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OBSERVABILITY INEQUALITY AND DECAY RATE FOR WAVE EQUATIONS WITH NONLINEAR BOUNDARY CONDITIONS

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Abstract. We study a class of wave propagation problems concerning the nonlinearity of dynamic evolution for boundary material. We establish an observability inequality for the related linear system, and make a connection between the linear system and the original nonlinear coupled system. Also, we obtain the desired energy decay rate for the original nonlinear boundary value problem.

1. Introduction

We are concerned with the nonlinear boundary value problem

$$u_{tt}(x,t) = \Delta u(x,t), \quad x \in \Omega,\, t > 0; \tag{1.1}$$

$$u(x,t) = 0, \quad x \in \Gamma_0,\, t > 0; \tag{1.2}$$

$$u_t(x,t) + f(z_t) + g(z) = 0 \quad x \in \Gamma_1,\, t > 0; \tag{1.3}$$

$$\frac{\partial u}{\partial \nu} = z_t \quad x \in \Gamma_1,\ t > 0; \tag{1.4}$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega, \quad z(x,0) = z_0(x), \quad x \in \Gamma_1; \tag{1.5}$$

where Δ is the Laplacian operator, Ω is a bounded domain in $\mathbb{R}^n$ with a boundary $\Gamma = \Gamma_0 \cup \Gamma_1$ (disjoint, closed, and nonempty) of class C^2, and f, g are given functions on $\mathbb{R}$.

For some similar systems with or without source terms in (1.1), there exist several results about uniform decay rate of the solutions to these systems. For instance, [6, 9, 10, 11] study the porous boundary condition with the interface described by

$$u_t + f(x)z_t + g(x)z = 0, \quad x \in \Gamma_1,\, t > 0;$$

$$\frac{\partial u}{\partial \nu} + \rho(u_t) = z_t, \quad x \in \Gamma_1,\, t > 0,$$

where ρ is a given function. In this paper, we focus on the investigation of the problem above concerning the nonlinearity of dynamic evolution for boundary material, which is always described by boundary displacement z. We allow for nonlinear damping $f(z_t)$ and nonlinear potential $g(z)$ (f and g may depend on x also, which

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can be handled similarly) in the boundary displacement equation (1.3). Such nonlinearity enables our results to possess wide applicability.

Since our system is a coupled system and we hope to control the whole coupled system by only using a single boundary damping, which is different from and much more complex than the case of single equations, we will make efforts to establish the observability of the related linear system, to find a useful connection between the linear system and the original nonlinear system, and finally to obtain the decay rate of the energy. We also would like to state that our idea is stimulated by the significant papers [1, 2, 4, 6, 7, 8, 10, 13, 14].

We first present some notation, basic definitions and assumptions (cf., e.g., [1, 8]). Throughout this paper, c, c_i are as generic constants whose values may change from line to line. We make the following assumptions:

- (H1) there exists $x_0 \in \mathbb{R}^n$ such that $m(x) \cdot \nu(x) \leq 0$ for $x \in \Gamma_0$, where $m(x) = x - x_0$ and $\nu(x)$ is the unit normal vector pointing to the exterior of Ω.
- (H2) The function $g \in C(\mathbb{R})$ is monotone nondecreasing such that $g(0) = 0$; the function $f \in C^1(\mathbb{R})$ satisfies $f(0) = 0$ and $\inf_{s\in\mathbb{R}} f'(s) > 0$, and there exists a continuous strictly increasing odd function $\rho \in C([-1, 1]; \mathbb{R})$, which is continuously differentiable in a neighbourhood of 0 with $\rho(0) = \rho'(0) = 0$, such that

$$\begin{gathered} c_1\rho(|v|) \leq |f(v)| \leq c_2\rho^{-1}(|v|), \quad |v| \leq 1, \text{ a.e. on } \Gamma_1, \\ c_1|v| \leq |f(v)| \leq c_2|v|, \quad |v| \geq 1, \text{ a.e. on } \Gamma_1. \end{gathered} \tag{1.6}$$

  Moreover, $g(s)$ is locally Lipschitz continuous such that

$$c_1|v| \leq |g(v)| \leq c_2|v|, \quad |v| \geq 1, \text{ a.e. } \Gamma_1. \tag{1.7}$$

Also we define

$$H(x) := \sqrt{x}\rho(\sqrt{x}), \quad x \in [0, r_0^2], \tag{1.8}$$

$r_0 > 0$ being small enough such that H is strictly convex on $[0, r_0^2]$. We define

$$L(y) := \begin{cases} \hat{H}^\star(y)/y, & \text{if } y \in (0, \infty), \\ 0, & \text{if } y = 0. \end{cases} \tag{1.9}$$

Here

$$\hat{H}^\star := \sup_{x\in\mathbb{R}}\{xy - \hat{H}(x)\}$$

stands for the convex conjugate function of $\hat{H}$ (the extension of H to $\mathbb{R}$ in which $\hat{H}(x) = +\infty$ for $x \in \mathbb{R} \setminus [0, r_0^2]$). Moreover, we define a function Λ_H on $(0, r_0^2]$ by

$$\Lambda_H(x) := \frac{H(x)}{xH'(x)},$$

and write

$$\psi(x) := \frac{1}{H'(r_0^2)} + \int_{1/x}^{H'(r_0^2)} \frac{1}{v^2(1 - \Lambda_H((H')^{-1}(v)))} dv, \quad x \geq \frac{1}{H'(r_0^2)}.$$

Then, there exists $\delta > 0$ such that ψ is strictly increasing on $[0, \delta]$.

Let

$$V(\Omega) = \{u(x) \in H^1(\Omega), u|_{\Gamma_0} = 0\},$$

and define the inner products and norms on $V(\Omega)$, $L^2(\Omega)$, and $L^2(\Gamma_1)$ respectively as follows

$$((u,v))_V = \int_\Omega \nabla u(x)\cdot\nabla v(x)dx, \quad \|u\|_V = \Big(\int_\Omega |\nabla u\,(x)\,|^2 dx\Big)^{1/2},$$

$$(u,v) = \int_\Omega u(x)v(x)dx, \quad |u| = (\int_\Omega (u(x))^2 dx)^{1/2},$$

$$\langle \phi,\psi\rangle = \int_{\Gamma_1} \phi(x)\psi(x)d\Gamma, \quad |\phi|_{\Gamma_1} = (\int_{\Gamma_1} (\phi(x))^2 dx)^{1/2}.$$

Clearly, the $\|\cdot\|_V$ is equivalent to the usual H^1 norm.

Define the "finite energy space" by

$$\mathcal{H} := V(\Omega)\times L^2(\Omega)\times L^2(\Gamma_1),$$

where the norm on $\mathcal{H}$ is given by

$$|(u,v,z)|^2_{\mathcal{H}} = \|u\|_V^2 + |v|^2 + |z|^2_{\Gamma_1}.$$

Define the energy of system (1.1)-(1.5) by

$$E(t) := \frac{1}{2}\int_\Omega |\nabla u|^2 + u_t^2 dx + \frac{1}{2}\int_{\Gamma_1} z_t^2 d\Gamma + \int_{\Gamma_1} G(z)d\Gamma,$$

where $G(x) = \int_0^x g(s)ds$ is the anti-derivative of g.

2. Main results and proofs

Rewrite the system (1.1)-(1.5) as

$$\frac{\partial}{\partial t}\begin{pmatrix} u \\ u_t \\ z \end{pmatrix} = \begin{pmatrix} u_t \\ \Delta u \\ f^{-1}(-u_t|_{\Gamma_1} - g(z)) \end{pmatrix} = \mathcal{A}\begin{pmatrix} u \\ u_t \\ z \end{pmatrix}. \tag{2.1}$$

The action of the operator $\mathcal{A}$ is given by the matrix of operators that appears in (2.1). The remaining boundary conditions are encoded in the domain of $\mathcal{A}$, given by

$$D(\mathcal{A}) = \Big\{ \begin{pmatrix} u \\ v \\ z \end{pmatrix} \in \mathcal{H}; v\in V(\Omega), \Delta u \in L^2(\Omega), \frac{\partial u}{\partial \nu}\Big|_{\Gamma_1} = f^{-1}(-v|_{\Gamma_1} - g(z))\Big\}.$$

From (H2), one knows that f is strictly increasing, and its inverse function f^{-1} is Lipschitz continuous. Thus, using the standard method of nonlinear monotone operators and the semigroup theory (cf. [3]), we can obtain wellposedness of the system.

To study the energy decay rates of (1.1)-(1.5), we first give an observability inequality of the following linear system, which has the same initial values as the original nonlinear system:

$$P_{tt}(x,t) = \Delta P(x,t), \quad x\in\Omega,\ t>0; \tag{2.2}$$

$$P(x,t) = 0, \quad x\in\Gamma_0,\ t>0; \tag{2.3}$$

$$P_t(x,t) + M_t(x,t) + M(x,t) = 0, \quad x\in\Gamma_1,\ t>0; \tag{2.4}$$

$$\frac{\partial P(x,t)}{\partial \nu} = M_t, \quad x\in\Gamma_1,\ t>0; \tag{2.5}$$

$$P(x,0) = u_0(x), \quad P_t(x,0) = u_1(x), \quad x\in\Omega; \tag{2.6}$$

$$M(x,0)=z_0(x),\quad x\in\Gamma_1. \tag{2.7}$$

Using the multiplier method (cf., e.g., [2, 14]), we can prove the following observability inequality.

Theorem 2.1 (Observability inequality)**.** *There is a constant $T_0>0$, depending only on Ω, such that for $T\geq T_0$, there corresponds a positive constant C_T satisfying*

$$C_TE_p(0)\leq\int_0^T\int_{\Gamma_1}M_t^2\,dx\,dt, \tag{2.8}$$

where

$$E_p(t):=\frac{1}{2}\int_\Omega P_t^2+|\nabla P|^2dx+\frac{1}{2}\int_{\Gamma_1}M^2d\Gamma$$

is the energy of (2.2)-(2.7).

Proof. The proof is divided into the following 5 steps.

Step 1: Let $\xi(t)\in C_0^\infty(\mathbb{R})$ be the cutoff function defined by

$$\xi(t)=\begin{cases}1, & t\in[\epsilon_0,T-\epsilon_0]\\ \text{a } C^\infty \text{ function with range in } (0,1), & t\in(0,\epsilon_0)\cup(T-\epsilon_0,T)\\ 0, & t\in(-\infty,0)\cup(T,\infty),\end{cases}$$

for $\epsilon_0\in(0,T/2)$.

Let h be a $[C^2(\bar\Omega)]^n$-vector field, which will be specified later. Then, multiplying (2.2) by $h\cdot\nabla P$, integrating in time and space and using the boundary condition, we obtain

$$\begin{aligned}
0&=\int_{\epsilon_0}^{T-\epsilon_0}\int_\Omega h\cdot\nabla P(P_{tt}-\Delta P)\,dx\,dt\\
&=(h\cdot\nabla P,P_t)_{L^2(\Omega)}\Big|_{\epsilon_0}^{T-\epsilon_0}-\int_{\epsilon_0}^{T-\epsilon_0}\int_\Omega\Big[\nabla\cdot(\frac{h}{2}(P_t^2))-\frac{\nabla\cdot h}{2}P_t^2\\
&\quad-\nabla\cdot(\frac{h}{2}|\nabla P|^2)\Big]\,dx\,dt-\int_{\Gamma_1}\int_{\epsilon_0}^{T-\epsilon_0}h\cdot\nabla PM_td\Gamma dt\\
&\quad+\int_{\epsilon_0}^{T-\epsilon_0}\int_\Omega J|\nabla P|^2\,dx\,dt-\int_{\epsilon_0}^{T-\epsilon_0}\int_\Omega\frac{\nabla\cdot h}{2}|\nabla P|^2\,dx\,dt\\
&=(h\cdot\nabla P,P_t)_{L^2(\Omega)}\Big|_{\epsilon_0}^{T-\epsilon_0}-\int_\Gamma\int_{\epsilon_0}^{T-\epsilon_0}\frac{h\cdot\nu}{2}(P_t^2-|\nabla P|^2)d\Gamma dt\\
&\quad+\int_{\epsilon_0}^{T-\epsilon_0}\int_\Omega\frac{\nabla\cdot h}{2}(P_t^2-|\nabla P|^2)\,dx\,dt+\int_{\epsilon_0}^{T-\epsilon_0}\int_\Omega J|\nabla P|^2\,dx\,dt\\
&\quad-\int_{\Gamma_1}\int_{\epsilon_0}^{T-\epsilon_0}h\cdot\nabla PM_td\Gamma dt,
\end{aligned}$$

where $J:=\frac{\partial h_i(x)}{\partial x_j}$.

By (H1) we can take h such that

$$h\cdot\nu=0\quad\text{on }\Gamma_0,$$

$$J=\frac{\partial h_i(x)}{\partial x_j}\geq\rho_0I\quad\text{on }\Omega,$$

for some constant $\rho_0 > 0$. Hence,

$$\begin{aligned}
&\rho_0 \int_{\epsilon_0}^{T-\epsilon_0} \int_\Omega |\nabla P|^2 \,dx\,dt \\
&\le \int_{\epsilon_0}^{T-\epsilon_0} \int_\Omega J|\nabla P|^2 \,dx\,dt \\
&\le \int_{\epsilon_0}^{T-\epsilon_0} \int_{\Gamma_1} h\cdot\nabla P M_t d\Gamma dt + \int_{\epsilon_0}^{T-\epsilon_0} \int_{\Gamma_1} \frac{h\cdot\nu}{2}(P_t^2 - |\nabla P|^2) d\Gamma dt \\
&\quad - (h\cdot\nabla P, P_t)_{L^2(\Omega)}\Big|_{\epsilon_0}^{T-\epsilon_0} - \int_{\epsilon_0}^{T-\epsilon_0} \int_\Omega \frac{\nabla\cdot h}{2}(P_t^2 - |\nabla P|^2) \,dx\,dt.
\end{aligned}$$

Since

$$|\nabla P|^2 = (M_t^2 + |\frac{\partial P}{\partial \tau}|^2), \quad E_p' = -\int_{\Gamma_1} M_t^2 d\Gamma \le 0,$$

we have

$$\begin{aligned}
&\rho_0 \int_{\epsilon_0}^{T-\epsilon_0} \int_\Omega |\nabla P|^2 \,dx\,dt \\
&\le \Big| \int_{\epsilon_0}^{T-\epsilon_0} \int_\Omega \frac{\nabla\cdot h}{2}(P_t^2 - |\nabla P|^2) \,dx\,dt \Big| \\
&\quad + C_h \Big[\int_{\Sigma_1} M_t^2 d\Gamma dt + \int_{\epsilon_0}^{T-\epsilon_0} \int_{\Gamma_1} P_t^2 + |\frac{\partial P}{\partial \tau}|^2 d\Gamma dt \Big] + C_h E_p(0),
\end{aligned} \tag{2.9}$$

where $\Sigma_1 := (0,T)\times\Gamma_1$, and C_h is a positive constant depending on h. Write

$$\text{l.o.t}(P,M) := \|(P,P_t,M)\|_{C([0,T];H^{1-\epsilon}(\Omega)\times H^{-\epsilon}(\Omega)\times H^{-\epsilon}(\Gamma_1))},$$

for $\epsilon > 0$.

Multiplying (2.2) by $P\nabla\cdot h$, integrating in time and space, and using the boundary condition and Sobolev Trace Theory, we obtain

$$\begin{aligned}
&\Big| \int_{\epsilon_0}^{T-\epsilon_0} \int_\Omega \nabla\cdot h(P_t^2 - |\nabla P|^2) \,dx\,dt \Big| \\
&= \Big| \langle P_t, P\nabla\cdot h\rangle_{H^{-\epsilon}(\Omega)\times H^{\epsilon}(\Omega)}\Big|_{\epsilon_0}^{T-\epsilon_0} + \int_{\epsilon_0}^{T-\epsilon_0} \int_\Omega P\nabla P\cdot\nabla(\nabla\cdot h) \,dx\,dt \\
&\quad - \int_{\epsilon_0}^{T-\epsilon_0} \int_{\Gamma_1} P\nabla\cdot h M_t d\Gamma dt \Big| \\
&\le C_\epsilon \int_{\Sigma_1} M_t^2 d\Gamma dt + \epsilon \int_{\epsilon_0}^{T-\epsilon_0} \int_\Omega |\nabla P|^2 \,dx\,dt + \text{l.o.t}(P,M).
\end{aligned} \tag{2.10}$$

Let $\min\{\nabla h\} = d_0 > 0$. Combining (2.10) and (2.9) gives

$$\begin{aligned}
&\int_{\epsilon_0}^{T-\epsilon_0} \int_\Omega |\nabla P|^2 + P_t^2 \,dx\,dt \\
&\le C_{\epsilon,h} \Big\{ \int_{\Sigma_1} M_t^2 d\Gamma dt + \int_{\epsilon_0}^{T-\epsilon_0} \int_{\Gamma_1} (P_t^2 + |\frac{\partial P}{\partial \tau}|^2) d\Gamma dt \Big\} \\
&\quad + C_h E_p(0) + \text{l.o.t}(P,M).
\end{aligned} \tag{2.11}$$

Using [2, Lemma 4] to estimate $\int_{\epsilon_0}^{T-\epsilon_0}\int_{\Gamma_1}|\frac{\partial P}{\partial \tau}|^2 d\Gamma dt$ in (2.11), we obtain

$$
\begin{aligned}
&\int_{\epsilon_0}^{T-\epsilon_0}\int_\Omega |\nabla P|^2 + P_t^2\,dx\,dt \\
&\le C_{T,\epsilon_0,h}\Big\{\int_{\Sigma_1} M_t^2 + \xi^2 P_t^2 d\Gamma dt + \int_{\epsilon_0}^{T-\epsilon_0}\int_{\Gamma_1} P_t^2 d\Gamma dt\Big\} \\
&\quad + C_h E_p(0) + \text{l.o.t}(P,M).
\end{aligned}
\tag{2.12}
$$

Step 2: We estimate $\int_{\epsilon_0}^{T-\epsilon_0}\int_{\Gamma_1} P_t^2 d\Gamma dt + \int_{\Sigma_1}\xi^2 P_t^2 d\Gamma dt$. The boundary condition on Γ_1 shows that

$$
\int_{\epsilon_0}^{T-\epsilon_0}\int_{\Gamma_1} P_t^2 d\Gamma dt \le \int_{\Sigma_1}\xi^2 P_t^2 d\Gamma dt \le 2\int_{\Sigma_1} M_t^2 + M^2 d\Gamma dt.
$$

By (2.12), we have

$$
\int_{\epsilon_0}^{T-\epsilon_0} E_p(t)dt \le C_{T,\epsilon_0,h,f}\int_{\Sigma_1}(M_t^2+M^2)d\Gamma dt + C_h E_p(0) + \text{l.o.t}(P,M). \tag{2.13}
$$

From $E_p' = -\int_{\Gamma_1} M_t^2 d\Gamma$, it follows that

$$
\begin{aligned}
&(T-2\epsilon_0)\Big[E_p(0) - \int_{\Sigma_1} M_t^2 d\Gamma dt\Big] \\
&\le (T-2\epsilon_0)E_p(T) \\
&\le \int_{\epsilon_0}^{T-\epsilon_0} E_p dt \\
&\le C_{T,\epsilon_0,h,f}\int_{\Sigma_1}(M_t^2+M^2)d\Gamma dt + C_h E_p(0) + \text{l.o.t}(P,M).
\end{aligned}
\tag{2.14}
$$

Step 3: We estimate $\int_{\Sigma_1} M^2 d\Gamma dt$. Multiplying (2.4) by M and integrating in time and space, we obtain

$$
\begin{aligned}
0 &= \int_{\Sigma_1} M(P_t - M_t + M)d\Gamma dt \\
&= \int_{\Gamma_1} MP d\Gamma\Big|_{t=0}^{t=T} - \int_{\Sigma_1}(M_tP + MM_t - M^2)d\Gamma dt.
\end{aligned}
$$

Hence

$$
\begin{aligned}
\int_{\Sigma_1} M^2 d\Gamma dt &= \Big|\int_{\Sigma_1} MM_t d\Gamma dt + \int_{\Sigma_1} M_tP d\Gamma dt - \int_{\Gamma_1} MP d\Gamma\Big|_{t=0}^{t=T}\Big| \\
&\le \epsilon_1\int_{\Sigma_1} M^2 d\Gamma dt + C_{\epsilon_1}\int_{\Sigma_1} M_t^2 d\Gamma dt + \text{l.o.t.}(P,M),
\end{aligned}
\tag{2.15}
$$

where ϵ_1 is arbitrarily small. Combining this with (2.14), we obtain

$$
(T-2\epsilon_0-C_h)E_p(0) \le C_{T,\epsilon_0,h}\int_{\Sigma_1} M_t^2 d\Gamma dt + \text{l.o.t}(P,M). \tag{2.16}
$$

Therefore, for $T > T_0 := 2\epsilon_0 - C_h$, we almost get (2.8) except for the lower-order terms l.o.t(P,M).

Step 5: We claim that for

$$
T > T_1 = \max\{T_0, 2\text{diam}(\Omega)\},
$$

there exists a constant $C_T > 0$ such that the solution of (2.2)-(2.7) satisfies the inequality

$$\text{l.o.t}(P, M) \le C_T \|M_t\|^2_{L^2(\Sigma_1)}. \tag{2.17}$$

Suppose this is false. Then there exists a sequence

$$(P(0)^n, P_t(0)^n, M(0)^n) \subset \mathcal{H},$$

and a corresponding solution sequence (P^n, P^n_t, M^n) of (2.2)-(2.7) such that

$$\text{l.o.t}(P^n, M^n) = 1 \quad \forall n,$$
$$\|M^n_t\|^2_{L^2(\Sigma_1)} \to 0 \quad n \to \infty.$$

Thus, by (2.16), we see that $\|(P(0)^n, P_t(0)^n, M(0)^n)\|_{\mathcal{H}}$ is bounded for T large enough. Hence there is a subsequence, still denoted by

$$(P(0)^n, P_t(0)^n, M(0)^n), \quad (P(0)^*, P_t(0)^*, M(0)^*),$$

such that

$$(P(0)^n, P_t(0)^n, M(0)^n) \to (P(0)^*, P_t(0)^*, M(0)^*), \quad \text{in } \mathcal{H} \text{ weakly.} \tag{2.18}$$

Let (P^*, P^*_t, M^*) be the solution corresponding to $(P(0)^*, P_t(0)^*, M(0)^*)$. Then from

$$E'_p = -\int_{\Gamma_1} M_t^2 d\Gamma < 0,$$

it follows that

$$(P^n, P^n_t, M^n) \to (P^*, P^*_t, M^*), \quad \text{weak star in } L^\infty(0,T;\mathcal{H}). \tag{2.19}$$

Clearly, $\|(P^n, P^n_t, M^n)\|_{C(0,T;\mathcal{H})}$ is bounded by the wellposedness of the system. Let

$$X := H^1(\Omega) \times L^2(\Omega) \times L^2(\Gamma_1),$$
$$B := H^{1-\epsilon}(\Omega) \times H^{-\epsilon}(\Omega) \times H^{-\epsilon}(\Gamma_1),$$
$$Y := H^{-\epsilon}(\Omega) \times (H^1(\Omega))' \times H^{-\epsilon}(\Gamma_1).$$

We claim that $X \hookrightarrow B$ compactly. Indeed, for all $s, t \in \mathbb{R}$ with $s > t$, for an arbitrary bounded set $\{\psi_n\} \subset H^s(\Omega)$, we can extend the domain of ψ_n to $\hat{\Omega}$, such that $\psi_n|_{\partial\hat{\Omega}} = 0$. It is known that $H^s_0(\hat{\Omega})$ is compactly embedded in $H^t_0(\hat{\Omega})$. Hence, there exists a $\psi \in H^t_0(\hat{\Omega})$ such that $\|\psi_{n_i} - \psi\|_{H^t_0(\hat{\Omega})} \to 0$. Hence $\|\psi_{n_i} - \psi\|_{H^t(\Omega)} \to 0$.

We also claim that

$$\|(P^n_t, P^n_{tt}, M^n_t)\|_{L^2(0,T;Y)} \le C \quad \text{uniformly.}$$

Indeed, it suffices to estimate $\|P^n_{tt}\|_{L^2(0,T;(H^1(\Omega))')}$. By (2.2) and the boundary condition, we see that for all $t \in (0,T)$ and $u \in H^1(\Omega)$,

$$\langle P_{tt}, u\rangle = \int_\Omega \Delta P u dx = \int_{\Gamma_1} M_t u d\Gamma - \int_\Omega \nabla P \cdot \nabla u dx \tag{2.20}$$

is meaningful. Hence $P_{tt} \in L^\infty(0,T;(H^1(\Omega))')$.

We deduce then by a classic compactness result (see [12]) that

$$(P^n, P^n_t, M^n) \to (P^*, P^*_t, M^*) \quad \text{in } L^\infty(0,T;B) \text{ strongly.}$$

Therefore,

$$\|(P^*, P^*_t, M^*)\|_{C([0,T];H^{1-\epsilon}(\Omega)\times H^{-\epsilon}(\Omega)\times H^{-\epsilon}(\Gamma_1))} = 1. \tag{2.21}$$

On the other hand, by (2.18), we have $M_t^*=0$. Differentiating (2.4) in time, we obtain $P_{tt}^*|_{\Gamma_1}=0$. Let $a(t,x)=P_{tt}^*(t,x)$ such that

$$
\begin{gathered}
a_{tt}=\Delta a, \quad \text{in } \Omega\times(0,T),\\
\frac{\partial a}{\partial \nu}=(\frac{\partial P}{\partial \nu})_{tt}=0, \quad \text{on } \Gamma\times(0,T),\\
a=0, \quad \text{on } \Gamma_1.
\end{gathered}
$$

Using Holmgren's Uniqueness Theorem [10], with $T>2\text{diam}(\Omega)$,

$$
a(t,x)=P_{tt}^*(t,x)=0, \quad \text{in } \Omega\times(0,T).
$$

Then from

$$
\begin{gathered}
\Delta P^*=0, \quad \text{in } \Omega,\\
P^*|_{\Gamma_0}=0, \quad \frac{\partial P^*}{\partial \nu}|_{\Gamma_1}=0,
\end{gathered}
$$

we know that $P^*=0$. So we obtain $M^*=0$ due to (2.4). Thus $(P^*,M^*)=(0,0)$ contradicts (2.21). A combination of Steps 1-5 completes the proof. □

Next we show a connection between linear and nonlinear systems.

Theorem 2.2. *Assume that* (u,u_t,z) *and* (P,P_t,M) *are solutions of system* (1.1)-(1.5) *and* (2.2)-(2.7) *respectively. Then*

$$
\int_{\Sigma_1} M_t^2 d\Gamma dt \le C\int_{\Sigma_1} z_t^2+f(z_t)^2 d\Gamma dt. \tag{2.22}
$$

Proof. Set $\xi=u-P$, $\eta=z-M$. Then (ξ,ξ_t,η) is the solution of

$$
\begin{gathered}
\xi_{tt}(x,t)=\Delta\xi(x,t), \quad x\in\Omega,\ t>0;\\
\frac{\partial \xi(x,t)}{\partial \nu}=0 \quad x\in\Gamma_0,\ t>0;\\
\xi_t(x,t)+f(z_t)-M_t+g(z)-M=0 \quad x\in\Gamma_1,\ t>0;\\
\frac{\partial \xi}{\partial \nu}(x,t)=\eta_t(x,t) \quad x\in\Gamma_1,\ t>0;\\
\xi(x,0)=0, \quad \xi_t(x,0)=0, \quad x\in\Omega;\\
\eta(x,0)=0, \quad x\in\Gamma_1.
\end{gathered} \tag{2.23}
$$

Multiplying (2.23) by ξ_t, integrating in time and space, we obtain

$$
\begin{aligned}
&\int_0^t\int_\Omega(\frac{\xi_t^2}{2}+\frac{|\nabla\xi|^2}{2})_t\,dx\,dt\\
&=\int_0^t\int_{\Gamma_1}\frac{\partial\xi}{\partial\nu}\xi_t d\Gamma dt\\
&=\int\int_{\Gamma_1}\eta(M_t-f(z_t)+M-g(z))d\Gamma dt\\
&=\int\int_{\Gamma_1}(z_t-M_t)[M_t-f(z_t)+M-g(z)]d\Gamma dt.
\end{aligned} \tag{2.24}
$$

Take $\epsilon>0$ small enough. (1.7) implies that there exist $c_1>0$, $c_2>0$ such that

$$
c_1|v|\le|g(v)|\le c_2|v|, \quad |v|\ge\epsilon, \text{a.e. } \Gamma_1.
$$

Assuming $z > \epsilon$, we have, by (2.24) and $\int_{\Gamma_1} -z_t f(z_t) d\Gamma \leq 0$,

$$\begin{aligned}\int_0^t \int_\Omega (\frac{\xi_t^2}{2} + \frac{|\nabla \xi|^2}{2})_t \,dx\,dt \leq & \int_0^t \int_{\Gamma_1} -M_t^2 + M_t f(z_t) + z_t M_t d\Gamma dt \\ & + \int_0^t \int_{\Gamma_1 \cap \{\eta_t \geq 0\}} \max\{-\eta\eta_t, -c_1 \eta\eta_t\} d\Gamma dt \\ & + \int_0^t \int_{\Gamma_1 \cap \{\eta_t < 0\}} \max\{-\eta\eta_t, -c_2 \eta\eta_t\} d\Gamma dt.\end{aligned}$$

Therefore

$$\begin{aligned}&\int_0^t \int_\Omega (\frac{\xi_t^2}{2} + \frac{|\nabla \xi|^2}{2})_t \,dx\,dt + \int_0^t \int_{\Gamma_1} M_t^2 d\Gamma dt \\ &\leq \int_0^t \int_{\Gamma_1} M_t f(z_t) + z_t M_t d\Gamma dt \\ &\quad + \int_0^t \int_{\Gamma_1 \cap \{\eta_t \geq 0\}} \max\{-(\frac{\eta^2}{2})_t, -c_1 (\frac{\eta^2}{2})_t\} d\Gamma dt \\ &\quad + \int_0^t \int_{\Gamma_1 \cap \{\eta_t < 0\}} \max\{-(\frac{\eta^2}{2})_t, -c_2 (\frac{\eta^2}{2})_t\} d\Gamma dt.\end{aligned}$$

Noting the initial values and using Young's inequality, we obtain

$$\begin{aligned}&\int_0^t \int_\Omega (\frac{\xi_t^2}{2} + \frac{|\nabla \xi|^2}{2})_t \,dx\,dt + \int_0^t \int_{\Gamma_1} M_t^2 d\Gamma dt \\ &\leq c \int_0^t \int_{\Gamma_1} f(z_t)^2 + z_t^2 d\Gamma dt\end{aligned}, \tag{2.25}$$

giving (2.22). Similarly, we obtain (2.22) for $z < -\epsilon$.

Finally, choose ϵ small enough such that $|g(z)| \leq c\epsilon$ and $|z| \leq \epsilon$. By (2.24) we have

$$\int_0^t \int_\Omega (\frac{\xi_t^2}{2} + \frac{|\nabla \xi|^2}{2})_t \,dx\,dt = \int_0^t \int_{\Gamma_1} (z_t - M_t)[M_t - f(z_t) + M - z + z - g(z)] d\Gamma dt,$$

and

$$\begin{aligned}&\int_0^t \int_\Omega (\frac{\xi_t^2}{2} + \frac{|\nabla \xi|^2}{2} + \frac{\eta^2}{2})_t \,dx\,dt \\ &\leq \int_0^t \int_{\Gamma_1} [-M_t^2 + z_t M_t + M_t f(z_t) + z z_t - M_t z - z_t g(z) + M_t g(z)] d\Gamma dt.\end{aligned}$$

By Young's inequality and Hölder's inequality, we obtain

$$\begin{aligned}&\int_0^t \int_\Omega (\frac{\xi_t^2}{2} + \frac{|\nabla \xi|^2}{2} + \frac{\eta^2}{2})_t \,dx\,dt + \int_0^t \int_{\Gamma_1} M_t^2 d\Gamma dt \\ &\leq \int_0^t \int_{\Gamma_1} [\epsilon_0 M_t^2 + C(\epsilon_0)(z_t^2 + f(z_t)^2 + \epsilon^2)] d\Gamma dt.\end{aligned} \tag{2.26}$$

Since the constant in (2.25) dose not depend on ϵ, we can let $\epsilon \to 0$ in (2.26). Noticing the initial values, we then obtain (2.22). □

Theorem 2.3 (Decay rate)**.** *Suppose that*

$$\lim_{x\to 0^+} \frac{H'(x)}{\Lambda_H(x)} = 0,$$

and T *is a time such that* (2.8) *holds. Then the energy of system* (1.1)-(1.5) *satisfies*

$$E(t) \le C(T, E(0)) L\Big(\frac{1}{\psi^{-1}(\frac{t-T}{T_\star})}\Big),$$

for t *large enough; moreover, if*

$$\limsup_{x\to 0^+} \Lambda_H(x) < 1,$$

then we have

$$E(t) \le C(T, E(0))(H')^{-1}\left(\frac{c_0}{t-T}\right), \quad \text{for } t \text{ large enough.}$$

Here, $C(T, E(0))$ *is a positive constant depending on* T *and* $E(0)$*, and* $T_\star > 0$ *depends on* T*.*

Proof. Clearly, we see that

$$\begin{aligned}
\int_{\Gamma_1} G(z)d\Gamma &= \int_{\Gamma_1}\int_0^z g(s)ds \\
&\le \int_{\Gamma_1\cap\{z\ge 1\}}\int_0^z c_2 s ds d\Gamma + \int_{\Gamma_1\cap\{z\le -1\}}\int_0^z c_1 s ds d\Gamma \\
&\quad + \int_{\Gamma_1\cap\{|z|\le 1\}}\int_0^z g(s) ds d\Gamma \\
&\le \frac{c}{2}\int_{\Gamma_1} z^2 d\Gamma.
\end{aligned}$$

Setting $c_0 = \max(c, 1)$, we have

$$E(0) \le c_0 E_p(0). \tag{2.27}$$

Let w satisfy

$$H^\star(w(s)) = \frac{sw(s)}{\beta}, \quad s \in [0, \beta r_0^2),$$

where

$$\beta > \max\Big\{\frac{E(0)}{c_0 L(H'(r_0^2))}, \frac{E(0)}{c_0\delta}\Big\}, \tag{2.28}$$

r_0 is as in (1.8), and $\delta > 0$ is a constant such that ψ is strictly increasing on $[0, \delta]$. Then the definition of L implies

$$w(s) = L^{-1}\left(\frac{s}{\beta}\right), \quad \forall s \in [0, \beta r_0^2). \tag{2.29}$$

From the property of L, it follows that w is a strictly increasing function from $[0, \beta r_0^2)$ onto $[0, +\infty)$. Thus, by using the optimal-weight convexity method (cf. [1, Lemma 2.1]), we deduce that

$$\begin{aligned}
&w(E_p(0))\int_{\Sigma_1} z_t^2 + f(z_t)^2 d\Gamma dt \\
&\le c_3 T H^\star(w(E_p(0))) + c_4(w(E_p(0)) + 1)\int_{\Sigma_1} z_t f(z_t) d\Gamma dt.
\end{aligned}$$

This and Theorems 2.1 and 2.2 yield

$$\begin{aligned}
&C_T E_p(0) w(E_p(0)) \\
&\le w(E_p(0)) \int_0^T \int_{\Gamma_1} M_t^2 d\Gamma dt \le C w(E_p(0)) \int_0^T \int_{\Gamma_1} z_t^2 + f(z_t)^2 d\Gamma dt \\
&\le T\tilde{c}_3 H^\star(w(E_p(0))) + c_6(w(E_p(0)) + 1) \int_{\Sigma_1} z_t f(z_t) d\Gamma dt \\
&\le T c_5 \frac{E_p(0) w(E_p(0))}{\beta} + c_6(H'(r_0^2) + 1) \int_{\Sigma_1} z_t f(z_t) d\Gamma dt,
\end{aligned}$$

where we used (2.29) and $\beta > \frac{E(0)}{c_0 L(H'(r_0^2))}$ in the last inequality. From this and (2.27), we have

$$\Big(\tilde{C}_T - \frac{\tilde{c}_5 T}{\beta}\Big) \frac{E(0)}{c_0} w\Big(\frac{E(0)}{c_0}\Big) \le E(0) - E(T).$$

Thanks to $\beta > \frac{T\tilde{c}_5}{\tilde{C}_T}$, we set

$$\rho_T := \frac{1}{c_0}\Big(\tilde{C}_T - \frac{T\tilde{c}_5}{\beta}\Big) > 0 \tag{2.30}$$

and deduce that

$$E(T) \le E(0)\Big[1 - \rho_T w(\frac{E(0)}{c_0})\Big] = E(0)\Big[1 - \rho_T L^{-1}(\frac{E(0)}{c_0 \beta})\Big].$$

Denoting $E_k := \frac{E(kT)}{c_0 \beta}$, we obtain

$$E_1 \le E_0[1 - \rho_T L^{-1}(E_0)].$$

From the invariance by time translation $t - kT$ for system (1.1)-(1.5) and (2.2)-(2.7), we have

$$E_{k+1} \le E_k[1 - \rho_T L^{-1}(E_k)].$$

Because $\beta > \frac{E(0)}{c_0 \delta}$, we can apply [1, Theorem 1.5] to complete the proof. □

Remark 2.4. Under the assumptions of Theorem 2.3, we have

$$L\Big(\frac{1}{\psi^{-1}(\frac{t-T}{T_\star})}\Big) \to 0, \text{ as } t \to 0.$$

Moreover, by taking special f and g, we can see clearly the meaning of the decay rate (please see the examples in [1, Section 4]).

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