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POSITIVE GROUND STATE SOLUTIONS FOR QUASICRITICAL KLEIN-GORDON-MAXWELL TYPE SYSTEMS WITH POTENTIAL VANISHING AT INFINITY

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Abstract. This article concerns the Klein-Gordon-Maxwell type system when the nonlinearity has a quasicritical growth at infinity, involving *zero mass* potential, that is, $V(x) \to 0$, as $|x| \to \infty$. The interaction of the behavior of the potential and nonlinearity recover the lack of the compactness of Sobolev embedding in whole space. The positive ground state solution is obtained by proving that the solution satisfies Mountain Pass level.

1. Introduction

This article concerns the existence of nontrivial solution to the Klein-Gordon-Maxwell system

$$\begin{gathered} -\Delta u + V(x)u - (2\omega + \phi)\phi u = K(x)f(u), \quad \text{in } \mathbb{R}^3, \\ \Delta\phi = (\omega + \phi)u^2, \quad \text{in } \mathbb{R}^3, \end{gathered} \tag{1.1}$$

where $u \in H^1(\mathbb{R}^3) := H$, $\omega > 0$ is a parameter, and we assume that $V, K : \mathbb{R}^3 \to \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ are continuous functions, with V, K nonnegative and f having a quasicritical growth at infinity. We will treat problem (1.1) with *zero mass* potential, that is, $V(x) \to 0$, as $|x| \to \infty$. Problems involving zero mass potential, with $\phi = 0$, have been studied by several researchers, and extended or improved in several ways; see for instance [1, 2, 3, 6, 9, 14, 15, 16, 23, 26] and reference therein. In all these papers above, there are restrictions on V and K to get some compact embedding into a weighted L^p space.

In a remarkable work, Benci and Fortunato in [11] considered problem (1.1), with $V(x) = m_0^2 - \omega^2$, as a model describing nonlinear Klein-Gordon fields in $\mathbb{R}^3$ interacting with the electromagnetic field. Thus the solution represents a solitary wave of the type $\Phi(x,t) = u(x)e^{i\omega t}$ in equilibrium with a purely electrostatic field $\mathbf{E} = -\nabla\phi(\mathbf{x})$. There are a lot of works devoted to system (1.1), and we would like to cite some of them. Benci and Fortunato [12] proved the existence of infinitely many radially symmetric solutions when $m_0 > \omega$ and $K(x)f(u) = |u|^{p-2}u$, $4 < p < 6$. D'Aprile and Mugnai [21, 22] covered the case $2 < p \leq 4$ and established some

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non-existence results for $p > 6$. For the critical nonlinearity $K(x)f(u) = |u|^{p-2}u$, with $p = 6$, Cassani [19] obtained a non-existence result for the above system, and he showed the existence of radially symmetric solution when $4 < p < 6$ or $p = 4$. In the critical case, radially symmetric solutions for this system were studied in [17, 18, 21, 28] and references therein. With respect to the existence of a *ground state solution,* that is, a couple (u, ϕ) which solves (1.1) and minimize the action functional associated to (1.1) among all possible nontrivial solutions, we mention [7, 8, 18, 27] and theirs references. In [24, 25] were considered (1.1) systems imposing a coercivity condition, as that in [10], to recover the lack of compactness of the Sobolev space embedding.

The interest in this kind of problem is twofold: on the one hand the vast range of applications, and on the other hand the mathematical challenge of solving a nonlocal problem and zero mass potential.

First of all, we would like to study the case in which V is bounded and then, in Section 5, we treat problem (1.1) with zero mass potential, that is, when $V(x) \to 0$, as $|x| \to \infty$.

We will work with the following assumptions:

(A1) $V, K : \mathbb{R}^3 \to \mathbb{R}$ are smooth functions, $K \in L^\infty(\mathbb{R}^3)$ and there are constants $\xi_0, a_1, a_2, V_0 > 0$ such that
$$0 < V_0 \le V(x) \le a_1, \quad \forall x \in \mathbb{R}^3 \tag{1.2}$$
and if $2 < \theta < 4$, then
$$0 < \frac{2(4-\theta)}{\theta - 2} \le V_0, \quad \forall x \in \mathbb{R}^3\,; \tag{1.3}$$
also
$$0 < K(x) \le \frac{a_2}{1 + |x|^{\xi_0}}, \quad \forall x \in \mathbb{R}^3. \tag{1.4}$$

(A2) If $\{A_n\} \subset \mathbb{R}^3$ is a sequence of Borel sets such that the Lebesgue measure of A_n is bounded uniformly, that is, $\mu(A_n) \le R$, for all n and some $R > 0$, then
$$\lim_{r\to+\infty} \int_{A_n \bigcap B_r^c(0)} K(x)\, dx = 0, \quad \text{uniformly for } n \in \mathbb{N}. \tag{1.5}$$

(A3) (behavior at zero) $\limsup_{s\to 0^+} f(s)/s = 0$,

(A3') (behavior at zero) there is a constant $p \in (2, 6)$ such that $\limsup_{s\to 0^+} \frac{f(s)}{s^{p-1}} < +\infty$,

(A4) (quasicritical growth) $\limsup_{s\to+\infty} f(s)/s^5 = 0$,

(A5) (Ambrosetti-Rabinowitz) there exists $\theta > 4$, such that $0 < \theta F(u) \le f(u)u$ for all $u > 0$, where $F(u) = \int_0^u f(s)\, ds$.

Remark 1.1. From (1.2), (1.4) and $p \in (2, 6)$, we have
$$\frac{K(x)}{[V(x)]^{(6-p/4}} \to 0, \quad \text{as } |x| \to +\infty. \tag{1.6}$$

Our main results are as follows.

Theorem 1.2. *Suppose that* (A1)–(A5) *hold. Then problem* (1.1) *possess a positive ground state solution.*

Theorem 1.3. *Suppose that* (A1), (A2), (A3'), (A4), (A5) *hold. Then problem* (1.1) *possess a positive ground state solution.*

Let us briefly sketch the contents of this article. In the next section we present some preliminaries. In Section 3, we prove the boundedness of the Cerami sequence and in the Section 4, we prove of the main results. In the Section 5, we analyze the case when $V(x) \to 0$, as $|x| \to \infty$.

2. Preliminary results

By the reduction method described in [13], the Euler-Lagrange functional associated with the system (1.1), $J : H \equiv H^1(\mathbb{R}^3) \to \mathbb{R}$, is

$$J(u) = \frac{1}{2}\|u\|^2 - \frac{1}{2}\int_{\mathbb{R}^3} \omega\phi_u u^2\,dx - \int_{\mathbb{R}^3} K(x)F(u)\,dx,$$

where $F(u) = \int_0^u f(s)\,ds$. From the conditions on f and by standard arguments, the functional $J \in C^1(H, \mathbb{R})$ has Frechet derivative

$$J'(u)v = \int_{\mathbb{R}^3} (\nabla u\nabla v + V(x)uv)\,dx - \int_{\mathbb{R}^3} (2\omega + \phi_u)\phi_u uv\,dx - \int_{\mathbb{R}^3} K(x)f(u)v\,dx,$$

for all $v \in H$. The norm in H given by

$$\|u\|^2 = \int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)u^2)dx$$

is equivalent to the usual norm in H. The induced inner product is

$$\langle u, v\rangle := \int_{\mathbb{R}^3} (\nabla u\nabla v + V(x)uv)\,dx,$$

We recall that the critical points of functional J are precisely the weak solutions of (1.1). We also assume that $f(s) = 0$ for all $s \in (-\infty, 0]$.

A fundamental tool in our analysis will be the following Lemma.

Lemma 2.1. *For every $u \in H$, there exists a unique $\phi_u \in D^{1,2}(\mathbb{R}^3)$ which solves*

$$\Delta\phi = (\omega + \phi)u^2. \tag{2.1}$$

Furthermore, in the set $\{x : u(x) \neq 0\}$ we have $-\omega \leq \phi_u \leq 0$ if $\omega > 0$.

For a proof of the above lemma, see [22, Proposition 2.1]. From assumption (A3) and (A4) [or (A3') and (A4)] and combining with Lemma 2.1 follows that the functional J satisfies the geometric conditions of the Mountain Pass Theorem of Ambrosetti and Rabinowitz in [5]. So, there is a sequence $(u_n) \subset H$ such that

$$J(u_n) \to c \quad \text{and} \quad (1 + \|u_n\|)\|J'(u_n)\| \to 0, \quad n \to \infty, \tag{2.2}$$

where

$$c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J(\gamma(t))$$

is the Mountain Pass level, with

$$\Gamma = \{\gamma \in C([0, 1], H^1(\mathbb{R}^3)); \gamma(0) = 0, J(\gamma(1)) \leq 0\}.$$

The second result in this section is the following Hardy-type inequality.

Lemma 2.2. *Suppose that* (A1)–(A4) *or* (A1), (A3), (A4) *hold. Then, H is compactly embedded into*

$$\Gamma^q(\mathbb{R}^3) := \{\varphi : \mathbb{R}^3 \to \mathbb{R}; \varphi \textit{ is measurable and } \int_{\mathbb{R}^3} K(x)|\varphi|^q\,dx < \infty\},$$

for all $q \in (2, 6)$.

Proof. Consider (A1), (A3) and (A4); thus fixed $q \in (2,6)$ and given $\varepsilon > 0$, there are $0 < s_0 < s_1$ and $C > 0$ such that

$$K(x)|s|^q \le \varepsilon C(V(x)|s|^2 + |s|^6) + CK(x)\mathcal{X}_{[s_0,s_1]}(|s|)|s|^6, \quad \forall s \in \mathbb{R}. \tag{2.3}$$

Hence,

$$\int_{B_r^c(0)} K(x)|u|^q\,dx \le \varepsilon CQ(u) + C\int_{A\cap B_r^c(0)} K(x)\,dx, \quad \forall u \in H \tag{2.4}$$

where

$$Q(u) = \int_{\mathbb{R}^3} V(x)|u|^2\,dx + \int_{\mathbb{R}^3} |u|^6\,dx,$$
$$A = \{x \in \mathbb{R}^3 : s_0 \le |u(x)| \le s_1\}.$$

If (v_n) is a sequence such that $v_n \rightharpoonup v$ weakly in H, as $n \to \infty$, there is some constant $M_1 > 0$ such that

$$\|v_n\|^2 = \int_{\mathbb{R}^3} (|\nabla v_n|^2 + V(x)|v_n|^2)\,dx \le M_1, \quad \int_{\mathbb{R}^3} |v_n|^6\,dx \le M_1, \quad \forall n \in \mathbb{N},$$

implying that $(Q(v_n))$ is bounded. On the other hand, setting

$$A_n = \{x \in \mathbb{R}^3 : s_0 \le |v_n(x)| \le s_1\},$$

the above inequality implies

$$s_0^6\mu(A_n) \le \int_{A_n} |v_n|^6\,dx \le M_1, \quad \forall n \in \mathbb{N},$$

showing that $\sup_{n\in\mathbb{N}}\mu(A_n) < +\infty$. Therefore, from (II), there is a $r > 0$ such that

$$\int_{A_n\cap B_r^c(0)} K(x)\,dx < \frac{\varepsilon}{s_1^6}, \quad \forall n \in \mathbb{N}. \tag{2.5}$$

Now, (2.4) and (2.5) lead to

$$\int_{B_r^c(0)} K(x)|v_n|^q\,dx \le \varepsilon CM_1 + s_1^6\int_{A_n\cap B_r^c(0)} K(x)\,dx < (CM_1+1)\varepsilon, \quad \forall n \in \mathbb{N}. \tag{2.6}$$

Since $q \in (2,6)$ and K is a continuous function, from the Sobolev embeddings it follows that

$$\lim_{n\to+\infty}\int_{B_r(0)} K(x)|v_n|^q\,dx = \int_{B_r(0)} K(x)|v|^q\,dx. \tag{2.7}$$

In light of $(2.6) and (2.7)$, we have

$$\lim_{n\to+\infty}\int_{\mathbb{R}^3} K(x)|v_n|^q\,dx = \int_{\mathbb{R}^3} K(x)|v|^q\,dx. \tag{2.8}$$

This means that

$$v_n \to v, \quad \text{in } \Gamma^q(\mathbb{R}^3), \quad n \to \infty,\ \forall q \in (2,6).$$

Now, we fix $x \in \mathbb{R}^3$ and $\forall s > 0$ there is a constant $C = C(p)$ such that

$$CV(x)^{\frac{6-p}{4}} \le V(x)s^{2-p} + s^{6-p};$$

it follows from the fact that the function

$$h(s) = V(x)s^{2-p} + s^{6-p}, \quad s > 0,$$

has the minimum value $CV(x)^{\frac{6-p}{4}}$.

Using (A1), (A3') and (A4),and choosing $\varepsilon \in (0, C)$ for some $C > 0$ we infer that

$$K(x)|s|^p \le \varepsilon(V(x)|s|^2 + |s|^6), \quad \forall s \in \mathbb{R}, |x| \ge r.$$

Consequently, for all $u \in H$ we have

$$\int_{B_r^c(0)} K(x)|s|^p\,dx \le \int_{B_r^c(0)} \varepsilon(V(x)|s|^2 + |s|^6)\,dx.$$

If (v_n) is a sequence such that $v_n \rightharpoonup v$ weakly in H, as $n \to \infty$, there is $M_2 > 0$ such that

$$\int_{B_r^c(0)} K(x)|v_n|^q\,dx \le 2\varepsilon M_2. \tag{2.9}$$

Since $q \in (2, 6)$ and K is a continuous function, it follows from the Sobolev embeddings

$$\lim_{n\to+\infty} \int_{B_r(0)} K(x)|v_n|^q\,dx = \int_{B_r(0)} K(x)|v|^q\,dx. \tag{2.10}$$

From (2.9) and (2.10), we obtain

$$\lim_{n\to+\infty} \int_{\mathbb{R}^3} K(x)|v_n|^q\,dx = \int_{\mathbb{R}^3} K(x)|v|^q\,dx.$$

implying that

$$v_n \to v \quad \text{in } \Gamma^q(\mathbb{R}^3), \quad n \to \infty,\ \forall q \in (2, 6).$$

□

Lemma 2.3. *Suppose that* (A1)–(A4) *are satisfied, and consider a sequence* (v_n) *in* H *such that* $v_n \rightharpoonup v$ *weakly in* H*, as* $n \to \infty$*. Then*

$$\lim_{n\to+\infty} \int_{\mathbb{R}^3} K(x)f(v_n)v_n\,dx = \int_{\mathbb{R}^3} K(x)f(v)v\,dx.$$

Proof. Assuming (A1), (A3) and (A4), for a fixed $q \in (2, 6)$ and $\varepsilon > 0$, there is $C > 0$ such that

$$|K(x)f(s)s| \le \varepsilon C(V(x)|s|^2 + |s|^6) + K(x)|s|^q, \quad \forall s \in \mathbb{R}. \tag{2.11}$$

From Lemma 2.2, we have

$$\int_{\mathbb{R}^3} K(x)|v_n|^q\,dx \to \int_{\mathbb{R}^3} K(x)|v|^q\,dx,$$

then there exists $r > 0$ such that

$$\int_{B_r^c(0)} K(x)|v_n|^q\,dx < \varepsilon, \quad \forall n \in \mathbb{N}. \tag{2.12}$$

Since (v_n) is bounded in H, there exists $M_3 > 0$ such that

$$\int_{\mathbb{R}^3} V(x)|v_n|^2\,dx \le M_3 \quad \text{and} \quad \int_{\mathbb{R}^3} V(x)|v_n|^6\,dx \le M_3.$$

Combining the last two inequalities with (2.11) and (2.12), we obtain

$$\Big| \int_{B_r^c(0)} K(x)f(v_n)v_n\,dx\Big| < (2CM_3 + 1)\varepsilon, \quad \forall n \in \mathbb{N}.$$

To complete the proof we need to show that

$$\lim_{n\to+\infty} \int_{B_r(0)} K(x)f(v_n)v_n\,dx = \int_{B_r(0)} K(x)f(v)v\,dx.$$

However, this limit is obtained by using hypothesis (A4) and arguing as in [20], setting

$$P(x,s) = K(x)f(s)s, \quad Q(x,u_n(x)) = |u_n(x)|^6.$$

□

Lemma 2.4. *Suppose that f satisfies* (A1), (A3'), (A4), *and consider a sequence (v_n) in H such that $v_n \rightharpoonup v$ weakly in H, as $n \to \infty$. Then*

$$\lim_{n\to+\infty} \int_{\mathbb{R}^3} K(x)f(v_n)v_n\,dx = \int_{\mathbb{R}^3} K(x)f(v)v\,dx.$$

Proof. Using the Lemma 2.2, for $r > 0$ sufficiently small, arguing as in (2.11) we infer that

$$K(x) \le \varepsilon(V(x)|s|^{2-p} + |s|^{6-p}), \quad \forall |x| \ge r.$$

The rest of the proof follows similarly to the proof of Lemma 2.3. □

3. Boundedness of Cerami sequence

Lemma 3.1. *The Cerami sequence $(u_n) \subset H$ given in* (2.2) *is bounded.*

Proof. We have a positive constant M such that

$$M + o_n(1)\|u_n\| \ge \theta J(u_n) - J'(u_n)u_n \tag{3.1}$$

for $2 < q < 6$. From (A1), (A5) and Lemma 2.1 the Cerami sequence (u_n) is such that

$$\begin{aligned}
\theta J(u_n) - J'(u_n)u_n &= \big(\frac{\theta-2}{2}\big)\|u_n\|^2 + \big(\frac{-\theta+4}{2}\big)\int_{\mathbb{R}^3} \omega\phi_{u_n}u_n^2\,dx + \int_{\mathbb{R}^3} \phi_{u_n}^2 u_n^2\,dx \\
&\quad + \int_{\mathbb{R}^3} K(x)(f(u_n)u_n - \theta F(u_n))\,dx \\
&\ge \big(\frac{\theta-2}{2}\big)\|u_n\|^2, \quad \text{if } \theta > 4.
\end{aligned}$$

Similarly, if $2 < \theta < 4$ we use the hypothesis

$$0 < \frac{2(4-\theta)}{\theta-2} \le V_0 \le V(x),$$

and Lemma 2.1 to obtain

$$\begin{aligned}
&\theta J(u_n) - J'(u_n)u_n \\
&\ge \big(\frac{\theta-2}{2}\big)\int_{\mathbb{R}^3} |\nabla u_n|^2\,dx + \big(\frac{\theta-2}{2}\big)\int_{\mathbb{R}^3} V(x)u_n^2\,dx + \omega\big(\frac{-\theta+4}{2}\big)\int_{\mathbb{R}^3} \phi_{u_n}u_n^2\,dx \\
&\ge \big(\frac{\theta-2}{2}\big)\int_{\mathbb{R}^3} |\nabla u_n|^2\,dx + \big(\frac{\theta-2}{2}\big)\int_{\mathbb{R}^3} V_0u_n^2\,dx + \omega\big(\frac{-\theta+4}{2}\big)\int_{\mathbb{R}^3} \phi_{u_n}u_n^2\,dx \\
&\ge \big(\frac{\theta-2}{2}\big)\int_{\mathbb{R}^3} |\nabla u_n|^2\,dx + \big(\frac{\theta-2}{2}\big)\int_{\mathbb{R}^3} V_0u_n^2\,dx + \omega^2\big(\frac{\theta-4}{2}\big)\int_{\mathbb{R}^3} u_n^2\,dx \\
&= \big(\frac{\theta-2}{2}\big)\int_{\mathbb{R}^3} |\nabla u_n|^2\,dx + \big[\frac{(\theta-2)V_0 + (\theta-4)\omega^2}{2}\big]\int_{\mathbb{R}^3} u_n^2\,dx \\
&\ge C\|u_n\|^2.
\end{aligned} \tag{3.2}$$

In light of (3.1) and (3.2) we conclude that (u_n) is bounded. □

Lemma 3.2. *If $u_n \rightharpoonup u$ weakly in H, as $n \to \infty$, then passing to a subsequence if necessary, $\phi_{u_n} \rightharpoonup \phi_u$ weakly in $D^{1,2}(\mathbb{R}^3)$, as $n \to \infty$.*

Proof. Consider $(u_n), u \in H$ such that $u_n \rightharpoonup u$ weakly in H, as $n \to \infty$. It follows that

$$\begin{gathered} u_n \rightharpoonup u \quad \text{weakly in } L^p(\mathbb{R}^3), \text{ as } n \to \infty,\ 2 \le p \le 6, \\ u_n \to u \quad \text{in } L^p_{\rm loc}(\mathbb{R}^3), \text{ as } n \to \infty,\ 2 \le p < 6. \end{gathered}$$

From Lemma 2.1, note that for all $n \ge 1$ we have

$$\begin{aligned} \|\phi_{u_n}\|^2_{D^{1,2}(\mathbb{R}^3)} &= -\int_{\mathbb{R}^3} \omega \phi_{u_n} u_n^2\,dx - \int_{\mathbb{R}^3} \phi_{u_n}^2 u_n^2\,dx \\ &\le -\int_{\mathbb{R}^3} \omega \phi_{u_n} u_n^2\,dx \le C\|\phi_{u_n}\|_{D^{1,2}(\mathbb{R}^3)} \|u_n\|^2_{\frac{12}{5}}. \end{aligned}$$

It means that (ϕ_{u_n}) is bounded in $D^{1,2}(\mathbb{R}^3)$. Since $D^{1,2}(\mathbb{R}^3)$ is a Hilbert space, there is a $\xi \in D^{1,2}(\mathbb{R}^3)$ such that

$$\begin{gathered} \phi_{u_n} \rightharpoonup \xi \quad \text{weakly in } L^p(\mathbb{R}^3),\ \text{as } n \to \infty,\ 2 \le p \le 6, \\ \phi_{u_n} \to \xi \quad \text{in } L^p_{\rm loc}(\mathbb{R}^3),\ \text{as } n \to \infty, \quad 2 \le p < 6. \end{gathered}$$

We desire to prove the following equality $\phi_u = \xi$. For this, it is necessary to show, in the sense of distributions,

$$\Delta \xi = (\omega + \xi) u^2$$

and use the uniqueness of the solution given in Lemma 2.1.

Consider a test function $\psi \in C_0^\infty(\mathbb{R}^3)$. We know by Lemma 2.1 we have

$$\Delta \phi_{u_n} = (\omega + \phi_{u_n}) u_n^2.$$

Then we just need to see how each term of the equality above converges. To verify that

$$\begin{gathered} \int_{\mathbb{R}^3} \nabla \phi_{u_n} \nabla \psi\,dx \to \int_{\mathbb{R}^3} \nabla \xi \nabla \psi\,dx, \quad \text{as } n \to \infty, \\ \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 \psi\,dx \to \int_{\mathbb{R}^3} \xi u^2 \psi\,dx, \quad \text{as } n \to \infty, \end{gathered}$$

it is sufficient to note that it is a consequence of the definition the weak convergence. By the strong convergence in $L^p_{\rm loc}(\mathbb{R}^3), 2 \le p < 6$, we obtain

$$\int_{\mathbb{R}^3} u_n^2 \psi\,dx \to \int_{\mathbb{R}^3} u^2 \psi\,dx, \quad \text{as } n \to \infty.$$

We consider a test function $\varphi \in C_0^\infty(\mathbb{R}^3)$. Using boundedness of (ϕ_{u_n}), the strong convergences in $L^p_{\rm loc}(\mathbb{R}^3), 2 \le p < 6$ and the Sobolev embeddings follows that as $n \to +\infty$, we have

$$\begin{aligned} \int_{\mathbb{R}^3} (\phi_{u_n} u_n - \xi u)\varphi\,dx &= \int_{\mathbb{R}^3} \phi_{u_n}(u_n - u)\varphi\,dx + \int_{\mathbb{R}^3} u(\phi_{u_n} - \xi)\varphi\,dx \\ &\le C\|\phi_{u_n}\|_{D^{1,2}(\mathbb{R}^3)} \Big(\int_{\mathbb{R}^3} |u_n - u|^{6/5} |\varphi|^{6/5}\,dx \Big)^{5/6} \\ &\quad + \int_{\mathbb{R}^3} (\phi_{u_n} - \xi) u \varphi\,dx \to 0, \quad \text{as } n \to \infty \to +\infty. \end{aligned}$$

For the same reasons, it follows that

$$\begin{aligned}\int_{\mathbb{R}^3}(\phi_{u_n}^2 u_n-\xi^2 u)\varphi\,dx &= \int_{\mathbb{R}^3}\phi_{u_n}^2(u_n-u)\varphi\,dx+\int_{\mathbb{R}^3}u(\phi_{u_n}^2-\xi^2)\varphi\,dx\\ &\le C\|\phi_{u_n}\|_{D^{1,2}(\mathbb{R}^3)}\Big(\int_{\mathbb{R}^3}|u_n-u|^{3/2}|\varphi|^{3/2}\,dx\Big)^{2/3}\\ &\quad+\int_{\mathbb{R}^3}(\phi_{u_n}^2-\xi^2)u\varphi\,dx\to 0,\quad \text{as } n\to+\infty.\end{aligned}$$

From density, for all $\varphi\in H$ we infer that

$$\int_{\mathbb{R}^3}(\nabla u_n\nabla\varphi+V(x)u_n\varphi)\,dx\to\int_{\mathbb{R}^N}(\nabla u\nabla\varphi+V(x)u\varphi)\,dx,$$

$$\int_{\mathbb{R}^3}(2\omega+\phi_{u_n})\phi_{u_n}u_n\varphi\,dx\to\int_{\mathbb{R}^3}(2\omega+\xi)\xi u\varphi\,dx,$$

as $n\to+\infty$, thus we prove the lemma. □

4. Proof of the main results

Proof of Theorem 1.2. Let (u_n) be a Cerami sequence as given in (2.2). From Lemma 3.1 follows that (u_n) is bounded and, up to subsequence, we can assume that there is $u\in H$, such that

$$u_n\rightharpoonup u,\quad \text{weakly in } H, \text{ as } n\to\infty.$$

We will show that $u_n\to u$, as $n\to+\infty$. From Lemma 2.3, we have

$$\lim_{n\to+\infty}\int_{\mathbb{R}^3}K(x)f(u_n)u_n\,dx=\int_{\mathbb{R}^3}K(x)f(u)u\,dx.$$

On the other hand, we know that

$$J'(u)v=\int_{\mathbb{R}^3}(\nabla u.\nabla v+V(x)uv)\,dx-\int_{\mathbb{R}^3}(2\omega+\phi_u)\phi_u uv\,dx-\int_{\mathbb{R}^3}K(x)f(u)v\,dx.$$

Since $J'(u_n)u_n=o_n(1)$, we get

$$\lim_{n\to+\infty}\|u_n\|^2=\lim_{n\to+\infty}\Big[\int_{\mathbb{R}^3}(2\omega+\phi_{u_n})\phi_{u_n}u_n^2\,dx+\int_{\mathbb{R}^3}K(x)f(u_n)u_n\,dx\Big].\tag{4.1}$$

By Lemma 2.3, we have

$$\lim_{n\to+\infty}\int_{\mathbb{R}^3}K(x)f(u_n)u\,dx=\int_{\mathbb{R}^3}K(x)f(u)u\,dx$$

and from Lemma 3.2, we obtain that

$$\lim_{n\to\infty}\int_{\mathbb{R}^3}(2\omega+\phi_{u_n})\phi_{u_n}u_n^2\,dx=\int_{\mathbb{R}^3}(2\omega+\xi)\xi u^2\,dx.$$

Then

$$\lim_{n\to+\infty}\|u_n\|^2=\int_{\mathbb{R}^3}(2\omega+\xi)\xi u^2\,dx+\int_{\mathbb{R}^3}K(x)f(u)u\,dx.\tag{4.2}$$

Moreover, since $J'(u_n)u=o_n(1)$, we have

$$\|u\|^2=\int_{\mathbb{R}^3}(2\omega+\xi)\xi u^2\,dx+\int_{\mathbb{R}^3}K(x)f(u)u\,dx.\tag{4.3}$$

Therefore, from (4.2) and (4.3), we obtain $\lim_{n\to+\infty}\|u_n\|^2=\|u\|^2$, showing that

$$u_n\to u,\quad \text{in } H,\ \text{as } n\to\infty.$$

Consequently,

$$J(u) = c \quad \text{and} \quad J'(u) = 0,$$

implying that u is a ground state solution for J. Since $u_n \geq 0$, we have that $u \geq 0$. The positivity of u follows by using the maximum principle. □

Proof of Theorem 1.3. It is similar to that of Theorem 1.2. However using the Lemma 2.4 instead of Lemma 2.3. We omit the proof here. □

5. Case $V(x) \to 0$, as $|x| \to \infty$

In this section, we study the problem (1.1), inspired by [4], replacing the hypothesis (A1) by

(A1') $V, K : \mathbb{R}^3 \to \mathbb{R}$ are smooth functions, $K \in L^\infty(\mathbb{R}^3)$ and there are constant $\tau, \xi_1, a_1, a_2, a_3 > 0$, such that

$$\frac{a_1}{1+|x|^\tau} \leq V(x) \leq a_2 \quad \text{and} \quad 0 < K(x) \leq \frac{a_3}{1+|x|^{\xi_1}}, \quad \forall x \in \mathbb{R}^3. \tag{5.1}$$

with τ, ξ_1 satisfying

$$5 - \frac{4\xi_1}{\tau} < p, \text{ if } 0 < \xi_1 < \tau, \quad \text{or} \quad 1 < p, \text{ if } \xi_1 \geq \tau\,.$$

Also we assume that $\frac{K}{V} \in L^\infty(\mathbb{R}^3)$.

In this case, the norm for H is

$$\|u\|_V^2 = \int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)u^2)\, dx$$

whose induced inner product is

$$\langle u, v\rangle_V = \int_{\mathbb{R}^3} (\nabla u \nabla v + V(x)uv)\, dx.$$

Remark 5.1. At this moment, it is important to observe that (1.5) is weaker than any one of the following conditions:

(a) there are $r \geq 1$ and $\rho \geq 0$ such that $K \in L^r(\mathbb{R}^3 \setminus B_\rho(0))$;
(b) $K(x) \to 0$ as $|x| \to \infty$;
(c) $K = H_1 + H_2$, with H_1 and H_2 verifying (a) and (b) respectively.

In this section, all the past results achieved follow naturally by using the hypothesis (A1') instead of (A1), except of Lemma 3.1. We would like to show another statement for the boundedness Cerami sequence.

Lemma 5.2. *The Cerami sequence $(u_n) \subset H$ given in* (2.2) *is bounded.*

Proof. Once that $(J(u_n))$ is bounded and $|J'(u_n)u_n| \leq \|u_n\|_V$ for n large enough, so there are some constant $M > 0$ and $n_0 \in \mathbb{N}$ such that

$$J(u_n) - \frac{1}{\theta}J'(u_n)u_n \leq M + \|u_n\|_V, \quad \forall n \geq n_0.$$

On the other hand, it is certain that $u_n > 0$ for each $x \in \mathbb{R}^3$ and using the assumption (f_2) for $\theta > 4$ combined with Lemma 2.1, we have

$$\begin{aligned} J(u_n) - \frac{1}{\theta}J'(u_n)u_n &\geq \big(\frac{1}{2} - \frac{1}{\theta}\big)\|u_n\|_V^2 - \frac{1}{2}\int_{\mathbb{R}^3} \omega\phi_{u_n}u_n^2\, dx + \frac{2\omega}{\theta}\int_{\mathbb{R}^3} \phi_{u_n}u_n^2\, dx \\ &\quad + \frac{1}{\theta}\int_{\mathbb{R}^3} \phi_{u_n}^2 u_n^2\, dx \end{aligned}$$

$$
\begin{aligned}
&\geq \big(\frac{1}{2}-\frac{1}{\theta}\big)\|u_n\|_V^2+\omega\big(\frac{4-\theta}{2\theta}\big)\int_{\mathbb{R}^3}\phi_{u_n}u_n^2\,dx \\
&\geq \big(\frac{1}{2}-\frac{1}{\theta}\big)\|u_n\|_V^2,
\end{aligned}
$$

which shows that (u_n) is bounded. □

In this way, we obtained the same results as those presented of Theorems 1.2 and 1.3, using (A1') instead of (A1).

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References

[1] C. O. Alves, M. Montenegro, M. A. S. Souto; *Existence of solution for three classes of elliptic problems in $\mathbb{R}^N$ with zero mass,* J. Differential Equations **252** (2012), 5735–5750.

[2] C. O. Alves, M. A. S. Souto; *Existence of solutions for a class of elliptic equations in $\mathbb{R}^N$ with vanishing potentials*, J. Differential Equations **252** (2012), 5555–5568.

[3] C. O. Alves, M. A. S. Souto; *Existence of solutions for a class of nonlinear Schrödinger equations with potential vanishing at infinity*, J. Differential Equations, **254** (2013), 1977–1991.

[4] A. Ambrosetti, V. Felli, A. Malchiodi; *Ground states of nonlinear Schrödinger equations with potentials vanishing at infinity*, J. Eur. Math. Soc. (JEMS), **7** (2005), 117–144.

[5] A. Ambrosetti, P. H. Rabinowitz; *Dual varational methods in critical point theorem and application*, J. Funct. Anal., **14** (1973), 349–381.

[6] A. Ambrosetti, Z.-Q. Wang; *Nonlinear Schrödinger equations with vanishing and decaying potentials,* Differential Integral Equations, **18** (2005), 1321–1332.

[7] A. Azzollini, A. Pomponio; *Ground state solutions for the nonlinear Klein-Gordon-Maxwell equations,* Topol. Methods Nonlinear Anal., **35** (2010), 33-42.

[8] A. Azzollini, L. Pisani, A. Pomponio; *Improved estimates and a limit case for the electrostatic Klein-Gordon-Maxwell system,* Proc. Roy. Soc. Edinburgh Sect. A., **141** (2011), 449–463.

[9] M. Badiale, L. Pisani, S. Rolando; *Sum of weighted Lebesgue spaces and nonlinear elliptic equations,* NoDEA Nonlinear Differential Equations Appl., **18** (2011), 369–405.

[10] T. Bartsch, Z.-Q. Wang, M. Willem; *The Dirichlet problem for superlinear elliptic equations,* in: Stationary Partial Differential Equations. Vol. II, in: Handb. Differ. Equ., Elsevier/North-Holland, Amsterdam, 2005, pp. 1–55.

[11] V. Benci, D. Fortunato; *The nonlinear Klein-Gordon equation coupled with the Maxwell equations,* Nonlinear Anal. **47** (2001), 6065–6072.

[12] V. Benci, D. Fortunato; *Solitary waves of the nonlinear Klein-Gordon equation coupled with the Maxwell equations,* Rev. Math. Phys., **14** (2002), 409-420.

[13] V. Benci, D. Fortunato, A. Masiello, L. Pisani; *Solitons and the electromagnetic field,* Math. A., **232** (1999), 73–102.

[14] V. Benci, C. R. Grisanti, A. M. Micheletti; *Existence of solutions for the nonlinear Schrödinger equation with $V(\infty)=0$*, Progr. Nonlinear Differential Equations Appl., **66** (2005), 53–65.

[15] H. Berestycki, P. L. Lions; *Nonlinear scalar field equations. I. Existence of a ground state,* Arch. Ration. Mech. Anal., **82** (1983), 313– 346.

[16] D. Bonheure, J. Van Schaftingen; *Ground states for nonlinear Schrödinger equation with potential vanishing at infinity,* Ann. Mat. Pura Appl., **189** (2010), 273–301.

[17] P. Carrião, P. Cunha, O. Miyagaki; *Existence results for the Klein-Gordon-Maxwell equations in higher dimensions with critical exponents,* Commun. Pure Appl. Anal., **10** (2011), 709-718.

[18] P. Carrião, P. Cunha, O. Miyagaki; *Positive and ground state solutions for the critical Klein-Gordon-Maxwell system with potentials,* arXiv:1005.4088v2 [math.AP].
[19] D. Cassani; *Existence and non-existence of solitary waves for the critical Klein-Gordon equation coupled with Maxwell's equations,* Nonlinear Analysis, **58** (2004), 733–747.
[20] J. Chabrowski; *Weak convergence methods for semilinear elliptic equations.* World Scientific, 1980.
[21] T. D'Aprile, D. Mugnai; *Solitary waves for nonlinear Klein-Gordon-Maxwell and Schrodinger-Maxwell equations,* Proc. R. Soc. Edinb., Sect. A, **134** (2004), 1-14.
[22] T. D'Aprile, D. Mugnai; *Non-existence results for the coupled Klein-Gordon-Maxwell equations,* Adv. Nonlinear Stud. **4** (2004) 307-322.
[23] M. Ghimenti, A. M. Micheletti; *Existence of minimal nodal solutions for the nonlinear Schrödinger equation with* $V(\infty) = 0$, Adv. Diff. Equations, **11** (2006), 1375–1396.
[24] X. He; *Multiplicity of solutions for a nonlinear Klein-Gordon-Maxwell system,* Acta Appl. Math., **130** (2014), 237–250.
[25] L, Li, C.-L. Tang; *Infinitely many solutions for a nonlinear Klein-Gordon-Maxwell System,* Nonlinear Analysis, **110** (2014), 157–169.
[26] Y. Liu, Z.-Q. Wang, J. Zhang; *Ground states of nonlinear Schrödinger equations with potentials,* Ann. Inst. H. Poincare Anal. Non Linéaire, **23** (2006), 829–837.
[27] F. Wang; *Ground-state solutions for the electrostatic nonlinear Klein-Gordon-Maxwell system,* Nonlinear Anal., **74** (14) (2011), 4796–4803.
[28] F. Wang; *Solitary waves for the Klein-Gordon-Maxwell system with critical exponent,* Nonlinear Anal. , **74 (3)** (2011), 827–835.

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