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AXISYMMETRIC SOLUTIONS OF A TWO-DIMENSIONAL NONLINEAR WAVE SYSTEM WITH A TWO-CONSTANT EQUATION OF STATE

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Abstract. We study a special class of Riemann problem with axisymmetry for two-dimensional nonlinear wave equations with the equation of state $p = A_1\rho^{\gamma_1} + A_2\rho^{\gamma_2}$, $A_i < 0$, $-3 < \gamma_i < -1$ $(i = 1, 2)$. The main difficulty lies in that the equations can not be directly reduced to an autonomous system of ordinary differential equations. To solve it, we use the axisymmetry and self-similarity assumptions to reduce the equations to a decoupled system which includes three components of solution. By solving the decoupled system, we obtain the structures of the corresponding solutions and their existence.

1. Introduction

This article concerns a special class of Riemann problem to the two-dimensional nonlinear wave system

$$
\begin{aligned}
\rho_t + (\rho u)_x + (\rho v)_y &= 0,\\
(\rho u)_t + p_x &= 0,\\
(\rho v)_t + p_y &= 0,
\end{aligned}
\tag{1.1}
$$

with the equation of state

$$
p(\rho) = A_1\rho^{\gamma_1} + A_2\rho^{\gamma_2}, \tag{1.2}
$$

where the variables (u, v), ρ represent the velocity and the density respectively, and $A_i < 0$, $-3 < \gamma_i < -1$ $(i = 1, 2)$. System (1.1) can be obtained either by starting with the isentropic gas dynamics equations and neglecting the quadratic terms in the velocity or by writing the nonlinear wave equation as a first-order system [1, 2]. If the equation of state is taken as the following form

$$
p = A\rho^{\gamma} \tag{1.3}
$$

with $-1 \le \gamma < 0$ and $A < 0$, (1.3) is called as the generalized Chaplygin gas. For the case $\gamma = -1$ and $A = -1$, (1.3) was introduced by Chaplygin[4], Tsien and von Karman [22, 24] as a suitable mathematical approximation for calculating the lifting force on a wing of an airplane in aerodynamics. Sen and Scherrer generalized this model to allow for the case $\gamma < -1$. The transient Chaplygin gas model provides a

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possible mechanism to allow for a currently accelerating universe without a future horizon and can be taken to be a model for dark energy alone [21].

The initial data for a general Riemann problem are constant along radial direction from the origin and piecewise constant as a function of angle. There are a set of conjectures offered by Zhang and Zheng in [26] for the solutions to the two-dimensional Riemann problem. Following Zhang and Zheng's work, many efforts have been made to prove these conjectures for the compressible Euler equations or some reduced systems in the past twenty years [19, 29]. Unfortunately, until now, none of them has been completely proved due to the complicated structure of solutions. For related results, we can consult the survey [20] and references cited therein.

The study of two-dimensional Riemann problem (or shock reflection) have been devoted for system (1.1) with general polytropic gas [2, 6, 11, 12, 14, 15, 16, 17, 18, 23]. Čanić, Keyfitz, Kim studied the Mach stem of shock solution by solving a free boundary problem [2]. Tesdall, Sanders and Keyfitz presented numerical solutions for the nonlinear wave system which is used to describe the Mach reflection of weak shock waves [23]. In particular, Kim and Lee studied the configuration that the transonic shock interacted with the sonic circle and obtained a global transonic solution to this configuration. Kim established a global transonic solution to the interaction of a transonic shock with a rarefaction wave[14, 15, 16, 17, 18]. Furthermore, Chen et al. [6] established a global theory of existence and optimal regularity for a shock diffraction problem. Hu ang Wang [11, 12] studied the semi-hyperbolic patches of solutions to the two-dimensional nonlinear wave system for Chaplygin gases and constructed a global classical solution to the interaction of two arbitrary planar rarefaction waves.

For a special class of Riemann problem with axisymmetry for the two-dimensional isentropic Euler equations with the equation of state $p = A\rho^\gamma$, $\gamma > 1$, Zhang and Zheng constructed rigorously a three-parameter family of self-similar, globally bounded, and continuous weak solutions for all positive time to the Euler equations [27, 28, 30]. Under the assumption that $v = 0$, Hu [10] constructed a family of self-similar and global bounded weak solutions to the two-dimensional isentropic Euler equations with the equation of state (1.2) and $A_i > 0$, $\gamma_i > 1$ $(i = 1, 2)$ for axisymmetry initial data. As for the related works, we refer the reader to [5, 7, 8, 9].

As a continuation of the paper [10], we will consider the system

$$\begin{gathered}
\rho_t + (\rho u)_x + (\rho v)_y = 0, \\
(\rho u)_t + p_x = 0, \\
\varphi_t + (\varphi u)_x + (\varphi v)_y = 0, \\
(\rho v)_t + p_y = 0,
\end{gathered} \tag{1.4}$$

with the equation of state

$$p = p_1 + p_2 = A_1\rho^{\gamma_1} + A_2\varphi^{\gamma_2}, \tag{1.5}$$

for axisymmetry initial data, where $p_1 = A_1\rho^{\gamma_1}, p_2 = A_2\varphi^{\gamma_2}$, $A_i < 0, -3 < \gamma_i < -1$ $(i = 1, 2)$, $\rho \geq 0$ and $\varphi \geq 0$. It is easy to see that, if $\varphi = \rho$, $\gamma_1 = \gamma_2$ and $A_1 = A_2$, then system (1.4) reduces to the system in [25]. The purpose of the present paper is to investigate the two-dimensional Riemann problem with axisymmetry to the nonlinear wave system (1.1) and (1.2). We will construct rigorously a family of self-similar and global solutions for all positive time to (1.4) with (1.5). The difference with this paper [10] is that we do not assume $v = 0$ here, since we may decouple the first three equations from the fourth equation in (1.4) by the axisymmetry and self-similarity assumptions. Moreover, we find that the shock solutions appear in the case $u_0 > 0$, instead of $u_0 < 0$. And then, we obtain the detailed structures of solutions as well as their existence for system (1.4) with (1.5). If $\varphi = \rho$ at time zero in the solutions obtained from system (1.4), we may obtain the existence of solutions and their structures for the special class of Riemann problem to system (1.1) with (1.2).

The rest of this paper is organized as follows. In section 2, we present some preliminaries, give axisymmetry induction, intermediate field equations and the Rankine-Hugoniot relations. The global solutions for two cases $u_0 \leq 0$ and $u_0 > 0$ are constructed in sections 3 and 4, respectively. The conclusion is delivered in Section 5.

2. Preliminaries

We impose axisymmetry to the system (1.4). That is that the solutions (ρ, u, v, φ) have the property

$$\begin{gathered}
\rho(t,r,\theta) = \rho(t,r,0), \\
\varphi(t,r,\theta) = \varphi(t,r,0), \\
\begin{pmatrix} u(t,r,\theta) \\ v(t,r,\theta) \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} u(t,r,0) \\ v(t,r,0) \end{pmatrix},
\end{gathered} \tag{2.1}$$

for all $t \geq 0$, $\theta \in \mathbb{R}$ and $r > 0$, where (r, θ) are the polar coordinates of the (x, y)-plane. Substituting (2.1) into (1.4) for the smooth solutions reduces

$$\begin{gathered}
\rho_t + (\rho u)_r + \frac{\rho u}{r} = 0, \\
(\rho u)_t + (p_1 + p_2)_r = 0, \\
\varphi_t + (\varphi u)_r + \frac{\varphi u}{r} = 0, \\
(\rho v)_t = 0,
\end{gathered} \tag{2.2}$$

where $u = u(t, r, 0)$ and $v = v(t, r, 0)$ are the radial and pure rotational velocities, respectively. We require the Riemann initial data as follows

$$\big(\rho(0,r,\theta), u(0,r,\theta), v(0,r,\theta), \varphi(0,r,\theta)\big) = \big(\rho_0(\theta), u_0(\theta), v_0(\theta), \varphi_0(\theta)\big),$$

which implies, by the axisymmetry condition (2.1), that our data are limited to

$$\begin{gathered}
\rho(0,r,\theta) = \rho_0, \\
\varphi(0,r,\theta) = \varphi_0, \\
u(0,r,\theta) = u_0\cos\theta - v_0\sin\theta, \\
v(0,r,\theta) = u_0\sin\theta + v_0\cos\theta,
\end{gathered} \tag{2.3}$$

where $\rho_0 > 0$, $\varphi_0 > 0$, u_0 and v_0 are arbitrary constants.

Since the problem (2.2) is invariant under self-similar transformations, we look for self-similar solutions $(\rho, u, v, \varphi)(\xi)(\xi = r/t)$. Thus, we obtain from (2.2) that

$$\begin{aligned} \rho_\xi &= -\frac{\rho u}{p_1' + \overline{\varphi} - \xi^2}, \\ u_\xi &= -\frac{u(p_1' + \overline{\varphi} - \xi u)}{\xi(p_1' + \overline{\varphi} - \xi^2)}, \\ \overline{\varphi}_\xi &= -\frac{(\gamma_2 - 1)\overline{\varphi} u}{p_1' + \overline{\varphi} - \xi^2}, \\ v_\xi &= \frac{uv}{p_1' + \overline{\varphi} - \xi^2}, \end{aligned} \tag{2.4}$$

and

$$\lim_{\xi \to +\infty} (\rho, u, \overline{\varphi}, v) = (\rho_0, u_0, \gamma_2 p_2(\varphi_0)/\rho_0, v_0), \tag{2.5}$$

where $\overline{\varphi} = \gamma_2 p_2(\varphi)/\rho$. It is easy to see that the first three equations of (2.4) do not involve the component v. Hence, we consider a subsystem for the initial pure radial flow

$$\begin{aligned} \rho_\xi &= -\frac{\rho u}{p_1' + \overline{\varphi} - \xi^2}, \\ u_\xi &= -\frac{u(p_1' + \overline{\varphi} - \xi u)}{\xi(p_1' + \overline{\varphi} - \xi^2)}, \\ \overline{\varphi}_\xi &= -\frac{(\gamma_2 - 1)\overline{\varphi} u}{p_1' + \overline{\varphi} - \xi^2}, \end{aligned} \tag{2.6}$$

and

$$\lim_{\xi \to +\infty} (\rho, u, \overline{\varphi}) = (\rho_0, u_0, \gamma_2 p_2(\varphi_0)/\rho_0). \tag{2.7}$$

In this paper, by (2.6) and (2.7), we will construct global solutions to the problem (2.4) and (2.5) for any $\rho_0 > 0$, $\varphi_0 > 0$ and $u_0 \in \mathbb{R}, v_0 \in \mathbb{R}$.

2.1. **Far-field solutions and intermediate field equations.** Let $s = 1/\xi$, then (2.6) and (2.7) become

$$\begin{aligned} \frac{d\rho}{ds} &= \frac{\rho u}{s^2(p_1' + \overline{\varphi}) - 1}, \\ \frac{du}{ds} &= \frac{u(sp_1' + s\overline{\varphi} - u)}{s^2(p_1' + \overline{\varphi}) - 1}, \\ \frac{d\overline{\varphi}}{ds} &= \frac{(\gamma_2 - 1)\overline{\varphi} u}{s^2(p_1' + \overline{\varphi}) - 1}, \end{aligned} \tag{2.8}$$

and

$$(\rho, u, \overline{\varphi})|_{s=0} = (\rho_0, u_0, \gamma_2 p_2(\varphi_0)/\rho_0). \tag{2.9}$$

We can see that (2.8) is well-posed and has a unique local solution for any initial datum with $\rho_0 > 0$ and $\varphi_0 > 0$.

By introducing the variables

$$I = su, \quad K = s\sqrt{p_1'(\rho)}, \quad H = s\sqrt{\overline{\varphi}}, \tag{2.10}$$

system (2.8) can be put into the form

$$
\begin{aligned}
s\frac{dI}{ds} &= \frac{I(1+I-2K^2-2H^2)}{1-K^2-H^2},\\
s\frac{dK}{ds} &= \frac{K(1-\frac{\gamma_1-1}{2}I-K^2-H^2)}{1-K^2-H^2},\\
s\frac{dH}{ds} &= \frac{H(1-\frac{\gamma_2-1}{2}I-K^2-H^2)}{1-K^2-H^2}.
\end{aligned}
\tag{2.11}
$$

Introducing a new parameter τ, it is easy to get from (2.11) that

$$
\begin{aligned}
\frac{dI}{d\tau} &= I(1+I-2K^2-2H^2),\\
\frac{dK}{d\tau} &= K(1-\frac{\gamma_1-1}{2}I-K^2-H^2),\\
\frac{dH}{d\tau} &= H(1-\frac{\gamma_2-1}{2}I-K^2-H^2).
\end{aligned}
\tag{2.12}
$$

and

$$
\frac{ds}{d\tau} = s(1-K^2-H^2). \tag{2.13}
$$

Corresponding to the initial data (2.10), we will look for solutions of (2.12) and (2.13) with the initial condition

$$
(I,K,H) \sim s(u_0, \sqrt{p_1'(\rho_0)}, \sqrt{\gamma_2 p_2(\varphi_0)/\rho_0}), \quad \text{as } s \to 0^+. \tag{2.14}
$$

2.2. **Rankine-Hugoniot relation.** In self-similar coordinates $(\xi,\eta) = (x/t, y/t)$, system (1.4) can be rewritten as

$$
\begin{aligned}
-\xi\rho_\xi - \eta\rho_\eta + (\rho u)_\xi + (\rho v)_\eta &= 0,\\
-\xi(\rho u)_\xi - \eta(\rho u)_\eta + (p_1+p_2)_\xi &= 0,\\
-\xi\varphi_\xi - \eta\varphi_\eta + (\varphi u)_\xi + (\varphi v)_\eta &= 0,\\
-\xi(\rho v)_\xi - \eta(\rho v)_\eta + (p_1+p_2)_\eta &= 0.
\end{aligned}
$$

Let a discontinuous curve be given $\eta = \eta(\xi)$ and the slope of the curve be denoted by $\sigma = \eta'(\xi)$, then the Rankine-Hugoniot relation is

$$
\begin{aligned}
\sigma &= [(\eta - v)\rho],\\
[\xi\rho u - p_1 - p_2]\sigma &= [\eta\rho u],\\
[(u-\xi)\varphi]\sigma &= [(\eta - v)\varphi],\\
[\xi\rho v]\sigma &= [\eta\rho v - p_1 - p_2],
\end{aligned}
\tag{2.15}
$$

where $\sigma = \eta'(\xi)$ and $[h] = h_r - h_\ell$, h_ℓ, h_r are the limit states on the left- and right-hand sides of the discontinuity curve $\eta = \eta(\xi)$, respectively. When axisymmetry is given, the discontinuity curve has an infinite slope at the ξ-axis, that is $\sigma = \infty$ in (2.15). So we can obtain

$$
\begin{aligned}
\xi[\rho] &= [\rho u],\\
\xi[\rho u] &= [p_1 + p_2],\\
\xi[\varphi] &= [\varphi u],\\
[\rho v] &= 0,
\end{aligned}
$$

which yields the slip line

$$\xi = 0, \quad p_{1\ell} + p_{2\ell} = p_{1r} + p_{2r},$$

and the forward or backward discontinuity

$$\begin{gathered}
\xi = \pm\Big(\frac{[p_1 + p_2]}{[\rho]}\Big)^{1/2}, \\
(u_r - \xi)\rho_r = (u_\ell - \xi)\rho_\ell, \\
(u_r - \xi)\varphi_r = (u_\ell - \xi)\varphi_\ell, \\
\rho_r v_r = \rho_1 v_\ell.
\end{gathered} \tag{2.16}$$

Recalling $\xi = 1/s$ and (2.10), the system (2.16) can be transformed to

$$\begin{gathered}
(1 - I_r)K_r^{\frac{2}{\gamma_1 - 1}} = (1 - I_\ell)K_\ell^{\frac{2}{\gamma_1 - 1}}, \\
(1 - I_r)H_r^{\frac{2}{\gamma_2}} K_r^{\frac{2}{\gamma_2(\gamma_1 - 1)}} = (1 - I_\ell)H_\ell^{\frac{2}{\gamma_2}} K_\ell^{\frac{2}{\gamma_2(\gamma_1 - 1)}}, \\
K_r^{\frac{2}{\gamma_1 - 1}} - K_\ell^{\frac{2}{\gamma_1 - 1}} = \frac{1}{\gamma_1}\big(K_r^{\frac{2\gamma_1}{\gamma_1 - 1}} - K_\ell^{\frac{2\gamma_1}{\gamma_1 - 1}}\big) + \frac{1}{\gamma_2}\big(H_r^2 K_r^{\frac{2}{\gamma_1 - 1}} - H_\ell^2 K_\ell^{\frac{2}{\gamma_1 - 1}}\big), \\
\rho_r v_r = \rho_\ell v_\ell.
\end{gathered} \tag{2.17}$$

3. Solutions for $u_0 \le 0$

In this section, we construct global bounded continuous solutions to problem (2.6) and (2.7) with initial negative radial velocity. Here we only consider the case $\gamma_1 \neq \gamma_2$, the result for case $\gamma_1 = \gamma_2$, which corresponds to the generalized Chaplygin gas, can be found in [25]. If $u_0 = 0$, we get from (2.8) that $\rho = \rho_0$, $\overline{\varphi} = \gamma_2 p_2(\varphi_0)/\rho_0$ and $u = 0$ is a trivial solution. So we only consider the case $u_0 < 0$ in the present section and divide this into two cases: $-3 < \gamma_1 < \gamma_2 < -1$ and $-3 < \gamma_2 < \gamma_1 < -1$.

We introduce

$$\begin{gathered}
A := 1 - \frac{\gamma_2 - 1}{2} I - K^2 - H^2, \\
B := 1 + I - 2K^2 - 2H^2, \\
C := 1 - \frac{\gamma_1 - 1}{2} I - K^2 - H^2, \\
D := 1 - K^2 - H^2.
\end{gathered}$$

Then (2.12) and (2.13) become

$$\begin{gathered}
\frac{dI}{d\tau} = IB, \quad \frac{dK}{d\tau} = KC, \quad \frac{dH}{d\tau} = HA, \\
\frac{ds}{d\tau} = D.
\end{gathered}$$

3.1. **Case one: $-3 < \gamma_1 < \gamma_2 < -1$.** We denote the set

$$\begin{aligned}
\Omega_1 : \Big\{ & I < 0,\ K > 0,\ H > 0; \quad B > 0 \text{ for } -1 < I \le 1/\gamma_2, \\
& A > 0 \text{ for } 1/\gamma_2 \le I < 0, \Big\}
\end{aligned} \tag{3.1}$$

see Figure 1.

It is not difficult to show that the far-fields solutions starting at $s = 0^+$ enter the region Ω_1 and do not leave Ω_1 as s increases. Notice that $s > 0$ is increasing

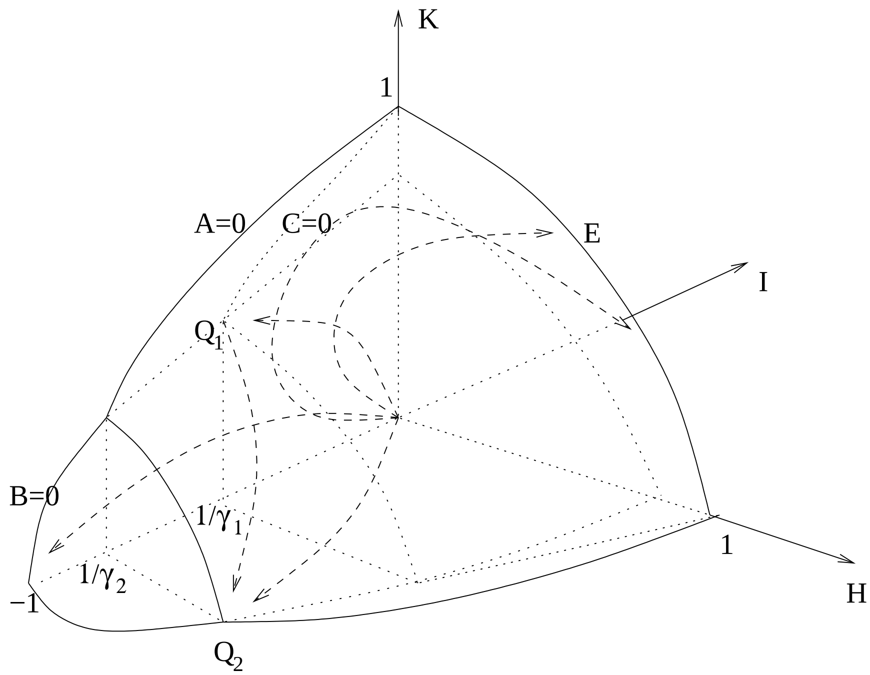

FIGURE 1. Integral curves for $u_0 < 0$ in Ω_1.

function of τ in Ω_1 by (3) and one can find that the stationary points of (3) in the closure $\overline{\Omega}_1$ are $(H, I, K) = (1, 0, 0)$, $(H, I, K) = (0, 0, 1)$, $(H, I, K) = (0, -1, 0)$, $Q_1 = (0, \frac{1}{\gamma_1}, \sqrt{\frac{1+\gamma_1}{2\gamma_1}})$, $Q_2 = (\sqrt{\frac{1+\gamma_2}{2\gamma_2}}, \frac{1}{\gamma_2}, 0)$, and the points on the edge

$$E: \quad K^2 + H^2 = 1, \quad I = 0, \quad K > 0, \quad H > 0.$$

Therefore, there is no stationary point in the open region Ω_1 and all the stationary points are on the boundary of Ω_1.

Lemma 3.1. *Solutions inside Ω_1 do not leave Ω_1 from its sides (excluding possibly edge or corners) as s increases.*

Proof. It is easy to see from (3) that the sides of Ω_1 in the surfaces $H = 0$, $I = 0$ or $K = 0$ are invariant regions. We need only to verify that no solution leaves Ω_1 from the following surfaces $\widetilde{A}$ and $\widetilde{B}$

$$\widetilde{A} := \big\{(H, I, K) : A = 0,\ H > 0,\ K > 0,\ 1/\gamma_2 < I < 0\big\},$$
$$\widetilde{B} := \big\{(H, I, K) : B = 0,\ H > 0,\ K > 0,\ -1 < I < 1/\gamma_2\big\}.$$

We easily obtain that in the coordinate order (H, I, K) the outward normal of surface $\widetilde{B}$ is given by

$$\vec{n}_{\widetilde{B}} = \big(4H, -1, 4K\big).$$

Noting that $A < 0$ and $C < 0$ and calculating the inner product of the normal $\vec{n}_{\widetilde{B}}$ with the tangent vector of an integral curve of (3) on the surface $\widetilde{B}$ yields

$$\vec{n}_{\widetilde{B}} \cdot \Big(\frac{dH}{d\tau}, \frac{dI}{d\tau}, \frac{dK}{d\tau}\Big) = 4H^2A - IB + 4K^2C < 0,$$

which implies that no solution leaves Ω_1 from the surface $\widetilde{B}$ as s increases.

An outward normal of the surface $\widetilde{A}$ is given by

$$\vec{n}_{\widetilde{A}} = \big(2H,\ \frac{\gamma_2 - 1}{2},\ 2K\big)$$

in the order (H, I, K). We similarly compute the inner product of the normal $\vec{n}_{\widetilde{A}}$ with the tangent vector of the integral curve on the surface $\vec{n}_{\widetilde{A}}$ to obtain

$$\vec{n}_{\widetilde{A}} \cdot \Big(\frac{dH}{d\tau}, \frac{dI}{d\tau}, \frac{dK}{d\tau}\Big) = 2H^2 A + \frac{\gamma_2 - 1}{2} IB + 2K^2 C < 0,$$

because $B < 0$, $C < 0$ and $\frac{\gamma_2 - 1}{2} < 0$ on the surface $\widetilde{A}$. Hence, no solution leaves Ω_1 from the surface $\widetilde{A}$. Thus, we complete the proof. $\square$

Now, we study the local structure of integral curves at the stationary points of (3). We first notice that no integral curve from inside Ω_1 goes to $(H, I, K) = (0, 0, 1)$ or Q_1, since the component H, by (3), is an increasing function of τ in Ω_1 for $I \in \big(\frac{1}{\gamma_2}, 0\big)$.

Setting

$$\hat{H} = H - \sqrt{\frac{1+\gamma_2}{2\gamma_2}}, \quad \hat{I} = I - \frac{1}{\gamma_2}, \quad \hat{K} = K,$$

and linearizing system (3) at point Q_2 yields

$$\begin{gathered}
\frac{d\hat{I}}{d\tau} = \frac{1}{\gamma_2}\hat{I} - \frac{4}{\gamma_2}\sqrt{\frac{1+\gamma_2}{2\gamma_2}}\hat{H}, \\
\frac{d\hat{K}}{d\tau} = \frac{\gamma_2 - \gamma_1}{2\gamma_2}\hat{K}, \\
\frac{d\hat{H}}{d\tau} = -\frac{\gamma_2 - 1}{2}\sqrt{\frac{1+\gamma_2}{2\gamma_2}}\hat{I} - \frac{1+\gamma_2}{\gamma_2}\hat{H}.
\end{gathered} \tag{3.2}$$

We compute the eigenvalues to get

$$\lambda^0_{Q_2} = \frac{\gamma_2 - \gamma_1}{2\gamma_2} < 0, \quad \lambda^-_{Q_2} = -\frac{1+\gamma_2}{\gamma_2} < 0, \quad \lambda^+_{Q_2} = 1,$$

which indicate that the stationary point Q_2 is hyperbolic. Similarly, we obtain that the stationary point Q_1 is also hyperbolic. For each integral curve ending at Q_2, we find that $s \to +\infty$ and thus $\xi \to 0^+$ as $\tau \to +\infty$ due to $D = \frac{\gamma_2 - 1}{2\gamma_2} > 0$. Hence, this family of integral curves (called the transitional solutions) yields smooth solutions of problem (2.6) and (2.7) in the entire region $\xi \in (0, +\infty)$.

Now, we linearize system (3) at stationary points on curve $E \cup \{(H, I, K) = (1, 0, 0)\}$. Setting

$$\hat{H} = H - \alpha, \quad \hat{I} = I, \quad \hat{K} = K - \sqrt{1-\alpha^2}$$

for $0 < \alpha \le 1$, one can obtain

$$\begin{gathered}
\frac{d\hat{I}}{d\tau} = -\hat{I}, \\
\frac{d\hat{K}}{d\tau} = -\frac{\gamma_1 - 1}{2}\sqrt{1-\alpha^2}\,\hat{I} - 2(1-\alpha^2)\hat{K} - 2\alpha\sqrt{1-\alpha^2}\,\hat{H}, \\
\frac{d\hat{H}}{d\tau} = -\frac{\gamma_2 - 1}{2}\alpha\hat{I} - 2\alpha\sqrt{1-\alpha^2}\,\hat{K} - 2\alpha^2\hat{H}.
\end{gathered} \tag{3.3}$$

which has three eigenvalues $\lambda_1 = -2$, $\lambda_2 = -1$ and $\lambda_3 = 0$. So, solutions of (3) near $E \cup \{(H, I, K) = (1, 0, 0)\}$ approach $E \cup \{(H, I, K) = (1, 0, 0)\}$ exponentially as $\tau \to +\infty$. From equation (3), we find

$$\ln\Big(\frac{s}{s_0}\Big) = \int_{\tau_0}^{\tau} \big(1 - K^2 - H^2\big) d\tau,$$

where $s_0 = s(\tau_0) > 0$. Hence, s approaches a finite number as $\tau \to +\infty$ since $1 - K^2 - H^2$ approaches zero exponentially. We linearize system (3) at point $(H, I, K) = (0, -1, 0)$ and get

$$\begin{aligned} \frac{d\hat{I}}{d\tau} &= -\hat{I}, \\ \frac{d\hat{K}}{d\tau} &= \frac{1+\gamma_1}{2}\hat{K}, \\ \frac{d\hat{H}}{d\tau} &= \frac{1+\gamma_2}{2}\hat{I}, \end{aligned}$$

which has three eigenvalues $\lambda_1 = -1$, $\lambda_2 = \frac{1+\gamma_1}{2} < 0$ and $\lambda_3 = \frac{1+\gamma_2}{2} < 0$. Similarly, we conclude that the parameter s approaches a finite number as $\tau \to +\infty$.

We now depict the integral curves inside Ω_1 in Figure 1. It is useful to observe that there exists a stable manifold of system (3) at the point Q_2 (see [13, 3] for details) which contains the transitional curve in $H = 0$, the transitional integral curve in $K = 0$ and the heteroclinic orbit from the point Q_1. There are three kinds of integral curves relative to this manifold. The first kind consists of integral curves that are below the manifold and go to the stationary point $(0, -1, 0)$. Each of the second king is right on the manifold and goes to the stationary point Q_2. The third kind consists of integral curves that are above the manifold and go to $E \cup \{(1, 0, 0)\}$. None of the integral curves from inside Ω_1 goes to point $(H, I, K) = (0, 0, 1)$ or Q_1.

We construct global solutions for (2.11) when $u_0 < 0$ and $-3 < \gamma_1 < \gamma_2 < -1$. For each integral curve which ends at point $(H, I, K) = (0, -1, 0)$, there exists a solution $(1/\rho, u, \overline{\varphi}) = \Big(\big(\frac{K^2}{-A_1\gamma s^2}\big)^{\frac{1}{1-\gamma_1}}, I/s, H^2/s^2\Big)$ defined for $s \in (0, s_1^*]$ for some positive number $s_1^* < +\infty$, and then continue the solution by $1/\rho = 0$, $\overline{\varphi} = 0$ in $s \in [s_1^*, +\infty)$. We here do not need to specify the function u in the above state since system (3) has $1/\rho$ or $\overline{\varphi}$ as a factor in every term. For each integral curve ending at a point on $E \cup \{(1, 0, 0)\}$, there exists a solution defined for $s \in (0, s_2^*]$ for some positive $s_2^* < +\infty$. We next continue the solution by the constant state $(1/\rho, u, \overline{\varphi}) = (1/\rho(s_2^*), 0, \overline{\varphi}(s_2^*))$ in $s \in [s_2^*, +\infty)$.

3.2. **Case two: $-3 < \gamma_2 < \gamma_1 < -1$.** It can be verified that all far-field solutions of system (2.11) with $u_0 < 0$, $\rho_0 > 0$ and $\overline{\varphi}_0 > 0$ enter the domain Ω_2 in $s > 0$ where $\Omega_2 \in \mathbb{R}^3$ be the set of points (H, I, K) satisfying

$$\Omega_2 : \begin{aligned} &\Big\{ I < 0,\ K > 0,\ H > 0; \quad B > 0 \text{ for } -1 < I \le 1/\gamma_1, \\ &A > 0 \text{ for } 1/\gamma_1 \le I < 0 \Big\}, \end{aligned} \tag{3.4}$$

see Figure 2. In this case, system (3) has similar stationary points with the case $\gamma_1 < \gamma_2$. Analogously, there is no stationary point in the interior of Ω_2. So, we can get the following lemma.

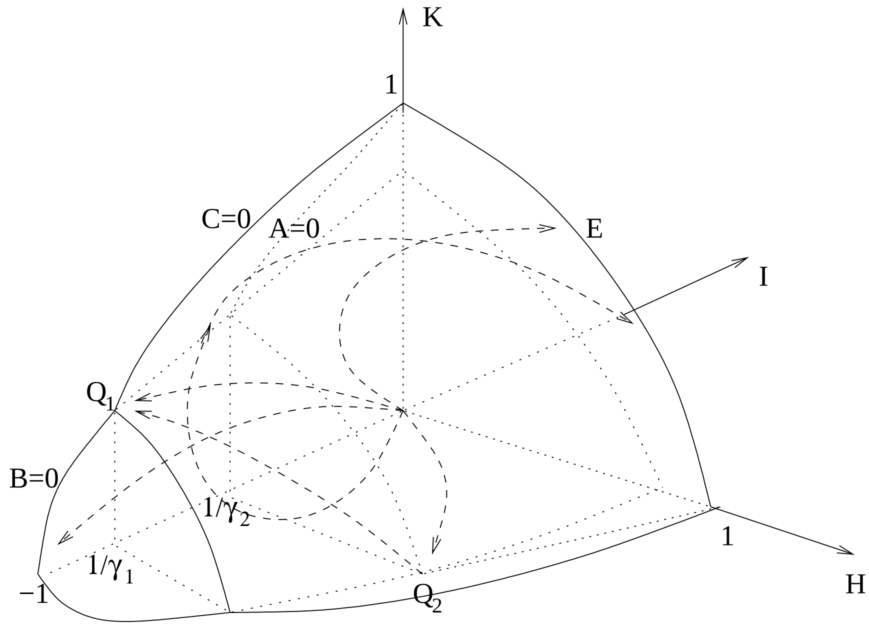

FIGURE 2. Integral curves for $u_0 < 0$ in Ω_2.

Lemma 3.2. *Solutions inside Ω_2 do not leave Ω_2 from its sides (excluding possibly edge or corners) as s increases.*

Proof. It is easy to see from (3) that the sides of Ω_2 in the surfaces $H = 0$, $I = 0$ or $K = 0$ are invariant regions. We need only to verify that no solution leaves Ω_2 from the following surfaces $\widetilde{B}$ and $\widetilde{C}$

$$\widetilde{B} := \Big\{(H, I, K) : B = 0,\ H > 0,\ K > 0,\ -1 < I < \frac{1}{\gamma_1}\Big\},$$
$$\widetilde{C} := \Big\{(H, I, K) : C = 0,\ H > 0,\ K > 0,\ \frac{1}{\gamma_1} < I < 0\Big\}.$$

Analogous to the proof of Lemma 3.1, we can obtain that no solution leaves Ω_2 from the surface $\widetilde{B}$ as s increases. An outward normal of the surface $\widetilde{C}$ is

$$\vec{n}_{\widetilde{C}} = \big(2H,\ \frac{\gamma_1 - 1}{2},\ 2K\big)$$

in the order (H, I, K). We directly compute on the surface $\widetilde{C}$

$$\vec{n}_{\widetilde{C}} \cdot \Big(\frac{dH}{d\tau}, \frac{dI}{d\tau}, \frac{dK}{d\tau}\Big) = 2H^2 A + \frac{\gamma_1 - 1}{2} IB + 2K^2 C < 0,$$

since $A < 0$, $B < 0$ and $\frac{\gamma_1 - 1}{2} < 0$ on the surface $\widetilde{C}$. Hence, no solution leaves Ω_2 from the surface $\widetilde{C}$. These above indicate that the Lemma is true. □

Similar to the case $\gamma_1 < \gamma_2$, no integral curve inside Ω_2 goes to points $(H, I, K) = (1, 0, 0)$ or Q_2, since the function K is an increasing function of τ in Ω_2 for $I \in (0, 1/\gamma_1)$ by (3). For the local structure of solutions at $E \cup \{(0, 0, 1)\}$ and the point $(H, I, K) = (0, -1, 0)$, we may see the related results for the case $\gamma_1 < \gamma_2$ and here omit the details. We next consider the local structure of solutions at the stationary point Q_1. Setting

$$\hat{H} = H, \quad \hat{I} = I - \frac{1}{\gamma_1}, \quad \hat{K} = K - \sqrt{\frac{1+\gamma_1}{2\gamma_1}},$$

and linearizing system (3) gives

$$\begin{gathered}
\frac{d\hat{I}}{d\tau} = \frac{1+\gamma_1}{2\gamma_1}\hat{I} - \frac{4}{\gamma_1}\sqrt{\frac{1+\gamma_1}{2\gamma_1}}\hat{K}, \\
\frac{d\hat{K}}{d\tau} = -\frac{\gamma_1 - 1}{2}\sqrt{\frac{1+\gamma_1}{2\gamma_1}}\hat{I} - \frac{1+\gamma_1}{\gamma_1}\hat{K}, \\
\frac{d\hat{H}}{d\tau} = \frac{\gamma_1 - \gamma_2}{2\gamma_1}\hat{H},
\end{gathered}$$

which three eigenvectors are

$$\lambda_{Q_1} = \frac{\gamma_1 - \gamma_2}{2\gamma_1}, \quad \lambda_{Q_1}^{\pm} = \frac{(1+\gamma_1) \pm \sqrt{(1+\gamma_1)(25\gamma_1 - 7)}}{-4\gamma_1}.$$

In this case, we easily see that $\lambda_{Q_1} < 0$, $\lambda_{Q_1}^- < 0$ and $\lambda_{Q_1}^+ > 0$, which indicates that the stationary point Q_1 is hyperbolic. The hyperbolicity of the stationary point Q_2 can be established in a similar way.

The integral curves in Ω_2 in Figure 2 can be depicted as follows. There also exists a stable manifold of system (3) for $-3 < \gamma_2 < \gamma_1 < -1$ at the point Q_1 which contains the transitional integral curve in $H = 0$, the transitional integral curve in $K = 0$ and the heteroclinic orbit from the point Q_2. There are three kinds of integral curves relative to this stable manifold. The first kind consists of integral curves that are below the manifold and go to the stationary point $(0, -1, 0)$. Each of the second kind is right on the manifold and goes to the stationary point Q_1. The third kind consists of integral curves that are above the manifold and go to $E \cup \{(0, 0, 1)\}$. None of the integral curves from inside Ω_2 goes to $(1, 0, 0)$ or Q_2.

Now the construction of the global solutions of system (3) for $u_0 < 0$ and $-3 < \gamma_2 < \gamma_1 < -1$. For each integral curve ending at $(H, I, K) = (0, -1, 0)$, we continue the solution by vacuum state $1/\rho = 0, \overline{\varphi} = 0$ in $s \in [s_3^*, +\infty)$ for some positive $s_3^* < +\infty$. We also do not specify the function u in the vacuum. For each solution curve ending on $E \cup \{(0, 0, 1)\}$, we continue the solution by the constat state $(1/\rho(s_4^*), 0, \overline{\varphi}(s_4^*))$ in $s \in [s_4^*, +\infty)$ for some positive finite number s_4^*. The integral curves ending at point Q_1 (that is the second kind) are already defined for all $s > 0$ since $\tau \to +\infty$ and the right hand side of equation (2.13)) does not vanish at Q_1.

Furthermore, from the last equation of (2.4), we have

$$v(s) = v_0 \exp\Big(\int_0^s \frac{u}{1 - K^2 - H^2} ds\Big), \tag{3.5}$$

which is a finite number for $u < 0$ and $1 - K^2 - H^2 > 0$.

In summary, we have constructed global bounded continuous solutions to system (2.4) and (2.5) in the case $u_0 \leq 0$.

4. Solutions for $u_0 > 0$

In this section, we construct global solutions for problem (2.6) and (2.7) with initial positive pure radial velocity for the case $\gamma_1 < \gamma_2$. For the other case $\gamma_1 > \gamma_2$, we can obtain the same results, similarly.

It is not difficult to prove that the integral curves of system (3) starting at the origin with $u_0 > 0, \rho_0 > 0$ and $\overline{\varphi}_0 > 0$ enter the region $\Omega_3 := \Omega_{31} \cup \Omega_{32}$, where

$$\Omega_{31} := \{(H, I, K) : H > 0,\ I > 0,\ K > 0,\ B > 0,\ 0 < I < 1\},$$
$$\Omega_{32} := \{(H, I, K) : H > 0,\ I > 0,\ K > 0,\ D > 0,\ I > 1\},$$

see Figure 3. Denote $\Omega_3' := \{H > 0, I > 0, K > 0, D < 0, B < 0\}$. Obviously, according to (3), the variable s is increasing in region Ω_3. We conclude, however, that the integral curves in Ω_3 do not always lie in it.

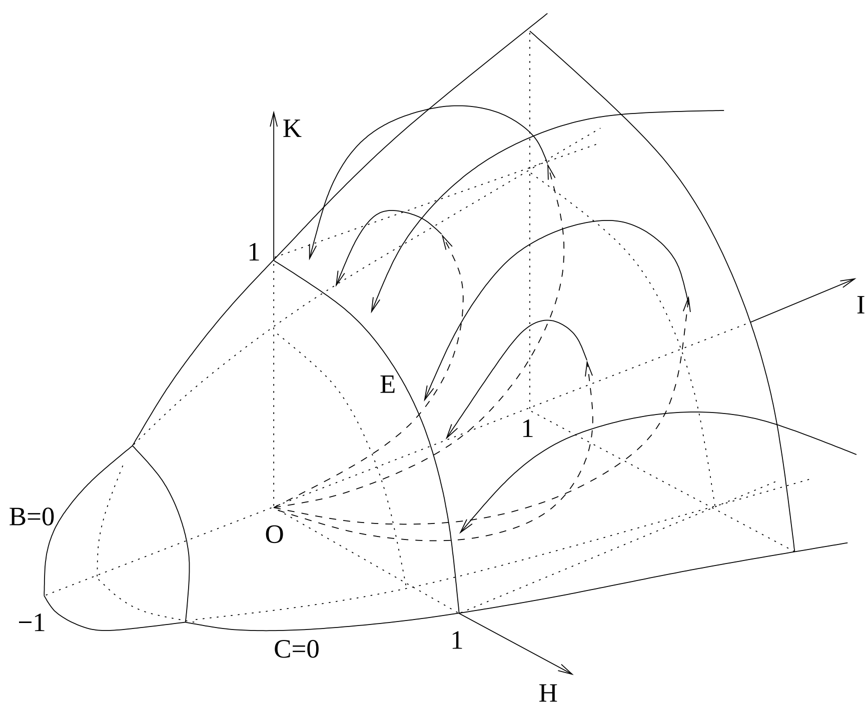

Figure 3. Integral curves for $u_0 > 0$ in the case $\gamma_1 < \gamma_2$.

Lemma 4.1. *Each integral curve of system* (3) *from the origin passes through the surface $D = 0$ or $B = 0$ at a finite point (τ, H, I, K) and enters the domain Ω_3', and then crosses the surface $C = 0$ at a finite τ from $C > 0$ to $C < 0$.*

Proof. To demonstrate the first part of the Lemma, we divide it into two parts.

(1) If the integral curves from the origin do not enter Ω_{32}. It is easy to obtain from $B > 0$ that $2(K^2 + H^2) < 1 + I$, from which, we find that

$$C = 1 - \frac{\gamma_1 - 1}{2} I - (K^2 + H^2) \geq (K^2 + H^2) - \frac{\gamma_1 + 1}{2} I \geq K^2.$$

From the equation for K, we find that

$$\frac{\mathrm{d}K}{\mathrm{d}\tau} = KC \geq K^3$$

which implies that K blows up at a finite $\tau = \hat{\tau}$. Combining this and the fact $0 < I < 1$, we conclude that the integral curve must pass through the surface $B = 0$ and $D = 0$, and then enters the domain Ω_3'.

(2) If the integral curves from the origin leave the region Ω_{31} and then enter the region Ω_{32}. Similar to the above procedure, the solution curves must pass through the surface $D = 0$ at finite $\overline{\tau}$. Now, we prove that the above curves must pass through the surface $B = 0$ at a finite τ, and then enter the domain Ω_3'.

Suppose, on the contrary, such an integral curve is always under the surface $B = 0$. It follows from the I equation that I is increasing for all τ. Using the K equation, $B > 0$ and $I > 1$, we have

$$\frac{d\ln K}{d\tau} = 1 - \frac{\gamma_1 - 1}{2} I - (K^2 + H^2) \leq 1 - \frac{\gamma_1 - 1}{2} I,$$

which means that

$$\int_{\tau_0}^{\tau_m} \big(1 - \frac{\gamma_1 - 1}{2} I\big) d\tau = +\infty,$$

for some positive number τ_m, from which one can achieve

$$\int_{\tau_0}^{\tau_m} \big(-\frac{1+\gamma_1}{2} I + K^2 + H^2\big) d\tau = +\infty. \tag{4.1}$$

We arrange the K equation in the following form

$$\frac{dK}{d\tau} = K\big(B - \frac{1+\gamma_1}{2} I + K^2 + H^2\big).$$

Integrating the above equality with respect to τ from τ_0 to τ yields

$$K = K_0 \exp\big(\int_{\tau_0}^{\tau} B d\tau\big) \cdot \exp\Big(\int_{\tau_0}^{\tau} \big(-\frac{1+\gamma_1}{2} I + K^2 + H^2\big) d\tau\Big), \tag{4.2}$$

where $K_0 = K(\tau_0)$. One can also obtain from the I equation that

$$I = I_0 \exp\Big(\int_{\tau_0}^{\tau} B d\tau\Big), \tag{4.3}$$

where $I_0 = I(\tau_0)$. Combining (4.1) with (4.2) and (4.3) gives

$$\frac{K}{I} = \frac{K_0}{I_0} \exp\Big(\int_{\tau_0}^{\tau} \big(-\frac{1+\gamma_1}{2} I + K^2 + H^2\big) d\tau\Big) \to +\infty, \quad \text{as } \tau \to \tau_m.$$

On the other hand, one can obtain from the assumption $B > 0$ and the fact $I > 1$ that

$$2K^2 < 2(K^2 + H^2) < 1 + I < (1 + I)^2,$$

which gives

$$\frac{\sqrt{2}K}{I} < 1 + \frac{1}{I}.$$

The right-hand side of the above inequality is bounded. So we get a contradiction.

Next, we prove the second part of the Lemma, that is that all integral curves from Ω_3' cross the surface $C = 0$ at a finite τ. One can obtain that all integral curves from Ω_3' in $I > 0$ must end at $\overline{E} := E \cup \{(1, 0, 0), (0, 0, 1)\}$. By the linearization equation (3.3) of (3), we get that the eigenvalues of (3.3) are $\lambda_1 = -2$, $\lambda_2 = -1$ and $\lambda_3 = 0$, and the corresponding eigenvectors are in turn

$$\vec{r}_1 = (\alpha, 0, \sqrt{1-\alpha^2}),$$
$$\vec{r}_2 = \Big(\alpha\big[(1+\gamma_2-2\gamma_1) - 2(\gamma_2-\gamma_1)\alpha^2\big],\ 2,\ [(1-\gamma_1) - 2(\gamma_2-\gamma_1)\alpha^2]\sqrt{1-\alpha^2}\Big),$$
$$\vec{r}_3 = \big(\sqrt{1-\alpha^2}, 0, -\alpha\big)$$

for $0 \le \alpha \le 1$. Since $2\gamma_2 < 1+\gamma_1$ for $-3 < \gamma_1 < \gamma_2 < -1$, we compute the inner product of the normal $\vec{n}_{\tilde{C}}$ with the vector $\vec{r}_2$ on $\overline{E}$ to obtain

$$\begin{aligned}\vec{n}_{\tilde{C}} \cdot \vec{r}_2 &= \alpha^2\big[2(\gamma_2-\gamma_1)(1-\alpha^2) + 1 - \gamma_2\big] + (1-\alpha^2)\big[(1-\gamma_1) - 2(\gamma_2-\gamma_1)\alpha^2\big] \\ &\quad + (\gamma_1 - 1)/2 \\ &= (\gamma_1-\gamma_2)\alpha^2 + (1-\gamma_1)/2 > 0,\end{aligned}$$

which indicates that each integral curve from Ω_3' will pass through the surface $C = 0$ at a finite τ and then go to a stationary point on $\overline{E}$ along the direction $\vec{r}_2$. $\square$

Using (3), we can see that the variable $s(\tau)$ starts to decrease as τ increases in the domain $\{D < 0\}$, which means that no continuous solutions exist in the case $u_0 > 0$. Hence we need to use discontinuous solutions to construct global solutions.

Denote the subscripts r and ℓ the front (the side close to $\xi = +\infty$) and behind (the side close to the origin) states, respectively. Here we have ignored the backward discontinuous solutions since we will not need it. The entropy condition requires that $\rho_r < \rho_\ell$, or equivalently, by (2.17),

$$K_r > K_\ell, \quad I_r < I_\ell \quad \text{or} \quad H_r > H_\ell. \tag{4.4}$$

Using the R-H relation (2.17), we get the following lemma.

Lemma 4.2. *For any state (H_ℓ, I_ℓ, K_ℓ) in the domain $\{D > 0\} \cap \{I_\ell > 0\}$, there exists a state (H_r, I_r, K_r) in the domain $\{D < 0\} \cap \{I_r > 0\}$ satisfying the R-H relation* (2.17) *and the entropy condition* (4.4).

Proof. It is easy to see from the third equation of (2.17) that

$$\begin{aligned}1 = &\underbrace{\frac{1}{\gamma_1}\big(K_r^{\frac{2\gamma_1}{\gamma_1-1}} - K_\ell^{\frac{2\gamma_1}{\gamma_1-1}}\big)\Big(K_r^{\frac{2}{\gamma_1-1}} - K_\ell^{\frac{2}{\gamma_1-1}}\Big)^{-1}}_{f(K_\ell,K_r)} \\ &+ \underbrace{\frac{1}{\gamma_2}\big(H_r^2 K_r^{\frac{2}{\gamma_1-1}} - H_\ell^2 K_\ell^{\frac{2}{\gamma_1-1}}\big)\Big(K_r^{\frac{2}{\gamma_1-1}} - K_\ell^{\frac{2}{\gamma_1-1}}\Big)^{-1}}_{g(K_\ell,K_r)}.\end{aligned}$$

Setting

$$x = K_r^{\frac{2}{\gamma_1-1}} \quad \text{and} \quad x_0 = K_\ell^{\frac{2}{\gamma_1-1}},$$

we have

$$f(K_\ell, K_r) := f(x_0, x) = \frac{1}{\gamma_1}\frac{x^{\gamma_1} - x_0^{\gamma_1}}{x - x_0}.$$

Since $\big(x^{\gamma_1}/\gamma_1\big)' = x^{\gamma_1-1} > 0$, $\big(x^{\gamma_1}/\gamma_1\big)'' = (\gamma_1-1)x^{\gamma_1-2} < 0$ for $0 < x < x_0$ and $-3 < \gamma_1 < -1$, we obtain from the entropy condition (4.4) that

$$K_\ell^2 < f(K_\ell, K_r) < K_r^2. \tag{4.5}$$

Now, we claim that

$$H_\ell^2 < g(K_\ell, K_r) < H_r^2. \tag{4.6}$$

In fact, from (2.17), we have

$$H_\ell^2 = \Big(\frac{K_\ell}{K_r}\Big)^{\frac{2(\gamma_2-1)}{\gamma_1-1}} H_r^2 := y^{-(\gamma_2-1)} H_r^2,$$

where $0 < y = \big(K_r/K_\ell\big)^{2/(\gamma_1-1)} < 1$. Thus

$$g(K_\ell, K_r) = y^{-(\gamma_2-1)} H_r^2 \cdot \frac{1}{\gamma_2} \frac{y^{\gamma_2} - 1}{y-1},$$

which gives

$$K_\ell^2 = y^{-(\gamma_2-1)} H_r^2 < g(K_\ell, K_r) < H_r^2.$$

Combining (4.5) and (4.5) and (4.6) gives

$$K_\ell^2 + H_\ell^2 < 1 < K_r^2 + H_r^2, \tag{4.7}$$

from which it follows that $(H_r, I_r, K_r) \in \{D < 0\} \cap \{I_r > 0\}$. □

From the R-H relation and the entropy condition, we only have three equations for four variables $(H(I_\ell), I_\ell, K(I_\ell))$ and (H_r, I_r, K_r), so we need one more condition to find a unique shock wave transition. Here $H_\ell = H(I_\ell), K_\ell = K(I_\ell)$ are determined by the integral curve of system (3).

According to (3), the direction of τ is opposite to that of s in the domain $\{D < 0\}$. Therefore, shock transition from a state (H_ℓ, I_ℓ, K_ℓ) in $D > 0$ to a state (H_r, I_r, K_r) on the plane $I = 0$ is the only one leading to a global solution, since the integral curve on the plane $I = 0$ yields stationary constant ρ and φ solutions to system (1.4). Thus, we require that

$$I_r = 0. \tag{4.8}$$

Lemma 4.3. *For any integral curve in the quadrant $\{H > 0, I > 0, K > 0\}$ of the autonomous system* (3)*, there exists a solution $(H_r, I_r, K_r, H_\ell, I_\ell, K_\ell)$ with $I_\ell > 0$ satisfying the R-H relation* (2.17)*, the entropy condition* (4.4) *and the condition* (4.8)*.*

Proof. We establish this lemma by the method of continuity. Since each integral curve of system (3) in the quadrant $\{H > 0, I > 0, K > 0\}$ crosses the surface $D = 0$ at a finite τ by Lemma 4.1, then if we take the cross point as (H_ℓ, I_ℓ, K_ℓ), the solution to the R-H relation (2.17) would be the same point, i.e., $(H_\ell, I_\ell, K_\ell) = (H_r, I_r, K_r)$. Now moving from this point down the integral curve, we notice by the R-H relation (2.17) that I_r will become minus when (H_ℓ, I_ℓ, K_ℓ) is near the origin. In fact, if not, we then have $K_r^2 + H_r^2 > 1$ by (4.7). Thus, one has by the R-H relation (2.17)

$$\begin{aligned}(1-I_\ell)(K_\ell^{\frac{2}{\gamma_1-1}} + H_\ell^{\frac{2}{\gamma_2}} K_\ell^{\frac{2}{\gamma_2(\gamma_1-1)}}) &= (1-I_r)(K_r^{\frac{2}{\gamma_1-1}} + H_r^{\frac{2}{\gamma_2}} K_r^{\frac{2}{\gamma_2(\gamma_1-1)}}) \\ &\le (K_r^{\frac{2}{\gamma_1-1}} + H_r^{\frac{2}{\gamma_2}} K_r^{\frac{2}{\gamma_2(\gamma_1-1)}}) < c_*\end{aligned}$$

for some positive constant c_*, which contradicts to the fact $(H_\ell, I_\ell, K_\ell) \to (0, 0, 0)$. Hence, we have $I_r > 0$ when the point (H_ℓ, I_ℓ, K_ℓ) near the origin. By continuity, we must have a solution $(H_r, I_r, K_r, H_\ell, I_\ell, K_\ell)$ with $I_\ell > 0$ for equations (2.17) and (4.8) and the entropy condition (4.4). □

From (3) and (3), we get a subsystem on the plane $I=0$, that reads

$$\begin{aligned}\frac{\mathrm{d}K}{\mathrm{d}\tau}&=K(1-H^2-K^2),\\ \frac{\mathrm{d}H}{\mathrm{d}\tau}&=H(1-H^2-K^2),\\ \frac{\mathrm{d}s}{\mathrm{d}\tau}&=s(1-H^2-K^2),\end{aligned}$$

which means that

$$\frac{\mathrm{d}K}{\mathrm{d}s}=\frac{K}{s},\quad \frac{\mathrm{d}H}{\mathrm{d}s}=\frac{H}{s},$$

which imply that $\frac{K}{s}$ and $\frac{H}{s}$ are constant. Then a point $(H,I,K)=(H_r,0,K_r)$ on the plane $I=0$ gives a constant solution to system (2.8)

$$u=0,\quad \frac{1}{\rho}=\big(-A_1\gamma_1\big)^{\frac{1}{\gamma_1-1}}(\xi^{**}K_r)^{\frac{2}{1-\gamma_1}},\quad \overline{\varphi}=(\xi^{**}H_r)^2,\quad (0<\xi<\xi^{**}),$$

where $\xi^{**}=1/s^{**}$ is the radial coordinate ξ of the shock location.

We are now ready to construct global solutions for (2.8) in the case $u_0>0$. For any integral curve of (3), there exists a solution $(H_\ell,I_\ell,K_\ell,H_r,I_r,K_r)$ satisfying Lemma 4.3 such that (H_ℓ,I_ℓ,K_ℓ) on the integral curve and (H_r,I_r,K_r) on the plane $I=0$. For this integral curve from $(H,I,K)=(0,0,0)$ to $(H,I,K)=(H_\ell,I_\ell,K_\ell)$, we have a solution $\big(1/\rho,u,\overline{\varphi}\big)=\big((K^2/(-A_1\gamma_1 s^2))^{\frac{1}{1-\gamma_1}},I/s,H^2/s^2\big)$ defined for $s\in(0,s^{**}]$ for some positive number $s^{**}<+\infty$ such that $(H_\ell,I_\ell,K_\ell)=(H(s^{**}),I(s^{**}),K(s^{**}))$. The solution for $s\in[s^{**},+\infty)$ is specified as $(1/\rho,u,\overline{\varphi})=((K_r^2/(-A_1\gamma_1 s^{**2}))^{\frac{1}{1-\gamma_1}},0,H_r^2/s^{**2})$. We use a shock wave to connect the two states $(1/\rho,u,\overline{\varphi})=((K_r^2/(-A_1\gamma_1 s^{**2}))^{\frac{1}{1-\gamma_1}},0,H_r^2/s^{**2})$ and

$$(1/\rho,u,\overline{\varphi})=((K_\ell^2/(-A_1\gamma_1 s^{**2}))^{\frac{1}{1-\gamma_1}},I_\ell/s^{**},H_\ell^2/s^{**2})$$

at $s=s^{**}$. Here s^{**} (i.e., the shock location ξ^{**}) is determined by the R-H relation (2.17), the entropy condition (4.4), the compatibility condition (4.8) and the autonomous system (3).

From the last equation in (2.17), we have the v component of the solution is

$$v=\frac{\rho_\ell v_\ell}{\rho}.$$

5. Conclusions

We first summarize the results for system (1.4).

Theorem 5.1. *For any datum $(\rho_0,u_0,v_0,\varphi_0)$ with $\rho_0>0$ and $\varphi_0>0$, there exists a global solution (ρ,u,v,φ) to the initial-value problem* (1.4) *and* (2.3).

Similar to [10], we obtain from the system (2.8) for $\rho>0,\varphi>0$ that

$$\frac{\mathrm{d}\overline{\varphi}}{\overline{\varphi}}=(\gamma_2-1)\frac{\mathrm{d}\rho}{\rho}\quad\text{or}\quad \frac{\mathrm{d}\varphi}{\varphi}=\frac{\mathrm{d}\rho}{\rho},$$

which means that our solutions satisfy $\varphi/\rho=\varphi_0/\rho_0$ in the smooth case. If we restrict $\varphi_0=\rho_0$ at initial time, then we have $\varphi=\rho$ in the smooth case. Moreover, the constant states continued the smooth parts of solutions for (2.4) are also the constant solutions to system (1.1). One can easily check that the shock solution of system (1.4) is also a shock solution to system (1.1) with (1.2) if the initial data satisfies $\varphi_0 = \rho_0$. Therefore, we have the following theorem.

Theorem 5.2. *For any datum* (ρ_0, u_0, v_0) *with* $\rho_0 > 0$*, there exists a global solution* (ρ, u, v) *to the initial-value problem* (1.1) *and* (2.3) *when* $p(\rho) = A_1\rho^{\gamma_1} + A_2\rho^{\gamma_2}$ *for any constants* $A_i < 0, -3 < \gamma_i < -1$ $(i = 1, 2)$.

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References

[1] S. Čanić, B. L. Keyfitz, E. H. Kim; *Mixed hyperbolic-elliptic systems in self-similar flows*, Bol. Soc. Brasil. Mat. 32 (2002), 1-23.

[2] S. Čanić, B. L. Keyfitz, E. H. Kim; *Free boundary problems for nonlinear wave systems: Mach stems for interacting shocks*, SIAM J. Math. Anal. 37 (2006), 1947-1977.

[3] J. Carr; *Applications of centre manifold theory*, Springer-Verlag, New York, 1981.

[4] S. Chaplygin; *On gas jets*, Sci. Mem. Moscow Univ. Math. Phys. 21 (1904), 1-121.

[5] G.-Q. Chen, J. Glimm; *Global solutions to the compressible Euler equations with geometrical structure*, Comm. Math. Phys. 180 (1996), 153-193.

[6] G.-Q. Chen, X. Deng, W. Xiang; *Shock Diffraction by Convex Cornered Wedges for the Nonlinear Wave System*, Arch. Ration. Mech. Anal. 211 (2014), 61-112.

[7] L. Guo, W. Sheng; *Axisymmetric solutions of the Chaplygin gas for initial negative radial velocity*, J. Shanghai University 14 (2010), 380-386.

[8] L. Guo, W. Sheng; *Axisymmetric solutions of the Euler system for the Chaplygin gas (in Chinese)*, China Ann. Math. 32A (2011), 193-204.

[9] Y. Hu; *Axisymmetric solutions of the pressure-gradient system*, J. Math. Phys. 53 (2012) 073703.

[10] Y. Hu; *Axisymmetric solutions of the two-dimensional Euler equations with a two-constant equation of state*, Nonlinear Anal. Real World Appl. 15 (2014), 67-79.

[11] Y. Hu, G. Wang; *Semi-hyperbolic patches of solutions to the two-dimensional nonlinear wave system for Chaplygin gases*, J. Differential Equations 257(5) (2014), 1567-1590.

[12] Y. Hu, G. Wang; *The interaction of rarefaction waves of a two-dimensional nonlinear wave system*, Nonlinear Anal. Real World Appl. 22 (2015), 1-15.

[13] A. Kelley; *The stable, center stable, center, center unstable and unstable manifolds*, J. Differential Equations 3 (1967), 546-570.

[14] E. H. Kim; *A global subsonic solution to an interaction transonic shock for the self-similar nonlinear wave equation*, J. Differential Equations 248 (2010), 2906-2930.

[15] E. H. Kim; *An Interaction of a Rarefaction Wave and a Transonic Shock for the Self-Similar Two-Dimensional Nonlinear Wave System*, Comm. Partial Differential Equations 37 (2012) 610–646.

[16] E. H. Kim, C.-M. Lee; *Numerical solutions to shock reflection and shock interaction problems for the self-similar transonic two-dimensional nonlinear wave system*, J. Comput. Sci. 4 (2013), 36-45.

[17] E. H. Kim, C.-M. Lee; *Transonic shock reflection problems for the self-similar two-dimensional nonlinear wave system*, Nonlinear Anal. 79 (2013), 85-102.

[18] E. H. Kim; *Transonic shock and rarefaction wave interactions of two-dimensional Riemann problems for the self-similar nonlinear wave system*, Bull. Braz. Math. Soc. 47 (2016), 431-444.

[19] J. Li, T. Zhang, S. Yang; *The two-dimensional Riemann problem in gas dynamics*, Longman, New York, 1998.

[20] J. Li, W. Sheng, T. Zhang, Y. Zheng; *Two-dimensional Riemann problems: from scalar conservation laws to compressible Euler equations*, Acta Math. Sci. 29 (2009), 777-802.

[21] A. A. Sen, R. J. Scherrer; *Generalizing the generalized Chaplygin gas*, Phys. Rev. D 72(6) (2005), 3511-1-3511-8.

[22] H. S. Tsien; *Two dimensional subsonic flow of compressible fluids*, J. Aeron. Sci. 6 (1939), 399-407.
[23] A. M. Tesdall, R. Sanders, B. L. Keyfitz; *The triple point paradox for the nonlinear wave system*, SIAM J. Appl. Math. 67 (2006) 321–336.
[24] T. von Kármán; *Compressibility effects in aerodynamics*, J. Aeron. Sci. 8 (1941) 337-365.
[25] G. Wang, Y. Hu; *Axisymmetric solutions of the two-dimensional nonlinear wave system with negative pressure*, Submitted, 2017.
[26] T. Zhang, Y. Zheng; *Conjecture on the structure of solutions of the Riemann problem for two-dimensional gas dynamics systems*, SIAM J. Math. Anal. 21 (1990), 593-630.
[27] T. Zhang, Y. Zheng; *Exact spiral solutions of the two-dimensional Euler equations*, Discrete Contin. Dyn. Syst. 3 (1997), 117-133.
[28] T. Zhang, Y. Zheng; *Axisymmetric solutions of the Euler equations for polytropic gases*, Arch. Ration. Mech. Anal. 142 (1998), 253-279.
[29] Y. Zheng; *Systems of conservation laws: two dimensional Riemann problems*, Birkhauser, Boston, 2001.
[30] Y. Zheng; *Absorption of characteristics by sonic curves of the two-dimensional Euler equations*, Discrete Contin. Dyn. Syst. 23 (2009), 605–616.

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