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ASYMPTOTIC BEHAVIOR FOR DIRICHLET PROBLEMS OF NONLINEAR SCHRÖDINGER EQUATIONS WITH LANDAU DAMPING ON A HALF LINE

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Abstract. This article is a continuation of the study in [5], where we proved the existence of solutions, global in time, for the initial-boundary value problem

$$\begin{gathered} u_t + iu_{xx} + i|u|^2u + |\partial_x|^{1/2}u = 0, \quad t \geq 0,\ x \geq 0; \\ u(x,0) = u_0(x), \quad x > 0 \\ u_x(0,t) = h(t), \quad t > 0, \end{gathered}$$

where $|\partial_x|^{1/2}$ is the module-fractional derivative operator defined by the modified Riesz Potential

$$|\partial_x|^{1/2} = \frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{\operatorname{sign}(x-y)}{\sqrt{|x-y|}} u_y(y)dy.$$

Here, we study the asymptotic behavior of the solution.

1. Introduction

Consider the initial-boundary value problem for a modified Schrödinger equation with Landau damping on a half-line

$$\begin{gathered} u_t + \mathbb{K}u + i|u|^2u = 0, \quad t \geq 0,\ x > 0; \\ u(x,0) = u_0(x), \quad x > 0 \\ u(0,t) = h(t), \quad t > 0, \end{gathered} \tag{1.1}$$

where the operator $\mathbb{K}$ is defined as

$$\mathbb{K} = \alpha u_{xx} + \lambda|\partial_x|^\gamma u, \tag{1.2}$$

with $\alpha, \lambda \in \mathbb{C}$, $\gamma \in \mathbb{R}$ and $|\partial_x|^\gamma$ is the module-fractional derivative operator given by $|\partial_x|^\gamma u = R^\gamma \partial_x u$. Here R^γ is the modified Riesz Potential

$$R^\gamma u = \frac{1}{2\Gamma(\gamma)\sin(\frac{\pi}{2}\gamma)} \int_0^\infty \frac{\operatorname{sign}(x-y)}{|x-y|^{1-\gamma}} u(y)dy.$$

In [5] we prove the existence solutions, global in time, to this initial-boundary value problem (IBV problem), as a continuation of this study in the present paper, we show the asymptotic expansion for the solutions to (1.1). More precisely, the principal result in [5] is the following.

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Theorem 1.1. *Suppose that $u \in L^{1,\mu}(\mathbb{R}^+) \cap L^\infty(\mathbb{R}^+)$ and $h \in Y_\beta = H^{1,\beta}_\infty$ with $\|u_0\|_Z + \|h\|_{Y_\beta} \le \epsilon$, where $\epsilon > 0$ is sufficiently small and $\beta > 1$. Then there exist a unique global solution*

$$u \in C(t[0,\infty); L^2(\mathbb{R}^+)) \cap C\Big((0,\infty); L^{2,\frac{1}{2}(\frac{1}{2}+\mu)}(\mathbb{R}^+) \cap L^\infty(\mathbb{R}^+)\Big),$$

with $\mu \in (0,1/2)$ to the initial-boundary value problem (1.1).

Here $L^{p,\mu}$ denote the function space $L^{p,\mu} := \{\phi \in \mathbf{S}' : \|\phi\|_{L^{p,\mu}} < \infty\}$, with the norm

$$\|\phi\|_{L^{p,\mu}} = \Big(\int_{\mathbb{R}^+} (1+|x|)^{\mu p} |\phi(x)|^p dx\Big)^{1/p}$$

for $1 \le p < \infty$ and $\|\phi\|_{L^{\infty,\mu}} = \sup_{x\in\mathbb{R}^+} |(1+|x|)^\mu \phi(x)|$ for $p = \infty$. We also use the notation $L^p = L^{p,0}$.

The theory of asymptotic methods for nonlinear evolution equations is relatively young and traditional questions of general theory are far from being answered. A description of the large time asymptotic behavior of solutions of nonlinear evolution equations requires principally new approaches and the reorientation of points of view in the asymptotic methods.

The difficulty of the asymptotic methods is explained by the fact that they need not only a global existence of solutions, but also a number of additional a priori estimates of the difference between the solution an the approximate solution (usually in the weighted norms). Some key developments can be found in the book [16], which is the first attempt to give a systematic approach for obtaining the large asymptotic representation of solutions to the nonlinear evolution equation with dissipation.

Some previous results concerning the nonlinear Schrödinger equation (NLS) $\alpha_2 = 0$, which is the most closely related to our problem, include [4], [22] and [24]. In [13] it was shown that (1.1) with $\alpha_1 = 0, \alpha_2 = i$ admits global solutions whose long-time behavior is not linear. For IBV-problems for the nonlinear Schrodinger equation, there are fewer amount of literature, in papers [2] and [17] with inhomogeneous Dirichlet boundary conditions there were certain results. Local existence in some Sobolev spaces. Weder [28] proved that the Dirichlet IBV-problem for the forced nonlinear Schrödinger equation with a potential on the half-line, is locally and (under stronger conditions) globally well posed. Bu and Strauss [3] proved the existence of global-in-time solution in the energy space for initial data in H^1 and the boundary data from $\mathbf{C}^3$ with a compact support.

Fokas[8], assuming that a solution of the nonlinear Schrödinger equation on the half-line exists, showed that the solution can be represented in terms of the solution of a matrix Riemann Hilbert, and in [9] the authors prove that given appropriate initial and boundary conditions, the solution of the nonlinear Schrödinger equation exists globally. However, in spite of the importance, few works have considered the IBV-problems for partial differential evolution equations with a fractional derivative. Some key developments include the book [14]. This book is the first attempt to develop systematically a general theory of IBV-problems for evolution equations with pseudo-differential operators on a half-line.The results of this book can be applied directly to study the initial-boundary value problem for differential equations with fractional Riemann-Liouville and Caputo derivatives.

A method for solving IBV-problems for linear partial differential evolution equations with a general fractional derivative operator, based on the Riemann-Hilbert theory, was introduced in [18] and further developed in [19]. It was proved in [5, 6] that the above approach can be used to the establish global existence in time of the solutions of (1.1) with Neumann and Dirichlet boundary data.

In this article, we use the factorization technique from paper [6] for the free Schrödinger evolution group

$$\mathcal{G}(t) = \mathcal{B}_s\{e^{K(z)t}B_s\}, \quad K(z) = iz^2 - \sqrt{z}. \tag{1.3}$$

Formula (1.3) is useful for studying the large time asymptotic behavior of solutions of Fractional Schrödinger equations. The distorted operators $\mathcal{B}_s^*$ and $\mathcal{B}_s$ will be defined in the following section. Formula (1.3) is obtained by using the Hilbert transform with respect to the space variable and by the use of techniques of complex analysis. Our main goal is to evaluate the influence of the boundary data on the asymptotic behavior of solutions. Theorem 1.1 shows that (1.1) admits global solutions and Theorem 2.1 shows that its long-time behavior essentially depends on the scattering properties of the boundary data.

We believe that the results of this paper could be applied to study a wide class of dissipative nonlinear equations with a fractional derivative on a half-line.

2. Preliminaries

2.1. **Notation and main results.** To state our results precisely, we introduce notation and function spaces. We denote the usual Fourier transform and inverse Fourier transform by $\mathcal{F}$ and $\mathcal{F}^{-1}$ respectively. The Fourier sine transform $\mathcal{F}_s$ and the Fourier cosine transform $\mathcal{F}_c$ are defined by

$$\mathcal{F}_s\phi = \sqrt{\frac{2}{\pi}}\int_{R^+}\phi(x)\sin px\,dx, \quad \mathcal{F}_c\phi = \sqrt{\frac{2}{\pi}}\int_{R^+}\phi(x)\cos px\,dx.$$

The usual direct and inverse Laplace transformation we denote by $\mathcal{L}$ and $\mathcal{L}^{-1}$

$$\mathcal{L}\phi = \widehat{\phi}(\xi) = \int_{\mathbf{0}}^{\infty} e^{-xp}\phi(x)dx, \mathcal{L}^{-}\phi = \frac{1}{2\pi i}\int_{i\mathbb{R}} e^{ix\xi}\widehat{\phi}(\xi)d\xi.$$

For a complex value function ϕ, which satisfies the Hölder condition on the imaginary axis, we define sectionally analytic function $\Phi(z)$ via the Cauchy type integral

$$\Phi(z) = \frac{1}{2\pi i}\int_{i\mathbb{R}}\frac{\phi(q)}{q-z}dq, \quad \operatorname{Re} z \neq 0.$$

We note that $\Phi(z)$ constitutes a function analytic in the left and right semi-planes. Here and below these functions will be denoted $\Phi^+(z)$ and $\Phi^-(z)$ respectively. These functions can be defined for all points of the imaginary axis $\operatorname{Re} p = 0$ via their limiting values $\Phi^+(p)$ and $\Phi^-(p)$, which are obtained on approaching to contour from the left and from the right, respectively. First, we define the sectionally analytic function

$$\mathcal{E}_w(x) = \frac{1}{2\pi i}\int_{i\mathbb{R}}\frac{e^{-qx}e^{-\Gamma(q,K(z))}}{q-w}dq, \tag{2.1}$$

for $\operatorname{Re} w \neq 0$, $K(z) = iz^2 - \sqrt{z}$, $z \geq 0$, where

$$\begin{gathered}\Gamma(w,\xi) = \frac{1}{2\pi i}\int_0^{\infty}\ln(q-w)\Big(\frac{K^{+\prime}(q)}{K^+(q)+\xi} - \frac{K^{-\prime}(q)}{K^-(q)+\xi}\Big)dq,\\ K^{\pm}(q) = iq^2 + \sqrt{\mp iq}.\end{gathered} \tag{2.2}$$

We make a cut along to negative axis $w < 0$. Denote by

$$\Gamma^{+}(s,\xi) = \lim_{w\to s, \operatorname{Im} w>0} \Gamma(w,\xi), s > 0$$
$$\Gamma^{-}(s,\xi) = \lim_{w\to s, \operatorname{Im} w<0} \Gamma(w,\xi), s > 0.$$

Define the "distorted" Fourier sine transform $\mathcal{B}_s$ and the inverse "distorted" Fourier sine transform $\mathcal{B}_s^*$ as follows

$$\widehat{\phi}(p) = \mathcal{B}_s\phi = \int_0^\infty \psi_s(z,x)\phi(x)dx, \quad \phi(x) = \mathcal{B}_s^*\widehat{\phi} = \frac{1}{2\pi}\int_0^\infty \psi_s^*(z,x)\widehat{\phi}(z)dz, \tag{2.3}$$

where

$$\psi_s(z,x) = \mathcal{E}_{iz}^{-}(x) - \mathcal{E}_{-iz}^{-}(x), \tag{2.4}$$
$$\psi_s^*(z,x) = e^{izx}e^{\Gamma(iz,K(z))} - e^{-izx}e^{\Gamma(-iz,K(z))} + K'(z)\Theta(x,z), \tag{2.5}$$
$$\Theta(x,z) = \frac{1}{2\pi}\int_0^\infty e^{-px}\psi(p,z)dp, \tag{2.6}$$
$$\psi(p,z) = \frac{\sqrt{2}}{2}\frac{e^{\Gamma(-p,K(z))}}{(ip^2+(ip)^{1/2}+K(z))(ip^2+(-ip)^{1/2}+K(z))}. \tag{2.7}$$

For a detailed study of properties of $\mathcal{B}_s\phi$ and $\mathcal{B}_s^*\widehat{\phi}$ see below in Lemmas 2.4. We introduce the Green operator on a half-line as

$$\mathcal{G}(t) = \mathcal{B}_s^*\{e^{tK(z)}\mathcal{B}_s\}, \tag{2.8}$$

Moreover, denoting

$$\mathring{\psi}_s(z,x) = e^{izx}e^{\Gamma(iz,K(z))} - e^{-izx}e^{\Gamma(-iz,K(z))} + \frac{z}{K(z)}(\frac{5}{2}\sqrt{|z|} - 2z^2)\Theta(x,z), \tag{2.9}$$

we introduce the operator

$$\mathring{\mathcal{B}}_s\phi = 2i\int_0^\infty \mathring{\psi}_s(x,p)\phi(x)dx, \tag{2.10}$$

and the Boundary operator on a half- line

$$\mathcal{H}(t)\phi = \mathring{\mathcal{B}}_s\Big\{\frac{K(z)}{z}\int_0^t e^{K(z)(t-\tau)}h(\tau)d\tau\Big\}. \tag{2.11}$$

For a Hölder continuous function ϕ on the imaginary axis, we define the operator

$$\mathbb{J}\{\phi\}(z) = -\frac{1}{\pi}\int_0^\infty \frac{\phi(p)}{p^2+z^2}(e^{-\Gamma^{+}(p,K(z))} - e^{-\Gamma^{-}(p,K(z))})dp, \tag{2.12}$$

To state the results of the present paper we give some notations. We denote $\langle t\rangle = 1+t$, $\{t\} = \frac{t}{\langle t\rangle}$. Moreover, we introduce the functional $\mathcal{S}$ on $L^{1,1}(\mathbb{R})$ as

$$\mathcal{S}u_0 = \int_0^\infty f(y)u_0(y)dy,$$

with $f(y) = y + \mathbb{J}\{e^{-py}-1\}|_{z=0}$. The weighted Sobolev space is

$$H_p^{k,s} = \big\{\phi \in S : \|\phi\|_{H_p^{k,s}} = \|\langle x\rangle^s\langle i\partial_x\rangle^k\phi\|_{L^p}\big\},$$

$k, s \in \mathbb{R}$, $1 \le p \le \infty$. We also use the notation $H^{k,s} = H_2^{k,s}$ and $H^k = H_2^{k,0}$.

Different constants might be denoted by the same letter C. For simplicity we put $\alpha_1 = 1, \alpha_2 = 1, \alpha_3 = 1$. Denote by

$$\theta(s) = \begin{cases} 1, & s > 1, \\ 0, & s \le 1 \end{cases}$$

Our main results read as follows.

Theorem 2.1. *Let $u_0 \in Z = H^1(\mathbb{R}^+) \cap H_1^{0,1}(\mathbb{R}^+)$, $h \in Y = H_\infty^{1,\beta}$, $\beta > 1/2$, be such that $\|u_0\|_Z + \|h\|_Y \le \epsilon$, where $\epsilon > 0$ is sufficiently small, and the compatibility condition $u_0(0) = h(0)$ is fulfilled. Then there exists a unique global solution*

$$u \in C([0,\infty); Z).$$

Moreover the following asymptotic statement is valid,

$$u(x,t) = h(t)\mathring{\mathcal{B}}_s\{z^{-1}\} + \theta(\beta)t^{-1}\hat{h}(0)\Psi(xt^{-2}) + t^{-3}A\Lambda(x) + R. \tag{2.13}$$

uniformly with respect $t \to \infty$, where $\Psi, \Lambda \in L^\infty(\mathbb{R}^+)$

$$\Psi(s) = -4\mathcal{F}_s\{e^{-\sqrt{z}}z^{-1/2}\}(s)$$

$$\Lambda(s) = \frac{\sqrt{2}}{8\pi}\Big[\int_0^\infty e^{-\sqrt{z}}\sqrt{z}dz\Big]\Big[\int_0^\infty e^{-ps}\frac{\sqrt{p}}{(ip^2 + (ip)^{1/2})(ip^2 + (-ip)^{1/2})}dp\Big]$$

$$A = \mathcal{S}\Big(u_0 + \int_0^\infty |u|^2 u(\tau)d\tau\Big),$$

$$R = O(t^{-(3+\delta)})(\|u_0\|_Z + \|u\|_{\mathbf{X}}^3 + t^{-(1+\beta)}\|h\|_Y).$$

From this Theorem we conclude that the solution possesses the following modified scattering behavior:

- If $\beta < 1$, then there exist a function $\Psi \in L^\infty$ such that

$$\sup_{t>0}\langle t\rangle^{\beta+\gamma}\|u - h(t)\mathring{\mathcal{B}}_s\{z^{-1}\}\|_{L^\infty} \le C\varepsilon.$$

- If $\beta \ge 1$ then there exist a constant B and a function $\widetilde{\Lambda}(\xi) \in L^\infty$ such that

$$\sup_{t>0}\langle t\rangle^{1+\gamma}\|u - t^{-1}B\widetilde{\Lambda}(xt^{-1/2})\|_{L^\infty} \le C\varepsilon,$$

2.2. **Linear problem.** Consider the linear fractional NLS equation posed on a half-line

$$\begin{gathered} u_t + \mathbb{K}u = 0, \quad t > 0,\ x > 0; \\ u(x,0) = u_0(x),\ x > 0, \quad u(0,t) = h(t),\ t > 0, \end{gathered} \tag{2.14}$$

In the next lemma we prove that $\mathcal{G}(t)$ and $\mathcal{H}(t)$ given by (2.8) and (2.11) are the Green and boundary operators of the problem (2.14).

Lemma 2.2. *Let the initial data $u_0 \in Z = H^1(\mathbb{R}^+) \cap H_1^{0,1}(\mathbb{R}^+)$, and boundary data $h \in Y = H_\infty^{1,\beta}$, $\beta > \frac{1}{2}$. Then the solution $u(x,t)$ of the initial-boundary value problem (2.14) has the following integral representation*

$$u(x,t) = \mathcal{G}(t)u_0 + \mathcal{H}(t)h,$$

where the operators $\mathcal{G}(t)$ and $\mathcal{H}(t)$ are given by (2.8) and (2.11).

Proof. In [5] we proved that the unique solution $u(x,t)$ to (2.14) has the integral representation

$$u(x,t) = \int_0^\infty G(x,y,t)u_0(y)dy + \int_0^t H(x,t-\tau)h(\tau)d\tau,$$

where

$$\begin{aligned} G(x,y,t) &= \big(\frac{1}{2\pi i}\big)^2 \int_{i\mathbb{R}} e^{\xi t} \int_{i\mathbb{R}} e^{px} \frac{Y(p,\xi)}{K(p)+\xi} \Big(\mathcal{J}^-_{\varphi(\xi)}(y,\xi) - \mathcal{J}^-_p(y,\xi)\Big)\, dy\, dp\, d\xi, \\ H(x,t) &= -\big(\frac{1}{2\pi i}\big)^2 \int_{i\mathbb{R}} e^{\xi t} \int_{i\mathbb{R}} e^{px} \frac{Y(p,\xi)}{K(p)+\xi} \Big(\mathcal{I}^-(\varphi(\xi),\xi) - \mathcal{I}^-(p,\xi)\Big)\, dp\, d\xi, \end{aligned} \tag{2.15}$$

with

$$\begin{gathered} \mathcal{J}_z(y,\xi) = \frac{1}{2\pi i}\int_{i\mathbb{R}} \frac{e^{-qy}}{q-z}\frac{1}{Y(q,\xi)}dq, \\ \mathcal{I}(z,\xi) = iz - \frac{1}{2\pi}\int_{i\mathbb{R}} \frac{\sqrt{|q|}}{q(q-z)}\frac{1}{Y(q,\xi)}dq, \end{gathered} \tag{2.16}$$

and the "analyticity switching" function $Y(w,\xi) = e^{\Gamma(w,\xi)}$, $\operatorname{Re}\xi > 0$, where Γ is defined in (2.2) and $\varphi(\xi)$ is the only one root of the equation $K(p)+\xi = 0$ in the right-half complex plane, with the analytic extension of the function $K(p)$ is given by

$$K(p) = \begin{cases} K^+(p) = ip^2 + \sqrt{-ip} & \text{if } \operatorname{Im} p > 0 \\ K^-(p) = ip^2 + \sqrt{ip} & \text{if } \operatorname{Im} p < 0. \end{cases}$$

Now we simplify the representation of $G(x,y,t)$. Via the Sokhotski-Plemelj formula we obtain

$$\begin{aligned} G(x,y,t) &= \big(\frac{1}{2\pi i}\big)^2 \int_{i\mathbb{R}} e^{\xi t} \int_{i\mathbb{R}} e^{px} \frac{Y(p,\xi)}{K(p)+\xi} (\mathcal{J}^-_{\varphi(\xi)}(y,\xi) - \mathcal{J}^+_p(y,\xi))dy\ dp\ d\xi \\ &\quad + \frac{1}{2\pi i}\int_{i\mathbb{R}} e^{px-K(p)t}dp. \end{aligned}$$

Remember that $\varphi(\xi)$ is the only root of the equation $K(p)+\xi = 0$ on the right half plane, using this, we change of variables $\xi = -K(z)$ and we obtain

$$\begin{aligned} G(x,y,t) &= -\big(\frac{1}{2\pi i}\big)^2 \int_{-i\infty e^{-i(\frac{\pi}{2})+\epsilon}}^{-i\infty e^{i(\frac{\pi}{2})-\epsilon}} e^{-K(z)t}K'(z) \int_{i\mathbb{R}} e^{px} \frac{Y(p,-K(z))}{K(p)-K(z)} \\ &\quad \times (\mathcal{E}^-_z(y,-K(z)) - \mathcal{E}^+_p(y,-K(z)))dy\, dp\, dz \\ &\quad + \frac{1}{2\pi i}\int_{i\mathbb{R}} e^{px-K(p)t}dp, \end{aligned}$$

with

$$\mathcal{E}_w(y) = \frac{1}{2\pi i}\int_{i\mathbb{R}} \frac{e^{-qy}}{q-w}\frac{1}{Y(p,-K(z))}dq, \quad \text{for } \operatorname{Re} w \neq 0.$$

To change the contour of integration with respect to p variable we apply Cauchy Theorem. Taking residue in the point $p = -z$ we obtain

$$\begin{aligned} G(x,y,t) = &-\big(\frac{1}{2\pi i}\big)^2 \int_{i\mathbb{R}} e^{-K(z)t} e^{-zx+\Gamma(-z,-K(z))} (\mathcal{E}_z^-(y) - \mathcal{E}_{-z}^-(y)) dz \\ &+ \big(\frac{1}{2\pi i}\big)^2 \int_{i\mathbb{R}} e^{-K(z)t} z K'(z) \mathcal{E}_z^-(y) \int_0^\infty e^{-px} \psi(p,z) dp\, dz \\ &+ \big(\frac{1}{2\pi i}\big)^2 \int_{i\mathbb{R}} e^{-K(z)t} K'(z) \int_0^\infty e^{-px} \mathcal{E}_p^+(y) \psi(p,z) dp\ dz, \end{aligned} \tag{2.17}$$

where

$$\psi(p,z) = \frac{\sqrt{2}}{2} \frac{e^{\Gamma(-p,K(z))}}{(K^+(p) - K(z))(K^-(p) - K(z))}. \tag{2.18}$$

Note that since integrand function is even with respect to z variable

$$\int_{i\mathbb{R}} e^{-K(z)t} K'(z) \int_0^\infty e^{-px} \mathcal{E}_p^+(y) \psi(p,z) dp\ dz = 0.$$

Consequently changing $z \mapsto iz$ into (2.17) we obtain

$$G(x,y,t) = \big(\frac{1}{2\pi}\big)^2 \int_0^\infty e^{-K(iz)t} \psi_s(z,y) \psi_s^*(x,z) dz, \tag{2.19}$$

where the functions ψ_s and ψ_s^* was defined in (2.4), (2.5). Therefore, we obtain

$$\mathcal{G}(t)\phi = \mathcal{B}_s^* \{ e^{K(p)t} \mathcal{B}_s \phi \}, \quad K(z) = ip^2 - \sqrt{p}. \tag{2.20}$$

For the operator $\mathcal{H}(t)$ we note that

$$\mathcal{I}(\varphi(\xi),\xi) - \mathcal{I}(p,\xi) = i(\varphi(\xi) - p)\Big[1 - \frac{1}{2\pi i} \int_{i\mathbb{R}} \frac{\sqrt{|q|}}{q(q-p)(q-\varphi(\xi))} \frac{1}{Y(q,\xi)} dq\Big].$$

Denoting $K_1(q) = iq^2 + \sqrt{|q|}$ we have

$$\begin{aligned} &\int_{i\mathbb{R}} \frac{\sqrt{|q|}}{q(q-p)(q-\varphi(\xi))} \frac{1}{Y(q,\xi)} dq \\ &= \int_{i\mathbb{R}} \frac{K_1(q) + \xi}{q(q-p)(q-\varphi(\xi))} \frac{1}{Y(q,\xi)} dq - \int_{i\mathbb{R}} \frac{iq^2 + \xi}{q(q-p)(q-\varphi(\xi))} \frac{1}{Y(q,\xi)}. \end{aligned}$$

Recalling that function $\frac{K(\cdot)+\xi}{Y(\cdot,\xi)}$ is analytic on the right half-plane, via the Cauchy theorem we have

$$\begin{gathered} \int_{i\mathbb{R}} \frac{K(q) + \xi}{q(q-p)(q-\varphi(\xi))} \frac{1}{Y(q,\xi)} dq = \frac{K(p) + \xi}{p(p-\varphi(\xi))} \frac{1}{Y(p,\xi)} + \frac{1}{2} \frac{\xi}{p\varphi(\xi)} \frac{1}{Y(0,\xi)} - \frac{1}{2}, \\ \int_{i\mathbb{R}} \frac{iq^2 + \xi}{q(q-p)(q-\varphi(\xi))} \frac{1}{Y(q,\xi)} dq = \frac{1}{2} - \frac{1}{2} \frac{\xi}{p\varphi(\xi)} \frac{1}{Y(0,\xi)}. \end{gathered}$$

Therefore,

$$\int_{i\mathbb{R}} \frac{\sqrt{|q|}}{q(q-p)(q-\varphi(\xi))} \frac{1}{Y(q,\xi)} dq = \frac{K(p) + \xi}{p(p-\varphi(\xi))} \frac{1}{Y(p,\xi)} + \frac{\xi}{p\varphi(\xi)} \frac{1}{Y(0,\xi)} - 1,$$

and as consequence

$$\mathcal{I}(\varphi(\xi),\xi) - \mathcal{I}(p,\xi) = i(\varphi(\xi) - p)\Big[\frac{K(p) + \xi}{p(p-\varphi(\xi))} \frac{1}{Y(p,\xi)} + \frac{\xi}{p\varphi(\xi)} \frac{1}{Y(0,\xi)}\Big]. \tag{2.21}$$

Thus using the Cauchy Theorem we obtain

$$H(x,t) = \widetilde{H}_1(x,t) + \widetilde{H}_2(x,t), \tag{2.22}$$

where

$$\begin{aligned} \widetilde{H}_1(x,t) &= \frac{1}{2\pi i}\int_{i\mathbb{R}} e^{\xi t}\,d\xi - \big(\frac{1}{2\pi i}\big)^2 \int_{i\mathbb{R}} e^{\xi t}\xi \fint_{i\mathbb{R}} e^{px}\frac{Y(p,\xi)}{Y(0,\xi)}\frac{1}{p(K(p)+\xi)}\,dp\,d\xi,\\ \widetilde{H}_2(x,t) &= \big(\frac{1}{2\pi i}\big)^2 \int_{i\mathbb{R}} e^{\xi t}\frac{\xi}{\varphi(\xi)} \fint_{i\mathbb{R}} e^{px}\frac{Y(p,\xi)}{Y(0,\xi)}\frac{1}{K(p)+\xi}\,dp\,d\xi. \end{aligned} \tag{2.23}$$

Using analytic properties of the integrand function via Jordan Lemma we have

$$\begin{aligned} \widetilde{H}_1(x,t) &= -\frac{1}{2\pi i}\int_{i\mathbb{R}} e^{\xi t} e^{-\varphi(\xi)x}\frac{Y(-\varphi(\xi),\xi)}{Y(0,\xi)}\frac{\xi}{\varphi(\xi)}\frac{1}{K'(\varphi(\xi))}d\xi\\ &\quad - \big(\frac{1}{2\pi i}\big)^2 \int_{i\mathbb{R}} e^{\xi t}\xi\int_{-\infty}^0 e^{px}\frac{Y(p,\xi)}{Y(0,\xi)}\frac{\sqrt{(-ip)}-\sqrt{(ip)}}{p(K^+(p)+\xi)(K^-(p)+\xi)}\,dp\,d\xi. \end{aligned} \tag{2.24}$$

Changing of variable $\xi = -K(z)$ and remembering $Y(0,-K(z)) = 1$ (see Lemma 2.7) we rewrite $\widetilde{H}_1$ as

$$\begin{aligned} \widetilde{H}_1(x,t) &= -\frac{1}{2\pi i}\int_{i\mathbb{R}} e^{-K(z)t} e^{-zx+\Gamma(-z,-K(z))}\frac{K(z)}{z}dz\\ &\quad - \big(\frac{1}{2\pi i}\big)^2 \int_{i\mathbb{R}} e^{-K(z)t}K(z)K'(z)\int_0^\infty e^{-px}\psi(p,z)dp\,dz, \end{aligned} \tag{2.25}$$

since the integrand in the second integral expression is an odd function with respect to z variables we conclude

$$\big(\frac{1}{2\pi i}\big)^2 \int_{i\mathbb{R}} e^{-K(z)t}K(z)K'(z)\int_0^\infty e^{-px}\psi(p,z)dp\;dz = 0,$$

and as a consequence

$$\widetilde{H}_1(x,t) = \frac{1}{2\pi i}\int_0^{i\infty} e^{-K(z)t}\frac{K(z)}{z}\Big[e^{zx+\Gamma(z,-K(z))} - e^{-zx+\Gamma(-z,-K(z))}\Big]\,dz. \tag{2.26}$$

In a similar form we obtain

$$\begin{aligned} \widetilde{H}_2(x,t) &= \frac{1}{2\pi i}\int_0^{i\infty} e^{-K(z)t}\frac{K(z)}{z}\big[e^{-zt+\Gamma(-z,-K(z))} - e^{zt+\Gamma(z,-K(z))}\big]dz\\ &\quad + \frac{1}{2\pi i}\fint_{-i\infty}^{i\infty} e^{-K(z)t}\frac{K(z)}{z}K'(z)\Theta(x,z)dz. \end{aligned} \tag{2.27}$$

Since $K(z)K'(z) = -2z^3 + \frac{5}{2}z\sqrt{|z|} + \frac{1}{2}$ and

$$\frac{1}{4\pi i}\fint_{-i\infty}^{i\infty} e^{-K(z)t}\frac{1}{z}\Theta(x,z)dz = 0,$$

we reduce the function $\widetilde{H}_2$ as

$$\begin{aligned} \widetilde{H}_2(x,t) &= \frac{1}{2\pi i}\int_0^{i\infty} e^{-K(z)t}\frac{K(z)}{z}[e^{-zt+\Gamma(-z,-K(z))} - e^{zt+\Gamma(z,-K(z))}]dz\\ &\quad + \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{-K(z)t}(\frac{5}{2}\sqrt{|z|} - 2z^2)\Theta(x,z)dz. \end{aligned} \tag{2.28}$$

Applying (2.26)–(2.28) into (2.22) we obtain

$$H(x,t) = 2i\int_0^\infty e^{-K(z)t}\frac{K(z)}{z}\mathring{\psi}_s(x,z)dz.$$

From this we conclude

$$\mathcal{H}(t)h = \mathring{\mathcal{B}}_s\Big\{\frac{K(p)}{p}\int_0^t e^{K(p)(t-\tau)}h(\tau)d\tau\Big\},$$

where $K(p) = ip^2 - \sqrt{p}$ and the operator $\mathring{\mathcal{B}}_s$ was defined in (2.10). □

2.3. Large time asymptotic behavior for the evolution group and the boundary operator.

Lemma 2.3. *For $K(z) = -\sqrt{z} + iz^2$, and ψ_s given by (2.4) we have*

$$\psi_s(z,x) = e^{-izx}e^{-\Gamma(-iz,K(z))} - e^{izx}e^{-\Gamma(iz,K(z))} + z\mathbb{J}\{e^{-px}\}(z),$$

where $\mathbb{J}$ was given by (2.12).

The proof of the above lemma is obtained using analytic properties of the integrand function via Jordan Lemma, and the Cauchy Theorem.

Lemma 2.4. *For $u_0 \in \mathbf{Z} = \mathbf{H}^1(\mathbb{R}^+) \cap H_1^{0,1}(\mathbb{R}^+)$, the Green operator $\mathcal{G}(t)$ satisfies the asymptotic expansion*

$$\mathcal{G}(t)u_0 = t^{-3}\Lambda(x)\mathcal{S}u_0 + O(t^{-(3+\gamma)})\|u_0\|_Z,$$

where $\Lambda \in L^\infty(\mathbb{R}^+)$,

$$\Lambda(x) = \frac{\sqrt{2}}{8\pi}\Big[\int_0^\infty e^{-\sqrt{z}}\sqrt{z}dz\Big]\Big[\int_0^\infty e^{-px}\frac{\sqrt{p}}{(ip^2+(ip)^{1/2})(ip^2+(-ip)^{1/2})}dp\Big], \tag{2.29}$$

and

$$\mathcal{S}u_0 = \int_0^\infty f(y)u_0(y)dy,$$

with $f(y) = y + \mathbb{J}\{e^{-py} - 1\}|_{z=0}$.

Proof. From the equality obtained in Lemma 2.3 for the function ψ_s, and Cauchy Theorem we obtain

$$\begin{aligned}\psi_s(z,y) &= \sin zy + (e^{-\Gamma(iz,K(z))} - 1)(e^{-izy} - 1)\\ &\quad - (e^{-\Gamma(-iz,K(z))} - 1)(e^{izy} - 1) + zW(z,y),\end{aligned}$$

where $W(z,y) = \mathbb{J}\{e^{-py} - 1\}(z)$. Via Taylor theorem we have

$$|W(0,y)| \le Cy\int_0^\infty \frac{1}{p}(\{p\}^\gamma\langle p\rangle^{-3/2})dp < C,$$

moreover

$$|W(z,y) - W(0,y)| \le Cyz. \tag{2.30}$$

Via Lemma (2.6) we have

$$e^{-\Gamma(\pm iz,K(z))} - 1 = O(z^\gamma), \quad \delta > 0.$$

Combining this with (2.30) we conclude

$$\psi_s(z,y) = zf(y) + O(yz^{1+\gamma}), \quad f(y) = y + W(0,y), \tag{2.31}$$

and as a consequence

$$\mathcal{B}_s u_0 = z\mathcal{S}u_0 + O(z^{1+\gamma})\|u_0\|_{L^{1,1}}, \tag{2.32}$$

where the functional $\mathcal{S}$ is given by

$$\mathcal{S}u_0 = \int_0^\infty f(y)u_0(y)dy.$$

On the other hand, from the definition of the function ψ_s^*, given by (2.4), we note

$$\begin{aligned}\psi_s^*(x,z) = &\frac{\sqrt{2}}{8\pi}\frac{1}{\sqrt{z}}\int_0^\infty e^{-px}\frac{\sqrt{p}}{(ip^2+(ip)^{1/2})(ip^2+(-ip)^{1/2})}dp \\ &+ R_1(x,z) + R_2(x,z),\end{aligned} \tag{2.33}$$

where

$$R_1(x,z) = e^{izx+\Gamma^+(z)} - e^{-izx+\Gamma^+(-z)} + z\Theta(x,z),$$

$$R_2(x,z) = \frac{1}{4\pi}\frac{1}{\sqrt{z}}\int_0^\infty e^{-px}p\left[e^{\Gamma^+(p,z)}\lambda(p,z) - e^{\Gamma^+(p,0)}\lambda(p,0)\right]dp.$$

Using

$$|\Theta(x,z)| \le C\int_0^\infty \frac{\sqrt{p}}{(ip^2+(-ip)^{1/2})(ip^2+(ip)^{1/2})} \le C,$$

and via Lemma 2.6 $|e^{izx+\Gamma^+(z)} - e^{-izx+\Gamma^+(-z)}| \le C$, we conclude

$$R_1(x,z) = O(\langle z\rangle).$$

Now, we estimate R_2. We have

$$\begin{aligned}&e^{\Gamma^+(p,z)}\lambda(p,z) - e^{\Gamma^+(p,0)}\lambda(p,0) \\ &= [e^{\Gamma^+(p,z)} - e^{\Gamma^+(p,0)}]\lambda(p,z) + e^{\Gamma^+(p,0)}\left[\lambda(p,z)-\lambda(p,0)\right],\end{aligned} \tag{2.34}$$

Using

$$|\lambda(p,z)-\lambda(p,0)| \le C\frac{|K(z)|^{1-\gamma}|p|}{|ip^2+(-ip)^{1/2}|^2|ip^2+(ip)^{1/2}|^{2-\gamma}},$$

and via Lemma 2.6

$$e^{\Gamma^+(p,z)} - e^{\Gamma^+(p,0)} = O(z^\gamma),$$

therefore

$$\begin{aligned}&R_2(x,z) \\ &= \frac{1}{4\pi}\frac{1}{\sqrt{z}}\int_0^\infty e^{-px}\Big([e^{\Gamma^+(p,z)} - e^{\Gamma^+(p,0)}]\psi(p,z) + e^{\Gamma^+(p,0)}[\psi(p,z)-\psi(p,0)]\Big)dp \\ &= O(z^{\gamma-\frac{1}{2}}).\end{aligned}$$

Thus via 2.33

$$\psi_s^*(x,z) = \frac{\sqrt{2}}{8\pi}\frac{1}{\sqrt{z}}\int_0^\infty e^{-px}\frac{\sqrt{p}}{(ip^2+(ip)^{1/2})(ip^2+(-ip)^{1/2})}dp + O(z^{\gamma-\frac{1}{2}}). \tag{2.35}$$

Combining (2.32) and (2.35) we conclude

$$\mathcal{G}(t)u_0 = t^{-3}\Lambda(x)\mathcal{S}u_0 + O(t^{-(3+\gamma)})\|u_0\|_{L^{1,1}}, \tag{2.36}$$

where $\Lambda \in L^\infty$,

$$\Lambda(x) = \frac{\sqrt{2}}{8\pi}\Big[\int_0^\infty e^{-\sqrt{z}}\sqrt{z}dz\Big]\Big[\int_0^\infty e^{-px}\frac{\sqrt{p}}{(ip^2+(ip)^{1/2})(ip^2+(-ip)^{1/2})}dp\Big].$$

$$\mathcal{S}u_0 = \int_0^\infty f(y)u_0(y)\,dy. \tag{2.37}$$

Thus, Lemma (2.4) is proved. □

Lemma 2.5. *Let $h \in Y = H^{1,\beta}_\infty(\mathbb{R}^+)$, $\beta > 1/2$, then the following asymptotic expansion for large time t holds*

$$\mathcal{H}(t)h = h(t)\mathring{\mathcal{B}}_s\{z^{-1}\} + \theta(\beta)t^{-1}\hat{h}(0)\Psi(xt^{-2}) + O(t^{-1-\beta})\|h\|_Y, \tag{2.38}$$

where θ is the characteristic function of the interval $[1,\infty)$ and $\Psi \in L^\infty(\mathbb{R}^+)$ is given by

$$\Psi(s) = -4\mathcal{F}_s\{e^{-\sqrt{z}}z^{-1/2}\}(s). \tag{2.39}$$

Proof. First, we recall the definition of the operator $\mathcal{H}$ given in (2.11):

$$\mathcal{H}(t)h = \mathring{\mathcal{B}}_s\Big\{\frac{K(z)}{z}[h_1(z,t)+h_2(z,t)]\Big\}, \tag{2.40}$$

where

$$h_1(z,t) = \int_{t/2}^t e^{K(z)(t-\tau)}h(\tau)d\tau, \quad h_2(z,t) = \int_0^{t/2} e^{K(z)(t-\tau)}h(\tau)d\tau.$$

Integrating by parts,

$$K(z)h_1(z,t) = h(t) - e^{K(z)\frac{t}{2}}h(\frac{t}{2}) - \int_{t/2}^t e^{K(z)(t-\tau)}h'(\tau)d\tau. \tag{2.41}$$

Recalling that $h \in H^{1,\beta}_\infty$,

$$\begin{gathered}\mathring{\mathcal{B}}_s\Big\{\frac{1}{z}e^{K(z)t}h(\frac{t}{2})\Big\} = O(\langle t\rangle^{-(1+\beta)})\|h\|_Y,\\ \int_{t/2}^t e^{K(z)(t-\tau)}h'(\tau)d\tau = O(h'(t)),\end{gathered} \tag{2.42}$$

and therefore

$$\mathring{\mathcal{B}}_s\Big\{\frac{1}{z}\int_{t/2}^t e^{K(z)(t-\tau)}h'(\tau)d\tau\Big\} = O(h'(t)). \tag{2.43}$$

Thus, (2.41)–(2.43) imply

$$\mathring{\mathcal{B}}_s\Big\{\frac{K(z)h_1(z,t)}{z}\Big\} = h(t)\mathring{\mathcal{B}}_s\{z^{-1}\} + O(\langle t\rangle^{-(1+\beta)})\|h\|_Y. \tag{2.44}$$

On the other hand

$$h_2(z,t) = e^{K(z)t}\int_0^{t/2}h(\tau)d\tau + \int_0^{t/2}e^{K(z)(t-\tau)}(1-e^{K(z)\tau})h(\tau)d\tau, \tag{2.45}$$

now, from the definition of the function $\mathring{\psi}_s(z,x)$ given by (2.10) we have

$$\frac{K(z)}{z}\mathring{\psi}_s(z,x) = \frac{2i\sin xz}{\sqrt{z}} + R_1(z,x) + R_2(z,x) + R_3(z,x),$$

with

$$R_1(z,x) = z(e^{ixz+\Gamma(iz,-K(z))} - e^{-ixz+\Gamma(-iz,-K(z))}),$$

$$R_2(z,x) = \frac{e^{ixz}(e^{\Gamma(iz,-K(z))}-1) - e^{-ixz}(e^{\Gamma(-iz,-K(z))}-1)}{\sqrt{z}},$$

$$R_3(z,x) = \big(\frac{5}{2}\sqrt{|z|} - 2z^2\big)\Theta(x,z)$$

From Lemma 2.6 $e^{\Gamma(\pm iz,-K(z))} = O(1)$ and $e^{\Gamma(\pm iz,-K(z))} - 1 = O(z^\gamma)$, $\gamma \in (0,1)$ and as consequence $R_1(z,x) + R_2(z,x) = O(z^{\gamma-\frac{1}{2}})$. Using

$$|\Theta(x,z)| \le C\int_0^\infty \frac{\sqrt{p}}{(ip^2+(-ip)^{1/2})(ip^2+(ip)^{1/2})} \le C,$$

we conclude $R_3(z,x) = O(z^\gamma)$. Therefore

$$\frac{K(z)}{z}\mathring{\psi}_s(z,x) = \frac{2i\sin xz}{\sqrt{z}} + O(z^{\gamma-\frac{1}{2}})\,. \tag{2.46}$$

From (2.45), (2.46) and $|1-e^{K(z)\tau}| \le Cz^{\frac{\gamma}{2}}\tau^\gamma$ we conclude

$$\begin{aligned}
&\mathring{\mathcal{B}}_s\Big\{\frac{K(z)h_2(z,t)}{z}\Big\}\\
&= t^{-1}\hat{h}(0)\Psi(xt^{-2}) + O(t^{-(1+\gamma)})\|h\|_Y + \int_0^{t/2} O((t-\tau)^{-2-\gamma})\tau^\gamma h(\tau)d\tau \\
&= t^{-1}\hat{h}(0)\Psi(xt^{-2}) + O(t^{-(1+\gamma)})\|h\|_Y + \max(t^{-2-\gamma}, t^{-1-\beta-\gamma})\|h\|_Y,
\end{aligned} \tag{2.47}$$

where $\Psi(s) = -4\mathcal{F}_s\{e^{-\sqrt{z}}z^{-1/2}\}(s)$. Finally, from (2.40) along to (2.41)-(2.47) we have

$$\mathcal{H}(t)h = h(t)\mathring{\mathcal{B}}_s\{z^{-1}\} + \theta(\beta)t^{-1}\hat{h}(0)\Psi(xt^{-2}) + O(t^{-1-\beta})\|h\|_Y,$$

where θ is the characteristic function of the interval $[1,\infty)$. The proof is complete. $\square$

In this Lemma we exhibit several properties of the "analyticity switching" function $Y(w,K(z)) = e^{\Gamma(w,K(z))}$, where

$$\Gamma(w,\xi) = \frac{1}{2\pi i}\int_0^\infty \ln(q-w)\Big(\frac{K^{+\prime}(q)}{K^+(q)+\xi} - \frac{K^{-\prime}(q)}{K^-(q)+\xi}\Big)dq.$$

We make a cut along to negative axis $w<0$. Denote by

$$\Gamma^+(s,\xi) = \lim_{w\to s, \operatorname{Im} w>0}\Gamma(w,\xi),\ \ s>0$$

$$\Gamma^-(s,\xi) = \lim_{w\to s, \operatorname{Im} w<0}\Gamma(w,\xi),\ \ s>0.$$

Lemma 2.6. *We have for $s>0$, $\arg\xi \in (-\frac{\pi}{2}-\frac{\pi}{4}, \frac{\pi}{2}+\frac{\pi}{4})$*

$$\frac{e^{\Gamma^+(s,\xi)}}{e^{\Gamma^-(s,\xi)}} = \frac{K^+(p)+\xi}{K^-(p)+\xi}.$$

Moreover the following formula is valid for $z \in \mathbb{R}$ and $w \in \mathbb{C}/w>0$,

$$|Y(w,K(z))| \le C, \quad |e^{-\Gamma^+(iz,K(z))}| \le C,$$

$$\Gamma(w,-K(z)) = O(\{w\}^\gamma + \{z\}^\gamma),$$

$$\partial_z\Gamma(iz,K(z)) = O(\{z\}^{-1}\langle z\rangle^{-2}).$$

The proof of the above Lemma can be found in [6].

Lemma 2.7. *For $w \in \mathbb{C}/w > 0$ we have*

$$\Gamma(-w, -K(z)) = \Gamma(w, -K(z)),$$

a s consequence $Y(0, -K(z)) = 1$.

Proof. Integrating by parts and via Cauchy Theorem we rewrite Γ as

$$\Gamma(w, -K(z)) = -\frac{1}{2\pi} \int_0^\infty \frac{1}{q-w} \ln\Big(\frac{K(q)-K(z)}{q^2-z^2}\Big) dq.$$

As consequence, via the change of variables $v = -q$, we obtain

$$\begin{aligned}\Gamma(-w, -K(z)) &= -\frac{1}{2\pi} \int_0^\infty \frac{1}{q+w} \ln(\frac{K(q)-K(z)}{q^2-z^2}) dq \\ &= \frac{1}{2\pi} \int_0^\infty \frac{1}{v-w} \ln(\frac{K(v)-K(z)}{v^2-z^2}) dq \\ &= -\Gamma(w, -K(z)).\end{aligned} \tag{2.48}$$

Therefore $\Gamma(0, -K(z)) = 0$, this guarantees that $Y(0, -K(z)) = 1$. $\square$

3. Proof of Theorem 2.1

It follows from Lemma 2.2 and Duhamel principle that the solution of (1.1) is given by

$$u(x,t) = \mathcal{G}(t)u_0 + \mathcal{H}(t)h + \int_0^t \mathcal{G}(t-\tau)\mathcal{N}(u)(\tau) d\tau, \tag{3.1}$$

Let us define the function spaces $Z = H^1(\mathbb{R}^+) \cap H^{0,1}(\mathbb{R}^+)$

$$X = \{\phi \in C([0,\infty); Z) : \|\phi\|_X < \infty\},$$

where

$$\|\phi\|_X = \sup_{0 \le t} \{\langle t\rangle^{1/2} \|\phi(t)\|_{H^1} + \|\phi(t)\|_{H^{0,1}} + \langle t\rangle^\rho \|\phi(t)\|_{L^\infty}\},$$

with $\rho = \min\{1, \beta\}$. By the contraction mapping principle we can prove that there exist an unique solution u to (1.1) in X, since

$$X \subset C([0,\infty); L^2(\mathbb{R}^+)) \cap C\Big((0,\infty); L^{2, \frac{1}{2}(\frac{1}{2}+\mu)}(\mathbb{R}^+) \cap L^\infty(\mathbb{R}^+)\Big)$$

the uniqueness guarantee that the solution given by Theorem 1.1 is the same solution $u \in X$.

Now we prove the asymptotic formula for the solution. From Lemmas 2.4 and 2.5 we obtain

$$\mathcal{G}(t)u_0 + \mathcal{H}(t)h = h(t)\mathring{\mathcal{B}}_s\{z^{-1}\} + \theta(\beta)t^{-1}\hat{h}(0)\Psi(xt^{-2}) + t^{-3}\Lambda(x)\mathcal{S}u_0 + R, \tag{3.2}$$

with $\theta(\beta) = 1$ for $\beta > 1$ and $\theta(\beta) = 0$ for $\beta \le 1$, $\widehat{h}(p) = \mathcal{L}h$, $\Lambda, \Psi \in L^\infty(\mathbb{R}^+)$ defined by (2.29), (2.39) respectively and

$$R = t^{-(3+\gamma)}\|u_0\|_Z + t^{-(1+\beta)}\|h\|_Y.$$

Thus we observe for $t > 1$,

$$\begin{aligned}
&\int_0^t \mathcal{G}(t-\tau)\mathcal{N}(u)(\tau)\,d\tau \\
&= \int_0^{t/2} \mathcal{G}(t)\mathcal{N}(u)(\tau)\,d\tau + \int_0^{t/2} [\mathcal{G}(t-\tau) - \mathcal{G}(t)]\mathcal{N}(u)(\tau)\,d\tau \\
&\quad + \int_{t/2}^t \mathcal{G}(t-\tau)\mathcal{N}(u)(\tau)\,d\tau.
\end{aligned} \tag{3.3}$$

Via 3.2 we note that

$$\begin{aligned}
&\int_0^{t/2} \mathcal{G}(t)\mathcal{N}(u)(\tau)\,d\tau \\
&= t^{-3}\Lambda(x)\int_0^\infty \mathcal{S}\mathcal{N}(u)(\tau)d\tau - t^{-3}\Lambda(x)\int_{t/2}^\infty \mathcal{S}\mathcal{N}(u)(\tau)d\tau \\
&\quad + O(t^{-(3+\delta)})\int_0^{t/2} (\|\mathcal{N}(u)(\tau)\|_{H_1^{0,1}} + \|\mathcal{N}(u)(\tau)\|_{H^1})d\tau,
\end{aligned}$$

since $|\mathcal{S}\phi| \le C\|\phi\|_{L^{1,1}}$ we observe

$$|\mathcal{S}\mathcal{N}u| \le C\|\mathcal{N}u\|_{L^{1,1}} \le C\|u\|_{L^\infty}\|u\|_{H^1} \le C\langle\tau\rangle^{-(1+\rho)}\|u\|_{\mathbf{X}}^3,$$

and as consequence

$$\int_0^{t/2} \mathcal{G}(t)\mathcal{N}(u)(\tau)\,d\tau = t^{-3}\Lambda(x)\int_0^\infty \mathcal{S}\mathcal{N}(u)(\tau)d\tau + O(t^{-(3+\gamma)})\|u\|_{\mathbf{X}}^3, \quad \gamma > 0 \tag{3.4}$$

By Lemma 2.4, we have

$$\mathcal{G}(t)u_0 = t^{-3}\Lambda(x)\mathcal{S}u_0 + O(t^{-(3+\gamma)})\|u_0\|_Z,$$

by properties of asymptotic representation we obtain $\|\partial_t\mathcal{G}(t)\phi\|_\infty \le Ct^{-4}(\|\phi\|_{H^{0,1}} + \|\phi\|_{H^1})$ we obtain

$$\int_0^{t/2} [\mathcal{G}(t-\tau) - \mathcal{G}(t)]\,\mathcal{N}(u)(\tau)\,d\tau = O(t^{-4})\|u\|_X^3. \tag{3.5}$$

By Lemma 2.4 we have

$$\begin{aligned}
\int_{t/2}^t \mathcal{G}(t-\tau)\mathcal{N}(u)(\tau)\,d\tau &= \|u\|_{\mathbf{X}}^3\int_{t/2}^t O(\langle t-\tau\rangle^{-3}\langle\tau\rangle^{-(1+\gamma)})d\tau \\
&= O(t^{-(3+\delta)})\|u\|_X^3
\end{aligned} \tag{3.6}$$

From (3.1)-(3.6) we obtain

$$u(x,t) = h(t)\mathring{\mathcal{B}}_s\{z^{-1}\} + \theta(\beta)t^{-1}\hat{h}(0)\Psi(xt^{-2}) + t^{-3}A\Lambda(x) + R. \tag{3.7}$$

where

$$A = \mathcal{S}(u_0 + \int_0^\infty |u|^2u(\tau)d\tau),$$

and

$$R = O(t^{-(3+\delta)})(\|u_0\|_Z + \|u\|_{\mathbf{X}}^3).$$

Hence, Theorem 2.1 is proved.

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