beta^{\frac{1-c^n}{1-c}} \alpha^{d((n-1)\frac{1-c^n}{1-c} - c\frac{1-nc^{n-1}+(n-1)c^n}{(1-c)^2}} z_i^{d \frac{1-c^n}{1-c}} w_i^{c^n} \\ &= \beta^{\frac{1-c^n}{1-c}} \alpha^{d \frac{c^n - nc + n - 1}{(1-c)^2}} z_i^{d \frac{1-c^n}{1-c}} w_i^{c^n}, \end{aligned} \tag{3.11}$$

for $n \in \mathbb{N}$ and $i = -1, 0$.

Case $a = c = 1$**.** From (3.7) and by a well-known formula, in this case, we have

$$\begin{aligned} w_{2n+i} &= \beta^{\sum_{l=0}^{n-1} 1} \alpha^{d \sum_{l=0}^{n-1} (n-l-1)} z_i^{d \sum_{l=0}^{n-1} 1} w_i \\ &= \beta^n \alpha^{d \frac{(n-1)n}{2}} z_i^{dn} w_i, \end{aligned} \tag{3.12}$$

for $n \in \mathbb{N}$ and $i = -1, 0$.

From all above mentioned the theorem follows. □

Corollary 3.2. *Assume that* $a, c, d \in \mathbb{Z}$*,* $b = 0$*,* $\alpha, \beta, z_{-1}, z_0, w_{-1}, w_0 \in \mathbb{C} \setminus \{0\}$*. Then the following statements are true.*

(a) *If* $a \neq 1 \neq c \neq a$*, then the general solution to system* (1.2) *is given by formulas* (3.4) *and* (3.8)*.*
(b) *If* $a = c \neq 1$*, then the general solution to system* (1.2) *is given by formulas* (3.4) *and* (3.9)*.*
(c) *If* $a \neq 1 = c$*, then the general solution to system* (1.2) *is given by formulas* (3.4) *and* (3.10)*.*
(d) *If* $a = 1 \neq c$*, then the general solution to system* (1.2) *is given by formulas* (3.5) *and* (3.11)*.*
(e) *If* $a = c = 1$*, then the general solution to system* (1.2) *is given by formulas* (3.5) *and* (3.12)*.*

Theorem 3.3. *Assume that* $a, b, c \in \mathbb{Z}$, $d = 0$, $\alpha, \beta, z_{-1}, z_0, w_{-1}, w_0 \in \mathbb{C} \setminus \{0\}$. *Then system* (1.2) *is solvable in closed form.*

Proof. Using the condition $d = 0$ in (1.2) we have

$$z_{n+1} = \alpha z_{n-1}^a w_n^b, \quad w_{n+1} = \beta w_{n-1}^c, \quad n \in \mathbb{N}_0. \tag{3.13}$$

This system corresponds the one in [33, Theorem 2.3], where parameters a and b, as well as c and d are interchanged, respectively. Hence, all the formulas for solutions to (3.13) are obtained from [33, Theorem 2.2 and Corollary 2.2]. □

Theorem 3.4. *Assume that* $b, c, d \in \mathbb{Z}$, $bd \neq 0$, $a = 0$, $\alpha, \beta, z_{-1}, z_0, w_{-1}, w_0 \in \mathbb{C} \setminus \{0\}$. *Then system* (1.2) *is solvable in closed form.*

Proof. Using the condition $a = 0$ in (1.2), we have

$$z_{n+1} = \alpha w_n^b, \quad w_{n+1} = \beta w_{n-1}^c z_{n-1}^d, \quad n \in \mathbb{N}_0. \tag{3.14}$$

From (3.14) we easily get

$$w_{n+1} = \alpha^d \beta w_{n-1}^c w_{n-2}^{bd}, \quad n \geq 2. \tag{3.15}$$

Let $\mu = \alpha^d \beta$,

$$a_1 = 0, \quad b_1 = c, \quad c_1 = bd, \quad y_1 = 1. \tag{3.16}$$

Then

$$w_{n+1} = \mu^{y_1} w_n^{a_1} w_{n-1}^{b_1} w_{n-2}^{c_1}, \quad n \geq 2. \tag{3.17}$$

Similarly, (3.17) implies

$$\begin{aligned} w_{n+1} &= \mu^{y_1} (\mu w_{n-1}^{a_1} w_{n-2}^{b_1} w_{n-3}^{c_1})^{a_1} w_{n-1}^{b_1} w_{n-2}^{c_1}, \\ &= \mu^{y_1 + a_1} w_{n-1}^{a_1 a_1 + b_1} w_{n-2}^{b_1 a_1 + c_1} w_{n-3}^{c_1 a_1} \\ &= \mu^{y_2} w_{n-1}^{a_2} w_{n-2}^{b_2} w_{n-3}^{c_2}, \end{aligned} \tag{3.18}$$

for $n \geq 3$, where

$$a_2 := a_1 a_1 + b_1, \quad b_2 := b_1 a_1 + c_1, \quad c_2 := c_1 a_1, \quad y_2 := y_1 + a_1. \tag{3.19}$$

Assume

$$w_{n+1} = \mu^{y_k} w_{n+1-k}^{a_k} w_{n-k}^{b_k} w_{n-k-1}^{c_k}, \tag{3.20}$$

for a $k \geq 2$ and all $n \geq k + 1$, and

$$a_k = a_1 a_{k-1} + b_{k-1}, \quad b_k = b_1 a_{k-1} + c_{k-1}, \quad c_k = c_1 a_{k-1}, \tag{3.21}$$

$$y_k = y_{k-1} + a_{k-1}. \tag{3.22}$$

If we replace n by $n - k$ in (3.17), and employ it in (3.20), we obtain

$$\begin{aligned} w_{n+1} &= \mu^{y_k} (\mu w_{n-k}^{a_1} w_{n-k-1}^{b_1} w_{n-k-2}^{c_1})^{a_k} w_{n-k}^{b_k} w_{n-k-1}^{c_k} \\ &= \mu^{y_k + a_k} w_{n-k}^{a_1 a_k + b_k} w_{n-k-1}^{b_1 a_k + c_k} w_{n-k-2}^{c_1 a_k} \\ &= \mu^{y_{k+1}} w_{n-k}^{a_{k+1}} w_{n-k-1}^{b_{k+1}} w_{n-k-2}^{c_{k+1}}, \end{aligned} \tag{3.23}$$

for $n \geq k + 2$, where

$$a_{k+1} := a_1 a_k + b_k, \quad b_{k+1} := b_1 a_k + c_k, \quad c_{k+1} := c_1 a_k, \quad y_{k+1} := y_k + a_k. \tag{3.24}$$

Equalities (3.18), (3.19), (3.23), (3.24) along with the induction show that (3.20)-(3.22) hold.

Setting $k = n-1$ in (3.20) and using (3.1), (3.21) and (3.22), we have

$$\begin{aligned} w_{n+1} &= \mu^{y_{n-1}} w_2^{a_{n-1}} w_1^{b_{n-1}} w_0^{c_{n-1}} \\ &= (\alpha^d\beta)^{y_{n-1}} (\beta w_0^c z_0^d)^{a_{n-1}} (\beta w_{-1}^c z_{-1}^d)^{b_{n-1}} w_0^{c_{n-1}} \\ &= \alpha^{dy_{n-1}} \beta^{y_{n-1}+a_{n-1}} z_{-1}^{db_{n-1}} z_0^{da_{n-1}} w_{-1}^{cb_{n-1}} w_0^{ca_{n-1}+c_{n-1}} \\ &= \alpha^{dy_{n-1}} \beta^{y_n} z_{-1}^{da_n} z_0^{da_{n-1}} w_{-1}^{ca_n} w_0^{a_{n+1}}, \end{aligned} \tag{3.25}$$

for $n \ge 2$.

From (3.21) we obtain

$$a_k = b_1 a_{k-2} + c_1 a_{k-3}, \quad \text{for } k \ge 4. \tag{3.26}$$

Since $c_1 = bd \ne 0$, from (3.26) it follows that

$$a_{k-3} = \frac{a_k - b_1 a_{k-2}}{c_1}.$$

Using this equality, a_j-s can be calculated for $j \le 0$, and, as in [28] and [35], we obtain

$$a_{-1} = a_{-2} = 0, \quad a_0 = 1. \tag{3.27}$$

From (3.16), (3.22) and (3.27), it follows that

$$y_{-2} = y_{-1} = y_0 = 0, \quad y_1 = 1, \tag{3.28}$$

$$y_k = \sum_{j=0}^{k-1} a_j, \quad k \in \mathbb{N}. \tag{3.29}$$

Since the initial-value problem (3.26)-(3.27) is solvable, formula for a_k can be found, from which along with (3.29) and by Lemma 2.2 is obtained a formula for y_k (the form of a_k is well-known, see [1, 12]). Using the formulas for a_k and y_k in (3.25) we have a closed form formula for the solution to equation (3.15).

Using (3.25) in the first equation in (3.14), we obtain

$$z_n = \alpha^{1+bdy_{n-3}} \beta^{by_{n-2}} z_{-1}^{bda_{n-2}} z_0^{bda_{n-3}} w_{-1}^{bca_{n-2}} w_0^{ba_{n-1}}, \tag{3.30}$$

for $n \in \mathbb{N}$.

It is not difficult to see that formulas (3.25) and (3.30) present a solution to system (3.14). This completes the proof. □

Now we specify the form of the solutions to (1.2) when $a = 0$ and $bd \ne 0$. The characteristic polynomial associated to (3.26) is

$$p_3(\lambda) = \lambda^3 - c\lambda - bd. \tag{3.31}$$

The zeros of (3.31) are

$$\lambda_j = \frac{1}{3\sqrt[3]{2}}\Big(\varepsilon^j \sqrt[3]{\Delta_1 - \sqrt{\Delta_1^2 - 4\Delta_0^3}} + \bar{\varepsilon}^j \sqrt[3]{\Delta_1 + \sqrt{\Delta_1^2 - 4\Delta_0^3}}\Big), \quad j = \overline{0,2}, \tag{3.32}$$

where

$$\Delta_0 = 3c =: -3p \quad \text{and} \quad \Delta_1 = 27bd =: -27q, \tag{3.33}$$

and $\varepsilon^3 = 1$, $\varepsilon \ne 1$ ([4]).

Zeros of p_3 are different and none of them is 1. In this case it must be $\Delta := (\Delta_1^2 - 4\Delta_0^3)/27 \ne 0$, that is, $27(bd)^2 \ne 4c^3$. If also $c + bd \ne 1$, then 1 is not a zero of p_3. These conditions are satisfied, for example, if $c = bd \in \mathbb{N}$.

Zeros of p_3 are different and one of them is 1. Since $p_3(1) = 0$, we have that $c + bd = 1$. Hence

$$p_3(\lambda) = \lambda^3 - c\lambda + c - 1 = (\lambda - 1)(\lambda^2 + \lambda + 1 - c),$$

and the zeros of p_3 are

$$\lambda_{1,2} = \frac{-1 \pm \sqrt{4c-3}}{2}, \quad \lambda_3 = 1. \tag{3.34}$$

Note also that it must be $c \neq 3$, from the condition $p_3'(1) = 3 - c \neq 0$.

Since in these two cases all the zeros of p_3 are different, the general solution to (3.26) must have the form

$$a_n = \alpha_1\lambda_1^n + \alpha_2\lambda_2^n + \alpha_3\lambda_3^n, \quad n \in \mathbb{N}, \tag{3.35}$$

where α_i, $i = \overline{1,3}$, are constants [1, 12].

As it was explained, for example, in [28], the solution to (3.26) satisfying (3.27) is given by

$$\begin{aligned} a_n &= \sum_{j=1}^{3} \frac{\lambda_j^{n+2}}{p_3'(\lambda_j)} \\ &= \frac{\lambda_1^{n+2}}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)} + \frac{\lambda_2^{n+2}}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)} + \frac{\lambda_3^{n+2}}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}, \end{aligned} \tag{3.36}$$

and the formula holds for every $n \in \mathbb{Z}$.

From (3.29) and (3.36), it follows that

$$y_n = \sum_{k=0}^{n-1}\sum_{j=1}^{3} \frac{\lambda_j^{k+2}}{p_3'(\lambda_j)}, \quad n \in \mathbb{N}. \tag{3.37}$$

The formula holds also for $n \geq -2$ ([28]).

If $\lambda_j \neq 1$, $j = \overline{1,3}$, then from (3.37) and Lemma 2.2, we obtain

$$\begin{aligned} y_n = &\frac{\lambda_1^2(\lambda_1^n - 1)}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_1-1)} + \frac{\lambda_2^2(\lambda_2^n - 1)}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)(\lambda_2-1)} \\ &+ \frac{\lambda_3^2(\lambda_3^n - 1)}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)(\lambda_3-1)}, \end{aligned} \tag{3.38}$$

while if one of the zeros is 1, say λ_3, then $1 \neq \lambda_1 \neq \lambda_2 \neq 1$, and we have

$$y_n = \frac{\lambda_1^2(\lambda_1^n - 1)}{(\lambda_1-\lambda_2)(\lambda_1-1)^2} + \frac{\lambda_2^2(\lambda_2^n - 1)}{(\lambda_2-\lambda_1)(\lambda_2-1)^2} + \frac{n}{(\lambda_1-1)(\lambda_2-1)}. \tag{3.39}$$

for $n \in \mathbb{N}$. Formulas (3.38) and (3.39) holds for every $n \geq -2$ ([28]).

Corollary 3.5. *Assume that $b, c, d \in \mathbb{Z}$, $a = 0$, $bd \neq 0$, $\alpha, \beta, z_{-1}, z_0, w_{-1}, w_0 \in \mathbb{C} \setminus \{0\}$ and $\Delta \neq 0$. Then the following statements are true.*

- (a) *If $c+bd \neq 1$, then the general solution to* (1.2) *is given by* (3.25) *and* (3.30)*, where $(a_n)_{n\geq -2}$ is given by* (3.36)*, $(y_n)_{n\geq -2}$ is given by* (3.38)*, while λ_j-s, $j = \overline{1,3}$, are given by* (3.32) *and* (3.33).
- (b) *If $c + bd = 1$ and $c \neq 3$, then p_3 has a unique zero equal to 1, say λ_3, and the general solution to* (1.2) *is given by formulas* (3.25) *and* (3.30)*, where $(a_n)_{n\geq -2}$ is given by* (3.36) *with $\lambda_3 = 1$, $(y_n)_{n\geq -2}$ is given by* (3.39)*, while λ_j-s, $j = \overline{1,3}$, are given by* (3.34).

One of the zeros is double. In this case it must be $\Delta = 0$, that is, $c^3 = 27(bd)^2/4$. If m is a double zero of p_3, then

$$m^3 - cm - bd = 0 \quad \text{and} \quad 3m^2 - c = 0.$$

Hence

$$p_3(\lambda) = \lambda^3 - 3m^2\lambda + 2m^3 = (\lambda - m)^2(\lambda + 2m). \tag{3.40}$$

Since $bd \neq 0$, we have $m \neq 0$. From this and (3.40) we see that p_3 cannot have a triple zero. It also cannot have a unique zero equal to 1, otherwise we would have $2m = 1$ and consequently $c = 3/4 \notin \mathbb{Z}$, which is impossible. Note also that 1 is a double zero if and only if $c = 3$ and $bd = -2$.

If $\lambda_1 \neq \lambda_2 = \lambda_3$, then the general solution to (3.26) has the form

$$a_n = \hat{\alpha}_1\lambda_1^n + (\hat{\alpha}_2 + \hat{\alpha}_3 n)\lambda_2^n, \quad n \in \mathbb{N}, \tag{3.41}$$

where $\hat{\alpha}_i$, $i = \overline{1,3}$, are constants.

The solution of the form in (3.41) satisfying (3.27) is

$$a_n = \frac{\lambda_1^{n+2} + \big(\lambda_2 - 2\lambda_1 + n(\lambda_2 - \lambda_1)\big)\lambda_2^{n+1}}{(\lambda_2 - \lambda_1)^2}, \tag{3.42}$$

and also holds for every $n \geq -2$ ([28]).

Since, in the case, $\Delta = 0$, from (3.32), we have

$$\lambda_1 = \frac{2}{3\sqrt[3]{2}}\sqrt[3]{\Delta_1}, \quad \lambda_{2,3} = -\frac{1}{3\sqrt[3]{2}}\sqrt[3]{\Delta_1}. \tag{3.43}$$

From (3.29) and (3.42), we have

$$y_n = \sum_{j=0}^{n-1} a_j = \sum_{j=0}^{n-1} \frac{\lambda_1^{j+2} + \big(\lambda_2 - 2\lambda_1 + j(\lambda_2 - \lambda_1)\big)\lambda_2^{j+1}}{(\lambda_2 - \lambda_1)^2}, \quad n \in \mathbb{N}. \tag{3.44}$$

The formula holds also for $n \geq -2$ ([28]).

From (3.44) and by Lemma 2.2, it follows that

$$\begin{aligned} y_n = &\frac{\lambda_1^2(\lambda_1^n - 1)}{(\lambda_2 - \lambda_1)^2(\lambda_1 - 1)} + \frac{(\lambda_2 - 2\lambda_1)\lambda_2(\lambda_2^n - 1)}{(\lambda_2 - \lambda_1)^2(\lambda_2 - 1)} \\ &+ \frac{\lambda_2^2(1 - n\lambda_2^{n-1} + (n-1)\lambda_2^n)}{(\lambda_2 - \lambda_1)(\lambda_2 - 1)^2}, \quad n \in \mathbb{N}. \end{aligned} \tag{3.45}$$

If we assume that $\lambda_1 \neq 1$ and $\lambda_2 = \lambda_3 = 1$, then from (3.44) it follows that

$$y_n = \frac{\lambda_1^2(\lambda_1^n - 1)}{(\lambda_1 - 1)^3} + \frac{(1 - 2\lambda_1)n}{(\lambda_1 - 1)^2} + \frac{(n-1)n}{2(1 - \lambda_1)}, \quad n \in \mathbb{N}. \tag{3.46}$$

Formulas (3.45) and (3.46) hold also for $n \geq -2$ ([28]).

Corollary 3.6. *Assume that $b, c, d \in \mathbb{Z}$, $a = 0$, $bd \neq 0$, $\alpha, \beta, z_{-1}, z_0, w_{-1}, w_0 \in \mathbb{C} \setminus \{0\}$ and $\Delta = 0$. Then the following statements are true.*

(a) *If $c + bd \neq 1$, then the general solution to* (1.2) *is given by* (3.25) *and* (3.30)*, where $(a_n)_{n\geq -2}$ is given by* (3.42)*, $(y_n)_{n\geq -2}$ is given by* (3.45)*, while λ_j-s, $j = \overline{1,3}$, are given by* (3.43) *and* (3.33)*.*

(b) *If $c = 3$ and $bd = -2$, then two zeros of* (3.31) *are equal to 1, say, λ_2 and λ_3, and the general solution to system* (1.2) *is given by* (3.25) *and* (3.30)*, where $(a_n)_{n\geq -2}$ is given by* (3.42) *with $\lambda_2 = 1$, $(y_n)_{n\geq -2}$ is given by* (3.46)*, while $\lambda_1 = -2$.*

(c) *Polynomial* (3.31) *cannot have 1 as a simple zero.*
(d) *Polynomial* (3.31) *cannot have a triple zero.*

Theorem 3.7. *Assume that* $a, b, d \in \mathbb{Z}$, $bd \neq 0$, $c = 0$, $\alpha, \beta, z_{-1}, z_0, w_0 \in \mathbb{C} \setminus \{0\}$. *Then system* (1.2) *is solvable in closed form.*

Proof. The condition $c = 0$ implies that

$$z_{n+1} = \alpha z_{n-1}^a w_n^b, \quad w_{n+1} = \beta z_{n-1}^d, \quad n \in \mathbb{N}_0. \tag{3.47}$$

From (3.47) we obtain

$$z_{n+1} = \alpha\beta^b z_{n-1}^a z_{n-2}^{bd}, \quad n \in \mathbb{N}. \tag{3.48}$$

Let $\mu = \alpha\beta^b$,

$$a_1 = 0, \quad b_1 = a, \quad c_1 = bd, \quad y_1 = 1. \tag{3.49}$$

Then

$$z_{n+1} = \mu^{y_1} z_n^{a_1} z_{n-1}^{b_1} z_{n-2}^{c_1}, \quad n \in \mathbb{N}. \tag{3.50}$$

As in the proof of Theorem 3.4 we obtain

$$z_{n+1} = \mu^{y_k} z_{n+1-k}^{a_k} z_{n-k}^{b_k} z_{n-k-1}^{c_k}, \tag{3.51}$$

$$a_k = a_1 a_{k-1} + b_{k-1}, \quad b_k = b_1 a_{k-1} + c_{k-1}, \quad c_k = c_1 a_{k-1}, \tag{3.52}$$

$$y_k = y_{k-1} + a_{k-1}, \tag{3.53}$$

for each $k \geq 2$ and all $n \geq k$.

Setting $k = n$ in (3.51) and using (3.1), (3.52) and (3.53), we have

$$\begin{aligned} z_{n+1} &= \mu^{y_n} z_1^{a_n} z_0^{b_n} z_{-1}^{c_n} && (3.54)\\ &= (\alpha\beta^b)^{y_n} (\alpha z_{-1}^a w_0^b)^{a_n} z_0^{b_n} z_{-1}^{c_n} && (3.55)\\ &= \alpha^{y_n + a_n} \beta^{by_n} z_{-1}^{aa_n + c_n} z_0^{b_n} w_0^{ba_n} && (3.56)\\ &= \alpha^{y_{n+1}} \beta^{by_n} z_{-1}^{a_{n+2}} z_0^{a_{n+1}} w_0^{ba_n}, && (3.57) \end{aligned}$$

for $n \in \mathbb{N}$.

From (3.52),

$$a_k = b_1 a_{k-2} + c_1 a_{k-3}, \qquad \text{for } k \geq 4. \tag{3.58}$$

Since $c_1 = bd \neq 0$, (3.58) yields

$$a_{k-3} = \frac{a_k - b_1 a_{k-2}}{c_1}. \tag{3.59}$$

Using (3.49), (3.53) and (3.59), we obtain

$$a_{-1} = a_{-2} = 0, \quad a_0 = 1, \tag{3.60}$$

$$y_{-2} = y_{-1} = y_0 = 0, \quad y_1 = 1, \tag{3.61}$$

$$y_k = \sum_{j=0}^{k-1} a_j, \quad k \in \mathbb{N}. \tag{3.62}$$

By using (3.58), (3.60), (3.62) and Lemma 2.2, it follows that closed form formulas for a_k and y_k can be found and consequently a closed form formula for the solution to equation (3.48).

Employing (3.57) in the second equation in (3.47) we obtain

$$w_n = \alpha^{dy_{n-2}} \beta^{1+bdy_{n-3}} z_{-1}^{da_{n-1}} z_0^{da_{n-2}} w_0^{bda_{n-3}}. \tag{3.63}$$

Some calculation show that formulas (3.57) and (3.63) present a solution to system (3.47), completing the proof of the theorem. □

The characteristic polynomial associated with equation (3.58) is

$$p_3(\lambda) = \lambda^3 - a\lambda - bd\,. \tag{3.64}$$

The zeros of the polynomial are given by (3.32) with

$$\Delta_0 = 3a =: -3p \quad \text{and} \quad \Delta_1 = 27bd =: -27q. \tag{3.65}$$

Hence, the analysis following Theorem 3.4 also holds here, with the difference parameter c replaced by a.

Zeros of p_3 are different and none of them is 1. In this case it must be $\Delta \neq 0$, that is, $27(bd)^2 \neq 4a^3$. If also $a + bd \neq 1$, then $p_3(1) \neq 0$. These conditions are satisfied, for example, if $a = bd \in \mathbb{N}$.

Zeros of p_3 are different and one of them is 1. Polynomial p_3 has a zero equal to 1 if $a + bd = 1$. From this we have

$$p_3(\lambda) = \lambda^3 - a\lambda + a - 1 = (\lambda - 1)(\lambda^2 + \lambda + 1 - a),$$

$$\lambda_{1,2} = \frac{-1 \pm \sqrt{4a-3}}{2}, \quad \lambda_3 = 1, \tag{3.66}$$

are the zeros of (3.64). Since 1 is a simple zero, then it must be $p_3'(1) = 3 - a \neq 0$, that is, we also have $a \neq 3$.

Corollary 3.8. *Assume that $a, b, d \in \mathbb{Z}$, $c = 0$, $bd \neq 0$, $\alpha, \beta, z_{-1}, z_0, w_0 \in \mathbb{C} \setminus \{0\}$ and $\Delta \neq 0$. Then the following statements are true.*

(a) *If $a+bd \neq 1$, then the general solution to (1.2) is given by (3.57) and (3.63), where $(a_n)_{n\geq -2}$ is given by (3.36), $(y_n)_{n\geq -2}$ is given by (3.38), while λ_j-s, $j = \overline{1,3}$, are given by (3.32) and (3.65).*

(b) *If $a + bd = 1$ and $a \neq 3$, then p_3 has a unique zero equal to 1, say λ_3, and the general solution to (1.2) is given by formulas (3.57) and (3.63), where $(a_n)_{n\geq -2}$ is given by (3.36) with $\lambda_3 = 1$, $(y_n)_{n\geq -2}$ is given by (3.39), while λ_j-s, $j = \overline{1,3}$, are given by (3.66).*

Corollary 3.9. *Assume that $a, b, d \in \mathbb{Z}$, $c = 0$, $bd \neq 0$, $\alpha, \beta, z_{-1}, z_0, w_0 \in \mathbb{C} \setminus \{0\}$ and $\Delta = 0$. Then the following statements are true.*

(a) *If $a+bd \neq 1$, then the general solution to (1.2) is given by (3.57) and (3.63), where $(a_n)_{n\geq -2}$ is given by (3.42), $(y_n)_{n\geq -2}$ is given by (3.45), while λ_j-s, $j = \overline{1,3}$, are given by (3.43) and (3.65).*

(b) *If $a = 3$ and $bd = -2$, then two zeros of (3.64) are equal to 1, say, λ_2 and λ_3, and the general solution to system (1.2) is given by (3.57) and (3.63), where $(a_n)_{n\geq -2}$ is given by (3.42) with $\lambda_2 = 1$, $(y_n)_{n\geq -2}$ is given by (3.46), while $\lambda_1 = -2$.*

(c) *Polynomial (3.64) cannot have 1 as a simple zero.*

(d) *Polynomial (3.64) cannot have a triple zero.*

Theorem 3.10. *Assume that $a, b, c, d \in \mathbb{Z}$, $abcd \neq 0$, $\alpha, \beta, z_{-1}, z_0, w_{-1}, w_0 \in \mathbb{C} \setminus \{0\}$. Then system (1.2) is solvable in closed form.*

Proof. The condition $\alpha, \beta, z_{-1}, z_0, w_{-1}, w_0 \in \mathbb{C}\setminus\{0\}$ along with (1.2) imply $z_n w_n \neq 0$ for $n \geq -1$. Hence

$$w_n^b = \frac{z_{n+1}}{\alpha z_{n-1}^a}, \quad n \in \mathbb{N}_0, \tag{3.67}$$

$$w_{n+1}^b = \beta^b w_{n-1}^{bc} z_{n-1}^{bd}, \quad n \in \mathbb{N}_0. \tag{3.68}$$

Using (3.67) in (3.68) we obtain

$$z_{n+2} = \alpha^{1-c}\beta^b z_n^{a+c} z_{n-1}^{bd} z_{n-2}^{-ac}, \quad n \in \mathbb{N}. \tag{3.69}$$

Let $\delta = \alpha^{1-c}\beta^b$,

$$a_1 = 0, \quad b_1 = a + c, \quad c_1 = bd, \quad d_1 = -ac, \quad y_1 = 1. \tag{3.70}$$

Then

$$z_{n+2} = \delta^{y_1} z_{n+1}^{a_1} z_n^{b_1} z_{n-1}^{c_1} z_{n-2}^{d_1}, \quad n \in \mathbb{N}. \tag{3.71}$$

We have

$$\begin{aligned} z_{n+2} &= \delta^{y_1}(\delta z_n^{a_1} z_{n-1}^{b_1} z_{n-2}^{c_1} z_{n-3}^{d_1})^{a_1} z_n^{b_1} z_{n-1}^{c_1} z_{n-2}^{d_1}, \\ &= \delta^{y_1+a_1} z_n^{a_1a_1+b_1} z_{n-1}^{b_1a_1+c_1} z_{n-2}^{c_1a_1+d_1} z_{n-3}^{d_1a_1} \\ &= \delta^{y_2} z_n^{a_2} z_{n-1}^{b_2} z_{n-2}^{c_2} z_{n-3}^{d_2}, \end{aligned} \tag{3.72}$$

for $n \geq 2$, where

$$a_2 := a_1a_1 + b_1, \; b_2 := b_1a_1 + c_1, \; c_2 := c_1a_1 + d_1, \; d_2 := d_1a_1, \; y_2 := y_1 + a_1. \tag{3.73}$$

Assume

$$z_{n+2} = \delta^{y_k} z_{n+2-k}^{a_k} z_{n+1-k}^{b_k} z_{n-k}^{c_k} z_{n-k-1}^{d_k}, \tag{3.74}$$

for a $k \geq 2$ and every $n \geq k$, and that

$$\begin{gathered} a_k = a_1a_{k-1} + b_{k-1}, \quad b_k = b_1a_{k-1} + c_{k-1}, \\ c_k = c_1a_{k-1} + d_{k-1}, \quad d_k = d_1a_{k-1}, \end{gathered} \tag{3.75}$$

$$y_k = y_{k-1} + a_{k-1}. \tag{3.76}$$

Using (3.71) in (3.74), it follows that

$$\begin{aligned} z_{n+2} &= \delta^{y_k}(\delta z_{n+1-k}^{a_1} z_{n-k}^{b_1} z_{n-k-1}^{c_1} z_{n-k-2}^{d_1})^{a_k} z_{n+1-k}^{b_k} z_{n-k}^{c_k} z_{n-k-1}^{d_k} \\ &= \delta^{y_k+a_k} z_{n+1-k}^{a_1a_k+b_k} z_{n-k}^{b_1a_k+c_k} z_{n-k-1}^{c_1a_k+d_k} z_{n-k-2}^{d_1a_k} \\ &= \delta^{y_{k+1}} z_{n+1-k}^{a_{k+1}} z_{n-k}^{b_{k+1}} z_{n-k-1}^{c_{k+1}} z_{n-k-2}^{d_{k+1}}, \end{aligned} \tag{3.77}$$

for $n \geq k+1$, where

$$\begin{gathered} a_{k+1} := a_1a_k + b_k, \quad b_{k+1} := b_1a_k + c_k, \\ c_{k+1} := c_1a_k + d_k, \quad d_{k+1} := d_1a_k, \end{gathered} \tag{3.78}$$

$$y_{k+1} := y_k + a_k. \tag{3.79}$$

From (3.72), (3.73), (3.77)-(3.79), the inductive argument shows that (3.74)-(3.76) hold for $2 \leq k \leq n$.

Setting $k = n$ in (3.74), then using (3.75) and (3.76) in the obtained equality, we have

$$\begin{aligned} z_{n+2} =& \delta^{y_n} z_2^{a_n} z_1^{b_n} z_0^{c_n} z_{-1}^{d_n} \\ =& (\alpha^{1-c}\beta^b)^{y_n} (\alpha\beta^b w_{-1}^{bc} z_{-1}^{bd} z_0^a)^{a_n} (\alpha z_{-1}^a w_0^b)^{b_n} z_0^{c_n} z_{-1}^{d_n} \\ =& \alpha^{(1-c)y_n+a_n+b_n} \beta^{by_n+ba_n} z_{-1}^{bda_n+ab_n+d_n} z_0^{aa_n+c_n} w_{-1}^{bca_n} w_0^{bb_n} \\ =& \alpha^{y_{n+2}-cy_n} \beta^{by_{n+1}} z_{-1}^{a_{n+3}-ca_{n+1}} z_0^{a_{n+2}-ca_n} w_{-1}^{bca_n} w_0^{ba_{n+1}}, \quad n \in \mathbb{N}. \end{aligned} \tag{3.80}$$

From (3.75) we obtain

$$a_k = b_1 a_{k-2} + c_1 a_{k-3} + d_1 a_{k-4}, \quad k \geq 5. \tag{3.81}$$

Equalities (3.76) and (3.81) yield

$$a_{-3} = a_{-2} = a_{-1} = 0, \quad a_0 = 1; \tag{3.82}$$

$$y_{-3} = y_{-2} = y_{-1} = y_0 = 0, \quad y_1 = 1, \tag{3.83}$$

$$y_k = \sum_{j=0}^{k-1} a_j. \tag{3.84}$$

Since the initial-value problem (3.81)-(3.82) is solvable, a formula for a_k can be found. Using it in (3.84) and applying Lemma 2.2, a formula for y_k is found. By using these two formulas in (3.80) we obtain a formula for z_n. So, (3.69) is solvable.

On the other hand, we have

$$z_{n-1}^d = \frac{w_{n+1}}{\beta w_{n-1}^c}, \quad n \in \mathbb{N}_0, \tag{3.85}$$

$$z_{n+1}^d = \alpha^d z_{n-1}^{ad} w_n^{bd}, \quad n \in \mathbb{N}_0, \tag{3.86}$$

and consequently

$$w_{n+3} = \alpha^d \beta^{1-a} w_{n+1}^{a+c} w_n^{bd} w_{n-1}^{-ac}, \quad n \in \mathbb{N}_0. \tag{3.87}$$

As above, we have

$$w_{n+3} = \eta^{y_k} w_{n+3-k}^{a_k} w_{n+2-k}^{b_k} w_{n+1-k}^{c_k} w_{n-k}^{d_k}, \quad n \geq k-1, \tag{3.88}$$

where $\eta = \alpha^d \beta^{1-a}$, $(a_k)_{k\in\mathbb{N}}$, $(b_k)_{k\in\mathbb{N}}$, $(c_k)_{k\in\mathbb{N}}$ and $(d_k)_{k\in\mathbb{N}}$ are defined by (3.70) and (3.75), while $(y_k)_{k\in\mathbb{N}}$ is defined by (3.76) and (3.83).

From (3.88) with $k = n+1$ and by using (3.1), we obtain

$$\begin{aligned} w_{n+3} &= \eta^{y_{n+1}} w_2^{a_{n+1}} w_1^{b_{n+1}} w_0^{c_{n+1}} w_{-1}^{d_{n+1}} \\ &= (\alpha^d \beta^{1-a})^{y_{n+1}} (\beta w_0^c z_0^d)^{a_{n+1}} (\beta w_{-1}^c z_{-1}^d)^{b_{n+1}} w_0^{c_{n+1}} w_{-1}^{d_{n+1}} \\ &= \alpha^{dy_{n+1}} \beta^{(1-a)y_{n+1}+a_{n+1}+b_{n+1}} z_{-1}^{db_{n+1}} z_0^{da_{n+1}} w_{-1}^{cb_{n+1}+d_{n+1}} w_0^{ca_{n+1}+c_{n+1}} \\ &= \alpha^{dy_{n+1}} \beta^{y_{n+3}-ay_{n+1}} z_{-1}^{da_{n+2}} z_0^{da_{n+1}} w_{-1}^{c(a_{n+2}-aa_n)} w_0^{a_{n+3}-aa_{n+1}}, \end{aligned} \tag{3.89}$$

for $n \in \mathbb{N}_0$.

The formulas for a_k and y_k are obtained as above. Using them in (3.89) is get a formula for a solution to (3.87). By some calculation it is checked that (3.80) and (3.89) are formulas for a solution to (1.2). Thus, the system is also solvable, as claimed. □

Corollary 3.11. *Assume that $a, b, c, d \in \mathbb{Z}$, $abcd \neq 0$, $\alpha, \beta, z_{-1}, z_0, w_{-1}, w_0 \in \mathbb{C} \setminus \{0\}$. Then the general solution to system* (1.2) *is given by* (3.80) *and* (3.89)*, where $(a_k)_{k\in\mathbb{N}}$ is defined by* (3.81) *and* (3.82)*, while $(y_k)_{k\in\mathbb{N}}$ is defined by* (3.83) *and* (3.84).

The characteristic polynomial associated to (3.81) is

$$p_4(\lambda) = \lambda^4 - (a+c)\lambda^2 - bd\lambda + ac. \tag{3.90}$$

Since $ac \neq 0$, it is of the forth order.

The equation $p_4(\lambda) = 0$ can be written as

$$\Big(\lambda^2 - \frac{a+c}{2} + \frac{s}{2}\Big)^2 - \Big(s\lambda^2 + bd\lambda + \Big(\frac{s-(a+c)}{2}\Big)^2 - ac\Big) = 0. \tag{3.91}$$

Let s satisfy $(bd)^2 = s(s-a-c)^2 - 4acs$, that is,

$$s^3 - 2(a+c)s^2 + (a-c)^2 s - (bd)^2 = 0. \tag{3.92}$$

For such s, (3.91) becomes

$$\Big(\lambda^2 - \frac{a+c}{2} + \frac{s}{2}\Big)^2 - \Big(\sqrt{s}\,\lambda + \frac{bd}{2\sqrt{s}}\Big)^2 = 0,$$

which is equivalent to the following two quadratic equations

$$\lambda^2 - \sqrt{s}\,\lambda + \frac{s-a-c}{2} - \frac{bd}{2\sqrt{s}} = 0, \tag{3.93}$$

$$\lambda^2 + \sqrt{s}\,\lambda + \frac{s-a-c}{2} + \frac{bd}{2\sqrt{s}} = 0. \tag{3.94}$$

By using the change of variables $s = t + \frac{2(a+c)}{3}$ in (3.92) we obtain

$$t^3 - \Big(\frac{4(a+c)^2}{3} - (a-c)^2\Big)t - \frac{16(a+c)^3}{27} + \frac{2(a+c)(a-c)^2}{3} - (bd)^2 = 0. \tag{3.95}$$

Let

$$p = -\frac{a^2 + 14ac + c^2}{3} \quad \text{and} \quad q = \frac{2a^3 + 2c^3 - 66ac(a+c) - 27(bd)^2}{27}.$$

Then, we can choose t as one of the three possible values of the quantity

$$t = \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}. \tag{3.96}$$

If we use the change $p = -\Delta_0/3$ and $q = -\Delta_1/27$ in (3.96) we obtain

$$t = \frac{1}{3\sqrt[3]{2}}\Big(\sqrt[3]{\Delta_1 - \sqrt{\Delta_1^2 - 4\Delta_0^3}} + \sqrt[3]{\Delta_1 + \sqrt{\Delta_1^2 - 4\Delta_0^3}}\Big).$$

Solutions to (3.93) and (3.94) are the zeros of polynomial (3.90) and they are

$$\lambda_1 = \frac{1}{2}\sqrt{t + \frac{2(a+c)}{3}} + \frac{1}{2}\sqrt{\frac{4(a+c)}{3} - t - \frac{Q}{4\sqrt{t + \frac{2(a+c)}{3}}}}, \tag{3.97}$$

$$\lambda_2 = \frac{1}{2}\sqrt{t + \frac{2(a+c)}{3}} - \frac{1}{2}\sqrt{\frac{4(a+c)}{3} - t - \frac{Q}{4\sqrt{t + \frac{2(a+c)}{3}}}}, \tag{3.98}$$

$$\lambda_3 = -\frac{1}{2}\sqrt{t + \frac{2(a+c)}{3}} + \frac{1}{2}\sqrt{\frac{4(a+c)}{3} - t + \frac{Q}{4\sqrt{t + \frac{2(a+c)}{3}}}}, \tag{3.99}$$

$$\lambda_4 = -\frac{1}{2}\sqrt{t + \frac{2(a+c)}{3}} - \frac{1}{2}\sqrt{\frac{4(a+c)}{3} - t + \frac{Q}{4\sqrt{t + \frac{2(a+c)}{3}}}}, \tag{3.100}$$

where

$$Q := -8bd. \tag{3.101}$$

The nature of λ_j, $j = \overline{1,4}$, depends on the sign of

$$\Delta := \frac{1}{27}(4\Delta_0^3 - \Delta_1^2), \tag{3.102}$$

where

$$\Delta_0 := a^2 + 14ac + c^2, \tag{3.103}$$

$$\Delta_1 := -2a^3 - 2c^3 + 66ac(a+c) + 27(bd)^2, \tag{3.104}$$

and the signs of

$$P := -8(a+c) \tag{3.105}$$

and

$$D := -16(a-c)^2. \tag{3.106}$$

Zeros of p_4 are different and none of them is 1. If a, b, c and d are chosen such that $\Delta_0 < 0$, that is, $a^2 + 14ac + c^2 < 0$, then it will be $\Delta < 0$, from which by Lemma 2.3 it follows that p_4 has four different zeros. Let $t_{1,2} = -7 \pm \sqrt{48}$. Then if we choose $a, c \in \mathbb{Z} \setminus \{0\}$ such that $a/c \in (t_1, t_2)$, then we have such a situation.

Zeros of p_4 are different and one of them is 1. Polynomial (3.90) has a zero equal to 1 if $p_4(1) = 1 - a - c - bd + ac = 0$, that is, if

$$(a-1)(c-1) = bd, \tag{3.107}$$

so that

$$p_4(\lambda) = \lambda^4 - (a+c)\lambda^2 - (a-1)(c-1)\lambda + ac. \tag{3.108}$$

Thus, if we choose a and c such that $p_4'(1) = 3-a-c-ac \neq 0$, that is, $(a+1)(c+1) \neq 4$, then p_4 will be such a polynomial if $\Delta \neq 0$. For example, if $a = -3$ and $c = 2$, then $bd = -4 \neq 0$, $\Delta \neq 0$ which means that the characteristic polynomial has all zeros mutually different and exactly one of them is equal to 1

$$p_4(\lambda) = \lambda^4 + \lambda^2 + 4\lambda - 6 = (\lambda - 1)(\lambda^3 + \lambda^2 + 2\lambda + 6). \tag{3.109}$$

Since in these two cases $\lambda_j \neq \lambda_i$, $i \neq j$, then the general solution to (3.81) is

$$a_n = \gamma_1\lambda_1^n + \gamma_2\lambda_2^n + \gamma_3\lambda_3^n + \gamma_4\lambda_4^n, \quad n \in \mathbb{N}, \tag{3.110}$$

where γ_i, $i = \overline{1,4}$, are arbitrary constants.

The solution to (3.81) satisfying (3.82) is

$$\begin{aligned} a_n &= \sum_{j=1}^{4} \frac{\lambda_j^{n+3}}{p_4'(\lambda_j)} \\ &= \frac{\lambda_1^{n+3}}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)} + \frac{\lambda_2^{n+3}}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)(\lambda_2-\lambda_4)} \\ &\quad + \frac{\lambda_3^{n+3}}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)(\lambda_3-\lambda_4)} + \frac{\lambda_4^{n+3}}{(\lambda_4-\lambda_1)(\lambda_4-\lambda_2)(\lambda_4-\lambda_3)}, \end{aligned} \tag{3.111}$$

for $n \ge -3$ ([32]).

From (3.84) and (3.111) it follows that

$$y_n = \sum_{j=0}^{n-1} \sum_{i=1}^{4} \frac{\lambda_i^{j+3}}{p_4'(\lambda_i)} = \sum_{i=1}^{4} \frac{\lambda_i^3(\lambda_i^n - 1)}{p_4'(\lambda_i)(\lambda_i - 1)}, \quad n \in \mathbb{N}, \tag{3.112}$$

when $\lambda_i \ne 1$, $i = \overline{1,4}$, and

$$y_n = \frac{n}{3-a-c-ac} + \sum_{i=2}^{4} \frac{\lambda_i^3(\lambda_i^n - 1)}{p_4'(\lambda_i)(\lambda_i - 1)}, \quad n \in \mathbb{N}, \tag{3.113}$$

when one of the zeros of p_4 is 1 (here $\lambda_1 = 1$).

Note that if one of the zeros of p_4 is 1, then we have

$$p_4(\lambda) = (\lambda - 1)(\lambda^3 + \lambda^2 - (a+c-1)\lambda - ac). \tag{3.114}$$

The change of variables $\lambda = t - \frac{1}{3}$ transforms the following equation

$$\lambda^3 + \lambda^2 - (a+c-1)\lambda - ac = 0,$$

into

$$t^3 + \tilde{p}t + \tilde{q} = 0, \tag{3.115}$$

where

$$\tilde{p} = \frac{2}{3} - a - c \quad \text{and} \quad \tilde{q} = \frac{9(a+c) - 7 - 27ac}{27}. \tag{3.116}$$

The zeros of (3.115) are

$$t_l = \varepsilon^{l-1} \sqrt[3]{-\frac{\tilde{q}}{2} - \sqrt{\frac{\tilde{q}^2}{4} + \frac{\tilde{p}^3}{27}}} + \varepsilon^{l-1} \sqrt[3]{-\frac{\tilde{q}}{2} + \sqrt{\frac{\tilde{q}^2}{4} + \frac{\tilde{p}^3}{27}}}, \quad l = \overline{1,3},$$

where $\varepsilon^3 = 1$ and $\varepsilon \ne 1$. Hence,

$$\lambda_j = -\frac{1}{3} + \varepsilon^{j-2} \sqrt[3]{-\frac{\tilde{q}}{2} - \sqrt{\frac{\tilde{q}^2}{4} + \frac{\tilde{p}^3}{27}}} + \varepsilon^{j-2} \sqrt[3]{-\frac{\tilde{q}}{2} + \sqrt{\frac{\tilde{q}^2}{4} + \frac{\tilde{p}^3}{27}}}, \tag{3.117}$$

for $j = \overline{2,4}$, are the other three zeros of p_4, in this case.

The previous analysis along with Corollary 3.11 implies the following corollary.

Corollary 3.12. *Assume that $a, b, c, d \in \mathbb{Z}$, $abcd \ne 0$, $\alpha, \beta, z_{-1}, z_0, w_{-1}, w_0 \in \mathbb{C} \setminus \{0\}$ and $\Delta \ne 0$. Then the following statements are true.*

(a) *If $(a-1)(c-1) \ne bd$, then the general solution to* (1.2) *is given by* (3.80) *and* (3.89)*, where $(a_n)_{n \ge -3}$ is given by* (3.111)*, $(y_n)_{n \ge -3}$ is given by* (3.112)*, while λ_j-s, $j = \overline{1,4}$, are given by* (3.97)-(3.100).

(b) *If* $(a-1)(c-1) = bd$ *and* $(a+1)(c+1) \neq 4$*, then the general solution to* (1.2) *is given by* (3.80) *and* (3.89)*, where* $(a_n)_{n\ge -3}$ *is given by* (3.111) *with* $\lambda_1 = 1$*,* $(y_n)_{n\ge -3}$ *is given by* (3.113)*,* $\lambda_1 = 1$*, while* λ_j*-s,* $j = \overline{2,4}$*, are given by* (3.117) *and* (3.116)*.*

p_4 has only one double zero which is equal to 1. Polynomial (3.90) will have a double zero equal to 1 if (3.107) holds and

$$(a+1)(c+1) = 4. \tag{3.118}$$

From (3.118) we have that one of the following cases must occur: (1) $a = 3$ and $c = 0$, (2) $a = 0$ and $c = 3$; (3) $a = c = 1$; (4) $a = -5$ and $c = -2$; (5) $a = -2$ and $c = -5$; (6) $a = c = -3$. If $a = 0$ or $c = 0$, then $ac = 0$, which is a contradiction.

If $a = c = 1$, then

$$p_4(\lambda) = \lambda^4 - 2\lambda^2 + 1 = (\lambda-1)^2(\lambda+1)^2,$$

and

$$\lambda_{1,2} = 1, \quad \lambda_{3,4} = -1. \tag{3.119}$$

If $a = -5$ and $c = -2$ or $a = -2$ and $c = -5$, then

$$p_4(\lambda) = \lambda^4 + 7\lambda^2 - 18\lambda^2 + 10 = (\lambda-1)^2(\lambda^2 + 2\lambda + 10),$$

and

$$\lambda_{1,2} = 1, \quad \lambda_{3,4} = -1 \pm 3i. \tag{3.120}$$

If $a = c = -3$, then

$$p_4(\lambda) = \lambda^4 + 6\lambda^2 - 16\lambda + 9 = (\lambda-1)^2(\lambda^2 + 2\lambda + 9),$$

and

$$\lambda_{1,2} = 1, \quad \lambda_{3,4} = -1 \pm 2\sqrt{2}i. \tag{3.121}$$

From this, we have proved in passing, that there are no such $a, c \in \mathbb{Z} \setminus \{0\}$, such that 1 is a triple zero of p_4, or that p_4 has two pairs of double zeros such that one of them is equal to 1.

In these four cases we have (see, for example, [30])

$$\begin{aligned} a_n = &\frac{n(1-\lambda_3)(1-\lambda_4) + 3\lambda_3\lambda_4 - 2\lambda_3 - 2\lambda_4 + 1}{(1-\lambda_3)^2(1-\lambda_4)^2} \\ &+ \frac{\lambda_3^{n+3}}{(\lambda_3-1)^2(\lambda_3-\lambda_4)} + \frac{\lambda_4^{n+3}}{(\lambda_4-1)^2(\lambda_4-\lambda_3)}, \end{aligned} \tag{3.122}$$

and

$$\begin{aligned} y_n &= \sum_{j=0}^{n-1} \Big(\frac{j(1-\lambda_3)(1-\lambda_4) + 3\lambda_3\lambda_4 - 2\lambda_3 - 2\lambda_4 + 1}{(1-\lambda_3)^2(1-\lambda_4)^2} \\ &\quad + \frac{\lambda_3^{j+3}}{(\lambda_3-1)^2(\lambda_3-\lambda_4)} + \frac{\lambda_4^{j+3}}{(\lambda_4-1)^2(\lambda_4-\lambda_3)} \Big) \\ &= \frac{(n-1)n}{2(1-\lambda_3)(1-\lambda_4)} + \frac{n(3\lambda_3\lambda_4 - 2\lambda_3 - 2\lambda_4 + 1)}{(1-\lambda_3)^2(1-\lambda_4)^2} \\ &\quad + \frac{\lambda_3^3(\lambda_3^n - 1)}{(\lambda_3-1)^3(\lambda_3-\lambda_4)} + \frac{\lambda_4^3(\lambda_4^n - 1)}{(\lambda_4-1)^3(\lambda_4-\lambda_3)}. \end{aligned} \tag{3.123}$$

Exactly one double zero which is different from 1. According to Lemma 2.3, in this case it must be $\Delta = 0$, that is,

$$(a^2 + 14ac + c^2)^3 = \Big(a^3 + c^3 - 33ac(a + c) - \frac{27}{2}(bd)^2\Big)^2, \tag{3.124}$$

and that

$$(a - 1)(c - 1) \neq bd, \quad a + c > 0, \quad a \neq c. \tag{3.125}$$

The problem of the existence $a, c \in \mathbb{Z} \setminus \{0\}$ such that (3.124) and (3.125) hold seems quite technical and we leave it to the reader as an open problem.

Since, in the case $\lambda_1 = \lambda_2$, $\lambda_i \neq \lambda_j$, $2 \le i, j \le 4$, we have that the general solution to (3.81) has the form

$$a_n = (\gamma_1 + \gamma_2 n)\lambda_2^n + \gamma_3\lambda_3^n + \gamma_4\lambda_4^n, \quad n \in \mathbb{N}, \tag{3.126}$$

where γ_i, $i = \overline{1, 4}$, are arbitrary constants, and the solution satisfying (3.82) can be obtained, for example, by letting $\lambda_1 \to \lambda_2$ in (3.111) [30],

$$\begin{aligned} a_n = &\frac{\lambda_2^{n+2}((n + 3)(\lambda_2 - \lambda_3)(\lambda_2 - \lambda_4) - \lambda_2(2\lambda_2 - \lambda_3 - \lambda_4))}{(\lambda_2 - \lambda_3)^2(\lambda_2 - \lambda_4)^2} \\ &+ \frac{\lambda_3^{n+3}}{(\lambda_3 - \lambda_2)^2(\lambda_3 - \lambda_4)} + \frac{\lambda_4^{n+3}}{(\lambda_4 - \lambda_2)^2(\lambda_4 - \lambda_3)}. \end{aligned} \tag{3.127}$$

From (3.84), (3.127) and by Lemma 2.2, we obtain

$$\begin{aligned} y_n = &\sum_{j=0}^{n-1}\Big(\frac{\lambda_2^{j+2}((j + 3)(\lambda_2 - \lambda_3)(\lambda_2 - \lambda_4) - \lambda_2(2\lambda_2 - \lambda_3 - \lambda_4))}{(\lambda_2 - \lambda_3)^2(\lambda_2 - \lambda_4)^2} \\ &+ \frac{\lambda_3^{j+3}}{(\lambda_3 - \lambda_2)^2(\lambda_3 - \lambda_4)} + \frac{\lambda_4^{j+3}}{(\lambda_4 - \lambda_2)^2(\lambda_4 - \lambda_3)}\Big) \\ = &\frac{\lambda_2^3 - n\lambda_2^{n+2} + (n - 1)\lambda_2^{n+3}}{(\lambda_2 - \lambda_3)(\lambda_2 - \lambda_4)(1 - \lambda_2)^2} + \frac{(\lambda_2^4 - 2\lambda_2^3\lambda_3 - 2\lambda_2^3\lambda_4 + 3\lambda_2^2\lambda_3\lambda_4)(\lambda_2^n - 1)}{(\lambda_2 - \lambda_3)^2(\lambda_2 - \lambda_4)^2(\lambda_2 - 1)} \\ &+ \frac{\lambda_3^3(\lambda_3^n - 1)}{(\lambda_3 - \lambda_2)^2(\lambda_3 - \lambda_4)(\lambda_3 - 1)} + \frac{\lambda_4^3(\lambda_4^n - 1)}{(\lambda_4 - \lambda_2)^2(\lambda_4 - \lambda_3)(\lambda_4 - 1)}. \end{aligned} \tag{3.128}$$

From the previous analysis and Corollary 3.11 we obtain the following result.

Corollary 3.13. *Assume that $a, b, c, d \in \mathbb{Z}$, $abcd \neq 0$ and $\alpha, \beta, z_{-1}, z_0, w_{-1}, w_0 \in \mathbb{C} \setminus \{0\}$. Then the following statements are true.*

(a) *If only one of the zeros of polynomial (3.90) is double and different from 1, then the general solution to (1.2) is given by (3.80) and (3.89), where $(a_n)_{n \ge -3}$ is given by (3.127), while $(y_n)_{n \ge -3}$ is given by (3.128).*

(b) *If 1 is a unique double zero of polynomial p_4, say $\lambda_1 = \lambda_2 = 1$, then the general solution to (1.2) is given by (3.80) and (3.89), where $(a_n)_{n \ge -3}$ is given by (3.122), $(y_n)_{n \ge -3}$ is given by (3.123), while $\lambda_{3,4}$ are given by (3.119) if $a = c = 1$, by (3.120) if $a = -5$, $c = -2$ or $a = -2$, $c = -5$, and by (3.121) if $a = c = -3$.*

Two pairs of different double zeros. Let $bd = 0$, then

$$p_4(\lambda) = \lambda^4 - (a + c)\lambda^2 + ac = (\lambda^2 - a)(\lambda^2 - c),$$
$$\lambda_{1,2} = \pm\sqrt{a}, \quad \lambda_{3,4} = \pm\sqrt{c}.$$

Hence, if $a = c \neq 0$, p_4 in this case has two pairs of different double zeros

$$\lambda_{1,3} = \sqrt{a}, \quad \lambda_{3,4} = -\sqrt{a}. \tag{3.129}$$

The general solution to (3.81) in this case has the form

$$a_n = (\gamma_1 + \gamma_2 n)\lambda_2^n + (\gamma_3 + \gamma_4 n)\lambda_4^n, \quad n \in \mathbb{N}, \tag{3.130}$$

where γ_i, $i = \overline{1,4}$ are constants.

The solution to (3.81) of the form in (3.130) and satisfying (3.82) is given by [30]

$$\begin{aligned} a_n = &\frac{\lambda_2^{n+2}(n(\lambda_2-\lambda_4)^2 + \lambda_2^2 - 4\lambda_2\lambda_4 + 3\lambda_4^2)}{(\lambda_2-\lambda_4)^4} \\ &+ \frac{\lambda_4^{n+2}(n(\lambda_4-\lambda_2)^2 + \lambda_4^2 - 4\lambda_2\lambda_4 + 3\lambda_2^2)}{(\lambda_4-\lambda_2)^4}. \end{aligned} \tag{3.131}$$

From (3.84), (3.131) and by Lemma 2.2, we obtain

$$\begin{aligned} y_n &= \sum_{j=0}^{n-1} \Big(\frac{\lambda_2^{j+2}(j(\lambda_2-\lambda_4)^2 + \lambda_2^2 - 4\lambda_2\lambda_4 + 3\lambda_4^2)}{(\lambda_2-\lambda_4)^4} \\ &\quad + \frac{\lambda_4^{j+2}(j(\lambda_4-\lambda_2)^2 + \lambda_4^2 - 4\lambda_2\lambda_4 + 3\lambda_2^2)}{(\lambda_4-\lambda_2)^4} \Big) \\ &= \frac{\lambda_2^3 - n\lambda_2^{n+2} + (n-1)\lambda_2^{n+3}}{(\lambda_2-\lambda_4)^2(1-\lambda_2)^2} + \frac{(\lambda_2^4 - 4\lambda_2^3\lambda_4 + 3\lambda_2^2\lambda_4^2)(\lambda_2^n - 1)}{(\lambda_2-\lambda_4)^4(\lambda_2-1)} \\ &\quad + \frac{\lambda_4^3 - n\lambda_4^{n+2} + (n-1)\lambda_4^{n+3}}{(\lambda_4-\lambda_2)^2(1-\lambda_4)^2} + \frac{(\lambda_4^4 - 4\lambda_2\lambda_4^3 + 3\lambda_2^2\lambda_4^2)(\lambda_4^n - 1)}{(\lambda_4-\lambda_2)^4(\lambda_4-1)}. \end{aligned} \tag{3.132}$$

Corollary 3.14. *Assume that $a, b, c, d \in \mathbb{Z}$, $ac \neq 0$ and $\alpha, \beta, z_{-1}, z_0, w_{-1}, w_0 \in \mathbb{C} \setminus \{0\}$. Then the following statements are true.*

(a) *If polynomial p_4 has two pairs of double zeros both different from 1, then the general solution to (1.2) is given by (3.80) and (3.89), where $(a_n)_{n\geq -3}$ is given by (3.131), while $(y_n)_{n\geq -3}$ is given by (3.132). If additionally $bd = 0$ and $a = c$, then λ_j-s, $j = \overline{1,4}$, are given by (3.129).*

(b) *For $a = c = 1$ and $bd = 0$ the polynomial in (3.90) can have two pairs of double zeros such that one of them is equal to 1, and the general solution to (1.2) is given by (3.80) and (3.89), where $(a_n)_{n\geq -3}$ is given by (3.131), while $(y_n)_{n\geq -3}$ is given by (3.132) with $\lambda_2 = 1$ and $\lambda_4 = -1$.*

Triple zero case. Since $\Delta_0 \neq 0$, when $a, c \in \mathbb{Z} \setminus \{0\}$, by Lemma 2.3 it follows that p_4 cannot have a triple zero. Consequently, it cannot have a quadruple zero.

References

[1] R. P. Agarwal; *Difference Equations and Inequalities: Theory, Methods, and Applications* 2nd Edition, Marcel Dekker Inc., New York, Basel, 2000.

[2] A. Andruch-Sobilo, M. Migda; Further properties of the rational recursive sequence $x_{n+1} = ax_{n-1}/(b + cx_n x_{n-1})$, *Opuscula Math.* **26** (3) (2006), 387-394.

[3] L. Berg, S. Stević; On some systems of difference equations, *Appl. Math. Comput.* **218** (2011), 1713-1718.

[4] D. K. Faddeyev; *Lectures on Algebra*, Nauka, Moscow, 1984 (in Russian).

[5] N. Fotiades, G. Papaschinopoulos; Asymptotic behavior of the positive solutions of a system of k difference equations of exponential form; *Dynam. Contin. Discrete Impuls. Systems Ser. A* **19** (5) (2012), 585-597.

[6] B. Iričanin, S. Stević; Eventually constant solutions of a rational difference equation, *Appl. Math. Comput.* **215** (2009), 854-856.
[7] C. Jordan; *Calculus of Finite Differences*, Chelsea Publishing Company, New York, 1956.
[8] G. L. Karakostas, Convergence of a difference equation via the full limiting sequences method; *Differ. Equ. Dyn. Syst.* **1** (4) (1993), pp. 289-294.
[9] V. A. Krechmar; *A Problem Book in Algebra*, Mir Publishers, Moscow, 1974.
[10] H. Levy, F. Lessman; *Finite Difference Equations*, Dover Publications, New York, 1992.
[11] J. Migda; Qualitative approximation of solutions to difference equations, *Electron. J. Qual. Theory Differ. Equ.* Vol. 2015, Article no. 32, (2015), 26 pages.
[12] D. S. Mitrinović, J. D. Kečkić; *Methods for Calculating Finite Sums*, Naučna Knjiga, Beograd, 1984 (in Serbian).
[13] G. Papaschinopoulos, N. Psarros, K. B. Papadopoulos; On a system of m difference equations having exponential terms, *Electron. J. Qual. Theory Differ. Equ.* Vol. 2015, Article no. 5, (2015), 13 pages.
[14] G. Papaschinopoulos, C. J. Schinas; On a system of two nonlinear difference equations, *J. Math. Anal. Appl.* **219** (2) (1998), 415-426.
[15] G. Papaschinopoulos, C. J. Schinas; On the behavior of the solutions of a system of two nonlinear difference equations; *Comm. Appl. Nonlinear Anal.* **5** (2) (1998), 47-59.
[16] G. Papaschinopoulos, C. J. Schinas; Invariants for systems of two nonlinear difference equations, *Differential Equations Dynam. Systems* **7** (2) (1999), 181-196.
[17] G. Papaschinopoulos, C. J. Schinas; Invariants and oscillation for systems of two nonlinear difference equations, *Nonlinear Anal. TMA* **46** (7) (2001), 967-978.
[18] G. Papaschinopoulos, C. J. Schinas; Oscillation and asymptotic stability of two systems of difference equations of rational form, *J. Difference Equat. Appl.* **7** (2001), 601-617.
[19] G. Papaschinopoulos, C. Schinas, V. Hatzifilippidis; Global behavior of the solutions of a max-equation and of a system of two max-equations, *J. Comput. Anal. Appl.* **5** (2) (2003), 237-254.
[20] G. Papaschinopoulos, G. Stefanidou; Asymptotic behavior of the solutions of a class of rational difference equations, *Inter. J. Difference Equations* **5** (2) (2010), 233-249.
[21] V. D. Rădulescu; Nonlinear elliptic equations with variable exponent: old and new, *Nonlinear Anal.* **121** (2015), 336-369.
[22] E. L. Rees; Graphical discussion of the roots of a quartic equation, *Amer. Math. Monthly* **29** (2) (1922), 51-55.
[23] M. Rosiu; Trajectory structure near critical points, *An. Univ. Craiova Ser. Mat. Inform.* **25** (1998), 35-44.
[24] G. Stefanidou, G. Papaschinopoulos, C. Schinas; On a system of max difference equations, *Dynam. Contin. Discrete Impuls. Systems Ser. A*, **14** (6) (2007), 885-903.
[25] G. Stefanidou, G. Papaschinopoulos, C. J. Schinas; On a system of two exponential type difference equations, *Commun. Appl. Nonlinear Anal.* **17** (2) (2010), 1-13.
[26] S. Stević; First-order product-type systems of difference equations solvable in closed form; *Electron. J. Differential Equations* Vol. 2015, Article No. 308, (2015), 14 pages.
[27] S. Stević; New solvable class of product-type systems of difference equations on the complex domain and a new method for proving the solvability, *Electron. J. Qual. Theory Differ. Equ.* Vol. 2016, Article No. 120, (2016), 19 pages.
[28] S. Stević; Solvability of a product-type system of difference equations with six parameters, *Adv. Nonlinear Anal.* (in press), doi:10.1515/anona-2016-0145.
[29] S. Stević; Solvable subclasses of a class of nonlinear second-order difference equations, *Adv. Nonlinear Anal.* **5** (2) (2016), 147-165.
[30] S. Stević; Solvable product-type system of difference equations whose associated polynomial is of the fourth order, *Electron. J. Qual. Theory Differ. Equ.* Vol. 2017, Article No. 13, (2017), 29 pages.
[31] S. Stević, M. A. Alghamdi, A. Alotaibi, E. M. Elsayed; Solvable product-type system of difference equations of second order, *Electron. J. Differential Equations* Vol. 2015, Article No. 169, (2015), 20 pages.
[32] S. Stević, B. Iričanin, Z. Šmarda; On a product-type system of difference equations of second order solvable in closed form, *J. Inequal. Appl.* Vol. 2015, Article No. 327, (2015), 15 pages.
[33] S. Stević, B. Iričanin, Z. Šmarda; Solvability of a close to symmetric system of difference equations; *Electron. J. Differential Equations* Vol. 2016, Article No. 159, (2016), 13 pages.

[34] S. Stević, B. Iričanin, Z. Šmarda; Two-dimensional product-type system of difference equations solvable in closed form; *Adv. Difference Equ.* Vol. 2016, Article No. 253, (2016), 20 pages.

[35] S. Stević and D. Ranković; On a practically solvable product-type system of difference equations of second order, *Electron. J. Qual. Theory Differ. Equ.* Vol. 2016, Article No. 56, (2016), 23 pages.

Stevo Stević
Mathematical Institute of the Serbian Academy of Sciences, Knez Mihailova 36/III, 11000 Beograd, Serbia.
Operator Theory and Applications Research Group, Department of Mathematics, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia
E-mail address: sstevic@ptt.rs