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ASYMPTOTIC BEHAVIOR OF TRAVELING WAVES FOR A NONLOCAL EPIDEMIC MODEL WITH DELAY

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Abstract. In this article we study the traveling wave solutions of a monostable nonlocal reaction-diffusion system with delay arising from the spread of an epidemic by oral-faecal transmission. From [23], there exists a minimal wave speed $c_* > 0$ such that a traveling wave solution exists if and only if the wave speed is above c_*. In this article, we first establish the exact asymptotic behavior of the traveling waves at $\pm\infty$. Then, we construct some annihilating-front entire solutions which behave like a traveling wave front propagating from the left side (or the right side) on the x-axis or two traveling wave fronts propagating from both sides on the x-axis as $t \to -\infty$ and converge to the unique positive equilibrium as $t \to +\infty$. From the viewpoint of epidemiology, these results provide some new spread ways of the epidemic.

1. Introduction

Capasso and Maddalena [2] proposed an epidemic model to describe the spatial spread of epidemics via the environmental pollution produced by the infective population. The model has been generalized in several directions: to include the latent period of a virus (e.g. [17]), to include the indirect transmission because of the infective population (e.g. [1, 28]) and to include both the two facts (e.g. [23]). For example, one of the above generalizations has the following form (e.g. [23]):

$$\begin{aligned} u_t(x,t) &= du_{xx} - \alpha u(x,t) + \int_{\mathbb{R}} J(x-y)v(y,t)dy, \\ v_t(x,t) &= -\beta v(x,t) + g(u(x,t-\tau)), \end{aligned} \tag{1.1}$$

where $u(x,t)$ and $v(x,t)$, respectively, represent the spatial densities of bacteria and infective population at a point x in the habitat $\Omega \subset \mathbb{R}$ and time t, $d > 0$ is the diffusion coefficient, $\tau > 0$ represents the latent period of a virus, $\alpha > 0$ is the natural death rate of bacteria, and $\beta > 0$ is the natural diminishing rate of infected individuals. The nonlinearity $g(u)$ gives the "force of infection" on human because of the concentration of bacteria. The function $J(x)$ describes the transfer kernel of the infective agents produced by the infective humans.

In epidemiology, one of the central issues is the traveling wave solution because of their significant roles in epidemic spreading. In the past decades, this topic has been widely studied for various evolution equations, see e.g. the survey paper

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[7] and the book [18]. In particular, the traveling wave problem of (1.1) have been widely discussed, see e.g. [27, 28, 17, 29, 23]. For example, in the case where $\tau = 0$ and $J(\cdot) = \delta(\cdot)$, Xu and Zhao [27] proved the existence, uniqueness and stability of bistable traveling wave fronts of (1.1), and Zhao and Wang [29] established the existence of the minimal wave speed of monostable traveling wave fronts. For the case $\tau = 0$, Xu and Zhao [28] considered the spreading speed and monostable traveling wave fronts. When $J(\cdot) = \delta(\cdot)$, Thieme and Zhao [17] obtained the existence of spreading speed and minimal wave speed of (1.1) with distributed delay by applying their theory for integral equations. Recently, Wu and Liu [23] extended the results in [29, 28, 17] to a general nonlocal reaction-diffusion model with distributed delay, which includes (1.1) as a particular case. However, to the best of our knowledge, there has been no results on the asymptotic behavior of the traveling waves of (1.1) at $\pm\infty$ which reflect important information of the traveling waves. This constitutes the first purpose of this paper.

The second purpose of this paper is to study solutions of (1.1) that are defined for all time $t \in \mathbb{R}$ and for all space points. In some publications these solutions are called entire solutions. (It does not mean entire functions in the sense of complex analysis). One of typical examples of entire solutions appear as traveling wave solution. Inspired by the work of Hamel and Nadirashvili [9], there have many significant works devoted to the entire solutions for various diffusion equations. We refer to [4, 8, 9, 11, 14, 12, 20, 16, 21, 15, 22, 24, 25, 26] and the references therein.

To this end, we impose the following assumptions on the functions $J(\cdot)$ and $g(\cdot)$:

(A1) $J \in L^1(\mathbb{R})$, $J(-x) = J(x) \geq 0$ for $x \in \mathbb{R}$, $\int_{-\infty}^{+\infty} J(y)dy = 1$ and there exists a $\lambda_0 > 0$ (λ_0 may be $+\infty$) such that

$$\int_{-\infty}^{+\infty} e^{-\lambda y} J(y)dy < +\infty \text{ for } \lambda \in [0, \lambda_0) \text{ and } \lim_{\lambda\to\lambda_0-0} \int_{-\infty}^{+\infty} e^{-\lambda y} J(y)dy = +\infty.$$

(A2) $\alpha, \beta > 0$, $g \in C^2([0, K_1], [0, +\infty))$, $g(0) = g(K_1) - \alpha\beta K_1 = 0$, $g'(K_1) < \alpha\beta$, $g(u) > \alpha\beta u$ for $u \in (0, K_1)$, and $g(u) \leq g'(0)u$ and $g'(u) \geq 0$ for $u \in [0, K_1]$, where $K_1 > 0$ is a constant.

Throughout this paper, we use the usual notation for the ordering in $\mathbb{R}^2$: Let $u = (u_1, u_2)$ and $v = (v_1, v_2)$. We write $u \leq v$ if $u_i \leq v_i$ for $i = 1, 2$; we write $u < v$ if $u_i leq v_i$, for $i = 1, 2$, with $u \neq v$; we write $u \ll v$ if $u_i < v_i$, for $i = 1, 2$. We also use $\|\cdot\|$ to denote the Euclidean norm in $\mathbb{R}^2$. In order to state our results, we first recall some known results on traveling wave solutions of (1.1). Let $\mathbf{K} := (K_1, K_2)$, where $K_2 = g(K_1)/\beta$. A solution $w(x,t) := \big(u(x,t), v(x,t)\big)$ of system (1.1) is called a traveling wave solution connecting $\mathbf{0} := (0, 0)$ and $\mathbf{K} := (K_1, K_2)$ with speed c, if $\big(u(x,t), v(x,t)\big) = \big(\phi_c(\xi), \psi_c(\xi)\big)$, $\xi := x + ct$ for some function $\Phi_c := (\phi_c, \psi_c) : \mathbb{R} \to [\mathbf{0}, \mathbf{K}] := [0, K_1] \times [0, K_2]$ which satisfies

$$\begin{gathered} c\phi_c'(\xi) = d\phi_c''(\xi) - \alpha\phi_c(\xi) + \int_{-\infty}^{\infty} J(y)\psi_c(\xi - y)dy, \\ c\psi_c'(\xi) = -\beta\psi_c(\xi) + g\big(\phi_c(\xi - c\tau)\big), \end{gathered} \tag{1.2}$$

and

$$\big(\phi_c(-\infty), \psi_c(-\infty)\big) = \mathbf{0}, \quad \big(\phi_c(+\infty), \psi_c(+\infty)\big) = \mathbf{K}. \tag{1.3}$$

Moreover, we say that (ϕ_c, ψ_c) is a *traveling (wave) front* if $(\phi_c(\cdot), \psi_c(\cdot))$ is monotone.

Proposition 1.1 ([23]). *Assume that* (A1), (A2) *hold. There exists a* $c_* > 0$ *such that for each* $c \geq c_*$, *system* (1.1) *has a traveling wave front* $\Phi_c(x+ct) = (\phi_c(x+ct), \psi_c(x+ct))$ *connecting* $\mathbf{0}$ *and* $\mathbf{K}$.

To ensure the strict positivity of $(\phi_c(\cdot), \psi_c(\cdot))$, we need the following additional assumption:

(A3) $J(0) > 0$ and $J(x)$ is continuous at $x = 0$.

Theorem 1.2. *Assume that* (A1)–(A3) *hold. Let* (ϕ, ψ) *be a traveling wave solution of* (1.1) *with speed* $c \geq c_*$. *Then,* $\phi(\xi) \in (0, K_1)$ *and* $\psi(\xi) \in (0, K_2)$ *for all* $\xi \in \mathbb{R}$.

As mentioned above, the asymptotic behavior of the traveling waves at $\pm\infty$ reflect important information of the traveling waves. By appealing to Ikehara's theorem (see [3]), we can obtain the following results on the asymptotic behavior of the traveling waves.

Theorem 1.3. *Assume that* (A1)–(A3) *hold. Let* $(\phi_c(\xi), \psi_c(\xi))$ *be a traveling wave solution of* (1.1) *with speed* $c \geq c_*$. *Then,*

(i) *for* $c > c_*$,

$$\lim_{\xi\to-\infty} \phi_c(\xi)e^{-\lambda_1(c)\xi} = a_0(c), \quad \lim_{\xi\to-\infty} \phi_c'(\xi)e^{-\lambda_1(c)\xi} = a_0(c)\lambda_1(c), \tag{1.4}$$

$$\lim_{\xi\to-\infty} \psi_c(\xi)e^{-\lambda_1(c)\xi} = A_c a_0(c), \quad \lim_{\xi\to-\infty} \psi_c'(\xi)e^{-\lambda_1(c)\xi} = A_c a_0(c)\lambda_1(c), \tag{1.5}$$

and for $c = c_*$,

$$\lim_{\xi\to-\infty} \phi_c(\xi)\xi^{-1}e^{-\lambda_1(c)\xi} = -a_0(c), \quad \lim_{\xi\to-\infty} \phi_c'(\xi)\xi^{-1}e^{-\lambda_1(c)\xi} = -a_0(c)\lambda_1(c), \tag{1.6}$$

$$\lim_{\xi\to-\infty} \psi_c(\xi)\xi^{-1}e^{-\lambda_1(c)\xi} = -A_c a_0(c), \quad \lim_{\xi\to-\infty} \psi_c'(\xi)\xi^{-1}e^{-\lambda_1(c)\xi} = -A_c a_0(c)\lambda_1(c), \tag{1.7}$$

(ii) *for* $c \geq c_*$,

$$\lim_{\xi\to+\infty} \big[K_1 - \phi_c(\xi)\big]e^{-\lambda_3(c)\xi} = a_1(c), \quad \lim_{\xi\to+\infty} \phi_c'(\xi)e^{-\lambda_3(c)\xi} = -a_1(c)\lambda_3(c), \tag{1.8}$$

$$\lim_{\xi\to+\infty} \big[K_2 - \psi_c(\xi)\big]e^{-\lambda_3(c)\xi} = B_c a_1(c), \quad \lim_{\xi\to+\infty} \psi_c'(\xi)e^{-\lambda_3(c)\xi} = -B_c a_1(c)\lambda_3(c), \tag{1.9}$$

where $\lambda_1(c)$ *is the smallest positive root of the characteristic equation of* (1.2) *at* $(0,0)$ *and* $\lambda_3(c)$ *is the unique negative root of the characteristic equation of* (1.2) *at* (K_1, K_2) *(see Proposition 2.3);* $a_0(c)$, $a_1(c)$ *are positive constants,*

$$A_c = \frac{g'(0)e^{-c\lambda_1(c)\tau}}{c\lambda_1(c)+\beta} > 0, \quad B_c = \frac{g'(K_1)e^{-c\lambda_3(c)\tau}}{c\lambda_3(c)+\beta} > 0 \quad \text{for } c \geq c_*.$$

To construct some new types of solutions, we also establish the following result on the spatially independent solution of (1.1) by applying the standard monotone iteration technique and the method of the sub- and super-solution.

Theorem 1.4. *Assume that* (A1)–(A3) *hold. System* (1.1) *has a spatially independent solution* $\Gamma(t) = \big(\Gamma_1(t), \Gamma_2(t)\big)$ *which satisfies*

$$\Gamma(+\infty) = \mathbf{K}, \quad \Gamma(t) \gg \mathbf{0}, \quad \lim_{t\to-\infty} \Gamma(t)e^{-\lambda^* t} = (1, b_*), \quad \Gamma(t) \leq (1, b_*)e^{\lambda^* t}$$

for $t \in \mathbb{R}$, *where* λ^* *is the unique positive root of the equation* $(\lambda+\alpha)(\lambda+\beta) - g'(0)e^{-\lambda\tau} = 0$ *(see Proposition 2.3) and* $b_* = g'(0)e^{-\lambda^*\tau}/(\lambda^*+\beta)$.

Based on the above results on the traveling wave solutions and spatially independent solutions of (1.1), we shall construct some new types of entire solutions which are different from the traveling wave solution and spatially independent solution. More precisely, these solutions behave like a traveling wave front propagating from left side (or right side) of the x-axis or two traveling wave front propagating from both sides of the x-axis as $t \to -\infty$ and converge to the unique positive equilibrium $\mathbf{K}$ as $t \to +\infty$. We call such solutions *annihilating-front* entire solutions. From the viewpoint of epidemiology, the results provide some new spread ways of the epidemic.

The main existence result on entire solutions is stated as follows. For the sake of convenience, we denote

$$\Pi_1(x,t) := \chi_1\Phi_{c_1}(x+c_1t+h_1) + \chi_2(1,A_{c_2})e^{\lambda_1(c_2)(-x+c_2t+h_2)} + \chi_3(1,b_*)e^{\lambda^*(t+h_3)},$$

$$\Pi_2(x,t) := \chi_1(1,A_{c_1})e^{\lambda_1(c_1)(x+c_1t+h_1)} + \chi_2\Phi_{c_2}(-x+c_2t+h_2) + \chi_3(1,b_*)e^{\lambda^*(t+h_3)},$$

$$\Pi_3(x,t) := \chi_1(1,A_{c_1})e^{\lambda_1(c_1)(x+c_1t+h_1)} + \chi_2(1,A_{c_2})e^{\lambda_1(c_2)(-x+c_2t+h_2)} + \chi_3\Gamma(t+h_3).$$

Theorem 1.5. *Let* (A1)–(A3) *hold. Assume $g'(u) \le g'(0)$ for $u \in [0,K_1]$. For any $h_1,h_2,h_3 \in \mathbb{R}$, $c_1,c_2 > c_*$ and $\chi_1,\chi_2,\chi_3 \in \{0,1\}$ with $\chi_1+\chi_2+\chi_3 \ge 2$, there exists an entire solution $W_p(x,t) = (U_p(x,t),V_p(x,t))$ of* (1.1) *such that*

$$\begin{aligned} &\max\big\{\chi_1\Phi_{c_1}(x+c_1t+h_1), \chi_2\Phi_{c_2}(-x+c_2t+h_2), \chi_3\Gamma(t+h_3)\big\} \\ &\le W_p(x,t) \le \min\big\{\mathbf{K}, \Pi_1(x,t), \Pi_2(x,t), \Pi_3(x,t)\big\} \end{aligned} \tag{1.10}$$

for $(x,t) \in \mathbb{R}^2$, where $p := p_{\chi_1,\chi_2,\chi_3} = \big(\chi_1c_1, \chi_2c_2, \chi_1h_1, \chi_2h_2, \chi_3h_3\big)$. Moreover, the following properties hold.

(1) $\lim_{t\to+\infty}\sup_{x\in\mathbb{R}}\big\|W_p(x,t) - \mathbf{K}\big\| = 0$.

(2) *If $\chi_1 = \chi_2 = 1$, then*

$$\lim_{t\to-\infty}\sup_{x\ge 0}\big\|W_p(x,t) - \Phi_{c_1}(x+c_1t+h_1)\big\| = 0, \tag{1.11}$$

$$\lim_{t\to-\infty}\sup_{x\le 0}\big\|W_p(x,t) - \Phi_{c_2}(-x+c_2t+h_2)\big\| = 0. \tag{1.12}$$

(3) *If $\chi_1 = \chi_3 = 1$ and $\chi_2 = 0$, then* (1.11) *holds and*

$$\lim_{t\to-\infty}\sup_{x\le 0}\big\|W_p(x,t) - \Gamma(t+h_3)\big\| = 0. \tag{1.13}$$

(4) *If $\chi_2 = \chi_3 = 1$ and $\chi_1 = 0$, then* (1.12) *holds and*

$$\lim_{t\to-\infty}\sup_{x\ge 0}\big\|W_p(x,t) - \Gamma(t+h_3)\big\| = 0. \tag{1.14}$$

Various other qualitative features of the entire solutions, such as the monotonicity and limit of $W_p(x,t)$ with respect to the variables x and t, and the shift parameters h_i, are also investigated in Section 3.

To prove Theorem 1.5, we use the comparison principle coupled with the method of super- and sub-solutions, which is inspired by [9, 21, 25]. The method includes the following steps. First, we study the Cauchy problems for (1.1) starting at times $-n$, where the combinations of the traveling wave fronts with speeds $c > c_*$ and a spatially independent solution are taken as the initial values. Then, we show that there exists a convergence subsequence of the solution sequence $\{W^n(x,t) = (u^n(x,t),v^n(x,t))\}_{n\in\mathbb{N}}$. Finally, by constructing appropriate subsolutions and supersolutions, the entire solution $W_p(x,t)$ are obtained by passing

$n \to \infty$. To prove $W_p(x,t)$ is a classical solution, it is crucial to establish some prior estimate for $W^n(x,t)$. However, since the diffusion coefficient in $v-$equation is zero, a lack of regularizing effect occurs for the system (1.1). In particular, the function $v^n(x,t)$ is not smooth enough with respect to the spatial variable x. To overcome this difficulty, we have to show that $v^n(x,t)$ possess a property which is similar to a global Lipschitz condition with respect to x (see Lemma 3.3).

The rest of this article is organized as follows. In Section 2, we first investigate two characteristic problems related to traveling wave solutions of (1.1). Then, we establish the asymptotic behavior of the traveling wave solutions. In Section 3, we first establish some existence and comparison theorems for solutions, supersolutions and subsolutions of (1.1) and the existence of the spatially independent solution. Then, we prove the existence result of entire solutions. Finally, some qualitative properties of the entire solutions are further investigated.

2. Asymptotic behavior of traveling wave solutions

In this section, we first investigate two characteristic problems related to the traveling wave solutions of (1.1). Then, we establish the asymptotic behavior of the traveling wave solutions. Define

$$k(x,t) = g'(0)\int_0^t \int_{\mathbb{R}} \Gamma_1(x-y,t-s)J(y)k_2(s)\,dy\,ds,$$

where

$$\Gamma_1(x,t) = \frac{1}{\sqrt{4\pi dt}}e^{-\frac{x^2}{4dt}-\alpha t}, \quad k_2(t) = \begin{cases} e^{-\beta(t-\tau)}, & t>\tau, \\ 0, & t\in[0,\tau]. \end{cases}$$

By solving the v-equation of (1.1), we can write the first equation of (1.1) as the integral equation (see [23])

$$u(x,t) = u_0(x,t) + \int_0^t \int_{\mathbb{R}} k(x-y,t-s)\frac{g\big(u(y,s)\big)}{g'(0)}\,dy\,ds,$$

where the function $u_0(x,t)$ only depends on the initial data $u(x,s)$ and $v(x,0)$, $x\in\mathbb{R}$, $s\in[-\tau,0]$. Motivated by the theory of the spreading speeds for integral equations developed in [17], we define

$$\mathcal{K}_k(c,\lambda) := \int_0^\infty \int_{\mathbb{R}} e^{-\lambda(cs+y)}k(y,s)\,dy\,ds, \quad \forall c,\lambda\ge 0.$$

One can verify that

$$\mathcal{K}_k(c,\lambda) = \frac{g'(0)e^{-\lambda c\tau}}{c\lambda+\beta}\int_{-\infty}^{\infty} J(y)e^{-\lambda y}dy \int_0^\infty e^{-(c\lambda-d\lambda^2+\alpha)s}ds.$$

Let

$$\lambda^\sharp(c) = \min\Big\{\lambda_0, \frac{c+\sqrt{c^2+4d\alpha}}{2d}\Big\}.$$

Then $\mathcal{K}_k(c,\lambda)<\infty$ for $\lambda\in[0,\lambda^\sharp(c))$ and $\lim_{\lambda\nearrow\lambda^\sharp(c)}\mathcal{K}_k(c,\lambda)=\infty$ for every $c\ge 0$. From assumption (A2), we have

$$g'(0)\frac{K_1}{2} \ge g(\frac{K_1}{2}) > \alpha\beta\frac{K_1}{2},$$

which implies that $g'(0) > \alpha\beta$. Moreover, it is easy to verify that $k(t,x)$ satisfies [17, assumption (B)]. Define

$$c^* := \inf\{c \geq 0 : \mathcal{K}_k(c,\lambda) < 1 \text{ for some } \lambda > 0\}.$$

By Thieme and Zhao [17, Lemmas 2.1 and 2.2 and Proposition 2.3], we have the following result.

Proposition 2.1. *The following statements are valid:*

(a) *For each $c \geq 0$, $\mathcal{K}_k(c,\lambda)$ is a convex function of $\lambda \in [0, \lambda^\sharp(c))$.*
(b) *$c_* \in (0,\infty)$ and for any $c > c_*$, there exists some $\lambda > 0$ such that $\mathcal{K}_k(c,\lambda) < 1$.*
(c) *There exists a unique $\lambda_* \in (0, \lambda^\sharp(c))$ such that c_* and λ_* are uniquely determined as the solutions of the system*

$$\mathcal{K}_k(c,\lambda) = 1, \quad \frac{d}{d\lambda}\mathcal{K}_k(c,\lambda) = 0. \tag{2.1}$$

Similarly, we define

$$k_1(x,t) = g'(K_1)\int_0^t \int_{\mathbb{R}} \Gamma_1(x-y,t-s)J(y)k_2(s)\,dy\,ds.$$

Since $\mathcal{K}_{k_1}(c,0) = g'(K_1)/(\alpha\beta) < 1$, $\mathcal{K}_k(c,\lambda)$ is convex for $\lambda \in (-\lambda^\sharp(c), \lambda^\sharp(c))$, and $\lim_{\lambda \nearrow \lambda^\sharp(c)} \mathcal{K}_k(c,\lambda) = \infty$ for every $c \geq 0$, we see that the following result holds.

Proposition 2.2. *The equation $\mathcal{K}_{k_1}(c,\lambda) = 1$ has a unique root $\lambda_3(c)$ in the interval $(-\lambda^\sharp(c), 0)$.*

Substituting $(u(t,x), v(x,t)) = e^{\lambda(x+ct)}(\phi_1,\phi_2)$ into the linearization of (1.1) at $(0,0)$ and (K_1,K_2), respectively, we obtain the following two characteristic functions:

$$\Delta_1(c,\lambda) = (c\lambda - d\lambda^2 + \alpha)(c\lambda + \beta) - g'(0)\int_{-\infty}^{\infty} J(y)e^{-\lambda(y+c\tau)}dy, \tag{2.2}$$

$$\Delta_2(c,\lambda) = (c\lambda - d\lambda^2 + \alpha)(c\lambda + \beta) - g'(K_1)\int_{-\infty}^{\infty} J(y)e^{-\lambda(y+c\tau)}dy, \tag{2.3}$$

for $\lambda \in \mathbb{R}$ and $c \geq 0$. Thus, Propositions 2.1 and 2.2 imply the following result.

Proposition 2.3. *The following statements hold:*

(a) *If $c \geq c_*$, the equation $\Delta_1(c,\lambda) = 0$ has two positive real roots $\lambda_1(c)$ and $\lambda_2(c)$ with $\lambda_1(c) \leq \lambda_2(c)$.*
(b) *If $c = c_*$, then $\lambda_1(c_*) = \lambda_2(c_*) := \lambda_*$, and if $c > c_*$, then $\lambda_1(c) < \lambda_* < \lambda_2(c)$, and*

$$\Delta_1(c,\lambda)\begin{cases} < 0 & \text{for } \lambda \in \mathbb{R} \setminus (\lambda_1(c), \lambda_2(c)), \\ > 0 & \text{for } \lambda \in (\lambda_1(c), \lambda_2(c)). \end{cases}$$

(c) *The equation $\Delta_2(c,\lambda) = 0$ has a unique root $\lambda_3(c)$ in $(-\lambda^\sharp(c), 0)$.*

Proof of Theorem 1.2. Set $H(\xi) = \int_{-\infty}^{\infty} J(y)\psi_c(\xi - y)dy$. From the first equation of (1.2), we have

$$d\phi_c''(\xi) - c\phi_c'(\xi) - \alpha\phi_c(\xi) + H(\xi) = 0. \tag{2.4}$$

By the theory of linear ordinary differential equations, we obtain

$$\phi_c(\xi) = \frac{1}{d(\lambda_4 - \lambda_3)} \Big[\int_{-\infty}^{\xi} e^{\lambda_3(\xi - s)} H(s) ds + \int_{\xi}^{+\infty} e^{\lambda_4(\xi - s)} H(s) ds \Big], \tag{2.5}$$

where

$$\lambda_3 := \big(c - \sqrt{c^2 + 4\alpha d}\big)/(2d) < 0 \quad \text{and} \quad \lambda_4 := \big(c + \sqrt{c^2 + 4\alpha d}\big)\Big/(2d) > 0.$$

We first show that $\phi_c(\cdot) > 0$ by a contradiction argument. Assume that there exists $\xi_1 \in \mathbb{R}$ such that $\phi_c(\xi_1) = 0$. Then

$$0 = \phi_c(\xi_1) = \frac{1}{d(\lambda_4 - \lambda_3)} \Big[\int_{-\infty}^{\xi_1} e^{\lambda_3(\xi_1 - s)} H(s) ds + \int_{\xi_1}^{+\infty} e^{\lambda_4(\xi_1 - s)} H(s) ds \Big].$$

Since $H(\xi) \geq 0$ for all $\xi \in \mathbb{R}$, $H(\xi) = 0$ for all $\xi \in \mathbb{R}$, and hence

$$0 = H(\xi) = \int_{-\infty}^{\infty} J(y) \psi_c(\xi - y) dy, \quad \forall \xi \in \mathbb{R}.$$

By (A3), we have $\psi_c(\xi) = 0$ for any $\xi \in \mathbb{R}$ which contradicts to $\psi_c(+\infty) = K_2$. Therefore $\phi_c(\cdot) > 0$. Similarly, we can prove that $\phi_c(\cdot) < K_1$.

Since $g(u) > 0$ for $u \in (0, K_1]$ and $\psi_c(\xi) = \frac{1}{c} \int_{-\infty}^{\xi} e^{-\frac{\beta}{c}(\xi - s)} g(\phi_c(s - c\tau)) ds$, it is easy to show that $\psi_c(\xi) \in (0, K_2)$ for all $\xi \in \mathbb{R}$. This completes the proof. $\square$

The following lemma is important for obtaining the the asymptotic behavior of the wave profiles, which can be found in Carr and Chmaj [3].

Lemma 2.4. *Let $u(\xi)$ be a positive decreasing function and*

$$H_1(\Lambda) := \int_0^{+\infty} e^{-\Lambda \xi} u(\xi) d\xi.$$

If H_1 can be written as $H_1(\Lambda) = H(\Lambda)(\Lambda + \Lambda_0)^{-(k+1)}$, where $k > -1$, $\Lambda_0 > 0$ are two constants and H is analytic in the strip $-\Lambda_0 \leq \mathrm{Re}\Lambda < 0$, then

$$\lim_{\xi \to +\infty} \frac{u(\xi)}{\xi^k e^{-\Lambda_0 \xi}} = \frac{H(-\Lambda_0)}{\Gamma(\Lambda_0 + 1)}.$$

Applying Lemma 2.4, we can prove Theorem 1.3.

Proof of Theorem 1.3. We only prove the assertion (i), since the assertion (ii) can be discussed similarly. First, we show that (1.4) and (1.6) hold. From (1.2) and (1.3), it is easy to verify that

$$c\phi_c'(\xi) = d\phi_c''(\xi) - \alpha \phi_c(\xi) + \frac{1}{c} \int_0^{\infty} \int_{-\infty}^{\infty} J(\xi - y) e^{-\frac{\beta}{c} s} g(\phi_c(y - s - c\tau)) \, dy \, ds. \tag{2.6}$$

The proofs of (1.4) and (1.6) are similar to those of [19, Theorem 4.8] and [20, Theorem 3.5], we only sketch the outline. The proof is divided into three steps.

Step 1. We show that $\phi_c(\xi)$ is integrable on $(-\infty, \xi']$ for some $\xi' \in \mathbb{R}$.

Step 2. We prove that $\phi_c(\xi) = O(e^{\gamma \xi})$ as $\xi \to -\infty$ for some $\gamma > 0$. To get the assertion, we first show that $W(\xi) = O(e^{\gamma \xi})$ as $\xi \to -\infty$, where $W(\xi) := \int_{-\infty}^{\xi} \phi_c(s) ds$.

Step 3. For $0 < \mathrm{Re}\lambda < \gamma$, define a two-sided Laplace transform of ϕ_c by

$$\mathcal{L}(\lambda) = \int_{-\infty}^{+\infty} \phi_c(\xi) e^{-\lambda \xi} d\xi.$$

Using Lemma 2.4, one can show that $\lim_{\xi\to-\infty}\phi_c(\xi)e^{-\lambda_1(c)\xi}=a_0(c)$ for $c>c_*$, and $\lim_{\xi\to-\infty}\phi_c(\xi)\xi^{-1}e^{-\lambda_1(c)\xi}=-a_0(c)$ for $c=c_*$.

Integrating the two sides of (2.6) from $-\infty$ to ξ, we obtain

$$\begin{aligned}
d\phi_c'(\xi) &= c\phi_c(\xi)+\alpha\int_{-\infty}^{\xi}\phi_c(z)dz\\
&\quad -\frac{1}{c}\int_{-\infty}^{\xi}\int_0^\infty\int_{-\infty}^\infty J(z-y)e^{-\frac{\beta}{c}s}g(\phi_c(y-s-c\tau))\,dy\,ds\,dz.
\end{aligned}$$

Since $g\in C^2([0,K_1],\mathbb{R})$ and $g(u)\le g'(0)u$ for $u\in[0,K_1]$, one can easily show that $\lim_{\xi\to-\infty}g(\phi_c(\xi))e^{-\lambda_1(c)\xi}=g'(0)a_0(c)$ for $c>c_*$. Moreover, we have

$$\begin{aligned}
&\lim_{\xi\to-\infty}e^{-\lambda_1(c)\xi}\int_0^\infty\int_{-\infty}^\infty J(y)e^{-\frac{\beta}{c}s}g(\phi_c(\xi-y-s-c\tau))\,dy\,ds\\
&=\lim_{\xi\to-\infty}\int_0^\infty\int_{-\infty}^\infty J(y)\\
&\quad\times e^{-\frac{\beta}{c}s}e^{-\lambda_1(c)(y+s+c\tau)}g(\phi_c(\xi-y-s-c\tau))e^{-\lambda_1(c)(\xi-y-s-c\tau)}\,dy\,ds\\
&=g'(0)a_0(c)\int_0^\infty\int_{-\infty}^\infty J(y)e^{-\frac{\beta}{c}s}e^{-\lambda_1(c)(y+s+c\tau)}\,dy\,ds\\
&=cg'(0)a_0(c)e^{-\lambda_1(c)c\tau}\int_{-\infty}^\infty J(y)e^{-\lambda_1(c)y}dy/(c\lambda_1(c)+\beta).
\end{aligned}$$

It then follows from the L'Hospital's rule that for $c>c_*$,

$$\begin{aligned}
&d\lim_{\xi\to-\infty}\phi_c'(\xi)e^{-\lambda_1(c)\xi}\\
&=c\lim_{\xi\to-\infty}\phi_c(\xi)e^{-\lambda_1(c)\xi}+\alpha\lim_{\xi\to-\infty}\frac{\int_{-\infty}^\xi\phi_c(z)dz}{e^{\lambda_1(c)\xi}}\\
&\quad-\frac{1}{c}\lim_{\xi\to-\infty}\frac{\int_{-\infty}^\xi\int_0^\infty\int_{-\infty}^\infty J(z-y)e^{-\frac{\beta}{c}s}g(\phi_c(y-s-c\tau))\,dy\,dsdz}{e^{\lambda_1(c)\xi}}\\
&=ca_0(c)+\alpha\frac{a_0(c)}{\lambda_1(c)}-\frac{1}{c}\lim_{\xi\to-\infty}\frac{\int_0^\infty\int_{-\infty}^\infty J(y)e^{-\frac{\beta}{c}s}g(\phi_c(\xi-y-s-c\tau))\,dy\,ds}{\lambda_1(c)e^{\lambda_1(c)\xi}}\\
&=a_0(c)[c+\frac{\alpha}{\lambda_1(c)}-\frac{g'(0)\int_{-\infty}^\infty J(y)e^{-\lambda_1(c)(y+c\tau)}dy}{\lambda_1(c)(c\lambda_1(c)+\beta)}]=da_0(c)\lambda_1(c).
\end{aligned}$$

Similarly, we can prove that for $c=c_*$, $\lim_{\xi\to-\infty}\phi_c'(\xi)\xi^{-1}e^{-\lambda_1(c)\xi}=-a_0(c)\lambda_1(c)$. Therefore, (1.4) and (1.6) hold.

Next, we prove (1.5) and (1.7). Note that

$$\psi_c(\xi)=\frac{1}{c}\int_{-\infty}^\xi e^{-\frac{\beta}{c}(\xi-s)}g(\phi_c(s-c\tau))ds.$$

Hence, for $c>c_*$,

$$\begin{aligned}
\lim_{\xi\to-\infty}\psi_c(\xi)e^{-\lambda_1(c)\xi}&=\lim_{\xi\to-\infty}\frac{\int_{-\infty}^\xi e^{\frac{\beta}{c}s}g(\phi_c(s-c\tau))ds}{ce^{\left(\lambda_1(c)+\frac{\beta}{c}\right)\xi}}\\
&=\frac{g'(0)e^{-c\lambda_1(c)\tau}}{c\lambda_1(c)+\beta}a_0(c)=A(c)a_0(c).
\end{aligned}$$

From the second equation of (1.2) it follows that

$$\lim_{\xi\to-\infty}\psi_c'(\xi)e^{-\lambda_1(c)\xi}=A(c)a_0(c)\lambda_1(c)$$

for $c>c_*$. Therefore, (1.5) holds. Similarly, one can show that (1.7) holds. This completes the proof. □

Corollary 2.5. *Let the assumptions of Theorem 1.3 be satisfied. Then, for all $c\ge c_*$,*

$$\lim_{\xi\to-\infty}\frac{\phi_c'(\xi)}{\phi_c(\xi)}=\lim_{\xi\to-\infty}\frac{\psi_c'(\xi)}{\psi_c(\xi)}=\lambda_1(c),$$
$$\lim_{\xi\to+\infty}\frac{\phi_c'(\xi)}{\phi_c(\xi)-K_1}=\lim_{\xi\to+\infty}\frac{\psi_c'(\xi)}{\psi_c(\xi)-K_2}=\lambda_3(c).$$

3. Existence and qualitative properties of entire solutions

This section is devoted to the study of entire solutions of (1.1). We first give some preliminaries. Then, we establish the existence of the spatially independent solution by transforming the system into a differential equation with an integral term. Further, we prove the existence of entire solutions. Finally, some qualitative properties of the solution are investigated.

3.1. **Preliminaries.**

In this subsection, we first give the well-posedness of initial value problem of (1.1), and establish some comparison theorems for supersolutions and subsolutions. Then, we establish two important lemmas which play important roles in investigating the existence and qualitative features of entire solutions.

Let $X=\mathrm{BUC}(\mathbb{R},\mathbb{R}^2)$ be the Banach space of all bounded and uniformly continuous functions from $\mathbb{R}$ into $\mathbb{R}^2$ with the supremum norm $\|\cdot\|_X$ and $\mathcal{C}=C([-\tau,0],X)$ be the Banach space of continuous functions from $[-\tau,0]$ into X with the supremum norm. Similarly, we define the space $\mathrm{BUC}(\mathbb{R},\mathbb{R})$. As usual, we identify an element $\phi\in\mathcal{C}$ as a function from $\mathbb{R}\times[-\tau,0]$ into $\mathbb{R}^2$ defined by $\phi(x,s)=\phi(s)(x)$. We further denote the following spaces:

$$X^+:=\{\varphi\in X:\varphi(x)\ge\mathbf{0},x\in\mathbb{R}\},$$
$$X_{[\mathbf{0},\mathbf{K}]}:=\{\varphi\in X:\varphi(x)\in[\mathbf{0},\mathbf{K}],x\in\mathbb{R}\},$$
$$\mathcal{C}_{[\mathbf{0},\mathbf{K}]}:=\{\varphi\in\mathcal{C}:\varphi(x,s)\in[\mathbf{0},\mathbf{K}],x\in\mathbb{R},s\in[-\tau,0]\}.$$

It is easy to see that X^+ is a closed cone of X.

For any continuous function $w:[-\tau,b)\to X$, $b>0$, we define $w^t\in\mathcal{C}$, $t\in[0,b)$ by $w^t(s)=w(t+s)$, $s\in[-\tau,0]$. Then $t\to w^t$ is a continuous function from $[0,b)$ to $\mathcal{C}$. For $\varphi\in\mathcal{C}$, we define $B(\varphi)=(B_1(\varphi),B_2(\varphi))$ by

$$B_1(\varphi)=\int_{\mathbb{R}}J(x-y)\varphi_2(y,0)dy,\quad B_2(\varphi)=g(\varphi_1(x,-\tau)).$$

Let $T_1(t)$ be the analytic semigroup on $\mathrm{BUC}(\mathbb{R},\mathbb{R})$ generated by $u_t=du_{xx}-\alpha u$ and $T_2(t)=e^{-\beta t}$. Clearly, $T(t)=\mathrm{diag}(T_1(t),T_2(t))$ is a linear semigroup on X.

Definition 3.1. A continuous function $w=(u,v):[s,T)\to X$, $s<T$, is called a supersolution (or a subsolution) of (1.1) on $[s,T)$ if

$$w(t)\ge(or\le)T(t-\tau)w(\tau)+\int_\tau^t T(t-r)B(w^r)dr$$

for any $s \leq \tau < t < T$.

A function $w : (-\infty, T) \to X$ is called a supersolution (or a subsolution) of (1.1) on $(-\infty, T)$, if for any $s < T$, w is a supersolution (or a subsolution) of (1.1) on $[s, T)$.

Using the theory of abstract functional differential equations [13, Corollary 5], it is easy to prove that the following result holds, see e.g., [25].

Lemma 3.2. (1) *For any $\varphi \in \mathcal{C}_{[\mathbf{0},\mathbf{K}]}$, (1.1) has a unique solution $w(x,t;\varphi)$ on $(x,t) \in \mathbb{R} \times [0,\infty)$ with $w(x,0;\varphi) = \varphi(x)$ and $\mathbf{0} \leq w(x,t;\varphi) \leq \mathbf{K}$ for $x \in \mathbb{R}$, $t \geq 0$. Moreover, $w(x,t;\varphi)$ is classical on $(\tau, +\infty)$.*

(2) *For any pair of supersolution $w^+(x,t)$ and subsolution $w^-(x,t)$ of (1.1) on $[0,\infty)$ with $\mathbf{0} \leq w^-(x,t), w^+(x,t) \leq \mathbf{K}$ for $(x,t) \in \mathbb{R} \times [0,\infty)$, and $w^+(x,s) \geq w^-(x,s)$ for $x \in \mathbb{R}$ and $s \in [-\tau, 0]$, there holds $\mathbf{0} \leq w^-(x,t) \leq w^+(x,t) \leq \mathbf{K}$ for $(x,t) \in \mathbb{R} \times [0,\infty)$.*

Next, we give the following two lemmas which play important roles in investigating the existence and qualitative features of entire solutions.

Lemma 3.3. *Suppose that $w(x,t) = (u(x,t), v(x,t))$ is a solution of (1.1) with initial value $\phi = (\phi_1, \phi_2) \in \mathcal{C}_{[\mathbf{0},\mathbf{K}]}$. Then there exists a positive constant $M > 0$, independent of ϕ, such that for any $\eta > 0$, $x \in \mathbb{R}$ and $t > 2(\tau + 1)$,*

$$\begin{gathered} \big|u_t(x,t)\big|, \big|u_{tx}(x,t)\big|, \big|u_{tt}(x,t)\big|, \big|u_x(x,t)\big|, \big|u_{xt}(x,t)\big| \leq M, \\ \big|u_{xx}(x,t)\big|, \big|u_{xxt}(x,t)\big|, \big|v_t(x,t)\big|, \big|v_{tt}(x,t)\big| \leq M. \end{gathered}$$

If, in addition, there exists a constant $L > 0$ such that for any $\eta > 0$,

$$\sup_{x \in \mathbb{R}} |\phi_2(x + \eta, 0) - \phi_2(x, 0)| \leq L\eta,$$

then for any $\eta > 0$, $x \in \mathbb{R}$ and $t > 2(\tau + 1)$, we have

$$\begin{gathered} \big|v(x+\eta,t) - v(x,t)\big|, \ |v_t(x+\eta,t) - v_t(x,t)| \leq \bar{M}\eta, \\ |u_{xx}(x+\eta,t) - u_{xx}(x,t)| \leq \bar{M}\eta, \end{gathered} \tag{3.1}$$

where $\bar{M} > 0$ is a constant which is independent of ϕ and η.

Proof. By Lemma 3.2, we see that $\mathbf{0} \leq (u(x,t), v(x,t)) \leq \mathbf{K}$ for all $x \in \mathbb{R}$ and $t \geq 0$. From the v-equation of (1.1), we have

$$|v_t(x,t)| \leq \beta K_2 + \max_{u \in [0,K_1]} g(u) := M_1 \quad \text{for } x \in \mathbb{R} \text{ and } t \geq 0.$$

Note that for any $s \geq 0$ and $t > s$,

$$u(x,t) = \int_{\mathbb{R}} J_1(x-y,t-s)u(y,s)dy + \int_s^t \int_{\mathbb{R}} J_1(x-y,t-r) \int_{\mathbb{R}} J(y-z)v(z,r)\,dz\,dy\,dr,$$

where $J_1(x,t) = \frac{e^{-\alpha t}}{\sqrt{4d\pi t}} \exp\Big\{-\frac{x^2}{4dt}\Big\}$. Consequently, for any $s \geq 0$ and $t \in [s+1, s+4]$,

$$\begin{aligned} u_x(x,t) = & \int_{\mathbb{R}} \frac{-(x-y)}{2d(t-s)} J_1(x-y,t-s)u(y,s)dy \\ & + \int_s^t \int_{\mathbb{R}} \frac{-(x-y)}{2d(t-r)} J_1(x-y,t-r) \int_{\mathbb{R}} J(y-z)v(z,r)\,dz\,dy\,dr. \end{aligned} \tag{3.2}$$

Direct computations show that

$$|u_x(x,t)| \le \frac{K_1}{\sqrt{\pi d(t-s)}} + \frac{2\sqrt{t-s}}{\sqrt{\pi d}} K_2 \le \frac{K_1}{\sqrt{\pi d}} + \frac{4K_2}{\sqrt{\pi d}} := M_2,$$

for any $x \in \mathbb{R}$ and $t \in [s+1, s+4]$. Since $s \ge 0$ is arbitrary, we have

$$|u_x(x,t)| \le M_2, \quad \text{for any } x \in \mathbb{R} \text{ and } t > 1.$$

Moreover, for any $s \ge 0$ and $t \in [s+1, s+4]$, we have

$$\begin{aligned}
|u_t(x,t)| &\le \Big| \int_{\mathbb{R}} J_1(y,t-s)\Big[-\alpha + \frac{|y|^2}{4d(t-s)^2} - \frac{1}{2(t-s)}\Big] u(x-y,s)dy \Big| \\
&\quad + \Big| \int_0^{t-s} \int_{\mathbb{R}} J_1(y,r) \int_{\mathbb{R}} J(z) v_t(x-y-z,t-r)\, dz\, dy\, dr \Big| \\
&\quad + \Big| \int_{\mathbb{R}} J_1(y,t-s) \int_{\mathbb{R}} J(z) v(x-y-z,s) dz\, dy \Big| \\
&\le K_1 \int_{\mathbb{R}} J_0(y) \Big[\alpha + \frac{|y|^2}{4d} + \frac{1}{2}\Big] dy \\
&\quad + M_1 \int_0^4 \int_{\mathbb{R}} J_1(y,r)\, dy\, dr + K_2 \int_{\mathbb{R}} J_0(y) dy := M_3,
\end{aligned}$$

where

$$J_0(x) := \frac{1}{(4d\pi t)^{1/2}} \exp\Big\{-\frac{|x|^2}{16d}\Big\}.$$

Hence, $|u_t(x,t)| \le M_3$ for any $x \in \mathbb{R}$ and $t > 1$. Similarly, using (3.2) and the estimate for v_t, we can show that a positive constant M_4, which is independent of ϕ, such that $\big|u_{xt}(x,t)\big| \le M_4$, for any $x \in \mathbb{R}$ and $t > 1$. Then, for $x \in \mathbb{R}$ and $t > \tau + 1$, $\big|u_{xx}(x,t)\big| \le (M_3 + \alpha K_1 + K_2)/d$ and

$$\begin{aligned}
\big|v_{tt}(x,t)\big| &= |-\beta v_t(x,t) + g'(u(x,t-\tau)) u_t(x,t-\tau)| \\
&\le \beta M_1 + M_3 \max_{u \in [0,K_1]} g'(u) := M_5.
\end{aligned}$$

Note that $u_t(x,t)$ satisfies the equation

$$z_t = d z_{xx} - \alpha z(x,t) + \int_{\mathbb{R}} J(x-y) v_t(y,t) dy, \quad t > \tau + 1$$

with initial value $z(x,\tau+1) = u_t(x,\tau+1)$. Using the estimate for v_t and applying a similar argument as in the previous part, we can find a positive constant M_6, which is independent of ϕ, such that for any $x \in \mathbb{R}$ and $t > 2(\tau+1)$,

$$|u_{tx}(x,t)|, |u_{tt}(x,t)|, |u_{xxt}(x,t)| \le M_6.$$

The first statement of this assertion follows by taking $M := \max\{M_1, \cdots, M_6\}$.

Next, we prove the estimates of (3.1). Note that

$$v(x,t) = \phi_2(x,0) e^{-\beta t} + \int_0^t g\big(u(x,s-\tau)\big) e^{-\beta(t-s)} ds, \quad \forall x \in \mathbb{R}, t > 0.$$

By our assumption, we have for any $\eta > 0$, $x \in \mathbb{R}$ and $t > \tau + 1$,

$$\begin{aligned}
&|v(x+\eta,t) - v(x,t)| \\
&\le |\phi_2(x+\eta,0) - \phi_2(x,0)| + \int_0^t \big|g\big(u(x+\eta,s-\tau) - g\big(u(x,s-\tau)\big)\big| e^{-\beta(t-s)} ds
\end{aligned}$$

$$\le L\eta + \frac{M}{\beta} \max_{u\in[0,K_1]} g'(u)\eta := M_1'\eta.$$

Moreover, one can easily verify that

$$|v_t(x+\eta,t) - v_t(x,t)| \le \big[\beta M_1' + M \max_{u\in[0,K_1]} g'(u)\big]\eta := M_2'\eta,$$

$$|u_{xx}(x+\eta,t) - u_{xx}(x,t)| \le \frac{1}{d}[M + \alpha M + K_2 M_1']\eta := M_3'\eta,$$

for any $\eta > 0$, $x \in \mathbb{R}$ and $t > 2(\tau+1)$. Let $\bar{M} := \max\{M_1', M_2', M_3'\}$, then (3.1) holds obviously. The proof is complete. □

Lemma 3.4. *Assume that* $w^+ = (u^+, v^+) \in C\big(\mathbb{R}\times[-\tau,+\infty), [0,+\infty)^2\big)$ *and* $w^- = (u^-, v^-) \in C\big(\mathbb{R}\times[-\tau,+\infty), (-\infty,K_1]\times(-\infty,K_2]\big)$ *satisfy* $w^+(x,s) \ge w^-(x,s)$ *for* $(x,s) \in \mathbb{R}\times[-\tau,0]$, *and*

$$u_t^+(x,t) \ge du_{xx}^+(x,t) - \alpha u^+(x,t) + \int_{-\infty}^{+\infty} J(x-y)v^+(y,t)dy,$$

$$v_t^+(x,t) \ge -\beta v^+(x,t) + g'(0)u^+(x,t-\tau),$$

and

$$u_t^-(x,t) \le du_{xx}^-(x,t) - \alpha u^-(x,t) + \int_{-\infty}^{+\infty} J(x-y)v^-(y,t)dy,$$

$$v_t^-(x,t) \le -\beta v^-(x,t) + g'(0)u^-(x,t-\tau),$$

for $x \in \mathbb{R}$ *and* $t > 0$. *Then* $w^+(x,t) \ge w^-(x,t)$ *for all* $x \in \mathbb{R}$ *and* $t \ge 0$.

Proof. Set $w(x,t) = \big(w_1(x,t), w_2(x,t)\big) := w^+(x,t) - w^-(x,t)$ for $x \in \mathbb{R}$ and $t \ge 0$, then $w(x,t)$ satisfies $w(x,0) \ge 0$ and

$$w_{1,t}(x,t) \ge dw_{1,xx}(x,t) - \alpha w_1(x,t) + \int_{-\infty}^{+\infty} J(x-y)w_2(y,t)dy, \tag{3.3}$$

$$w_{2,t}(x,t) \ge -\beta w_2(x,t) + g'(0)w_1(x,t-\tau) \tag{3.4}$$

for $x \in \mathbb{R}$ and $t > 0$. Note that $w(x,s) \ge 0$ for $x \in \mathbb{R}$ and $s \in [-\tau,0]$. Then, we have

$$w_{2,t}(x,t) \ge -\beta w_2(x,t), \quad \text{for } x \in \mathbb{R} \text{ and } t \in [0,\tau],$$

which implies that

$$w_2(x,t) \ge e^{-\beta t}w_2(x,0) \ge 0 \quad \text{for } x \in \mathbb{R} \text{ and } t \in [0,\tau].$$

Hence, for $x \in \mathbb{R}$ and $t \in [0,\tau]$, it follows from (3.3) that

$$\begin{aligned} w_{1,t}(x,t) &\ge dw_{1,xx}(x,t) - \alpha w_1(x,t) + \int_{-\infty}^{+\infty} J(x-y)w_2(y,t)dy \\ &\ge dw_{1,xx}(x,t) - \alpha w_1(x,t), \end{aligned}$$

which yields

$$w_1(x,t) \ge \int_{-\infty}^{+\infty} J_d(x-y,t)e^{-\alpha t}w_1(y,0)dy \ge 0,$$

where $J_d(x,t) = \frac{1}{\sqrt{4d\pi t}} \exp\big\{-\frac{x^2}{4dt}\big\}$. Therefore, $w(x,t) \ge 0$ for $x \in \mathbb{R}$ and $t \in [0,\tau]$. Inductively, we obtain that $w(x,t) \ge 0$ for all $x \in \mathbb{R}$ and $t \ge 0$. Therefore, $w^+(x,t) \ge w^-(x,t)$ for $x \in \mathbb{R}$ and $t \ge 0$. This completes the proof. □

3.2. **Existence of spatially independent solutions.** In this subsection, we prove the existence of the spatially independent solution $\Gamma = (\Gamma_1, \Gamma_2)$ of (1.1) connecting $\mathbf{0}$ and $\mathbf{K}$, i.e. solutions of the system

$$\begin{aligned} \Gamma_1'(t) &= -\alpha\Gamma_1(t) + \Gamma_2(t), \\ \Gamma_2'(t) &= -\beta\Gamma_2(t) + g\big(\Gamma_1(t-\tau)\big) \end{aligned} \tag{3.5}$$

with

$$\Gamma(-\infty) = \mathbf{0} \text{ and } \Gamma(+\infty) = \mathbf{K}. \tag{3.6}$$

We first transform the system (3.5) into a scalar differential equation with an integral term. In fact, from the second equation of (3.5) and $\Gamma_2(-\infty) = 0$, we obtain

$$\Gamma_2(t) = \int_{-\infty}^{t} e^{-\beta(t-s)} g(\Gamma_1(s-\tau))ds. \tag{3.7}$$

Then, Γ_1 satisfies

$$\Gamma_1'(t) = -\alpha\Gamma_1(t) + \int_{-\infty}^{t} e^{-\beta(t-s)} g(\Gamma_1(s-\tau))ds. \tag{3.8}$$

Conversely, if $\Gamma_1(t)$ is a non-decreasing solution of (3.8) with $\Gamma_1(-\infty) = 0$, and $\Gamma_1(+\infty) = K_1$, and $\Gamma_2(t)$ is defined by (3.7), then $(\Gamma_1(t), \Gamma_2(t))$ is a non-decreasing solution of (3.5), and satisfies (3.6).

By above discussions, to prove the existence of the spatially independent solution $\Gamma = (\Gamma_1, \Gamma_2)$ of (1.1) connecting $\mathbf{0}$ and $\mathbf{K}$, we only need to prove the existence of solutions of (3.8) satisfying $\Gamma_1(-\infty) = 0$, and $\Gamma_1(+\infty) = K_1$.

It is clear that the characteristic function of (3.8) at $(0, 0)$ has the form

$$\Delta_2(\lambda) = (\lambda + \alpha)(\lambda + \beta) - g'(0)e^{-\lambda\tau}. \tag{3.9}$$

Proposition 3.5. *The equation $\Delta_2(\lambda) = 0$ has two real roots $\lambda_1^* < 0$ and $\lambda^* > 0$. In particular, for any $c \geq c_*$, $c\lambda_1(c) > \lambda^*$, where $\lambda_1(c)$ is given as in Proposition 2.3.*

Proof. Since $g'(0) > \alpha\beta$, it is easy to see that the first part of the assertion holds. Now, we show that $c \geq c_*$, $c\lambda_1(c) > \lambda^*$. Suppose for the contrary that there exists $c_1 \geq c_*$ such that $c_1\lambda_1(c_1) \leq \lambda^*$. It follows from (2.2) and (3.9) that

$$\begin{aligned} c_1\lambda_1(c_1) &= d\lambda_1^2(c_1) - \alpha + \frac{g'(0)e^{-c_1\lambda_1(c_1)\tau}\int_{-\infty}^{\infty} J(y)e^{-\lambda_1(c_1)y}dy}{c_1\lambda_1(c_1) + \beta} \\ &> -\alpha + \frac{g'(0)e^{-\lambda^*\tau}}{\lambda^* + \beta} = \lambda^*. \end{aligned}$$

This contradiction implies that $c\lambda_1(c) > \lambda^*$ for any $c \geq c_*$. This completes the proof. $\square$

We now consider the space $C(\mathbb{R}, \mathbb{R})$ of continuous real functions on $\mathbb{R}$, and the operator $T : C(\mathbb{R}, [0, K_1]) \to C(\mathbb{R}, \mathbb{R})$ defined by

$$T(\phi)(t) = \int_{-\infty}^{t} e^{-\alpha(t-s)} h(\phi)(s)ds,$$

where $h(\phi)(t) = \int_{-\infty}^{t} e^{-\beta(t-s)} g(\phi(s-\tau))ds$. Since g is non-decreasing on $[0, K_1]$, it is easy to verify the following statements.

Lemma 3.6. (i) *$T : C(\mathbb{R}, [0, K_1]) \to C(\mathbb{R}, [0, K_1])$;*

(ii) *$T(\phi)(t) \geq T(\psi)(t)$ for $\phi, \psi \in C(\mathbb{R}, [0, K_1])$ with $\phi(t) \geq \psi(t)$;*
(iii) *$T(\phi)(t)$ is increasing in $\mathbb{R}$ for $\phi \in C(\mathbb{R}, [0, K_1])$ with $\phi(t)$ is increasing in $\mathbb{R}$.*

For any fixed $\epsilon \in \big(0, \min\{1, K_1\}\big)$ and sufficiently large $q > 1$, define the following two functions:

$$\overline{\phi}(t) = \min\big\{K_1, e^{\lambda^* t}\big\}, \quad \underline{\phi}(t) = \max\big\{0, \big(1 - qe^{\epsilon\lambda^* t}\big)e^{\lambda^* t}\big\}, \quad t \in \mathbb{R}.$$

By direct computations, one can easily verify that the following result holds.

Lemma 3.7. (i) *$0 \leq \underline{\phi}(t) \leq \overline{\phi}(t) \leq K_1$ for all $t \in \mathbb{R}$;*
(ii) *$T(\overline{\phi})(t) \leq \overline{\phi}(t)$ and $T(\underline{\phi})(t) \geq \underline{\phi}(t)$ for all $t \in \mathbb{R}$.*

Using the monotone iteration technique, the existence of the spatially independent solution follows from Lemmas 3.6-3.7. Moreover, using the similar method as in the proof of Theorem 1.2, we can show that $\Gamma(t) \gg \mathbf{0}$ for any $t \in \mathbb{R}$. We omit the details here.

3.3. **Existence of entire solutions.** In this section, we will use the results of previous sections to obtain an appropriate upper estimate for solutions of (1.1) and then prove the existence result of Theorem 1.5. For any $n \in \mathbb{Z}^+$, $h_1, h_2, h_3 \in \mathbb{R}$, $c_1, c_2 > c_*$ and $\chi_1, \chi_2, \chi_3 \in \{0, 1\}$ with $\chi_1 + \chi_2 + \chi_3 \geq 2$, we denote

$$\begin{aligned}\varphi^n(x, s) &:= \max\big\{\chi_1\Phi_{c_1}(x + c_1 s + h_1), \chi_2\Phi_{c_2}(-x + c_2 s + h_2), \chi_3\Gamma(s + h_3)\big\},\\ \underline{w}(x, t) &:= \max\big\{\chi_1\Phi_{c_1}(x + c_1 t + h_1), \chi_2\Phi_{c_2}(-x + c_2 t + h_2), \chi_3\Gamma(t + h_3)\big\},\end{aligned}$$

where $s \in [-n-\tau, -n]$ and $t > -n$. Let $w^n(x, t) = (u^n(x, t), v^n(x, t))$ be the unique solution of the initial value problem of (1.1) with the initial value

$$w^n(x, s) = \varphi^n(x, s), \quad x \in \mathbb{R},\ s \in [-n - \tau, -n]. \tag{3.10}$$

Then, by Lemma 3.2, we have

$$\underline{w}(x, t) \leq w^n(x, t) \leq \mathbf{K} \quad \text{for all } x \in \mathbb{R} \text{ and } t \geq -n.$$

The following result provides the appropriate upper estimate of $w^n(x, t)$.

Lemma 3.8. *The unique solution $w^n(x, t)$ of (3.10) satisfies*

$$\underline{w}(x, t) \leq w^n(x, t) \leq \min\big\{\mathbf{K}, \Pi_1(x, t), \Pi_2(x, t), \Pi_3(x, t)\big\}$$

for any $x \in \mathbb{R}$ and $t \geq -n - \tau$, where $\Pi_1(x, t)$, $\Pi_2(x, t)$ and $\Pi_3(x, t)$ are defined in Theorem 1.5.

Proof. We only prove $w^n(x, t) \leq \Pi_1(x, t)$ for all $x \in \mathbb{R}$ and $t \geq -n - \tau$. The other cases can be proved in the same way. Assume $\chi_1 = 1$ and set

$$Z^n(x, t) := (Z_1^n(x, t), Z_2^n(x, t)) = w^n(x, t) - \Phi_{c_1}(x + c_1 t + h_1), \quad x \in \mathbb{R}, t \geq -n - \tau.$$

Clearly, $Z^n(x, t) \geq \mathbf{0}$ for all $x \in \mathbb{R}$ and $t \geq -n-\tau$. By the assumption $g'(u) \leq g'(0)$ for $u \in [0, K_1]$, we obtain

$$\begin{gathered}(Z_1^n)_t(x, t) = d(Z_1^n)_{xx} - \alpha Z_1^n(x, t) + \int_{\mathbb{R}} J(x - y)Z_2^n(y, t)dy,\\ (Z_2^n)_t(x, t) \leq -\beta Z_2^n(x, t) + g'(0)Z_1^n(x, t - \tau),\\ Z^n(x, s) = \varphi^n(x, s) - \Phi_{c_1}(x + c_1 s + h_1),\end{gathered} \tag{3.11}$$

where $x \in \mathbb{R}$, $t > -n$, $s \in [-n-\tau, -n]$. Taking

$$V(x,t) := (V_1(x,t), V_2(x,t)) = \chi_2(1, A_{c_2})e^{\lambda_1(c_2)(-x+c_2t+h_2)} + \chi_3(1, b_*)e^{\lambda^*(t+h_3)},$$

it is easy to verify that

$$\begin{aligned}(V_1)_t(x,t) &= d(V_1)_{xx} - \alpha V_1(x,t) + \int_{\mathbb{R}} J(x-y)V_2(y,t)dy,\\ (V_2)_t(x,t) &= -\beta V_2(x,t) + g'(0)V_1(x,t-\tau),\end{aligned}$$

where $x \in \mathbb{R}$, $t > -n$. By Theorems 1.3 and 1.4, we have

$$\Phi_{c_2}(z) \le (1, A_{c_2})e^{\lambda_1(c_2)z} \quad \text{and} \quad \Gamma(z) \le (1, b_*)e^{\lambda^* z} \quad \text{for all } z \in \mathbb{R},$$

which implies that

$$\begin{aligned}V(x,s) &= \chi_2(1, A_{c_2})e^{\lambda_1(c_2)(-x+c_2s+h_2)} + \chi_3(1, b_*)e^{\lambda^*(s+h_3)}\\ &\ge \varphi^n(x,s) - \Phi_{c_1}(x + c_1s + h_1)\\ &= Z^n(x,s) \quad \text{for } s \in [-n-\tau, -n].\end{aligned}$$

It then follows from Lemma 3.4 that

$$Z^n(x,t) \le V(x,t) \quad \text{for all } x \in \mathbb{R} \text{ and } t > -n-\tau,$$

that is,

$$\begin{aligned}w^n(x,t) \le \Phi_{c_1}(x + c_1t + h_1) &+ \chi_2(1, A_{c_2})e^{\lambda_1(c_2)(-x+c_2t+h_2)}\\ &+ \chi_3(1, b_*)e^{\lambda^*(t+h_3)} = \Pi_1(x,t).\end{aligned}$$

If $\chi_1 = 0$, then the assertion $w^n(x,t) \le \Pi_1(x,t)$ reduces to

$$w^n(x,t) \le \chi_2(1, A_{c_2})e^{\lambda_1(c_2)(-x+c_2t+h_2)} + \chi_3(1, b_*)e^{\lambda^*(t+h_3)}$$

which holds obviously. The proof is complete. □

Definition 3.9. Let $k \in \mathbb{N}$ and $p, p_0 \in \mathbb{R}^k$. We say that the functions $W_p(x,t) = \big(U_p(x,t), V_p(x,t)\big)$ converge to $W_{p_0}(x,t) = \big(U_{p_0}(x,t), V_{p_0}(x,t)\big)$ as $p \to p_0$ in the sense of topology $\mathcal{T}$ if the functions W_p, $\partial_t W_p$ and $\partial_{xx} W_p$ converge uniformly in any compact set $S \subset \mathbb{R}^2$ to W_{p_0}, $\partial_t W_{p_0}$ and $\partial_{xx} W_{p_0}$, as $p \to p_0$.

Proof of Theorem 1.5. By Lemmas 3.2 and 3.8, we have

$$\underline{w}(x,t) \le w^n(x,t) \le w^{n+1}(x,t) \le \min\big\{\mathbf{K}, \Pi_1(x,t), \Pi_2(x,t), \Pi_3(x,t)\big\}$$

for any $x \in \mathbb{R}$ and $t \ge -n-\tau$. It is easy to see that there exists $L' > 0$, which is independent of n, such that

$$\sup_{x\in\mathbb{R}} |\varphi_2^n(x+\eta, 0) - \varphi_2^n(x,0)| \le L'\eta, \quad \forall \eta > 0. \tag{3.12}$$

Thus, using Lemma 3.3 and the diagonal extraction process, there exists a subsequence $w^{n_l}(x,t)$ of $w^n(x,t)$ such that $w^{n_l}(x,t)$ converges to a function $W_p(x,t)$ in the sense of topology $\mathcal{T}$. Since $w^n(x,t) \le w^{n+1}(x,t)$ for any $t > -n$, we have

$$\lim_{n\to+\infty} w^n(x,t) = W_p(x,t) \quad \text{for any } (x,t) \in \mathbb{R}^2.$$

The limit function is unique, whence all of the functions $w^n(x,t)$ converge to the function $W_p(x,t)$ in the sense of topology $\mathcal{T}$ as $n \to +\infty$. Clearly, $W_p(x,t)$ is an entire solution of (1.1) satisfying (1.10).

Using (1.10) and the facts $\Phi_c(-\infty) = \mathbf{0}$ and $\Phi_c(+\infty) = \mathbf{K}$, it is easy to show that assertions (1)-(4) hold. This completes the proof. □

3.4. **Qualitative properties of the entire solutions.** In addition to the existence result of Theorem 1.5, in this section we further investigate some qualitative properties of the entire solution $W_p(x,t)$, such as the monotonicity and limit of $W_p(x,t)$ with respect to the variables x and t, and the shift parameters h_i.

For any $A,\gamma\in\mathbb{R}$, denote the regions $T^i_{A,\gamma}$, $i=1,\dots,6$ by

$$\begin{gathered} T^1_{A,\gamma}:=[A,\infty)\times[\gamma,\infty),\quad T^2_{A,\gamma}:=(-\infty,A]\times[\gamma,\infty),\\ T^3_{A,\gamma}:=\mathbb{R}\times[\gamma,\infty),\quad T^4_{A,\gamma}:=(-\infty,A]\times(-\infty,\gamma],\\ T^5_{A,\gamma}:=[A,\infty)\times(-\infty,\gamma],\quad T^6_{A,\gamma}:=\mathbb{R}\times(-\infty,\gamma]. \end{gathered}$$

Various qualitative properties of the entire solutions are stated in the sequel.

Theorem 3.10. *Let $W_p(x,t)$ be the entire solution of* (1.1) *as stated in Theorem 1.5, then the following properties hold.*

(1) *$W_p(x,t)\gg 0$ and $\partial_t W_p(x,t)\gg\mathbf{0}$ for all $(x,t)\in\mathbb{R}^2$.*
(2) *$\lim_{t\to+\infty}\sup_{x\in\mathbb{R}}\big\|W_p(x,t)-\mathbf{K}\big\|=0$ and $\lim_{t\to-\infty}\sup_{|x|\le A}\|W_p(x,t)\|=0$ for any $A\in\mathbb{R}_+$.*
(3) *If $\chi_1=1$ then $\lim_{x\to+\infty}\sup_{t\ge T}\big\|W_p(x,t)-\mathbf{K}\big\|=0$ for any $T\in\mathbb{R}$. If, in addition $\chi_2=0$, then $\partial_x W_p(x,t)\gg\mathbf{0}$ for all $(x,t)\in\mathbb{R}^2$.*
(4) *If $\chi_2=1$ then $\lim_{x\to-\infty}\sup_{t\ge T}\big\|W_p(x,t)-\mathbf{K}\big\|=0$ for any $T\in\mathbb{R}$. If, in addition $\chi_1=0$, then $\partial_x W_p(x,t)\ll\mathbf{0}$ for all $(x,t)\in\mathbb{R}^2$.*
(5) *If $\chi_3=1$, then for every $x\in\mathbb{R}$, $W_p(x,t)\sim\Gamma(t+h_3)\sim(1,b_*)e^{\lambda^*(t+h_3)}$ as $t\to-\infty$.*
(6) *If $\chi_3=0$ then for any $x\in\mathbb{R}$, there exist constants $D(x)>C(x)\gg\mathbf{0}$ such that*
$$C(x)e^{\vartheta(c_1,c_2)t}\le W_p(x,t)\le D(x)e^{\vartheta(c_1,c_2)t}$$
for $t\ll-1$, here $\vartheta(c_1,c_2):=\min\{c_1\lambda_1(c_1),c_2\lambda_1(c_2)\}$.
(7) *For any $x\in\mathbb{R}$, $W_p(x,t)$ is increasing with respect to h_i, $i=1,2,3$.*
(8) *For any $x\in\mathbb{R}$ and $\gamma\in\mathbb{R}$, $W_p(x,t)$ converges to K in the sense of topology $\mathcal{T}$ as $h_i\to+\infty$ and uniformly on $(x,t)\in T^i_{A,\gamma}$ for $i=1,2,3$.*

Proof. The assertions for parts (2)-(4) and (6)-(8) are direct consequences of (1.10). Therefore, we only prove the results of parts (1) and (5).

(1) From (1.10), one can see that $W_p(x,t)\gg 0$ for all $x\in\mathbb{R}$ and $t\in\mathbb{R}$. Since
$$w^n(x,t)\ge\underline{w}(x,t)\ge\underline{w}(x,s)=\varphi^n(x,s)$$
for all $(x,t)\in\mathbb{R}\times[-n,+\infty)$ and $s\in[-n-\tau,-n]$, by Lemma 3.2, we have $\frac{\partial}{\partial t}W_p(x,t)\ge 0$ for $(x,t)\in\mathbb{R}\times(-n,+\infty)$, which yields to $\frac{\partial}{\partial t}W_p(x,t)\ge 0$ for all $(x,t)\in\mathbb{R}^2$. Note that

$$\begin{aligned} \partial_{tt}U_p(x,t)&=d(\partial_t U_p)_{xx}-\alpha\partial_t U_p(x,t)+\int_{\mathbb{R}}J(x-y)\partial_t V_p(y,t)dy\\ &\ge d(\partial_t U_p)_{xx}-\alpha\partial_t U_p(x,t),\\ \partial_{tt}V_p(x,t)&=-\beta\partial_t V_p(x,t)+g'(U_p(x,t-\tau))\partial_t U_p(x,t-\tau)\\ &\ge-\beta\partial_t V_p(x,t), \end{aligned}$$

where $x\in\mathbb{R}$ and $t\in\mathbb{R}$. Hence, for any $r<t$, we have

$$\partial_t U_p(x,t)\ge\int_{-\infty}^{+\infty}J_d(x-y,t)e^{-\alpha(t-r)}\partial_t U_p(y,r)dy, \tag{3.13}$$

$$\partial_t V_p(x,t) \geq e^{-\beta(t-r)}\partial_t V_p(x,r) \tag{3.14}$$

for $x \in \mathbb{R}$ and $t > 0$, where $J_d(x,t) = \frac{1}{\sqrt{4d\pi t}}\exp\big\{-\frac{x^2}{4dt}\big\}$. Suppose for the contrary that there exist $(x_0,t_0) \in \mathbb{R}^2$ such that $\partial_t U_p(x_0,t_0) = 0$, then it follows from (3.13) that $\partial_t U_p(x_0,r) = 0$ for all $r \leq t_0$. Hence, $\lim_{t\to-\infty} U_p(x_0,t) = U_p(x_0,t_0)$. However, from (1.10),

$$\lim_{t\to-\infty} U_p(x_0,t) = 0 \quad \text{and } U_p(x_0,t_0) > 0.$$

This contradiction yields that $\partial_t U_p(x,t) > 0$ for $(x,t) \in \mathbb{R}^2$. Similarly, we can show that $\partial_t V_p(x,t) > 0$ for $(x,t) \in \mathbb{R}^2$. Therefore, $\partial_t W_p(x,t) \gg \mathbf{0}$ for all $(x,t) \in \mathbb{R}^2$.

(5) By Proposition 3.5, we know that

$$\min\big\{c_1\lambda_1(c_1), c_2\lambda_1(c_2)\big\} > \lambda^* \quad \text{for any } c_1, c_2 > c_*.$$

Then (1.10) implies

$$\begin{aligned}
\Gamma(t+h_3) &\leq W_p(x,t) \\
&\leq \chi_1(1,A_{c_1})e^{\lambda_1(c_1)(x+c_1t+h_1)} + \chi_2(1,A_{c_2})e^{\lambda_1(c_2)(-x+c_2t+h_2)} + \Gamma(t+h_3) \\
&\leq \chi_1(1,A_{c_1})e^{\lambda_1(c_1)(x+c_1t+h_1)} + \chi_2(1,A_{c_2})e^{\lambda_1(c_2)(-x+c_2t+h_2)} + (1,b_*)e^{\lambda^* t}.
\end{aligned}$$

Since $\lim_{t\to-\infty}\Gamma(t)e^{-\lambda^* t} = (1,b_*)$, the statement of (5) holds obviously. This completes the proof. $\square$

Moreover, according to the assumption $\chi_1,\chi_2,\chi_3 \in \{0,1\}$ with $\chi_1+\chi_2+\chi_3 \geq 2$ in Theorem 1.5, we further denote the entire solution $W_p(x,t)$ of (1.1) by

$$W_p(x,t) := \begin{cases} W_{p_0}(x,t), & \text{if } (\chi_1,\chi_2,\chi_3) = (1,1,1), \\ W_{p_1}(x,t), & \text{if } (\chi_1,\chi_2,\chi_3) = (0,1,1), \\ W_{p_2}(x,t), & \text{if } (\chi_1,\chi_2,\chi_3) = (1,0,1), \\ W_{p_3}(x,t), & \text{if } (\chi_1,\chi_2,\chi_3) = (1,1,0), \end{cases} \tag{3.15}$$

where $p = p_{\chi_1,\chi_2,\chi_3} = (\chi_1c_1, \chi_2c_2, \chi_1h_1, \chi_2h_2, \chi_3h_3)$, $p_0 = (c_1,c_2,h_1,h_2,h_3)$, and

$$p_1 = (0,c_2,0,h_2,h_3),\ p_2 = (c_1,0,h_1,0,h_3) \text{ and } p_3 = (c_1,c_2,h_1,h_2,0).$$

Then we have the following convergence results.

Theorem 3.11. *From* (3.15)*, we have the following properties.*

(1) *For any $x \in \mathbb{R}$ and $\gamma \in \mathbb{R}$, $W_{p_0}(x,t)$ converges (in the sense of topology $\mathcal{T}$) to $W_{p_i}(x,t)$ as $h_i \to -\infty$, and uniformly on $(x,t) \in T^{3+i}_{A,\gamma}$, $i = 1,2,3$.*

(2) *For any $x \in \mathbb{R}$ and $\gamma \in \mathbb{R}$, $W_{p_1}(x,t)$ converges (in the sense of topology $\mathcal{T}$) to $\Gamma(t+h_3)$ as $h_2 \to -\infty$, and uniformly on $(x,t) \in T^5_{A,\gamma}$; to $\Phi_{c_2}(-x+c_2t+h_2)$ as $h_3 \to -\infty$, and uniformly on $(x,t) \in T^6_{A,\gamma}$.*

(3) *For any $x \in \mathbb{R}$ and $\gamma \in \mathbb{R}$, $W_{p_2}(x,t)$ converges (in the sense of topology $\mathcal{T}$) to $\Gamma(t+h_3)$ as $h_1 \to -\infty$, and uniformly on $(x,t) \in T^4_{A,\gamma}$; to $\Phi_{c_1}(x+c_1t+h_1)$ as $h_3 \to -\infty$, and uniformly on $(x,t) \in T^6_{A,\gamma}$.*

(4) *For any $x \in \mathbb{R}$ and $\gamma \in \mathbb{R}$, $W_{p_3}(x,t)$ converges (in the sense of topology $\mathcal{T}$) to $\Phi_{c_2}(-x+c_2t+h_2)$ as $h_1 \to -\infty$, and uniformly on $(x,t) \in T^4_{A,\gamma}$; to $\Phi_{c_1}(x+c_1t+h_1)$ as $h_2 \to -\infty$, and uniformly on $(x,t) \in T^5_{A,\gamma}$.*

Proof. (1) We only prove the case that $W_{p_0}(x,t)$ converges to $W_{p_3}(x,t)$ in the sense of topology $\mathcal{T}$ and uniformly on $(x,t) \in T^6_{A,\gamma}$, as $h_3 \to -\infty$. The proofs for the other cases are similar.

For $(\chi_1,\chi_2,\chi_3) = (1,1,1)$, we denote $\varphi^n(x,s)$ by $\varphi^n_{p_0}(x,s)$ and $w^n(x,t)$ by $w^n_{p_0}(x,t)$, respectively. Similarly, when $(\chi_1,\chi_2,\chi_3) = (1,1,0)$, we denote $\varphi^n(x,s)$ by $\varphi^n_{p_3}(x,s)$ and $w^n(x,t)$ by $w^n_{p_3}(x,t)$, respectively. Let

$$Z^n(x,t) := (Z^n_1(x,t), Z^n_2(x,t)) = w^n_{p_0}(x,t) - w^n_{p_3}(x,t),$$

then $\mathbf{0} \le Z^n(x,t) \le \mathbf{K}$ for all $(x,t) \in \mathbb{R}\times(-n,+\infty)$ and

$$\begin{gathered}(Z^n_1)_t(x,t) = d(Z^n_1)_{xx} - \alpha Z^n_1(x,t) + \int_{\mathbb{R}} J(x-y)Z^n_2(y,t)dy,\\ (Z^n_2)_t(x,t) \le -\beta Z^n_2(x,t) + g'(0)Z^n_1(x,t-\tau),\end{gathered}$$

for $x\in\mathbb{R}$ and $t > -n$. Note that

$$Z^n(x,s) = \varphi^n_{p_0}(x,s) - \varphi^n_{p_3}(x,s) \le \Gamma(s+h_3) \le (1,b_*)e^{\lambda^*(s+h_3)} \quad \text{for } s\in[-n-\tau,-n]$$

and the function $V(x,t) := (V_1(x,t),V_2(x,t)) = (1,b_*)e^{\lambda^*(t+h_3)}$ satisfies

$$\begin{gathered}(V_1)_t(x,t) = d(V_1)_{xx} - \alpha V_1(x,t) + \int_{\mathbb{R}} J(x-y)V_2(y,t)dy,\\ (V_2)_t(x,t) = -\beta V_2(x,t) + g'(0)V_1(x,t-\tau),\end{gathered}$$

It then follows from Lemma 3.4 that

$$\mathbf{0} \le Z^n(x,t) \le (1,b_*)e^{\lambda^*(t+h_3)} \quad \text{for all } (x,t)\in\mathbb{R}\times[-n,+\infty).$$

Since $\lim_{n\to+\infty} w^n_{p_0}(x,t) = W_{p_0}(x,t)$ and $\lim_{n\to+\infty} w^n_{p_3}(x,t) = W_{p_3}(x,t)$, we obtain

$$\mathbf{0} \le W_{p_0}(x,t) - W_{p_3}(x,t) \le (1,b_*)e^{\lambda^*(t+h_3)} \quad \text{for all } (x,t)\in\mathbb{R}^2,$$

which implies that $W_{p_0}(x,t)$ converges to $W_{p_3}(x,t)$ as $h_3 \to -\infty$ uniformly on $(x,t)\in T^6_{A,\gamma}$ for any $\gamma\in\mathbb{R}$. For any sequence h^ℓ_3 with $h^\ell_3 \to -\infty$ as $\ell\to+\infty$, the functions $W_{p^\ell}(x,t)$ (here $p^\ell := (c_1,c_2,h_1,h_2,h^\ell_3)$) converge to a solution of (1.1) (up to extraction of some subsequence) in the sense of topology $\mathcal{T}$, which turns out to be $W_{p_3}(x,t)$. The limit does not depend on the sequence h^ℓ_3, whence all of the functions $W_{p_0}(x,t)$ converge to $W_{p_3}(x,t)$ in the sense of topology $\mathcal{T}$ as $h_3\to-\infty$. Hence the assertion of this part follows.

The proofs of parts (2)-(4) are similar to that of part (1), and omitted. This completes the proof. □

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