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OSCILLATORY AND ASYMPTOTIC BEHAVIOR OF THIRD-ORDER NEUTRAL DIFFERENTIAL EQUATIONS WITH DISTRIBUTED DEVIATING ARGUMENTS

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Abstract. This article concerns the oscillatory and asymptotic properties of solutions of a class of third-order neutral differential equations with distributed deviating arguments. We give sufficient conditions for every solution to be either oscillatory or to converges to zero. The results obtained can easily be extended to more general neutral differential equations and neutral dynamic equations on time scales. Two examples are also provided to illustrate the results.

1. Introduction

We are interested in the oscillation and asymptotic behavior of solutions to the third-order neutral differential equations with distributed deviating arguments

$$\Big(r(t)\big((x(t)+p(t)x(\tau(t)))''\big)^{\alpha}\Big)' + \int_a^b q(t,\xi)x^{\alpha}(\phi(t,\xi))d\xi = 0, \quad t \ge t_0 > 0, \quad (1.1)$$

where α is a quotient of odd positive integers and $0 < a < b$.

In the remainder of the paper we assume that:

(i) $r \in C([t_0,\infty),(0,\infty))$ and $\int_{t_0}^{\infty} r^{-1/\alpha}(s)ds = \infty$;
(ii) $p \in C([t_0,\infty),\mathbb{R})$ with $p(t) \ge 1$, and $p(t) \not\equiv 1$, eventually;
(iii) $q(t,\xi) \in C([t_0,\infty)\times[a,b],[0,\infty))$;
(iv) $\tau \in C([t_0,\infty),\mathbb{R})$ is strictly increasing, $\tau(t) < t$, and $\lim_{t\to\infty}\tau(t) = \infty$;
(v) $\phi(t,\xi) \in C([t_0,\infty)\times[a,b],\mathbb{R})$ is nonincreasing in ξ, and

$$\lim_{t\to\infty}\phi(t,\xi) = \infty, \quad \xi \in [a,b].$$

The cases

$$\tau(t) \ge \phi(t,\xi), \quad \xi \in [a,b], \quad (1.2)$$

and

$$\tau(t) \le \phi(t,\xi), \quad \xi \in [a,b], \quad (1.3)$$

are both considered.

By defining the function

$$z(t) = x(t) + p(t)x(\tau(t)), \quad (1.4)$$

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equation (1.1) can be written as

$$\big(r(t)(z''(t))^{\alpha}\big)' + \int_a^b q(t,\xi)x^{\alpha}(\phi(t,\xi))d\xi = 0. \tag{1.5}$$

By a solution of (1.1) we mean a function $x : [t_x,\infty) \to \mathbb{R}$ such that $z(t) \in C^2([t_x,\infty),\mathbb{R})$ and $r(t)(z''(t))^{\alpha} \in C^1([t_x,\infty),\mathbb{R})$, and which satisfies equation (1.1) on $[t_x,\infty)$. Without further mention, we will assume throughout that every solution $x(t)$ of (1.1) under consideration here is continuable to the right and nontrivial, i.e., $x(t)$ is defined on some ray $[t_x,\infty)$, for some $t_x \ge t_0$, and $\sup\{|x(t)| : t \ge T\} > 0$ for every $T \ge t_x$. Moreover, we tacitly assume that (1.1) possesses such solutions. Such a solution is said to be *oscillatory* if it has arbitrarily large zeros on $[t_x,\infty)$; otherwise it is called *nonoscillatory*.

The oscillatory behavior of solutions of various classes of functional differential equations and functional dynamic equations on time scales is an active and important area of research, and we refer the reader to the papers [1, 2, 3, 4, 8, 9, 10, 11, 14, 16, 17, 20] and the references therein as examples of recent results on this topic. However, oscillation results for third order neutral differential equations and/or third order neutral dynamic equations on time scales with distributed deviating arguments are relatively scarce in the literature; some results can be found, for example, in [5, 6, 7, 15, 18, 19, 21, 22] and the references contained therein.

The asymptotic and oscillatory behavior of solutions of neutral differential equations is of both theoretical and practical interest. One reason for this is that they arise, for example, in applications to electric networks containing lossless transmission lines. Such networks appear in high speed computers where lossless transmission lines are used to interconnect switching circuits. They also occur in problems dealing with vibrating masses attached to an elastic bar and in the solution of variational problems with time delays. Interested readers can refer to the book by Hale [12] for some applications in science and technology.

Types of third-order neutral differential equations and/or third order neutral dynamic equations on time scales with distributed deviating arguments that have been dealt with in the relevant literature have generally the forms

$$\Big(r_2(t)\Big(\big(r_1(t)(x(t)+p(t)x(\tau(t))')^{\alpha_1}\big)'\Big)^{\alpha_2}\Big)' + \int_a^b q(t,\xi)f(x(g(t,\xi)))d\sigma(\xi) = 0, \tag{1.6}$$

$$\Big(r(t)\Big(\big(x(t)+p(t)x(\tau(t))\big)^{\Delta\Delta}\Big)^{\alpha}\Big)^{\Delta} + \int_c^d f(t,x(\phi(t,\xi)))\Delta\xi = 0, \tag{1.7}$$

$$\big\{r(t)\big[a(t)(x(t)+p(t)x(\tau(t)))^{\Delta}\big]^{\Delta}\big\}^{\Delta} + \int_a^b F(t,\xi,x(\phi(t,\xi)))\Delta\xi = 0, \tag{1.8}$$

$$\Big[r(t)\Big(\Big[x(t)+\int_a^b p(t,\mu)x(\tau(t,\mu))d\mu\Big]''\Big)^{\alpha}\Big]' + \int_c^d q(t,\xi)f(x(\phi(t,\xi)))d\xi = 0, \tag{1.9}$$

$$\Big[r(t)\Big(\Big[x(t)+\int_a^b p(t,\mu)x(\tau(t,\mu))\Delta\mu\Big]^{\Delta\Delta}\Big)^{\alpha}\Big]^{\Delta} + \int_c^d q(t,\xi)x^{\lambda}(\phi(t,\xi))\Delta\xi = 0, \tag{1.10}$$

and the results obtained are for the cases where $0 \le p(t) \le p_0 < 1$ or $0 \le \int_a^b p(t,\mu)d\mu \le p_0 < 1$, and $0 \le \int_a^b p(t,\mu)\Delta\mu \le p_0 < 1$, see, for example, [5, 6, 7, 15, 18, 19, 21, 22].

However, to the best of our knowledge, there does not appear to be any results for third order neutral differential equations and/or third order neutral dynamic equations on time scales with distributed deviating arguments in the case $p(t) \geq 1$. The main objective of this paper is to establish some new criteria for the oscillation and asymptotic behavior of solutions of (1.1) in the case $p(t) \geq 1$. It should be noted that the results in this paper are new even for the $\alpha = 1$, and for the constant delays such as $\tau(t) = t - c$ with $c > 0$ and $\phi(t, \xi) = t \pm \xi$. Furthermore, the results in this paper can easily be extended to more general equations (1.6)-(1.8) as well as the more general third order neutral differential equations and/or third order neutral dynamic equations with distributed deviating arguments of the type (1.1). It is therefore hoped that the present paper will contribute significantly to the study of oscillatory and asymptotic behavior of solutions of third order neutral differential equations and neutral dynamic equations on time scales with distributed deviating arguments.

2. Main results

We begin with the following lemmas that are essential in the proofs of our theorems. For simplicity in what follows, it will be convenient to set:

$$\theta_1(t) := \phi(t, a), \quad \theta_2(t) := \phi(t, b), \quad \eta'_+(t) := \max\{0, \eta'(t)\},$$

$$R_1(t, t_1) := \int_{t_1}^{t} \frac{ds}{r^{1/\alpha}(s)} \text{ for } t \geq t_1, \quad R_2(t, t_2) := \int_{t_2}^{t} R_1(s, t_1) ds \text{ for } t \geq t_2 > t_1.$$

Throughout this paper, we assume that

$$p^*(t) := \frac{1}{p(\tau^{-1}(t))}\big(1 - \frac{1}{p(\tau^{-1}(\tau^{-1}(t)))}\big) > 0 \tag{2.1}$$

and

$$p_*(t) := \frac{1}{p(\tau^{-1}(t))}\Big(1 - \frac{1}{p(\tau^{-1}(\tau^{-1}(t)))} \frac{R_2(\tau^{-1}(\tau^{-1}(t)), t_2)}{R_2(\tau^{-1}(t), t_2)}\Big) > 0, \tag{2.2}$$

for all sufficiently large t, where τ^{-1} is the inverse of τ, and we let

$$q_1(t) := \int_a^b q(t, \xi)(p^*(\phi(t, \xi)))^\alpha d\xi, \quad q_2(t) := \int_a^b q(t, \xi)(p_*(\phi(t, \xi)))^\alpha d\xi.$$

Lemma 2.1 ([13]). *If X and Y are nonnegative and $\lambda > 1$, then*

$$\lambda X Y^{\lambda - 1} - X^\lambda \leq (\lambda - 1) Y^\lambda,$$

where equality holds if and only if $X = Y$.

Lemma 2.2. *Assume that conditions* (i)-(v) *hold and let $x(t)$ be an eventually positive solution of* (1.1). *Then for sufficiently large t, either*

(I) *$z(t) > 0$, $z'(t) > 0$, $z''(t) > 0$, and $(r(t)(z''(t))^\alpha)' \leq 0$, or*
(II) *$z(t) > 0$, $z'(t) < 0$, $z''(t) > 0$, and $(r(t)(z''(t))^\alpha)' \leq 0$.*

The proof of the above lemma is standard and so it is omitted.

Lemma 2.3. *Suppose that conditions* (i)-(v) *and* (2.1) *hold, and let $x(t)$ be an eventually positive solution of* (1.1) *with $z(t)$ satisfying Case (II) of Lemma 2.2. If*

$$\int_{t_0}^{\infty} \int_{v}^{\infty} \frac{1}{r^{1/\alpha}(u)} \Big(\int_u^\infty q_1(s) ds\Big)^{1/\alpha} du\, dv = \infty, \tag{2.3}$$

then $\lim_{t \to \infty} x(t) = 0$.

Proof. Let $x(t)$ be an eventually positive solution of (1.1). Then, there exists $t_1 \in [t_0,\infty)$ such that $x(t) > 0$, $x(\tau(t)) > 0$, and $x(\phi(t,\xi)) > 0$ for $t \geq t_1$ and $\xi \in [a,b]$. From (1.4), we have (see also [1, (8.6)]),

$$
\begin{aligned}
x(t) &= \frac{1}{p(\tau^{-1}(t))}(z(\tau^{-1}(t)) - x(\tau^{-1}(t))) \\
&= \frac{z(\tau^{-1}(t))}{p(\tau^{-1}(t))} - \frac{1}{p(\tau^{-1}(t))p(\tau^{-1}(\tau^{-1}(t)))} \\
&\quad \times \big(z(\tau^{-1}(\tau^{-1}(t))) - x(\tau^{-1}(\tau^{-1}(t)))\big) \\
&\geq \frac{z(\tau^{-1}(t))}{p(\tau^{-1}(t))} - \frac{1}{p(\tau^{-1}(t))p(\tau^{-1}(\tau^{-1}(t)))} z(\tau^{-1}(\tau^{-1}(t))).
\end{aligned}
\tag{2.4}
$$

From $\tau(t) < t$, (iv) and the fact that $z(t)$ is decreasing, we have

$$z(\tau^{-1}(t)) \geq z(\tau^{-1}(\tau^{-1}(t))),$$

using this in (2.4), we obtain

$$x(t) \geq p^*(t)z(\tau^{-1}(t)),$$

so

$$x(\phi(t,\xi)) \geq p^*(\phi(t,\xi))z(\tau^{-1}(\phi(t,\xi))) \quad \text{for } t \geq t_2. \tag{2.5}$$

In view of (2.5), equation (1.1) or (1.5) can be written as

$$(r(t)(z''(t))^\alpha)' + \int_a^b q(t,\xi)(p^*(\phi(t,\xi)))^\alpha z^\alpha(\tau^{-1}(\phi(t,\xi)))d\xi \leq 0 \tag{2.6}$$

for $t \geq t_2$. From (iv)-(v) and the fact that $z(t)$ is decreasing, (2.6) yields

$$(r(t)(z''(t))^\alpha)' + z^\alpha(\tau^{-1}(\theta_1(t)))q_1(t) \leq 0 \quad \text{for } t \geq t_2. \tag{2.7}$$

Since $z(t) > 0$ and $z'(t) < 0$, there exists a constant κ such that

$$\lim_{t\to\infty} z(t) = \kappa < \infty,$$

where $\kappa \geq 0$. If $\kappa > 0$, then there exists $t_3 \geq t_2$ such that $\tau^{-1}(\theta_1(t)) > t_2$ and

$$z(t) \geq \kappa \quad \text{for } t \geq t_3. \tag{2.8}$$

Integrating (2.7) from t to ∞ two times gives

$$-z'(t) \geq \kappa \int_t^\infty \frac{1}{r^{1/\alpha}(u)}\Big(\int_u^\infty q_1(s)ds\Big)^{1/\alpha} du.$$

An integration of the last inequality from t_3 to t yields

$$z(t_3) \geq \kappa \int_{t_3}^t \int_v^\infty \frac{1}{r^{1/\alpha}(u)}\Big(\int_u^\infty q_1(s)ds\Big)^{1/\alpha} du\, dv,$$

which contradicts (2.3), and so we have $\kappa = 0$. Therefore, $\lim_{t\to\infty} z(t) = 0$. Since $0 < x(t) \leq z(t)$ on $[t_1,\infty)$, we obtain $\lim_{t\to\infty} x(t) = 0$. This completes the proof of Lemma 2.3. □

Lemma 2.4. *Assume that conditions* (i)-(v) *and* (2.2) *hold, and that* $x(t)$ *is an eventually positive solution of* (1.1) *with* $z(t)$ *satisfying Case (I) of Lemma 2.2. Then,* $z(t)$ *satisfies the inequality*

$$(r(t)(z''(t))^\alpha)' + z^\alpha(\tau^{-1}(\theta_2(t)))q_2(t) \leq 0, \tag{2.9}$$

for large t.

Proof. Let $x(t)$ be an eventually positive solution of (1.1) such that $x(t) > 0$, $x(\tau(t)) > 0$, and $x(\phi(t,\xi)) > 0$, $z(t)$ satisfies Case (I), and (2.2) holds for $t \geq t_1$ for some $t_1 \geq t_0$ and $\xi \in [a,b]$. Proceeding as in the proof of Lemma 2.3, we again arrive at (2.4). Since $r(t)(z''(t))^\alpha$ is decreasing, we see that

$$\begin{aligned} z'(t) &= z'(t_1) + \int_{t_1}^t \frac{(r(s)(z''(s))^\alpha)^{1/\alpha}}{r^{1/\alpha}(s)} ds \\ &\geq (r(t)(z''(t))^\alpha)^{1/\alpha} R_1(t,t_1) \quad \text{for } t \geq t_1. \end{aligned} \tag{2.10}$$

From (2.10), we have for all $t \geq t_2 := t_1 + 1$ that

$$\Big(\frac{z'(t)}{R_1(t,t_1)}\Big)' = \frac{r^{-1/\alpha}(t)[r^{1/\alpha}(t)z''(t)R_1(t,t_1) - z'(t)]}{(R_1(t,t_1))^2} \leq 0,$$

so $z'(t)/R_1(t,t_1)$ is decreasing for $t \geq t_2$. Next, using that $z'(t)/R_1(t,t_1)$ is decreasing for $t \geq t_2$, we obtain

$$\begin{aligned} z(t) &= z(t_2) + \int_{t_2}^t \frac{z'(s)}{R_1(s,t_1)} R_1(s,t_1) ds \\ &\geq \frac{z'(t)}{R_1(t,t_1)} \int_{t_2}^t R_1(s,t_1) ds \\ &= \frac{R_2(t,t_2)}{R_1(t,t_1)} z'(t) \quad \text{for } t \geq t_2. \end{aligned} \tag{2.11}$$

From (2.11), for all $t \geq t_3 := t_2 + 1$ we have that

$$\Big(\frac{z(t)}{R_2(t,t_2)}\Big)' = \frac{z'(t)R_2(t,t_2) - z(t)R_1(t,t_1)}{(R_2(t,t_2))^2} \leq 0,$$

so $z(t)/R_2(t,t_2)$ is decreasing for $t \geq t_3$. Next, in view of the fact that $z(t)/R_2(t,t_2)$ is decreasing for $t \geq t_3$ and $\tau(t) < t$ or $\tau^{-1}(t) \leq \tau^{-1}(\tau^{-1}(t))$, we obtain

$$\frac{R_2(\tau^{-1}(\tau^{-1}(t)), t_2) z(\tau^{-1}(t))}{R_2(\tau^{-1}(t), t_2)} \geq z(\tau^{-1}(\tau^{-1}(t))). \tag{2.12}$$

Using (2.12) in (2.4), we obtain

$$x(t) \geq p_*(t) z(\tau^{-1}(t)),$$

so

$$x(\phi(t,\xi)) \geq p_*(\phi(t,\xi)) z(\tau^{-1}(\phi(t,\xi))) \quad \text{for } t \geq t_3. \tag{2.13}$$

Substituting (2.13) into (1.1), we arrive at (2.9) and completes the proof. □

We now give oscillation results when (1.2) holds.

Theorem 2.5. *Assume that conditions* (i)–(v), (1.2)*, and* (2.1)-(2.3) *hold. If there exists a positive function* $\eta \in C^1([t_0,\infty),\mathbb{R})$ *such that*

$$\limsup_{t\to\infty} \int_T^t \Big[\eta(s) q_2(s) \Big(\frac{R_2(\tau^{-1}(\theta_2(s)), t_2)}{R_1(s,t_1)}\Big)^\alpha - \frac{\eta'_+(s)}{(R_1(s,t_1))^\alpha}\Big] ds = \infty, \tag{2.14}$$

for all $t_1, t_2, T \in [t_0,\infty)$*, where* $T > t_2 > t_1$*, then any solution of* (1.1) *is either oscillatory or tends to zero as* $t \to \infty$.

Proof. Let x be a nonoscillatory solution of (1.1). Without loss of generality, we may assume that there exists $t_1 \in [t_0, \infty)$ such that $x(t) > 0$, $x(\tau(t)) > 0$, and $x(\phi(t,\xi)) > 0$, (2.1)-(2.2) hold, and $z(t)$ satisfies either Case (I) or Case (II) for $t \geq t_1$ and $\xi \in [a, b]$. Assume that Case (I) holds and define

$$w(t) = \eta(t)\frac{r(t)(z''(t))^\alpha}{(z'(t))^\alpha} \quad \text{for } t \geq t_1. \tag{2.15}$$

Then $w(t) > 0$, and from (2.9), we see that

$$\begin{aligned} w'(t) &= \eta'(t)\frac{r(t)(z''(t))^\alpha}{(z'(t))^\alpha} + \eta(t)[\frac{(r(t)(z''(t))^\alpha)'}{(z'(t))^\alpha} - \frac{r(t)(z''(t))^\alpha((z'(t))^\alpha)'}{(z'(t))^{2\alpha}}] \\ &\leq \eta'_+(t)\frac{r(t)(z''(t))^\alpha}{(z'(t))^\alpha} - \eta(t)q_2(t)\frac{z^\alpha(\tau^{-1}(\theta_2(t)))}{(z'(t))^\alpha} - \alpha\eta(t)r(t)\frac{(z''(t))^{\alpha+1}}{(z'(t))^{\alpha+1}}. \end{aligned} \tag{2.16}$$

for $t \geq t_3$ with $t_3 \in (t_2, \infty)$ and $t_2 \in (t_1, \infty)$.

From (2.10), $z'(t) > 0$ and $z''(t) > 0$, (2.16) yields

$$w'(t) \leq \frac{\eta'_+(t)}{(R_1(t,t_1))^\alpha} - \eta(t)q_2(t)\frac{z^\alpha(\tau^{-1}(\theta_2(t)))}{z^\alpha(t)}\frac{z^\alpha(t)}{(z'(t))^\alpha} \quad \text{for } t \geq t_3. \tag{2.17}$$

From (iv) and (1.2), we have

$$\tau^{-1}(\theta_2(t)) \leq t,$$

and thus, in view of the fact that $z(t)/R_2(t, t_2)$ is decreasing for $t \geq t_3$, we obtain

$$\frac{z(\tau^{-1}(\theta_2(t)))}{z(t)} \geq \frac{R_2(\tau^{-1}(\theta_2(t)), t_2)}{R_2(t, t_2)} \quad \text{for } t \geq t_3. \tag{2.18}$$

Using (2.18) and (2.11) in (2.17), we obtain

$$w'(t) \leq \frac{\eta'_+(t)}{(R_1(t,t_1))^\alpha} - \eta(t)q_2(t)(\frac{R_2(\tau^{-1}(\theta_2(t)), t_2)}{R_1(t,t_1)})^\alpha \quad \text{for } t \geq t_3. \tag{2.19}$$

An integration of (2.19) from t_3 to t yields

$$\int_{t_3}^t \Big[\eta(s)q_2(s)\Big(\frac{R_2(\tau^{-1}(\theta_2(s)), t_2)}{R_1(s,t_1)}\Big)^\alpha - \frac{\eta'_+(s)}{(R_1(s,t_1))^\alpha}\Big]ds \leq w(t_3),$$

which contradicts (2.14).

This implies that Case (II) holds, and so from Lemma 2.3, we have $\lim_{t\to\infty} x(t) = 0$. This completes the proof. □

Theorem 2.6. *Assume that conditions* (i)–(v), (1.2)*, and* (2.1)-(2.3) *hold. If there exists a positive function* $\eta \in C^1([t_0, \infty), \mathbb{R})$ *such that,*

$$\limsup_{t\to\infty} \int_T^t \Big[\eta(s)q_2(s)\Big(\frac{R_2(\tau^{-1}(\theta_2(s)), t_2)}{R_1(s,t_1)}\Big)^\alpha - \frac{r(s)(\eta'_+(s))^{\alpha+1}}{(\alpha+1)^{\alpha+1}\eta^\alpha(s)}\Big]ds = \infty, \tag{2.20}$$

for all $t_1, t_2, T \in [t_0, \infty)$*, where* $T > t_2 > t_1$*, then any solution of* (1.1) *is either oscillatory or tends to zero as* $t \to \infty$.

Proof. Let x be a nonoscillatory solution of (1.1). Without loss of generality, we may assume that there exists $t_1 \in [t_0, \infty)$ such that $x(t) > 0$, $x(\tau(t)) > 0$, and $x(\phi(t,\xi)) > 0$, (2.1)-(2.2) hold, and $z(t)$ satisfies either Case (I) or Case (II) for $t \geq t_1$ and $\xi \in [a, b]$. Assume that Case (I) holds. Proceeding as in the proof of

Theorem 2.5, we again arrive at (2.16). In view of (2.15), inequality (2.16) takes the form

$$w'(t) \le \frac{\eta'_+(t)}{\eta(t)} w(t) - \eta(t) q_2(t) \frac{z^\alpha(\tau^{-1}(\theta_2(t)))}{z^\alpha(t)} \frac{z^\alpha(t)}{(z'(t))^\alpha} - \frac{\alpha w^{(\alpha+1)/\alpha}(t)}{(\eta(t) r(t))^{1/\alpha}}. \tag{2.21}$$

Using (2.11) and (2.18) in (2.21), for $t \ge t_3$, we obtain

$$w'(t) \le \frac{\eta'_+(t)}{\eta(t)} w(t) - \eta(t) q_2(t) \Big(\frac{R_2(\tau^{-1}(\theta_2(t)), t_2)}{R_1(t, t_1)}\Big)^\alpha - \frac{\alpha w^{(\alpha+1)/\alpha}(t)}{(\eta(t) r(t))^{1/\alpha}}. \tag{2.22}$$

Applying Lemma 2.1 with

$$\begin{gathered} X = \frac{\alpha^{1/\lambda}}{[(\eta(t) r(t))^{1/\alpha}]^{1/\lambda}} w(t), \quad \lambda = \frac{\alpha+1}{\alpha}, \\ Y = \Big[\frac{\alpha}{\alpha+1} \frac{[(\eta(t) r(t))^{1/\alpha}]^{1/\lambda}}{\alpha^{1/\lambda}} \frac{\eta'_+(t)}{\eta(t)}\Big]^\alpha, \end{gathered}$$

we see that

$$\frac{\eta'_+(t)}{\eta(t)} w(t) - \frac{\alpha}{(\eta(t) r(t))^{1/\alpha}} w^{(\alpha+1)/\alpha}(t) \le \frac{1}{(\alpha+1)^{\alpha+1}} \frac{r(t)(\eta'_+(t))^{\alpha+1}}{\eta^\alpha(t)}.$$

Substituting this into (2.22), we obtain

$$w'(t) \le -\eta(t) q_2(t) \Big(\frac{R_2(\tau^{-1}(\theta_2(t)), t_2)}{R_1(t, t_1)}\Big)^\alpha + \frac{1}{(\alpha+1)^{\alpha+1}} \frac{r(t)(\eta'_+(t))^{\alpha+1}}{\eta^\alpha(t)}.$$

Integrating the above inequality from t_3 to t gives

$$\int_{t_3}^t \Big[\eta(s) q_2(s) \Big(\frac{R_2(\tau^{-1}(\theta_2(s)), t_2)}{R_1(s, t_1)}\Big)^\alpha - \frac{1}{(\alpha+1)^{\alpha+1}} \frac{r(s)(\eta'_+(s))^{\alpha+1}}{\eta^\alpha(s)}\Big] ds \le w(t_3),$$

which contradicts (2.20). Therefore Case (II) holds, and so $\lim_{t\to\infty} x(t) = 0$ by Lemma 2.3. This completes the proof. □

Theorem 2.7. *Let $\alpha \ge 1$. Assume that conditions* (i)–(v), (1.2), *and* (2.1)-(2.3) *hold. If there exists a positive function $\eta \in C^1([t_0, \infty), \mathbb{R})$ such that*

$$\begin{aligned} \limsup_{t\to\infty} \int_T^t & \Big[\eta(s) q_2(s) \Big(\frac{R_2(\tau^{-1}(\theta_2(s)), t_2)}{R_1(s, t_1)}\Big)^\alpha \\ & - \frac{r^{1/\alpha}(s)}{4\alpha[R_1(s, t_1)]^{\alpha-1}} \frac{(\eta'_+(s))^2}{\eta(s)}\Big] ds = \infty, \end{aligned} \tag{2.23}$$

for all $t_1, t_2, T \in [t_0, \infty)$, where $T > t_2 > t_1$, then any solution of (1.1) *is either oscillatory or tends to zero as $t \to \infty$.*

Proof. Let x be a nonoscillatory solution of (1.1). Without loss of generality, we may assume that there exists $t_1 \in [t_{0,}\infty)$ such that $x(t) > 0$, $x(\tau(t)) > 0$, and $x(\phi(t, \xi)) > 0$, (2.1)-(2.2) hold, and $z(t)$ satisfies either Case (I) or Case (II) for $t \ge t_1$ and $\xi \in [a, b]$. Assume Case (I) holds. Proceeding as in the proof of Theorem 2.6, we again arrive at (2.22) which can be rewritten as

$$w'(t) \le \frac{\eta'_+(t)}{\eta(t)} w(t) - \eta(t) q_2(t) \Big(\frac{R_2(\tau^{-1}(\theta_2(t)), t_2)}{R_1(t, t_1)}\Big)^\alpha - \frac{\alpha w^2(t) w^{\frac{1}{\alpha}-1}(t)}{(\eta(t) r(t))^{1/\alpha}}. \tag{2.24}$$

From (2.10) and (2.15) , we see that

$$
\begin{aligned}
w^{\frac{1}{\alpha}-1}(t) &= (\eta(t)r(t))^{\frac{1}{\alpha}-1}\Big(\frac{(z''(t))^{\alpha}}{(z'(t))^{\alpha}}\Big)^{\frac{1}{\alpha}-1} \\
&= (\eta(t)r(t))^{\frac{1}{\alpha}-1}\Big(\frac{z'(t)}{z''(t)}\Big)^{\alpha-1} \\
&\geq (\eta(t)r(t))^{\frac{1}{\alpha}-1}\big[r^{1/\alpha}(t)R_1(t,t_1)\big]^{\alpha-1} \\
&= \eta^{\frac{1}{\alpha}-1}(t)[R_1(t,t_1)]^{\alpha-1}.
\end{aligned} \tag{2.25}
$$

Using (2.25) in (2.24), for $t \geq t_3$, we obtain

$$
w'(t) \leq \frac{\eta'_+(t)}{\eta(t)}w(t) - \eta(t)q_2(t)\Big(\frac{R_2(\tau^{-1}(\theta_2(t)),t_2)}{R_1(t,t_1)}\Big)^{\alpha} - \frac{\alpha[R_1(t,t_1)]^{\alpha-1}}{\eta(t)r^{1/\alpha}(t)}w^2(t). \tag{2.26}
$$

Completing the square with respect to w, from (2.26) it follows that

$$
w'(t) \leq -\eta(t)q_2(t)\Big(\frac{R_2(\tau^{-1}(\theta_2(t)),t_2)}{R_1(t,t_1)}\Big)^{\alpha} + \frac{r^{1/\alpha}(t)}{4\alpha[R_1(t,t_1)]^{\alpha-1}}\frac{(\eta'_+(t))^2}{\eta(t)}.
$$

Integrating this inequality from t_3 to t gives

$$
\int_{t_3}^{t}\Big[\eta(s)q_2(s)\Big(\frac{R_2(\tau^{-1}(\theta_2(s)),t_2)}{R_1(s,t_1)}\Big)^{\alpha} - \frac{r^{1/\alpha}(s)}{4\alpha[R_1(s,t_1)]^{\alpha-1}}\frac{(\eta'_+(s))^2}{\eta(s)}\Big]ds \leq w(t_3),
$$

which contradicts (2.23).

If Case (II) holds, then again from Lemma 2.3, we have $\lim_{t\to\infty} x(t) = 0$. The proof is complete. $\square$

Next, we give oscillation results in the case when (1.3) holds.

Theorem 2.8. *Assume that conditions* (i)–(v), (1.3)*, and* (2.1)-(2.3) *hold. If there exists a positive function $\eta \in C^1([t_0,\infty),\mathbb{R})$ such that*

$$
\limsup_{t\to\infty}\int_{T}^{t}\Big[\eta(s)q_2(s)\Big(\frac{R_2(s,t_2)}{R_1(s,t_1)}\Big)^{\alpha} - \frac{\eta'_+(s)}{(R_1(s,t_1))^{\alpha}}\Big]ds = \infty, \tag{2.27}
$$

for all $t_1, t_2, T \in [t_0,\infty)$, where $T > t_2 > t_1$, then any solution of (1.1) *is either oscillatory or tends to zero as $t \to \infty$.*

Proof. Let x be a nonoscillatory solution of (1.1). Without loss of generality, we may assume that there exists $t_1 \in [t_0,\infty)$ such that $x(t) > 0$, $x(\tau(t)) > 0$, and $x(\phi(t,\xi)) > 0$, (2.1)-(2.2) hold, and $z(t)$ satisfies either Case (I) or Case (II) for $t \geq t_1$ and $\xi \in [a,b]$. Assume that Case (I) holds. Proceeding as in the proof of Theorem 2.5, we again arrive at (2.17). In view of (iv) and (1.3), we have

$$
t \leq \tau^{-1}(\theta_2(t)),
$$

thus, in view of the fact that $z(t)$ is increasing, we obtain

$$
\frac{z(\tau^{-1}(\theta_2(t)))}{z(t)} \geq 1. \tag{2.28}
$$

Using (2.28) in (2.17), we obtain that

$$
w'(t) \leq \frac{\eta'_+(t)}{(R_1(t,t_1))^{\alpha}} - \eta(t)q_2(t)\frac{z^{\alpha}(t)}{(z'(t))^{\alpha}} \quad \text{for } t \geq t_3. \tag{2.29}
$$

In view of (2.11), (2.29) takes the form

$$w'(t) \le \frac{\eta'_+(t)}{(R_1(t,t_1))^\alpha} - \eta(t)q_2(t)\Big(\frac{R_2(t,t_2)}{R_1(t,t_1)}\Big)^\alpha \quad \text{for } t \ge t_3. \tag{2.30}$$

The remainder of the proof is similar to that of Theorem 2.5 and so we omit it. □

Theorem 2.9. *Assume that conditions* (i)–(v), (1.3)*, and* (2.1)-(2.3) *hold. If there exists a positive function* $\eta \in C^1([t_0,\infty),\mathbb{R})$ *such that*

$$\limsup_{t\to\infty} \int_T^t \Big[\eta(s)q_2(s)\Big(\frac{R_2(s,t_2)}{R_1(s,t_1)}\Big)^\alpha - \frac{1}{(\alpha+1)^{\alpha+1}}\frac{r(s)(\eta'_+(s))^{\alpha+1}}{\eta^\alpha(s)}\Big]ds = \infty, \tag{2.31}$$

for all $t_1, t_2, T \in [t_0,\infty)$*, where* $T > t_2 > t_1$*, then every solution of* (1.1) *is either oscillatory or tends to zero as* $t \to \infty$*.*

The above theorem follows from (2.28) and Theorem 2.6; we omit its proof.

Theorem 2.10. *Let* $\alpha \ge 1$*. Assume that conditions* (i)–(v), (1.3)*, and* (2.1)-(2.3) *hold. If there exists a positive function* $\eta \in C^1([t_0,\infty),\mathbb{R})$ *such that*

$$\limsup_{t\to\infty} \int_T^t \Big[\eta(s)q_2(s)\Big(\frac{R_2(s,t_2)}{R_1(s,t_1)}\Big)^\alpha - \frac{r^{1/\alpha}(s)}{4\alpha[R_1(s,t_1)]^{\alpha-1}}\frac{(\eta'_+(s))^2}{\eta(s)}\Big]ds = \infty, \tag{2.32}$$

for all $t_1, t_2, T \in [t_0,\infty)$*, where* $T > t_2 > t_1$*, then every solution of* (1.1) *is either oscillatory or tends to zero as* $t \to \infty$*.*

The above theorem follows from (2.28) and Theorem 2.7; we omit its proof.

Example 2.11. Consider the neutral differential equation with distributed deviating arguments

$$\Big(\Big(\big(x(t) + 9x(\frac{t}{2})\big)''\Big)^3\Big)' + \int_1^2 (t^2+\xi)x^3(\frac{t}{2}-\xi)d\xi = 0, \quad t \ge 1. \tag{2.33}$$

Here we have $\alpha = 3$, $\tau(t) = t/2$, $\phi(t,\xi) = t/2 - \xi$, $q(t,\xi) = t^2 + \xi$, $r(t) = 1$, and $p(t) = 9$. Then, we obtain

$$\begin{gathered} R_1(t,t_1) = R_1(t,1) = t-1, \\ R_2(t,t_2) = R_2(t,2) = (t^2-2t)/2, \\ R_2(\tau^{-1}(t),t_2) = R_2(2t,2) = 2t^2-2t, \\ R_2(\tau^{-1}(\tau^{-1}(t)),t_2) = R_2(4t,2) = 8t^2-4t, \\ R_2(\tau^{-1}(\theta_2(t)),t_2) = R_2(t-4,2) = (t^2-10t+24)/2, \end{gathered}$$

and

$$p^*(t) = 8/81 > 0, \tag{2.34}$$

$$p_*(t) = \frac{1}{9}(1 - \frac{1}{9}\frac{8t^2-4t}{2t^2-2t}) = \frac{1}{81}(5 - \frac{2}{t-1}) \ge \frac{1}{27} > 0, \quad \text{for } t \ge t_2 = 2. \tag{2.35}$$

In view of (2.34) and (2.35) , we see that

$$q_1(t) = \int_1^2 (t^2+\xi)\big(\frac{8}{81}\big)^3 d\xi = \big(\frac{8}{81}\big)^3(t^2+3/2), \tag{2.36}$$

$$q_2(t) \ge \int_1^2 (t^2+\xi)\big(\frac{1}{27}\big)^3 d\xi \ge \big(\frac{1}{27}\big)^3(t^2+3/2) \quad \text{for } t \ge t_2 = 2, \tag{2.37}$$

respectively. With (2.36), condition (2.3) becomes

$$\begin{aligned}
&\int_{t_0}^{\infty}\int_{v}^{\infty}\frac{1}{r^{1/\alpha}(u)}\Big(\int_{u}^{\infty}q_1(s)ds\Big)^{1/\alpha}du\,dv\\
&=\int_{1}^{\infty}\int_{v}^{\infty}\Big(\int_{u}^{\infty}\Big(\frac{8}{81}\Big)^3(s^2+3/2)ds\Big)^{1/3}du\,dv=\infty
\end{aligned}$$

because $\int_u^\infty (s^2+3/2)ds=\infty$ for $u\ge 1$, and so condition (2.3) holds. With $\eta(t)=t$ and (2.37), we see that

$$\begin{aligned}
&\int_{T}^{t}\Big[\eta(s)q_2(s)(\frac{R_2(\tau^{-1}(\theta_2(s)),t_2)}{R_1(s,t_1)})^{\alpha}-\frac{\eta'_+(s)}{(R_1(s,t_1))^{\alpha}}\Big]ds\\
&\ge\int_{3}^{t}\Big[s(\frac{1}{27})^3(s^2+3/2)\Big(\frac{s^2-10s+24}{2(s-1)}\Big)^3-\frac{1}{(s-1)^3}\Big]ds=\infty,
\end{aligned}$$

because $\int_3^t\frac{1}{(s-1)^3}ds<\infty$ and

$$\int_{3}^{t}\Big[s\big(\frac{1}{27}\big)^3(s^2+3/2)\big(\frac{s^2-10s+24}{2(s-1)}\big)^3\Big]ds=\infty,$$

so condition (2.14) holds. Thus, all conditions of Theorem 2.5 are satisfied. Therefore, by Theorem 2.5, any solution of (2.33) is either oscillatory or converges to zero.

Example 2.12. Consider the neutral differential equation with distributed deviating arguments

$$\Big(\Big(\big(x(t)+\frac{7t+8}{t+1}x(t-2)\big)''\Big)^{1/5}\Big)'+\int_1^2(t+\xi)x^{1/5}(t-2+\frac{1}{\xi})d\xi=0,\quad t\ge 2. \tag{2.38}$$

Here we have $\alpha=1/5$, $\tau(t)=t-2$, $\phi(t,\xi)=t-2+1/\xi$, $q(t,\xi)=t+\xi$, $r(t)=1$, and $p(t)=(7t+8)/(t+1)$. Then, we obtain

$$\begin{gathered}
7\le p(t)<8,\\
R_1(t,t_1)=R_1(t,2)=t-2,\\
R_2(t,t_2)=R_2(t,3)=(t^2-4t+3)/2,\\
R_2(\tau^{-1}(t),t_2)=R_2(t+2,3)=(t^2-1)/2,\\
R_2(\tau^{-1}(\tau^{-1}(t)),t_2)=R_2(t+4,3)=(t^2+4t+3)/2,
\end{gathered}$$

and

$$p^*(t)\ge 3/28>0, \tag{2.39}$$

$$p_*(t)\ge\frac{1}{8}\big(1-\frac{1}{7}\frac{t^2+4t+3}{t^2-1}\big)=\frac{1}{28}\big(3-\frac{2}{t-1}\big)\ge\frac{1}{14}>0,\quad t\ge t_2=3. \tag{2.40}$$

In view of (2.39) and (2.40), we see that

$$q_1(t)\ge\int_1^2(t+\xi)\big(\frac{3}{28}\big)^{1/5}d\xi=(\frac{3}{28})^{1/5}(t+3/2), \tag{2.41}$$

$$q_2(t)\ge\int_1^2(t+\xi)\big(\frac{1}{14}\big)^{1/5}d\xi\ge\big(\frac{1}{14}\big)^{1/5}(t+3/2)\quad\text{for } t\ge t_2=3, \tag{2.42}$$

respectively. By (2.41), condition (2.3) becomes

$$\begin{aligned}&\int_{t_0}^{\infty}\int_{v}^{\infty}\frac{1}{r^{1/\alpha}(u)}\Big(\int_{u}^{\infty}q_1(s)ds\Big)^{1/\alpha}du\,dv\\&\geq\int_{2}^{\infty}\int_{v}^{\infty}\Big(\int_{u}^{\infty}(\frac{3}{28})^{1/5}(s+3/2)ds\Big)^{5}du\,dv=\infty\end{aligned}$$

because $\int_u^\infty(s+3/2)ds=\infty$ for $u\geq 2$; so condition (2.3) holds.

With $\eta(t)=c>0$, where c is a constant, and (2.42), we see that

$$\begin{aligned}&\int_{T}^{t}\Big[\eta(s)q_2(s)\Big(\frac{R_2(s,t_2)}{R_1(s,t_1)}\Big)^{\alpha}-\frac{1}{(\alpha+1)^{\alpha+1}}\frac{r(s)(\eta'_+(s))^{\alpha+1}}{\eta^{\alpha}(s)}\Big]ds\\&\geq\int_{4}^{t}\Big[c(\frac{1}{14})^{1/5}(s+3/2)\Big(\frac{s^2-4s+3}{2(s-2)}\Big)^{1/5}\Big]ds\\&>\int_{4}^{t}\Big[c(\frac{1}{14})^{1/5}(s-2)\Big(\frac{s^2-4s+3}{2(s-2)}\Big)^{1/5}\Big]ds=\infty;\end{aligned}$$

so condition (2.31) holds. Now, all conditions of Theorem 2.9 are satisfied. Therefore, by Theorem 2.9, a solution of (2.38) is either oscillatory or converges to zero.

Remark 2.13. The results of this paper can easily be extended to the third order neutral dynamic equations with distributed deviating arguments of the form

$$\Big(r(t)\Big((x(t)+p(t)x(\tau(t)))^{\Delta\Delta}\Big)^{\alpha}\Big)^{\Delta}+\int_a^b q(t,\xi)x^{\alpha}(\phi(t,\xi))\Delta\xi=0,$$

on an arbitrary time scale $\mathbb{T}$ with $\sup\mathbb{T}=\infty$. Where, $\alpha>0$ is the ratio of odd positive integers, $r\in C_{rd}(\mathbb{T},(0,\infty))$ with $\int_{t_0}^{\infty}r^{-1/\alpha}(s)\Delta s=\infty$, $p\in C_{rd}(\mathbb{T},(0,\infty))$ with $p(t)\geq 1$ and $p(t)\not\equiv 1$ eventually, $\tau:\mathbb{T}\to\mathbb{T}$ is strictly increasing and $\lim_{t\to\infty}\tau(t)=\infty$, $q(t,\xi)\in C_{rd}(\mathbb{T}\times[a,b]_{\mathbb{T}},[0,\infty))$, $[a,b]_{\mathbb{T}}=\{\xi\in\mathbb{T}:a\leq\xi\leq b\}$, $\phi(t,\xi)\in C_{rd}(\mathbb{T}\times[a,b]_{\mathbb{T}},\mathbb{T})$ is nonincreasing in ξ, and $\lim_{t\to\infty}\phi(t,\xi)=\infty$, $\xi\in[a,b]$.

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