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MULTIPLICITY OF SOLUTIONS TO FOURTH-ORDER SUPERLINEAR ELLIPTIC PROBLEMS UNDER NAVIER CONDITIONS

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Abstract. We establish the existence and multiplicity of solutions for a class of fourth-order superlinear elliptic problems under Navier conditions on the boundary. Here we do not use the Ambrosetti-Rabinowitz condition; instead we assume that the nonlinear term is a nonlinear function which is nonquadratic at infinity.

1. Introduction

In this work we shall consider the fourth-order elliptic problem

$$\begin{gathered} \alpha\Delta^2 u + \beta\Delta u = f(x,u) \quad \text{in } \Omega, \\ u = \Delta u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where $\Delta^2 = \Delta \circ \Delta$ is the biharmonic operator, $N \geq 4, \Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $\alpha > 0$, $\beta \in (-\infty, \alpha\lambda_1)$. Problem (1.1) is called fourth-order elliptic problem under Navier boundary conditions. Here and throughout this paper λ_1 denotes the first eigenvalue problem on $(-\Delta, H_0^1(\Omega))$. The nonlinear term f is a continuous function which is superlinear at infinity and at the origin. Latter on, we shall consider the assumptions on the nonlinear term f.

Semilinear elliptic problems involving operators of fourth order have been considered since the pioneer paper Lazer and Mckenna [23]. In that work Lazer and Mackenna modeled nonlinear oscillations for suspensions bridges. It is worthwhile to mention that problem (1.1) models static deflection of an elastic plate in a fluid, see [21, 22, 23, 24, 25]. The same problems can be used to describe the static form change of beam or the motion of rigid body. Equations of this type have been extensively studied during the last years. Here we refer the reader to [3, 6, 9, 11, 15, 16, 26, 28, 32, 33, 35, 37] and references therein. In these papers existence and multiplicity of solutions have been considered using several assumptions on the nonlinear term f. Most of them considered the case $f(x,t) = b[(t+1)^+ - 1]$ or f satisfying the well known Ambrosetti-Rabinowitz superlinear condition at infinity.

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The main goal in this work is to consider fourth-order elliptic problems without the Ambrosetti-Rabinowitz condition introduced in [1]. The main difficulty arises from the fact that Palais-Smale sequences are not necessary bounded under our assumptions. To overcome this difficulty we apply the nonquadraticity condition introduced by Costa-Magalhẽs [8] proving that any Cerami sequences are necessary bounded, see Section 2 ahead. It is important to emphasize that compactness results such as Cerami condition is a powerful tool in order to apply variational methods.

Notice that fourth-order elliptic problems are modeled in the working space $\mathcal{H} = H_0^1(\Omega) \cap H^2(\Omega)$. This space is an Hilbert space endowed with the norm

$$\|u\| = \Big(\int_\Omega \big(\alpha|\Delta u|^2 - \beta|\nabla u|^2\big)\, dx\Big)^{1/2}, \quad u \in \mathcal{H},$$

and the inner product

$$\langle u, v\rangle = \int_\Omega (\alpha \Delta u \Delta v - \beta \nabla u \nabla v)\, dx, \quad u, v \in \mathcal{H}.$$

The weak solutions for problem (1.1) are precisely the critical points for the functional of C^1 class $I : \mathcal{H} \to \mathbb{R}$ given by

$$I(u) = \frac{1}{2}\int_\Omega \big(\alpha|\Delta u|^2 - \beta|\nabla u|^2\big)\ dx - \int_\Omega F(x,u)\, dx, \tag{1.2}$$

where the primitive for f is denoted by $F(x,t) = \int_0^t f(x,s)ds$, $x \in \Omega$, $t \in \mathbb{R}$. More specifically, given $u \in \mathcal{H}$ we have that $I'(u)$ belongs to $\mathcal{H}'$ and

$$I'(u)v = \langle u, v\rangle - \int_\Omega f(x,u) v dx \quad \text{for any } u, v \in \mathcal{H}.$$

Here $I'(u)v$ is standard for the duality product between $\mathcal{H}$ and $\mathcal{H}'$. Furthermore, $u \in \mathcal{H}$ is a critical point for I if and only if u is a weak solution to the elliptic problem (1.1).

In this work we denote (λ_i) the sequence of eigenvalues on $(-\Delta, H_0^1(\Omega))$. Consider the eigenvalue problem

$$\begin{gathered} \alpha\Delta^2 u + \beta\Delta u = \mu u \quad \text{in } \Omega,\\ u = \Delta u = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{1.3}$$

It is easy to verify that $\mu \in \mathbb{R}$ is an eigenvalue for problem (1.3) if only if

$$\mu_i = \lambda_i\left(\lambda_i \alpha - \beta\right), \quad i \in \mathbb{N}.$$

Furthermore, the eigenfunctions for the problem (1.3) are the eigenfunctions for the problem $(-\Delta, H_0^1(\Omega))$.

Throughout this work we assume that $f \in C^0(\bar{\Omega} \times \mathbb{R}, \mathbb{R})$. Furthermore, we shall consider the following hypotheses

(H1) There exist $a_1 > 0$ and $p \in (2, 2_*)$ such that

$$|f(x,t)| \le a_1(1 + |t|^{p-1}), \quad \text{for any } (x,t) \in \Omega \times \mathbb{R}$$

where $2_* = 2N/(N-4)$.

(H2) $\lim_{|t|\to\infty} \frac{f(x,t)}{t} = +\infty$ uniformly in Ω;

(H3) There exists $f_0 \in [0, \mu_1)$ such that

$$\lim_{|t|\to 0} \frac{f(x,t)}{t} = f_0 \quad \text{uniformly in } \Omega,$$

where $\mu_1 = \lambda_1(\alpha\lambda_1 - \beta) > 0$.

It is worthwhile to mention that hypothesis (H3) implies that

$$f(x, 0) = 0, \quad \forall x \in \Omega.$$

As a consequence $u \equiv 0$ is a trivial solution to the elliptic problem (1.1). Hence, applying (H1)–(H3), the main objective in the present work is to verify the existence of nontrivial solutions.

It is important to mention that $2_* = 2N/(N-4)$ is the critical Sobolev exponent for fourth order elliptic equations for any $N \geq 4$. More precisely, we have that $\mathcal{H}$ is continuous embedding into $L^s(\Omega)$ for any $s \in [1, 2_*]$ where $N \geq 5$. For the case $N = 4$ we have that $\mathcal{H}$ is not included in $L^\infty(\Omega)$ and for this case we consider $p \in (2, 2_*)$ in the following form $2 < p < \infty$. Notice also that the embedding $\mathcal{H} \subset L^s(\Omega)$ is compact for any $s \in [1, 2_*)$ where we put $N \geq 4$.

Under hypotheses (H1)–(H3) problem (1.1) is superlinear at infinity and at the origin. In [26, 28] the authors considered fourth order elliptic problems where the nonlinear term is a function that satisfies the well known Ambrosetti-Rabinowitz condition at infinity. Namely, the Ambrosetti-Rabinowitz condition, in short (AR) condition, says that: There are $\theta > 2$ and $R > 0$ such that

$$0 < \theta F(x, t) \leq t f(x, t), \quad |t| \geq R, \ x \in \Omega.$$

Using a standard procedure recall that (AR) condition implies

$$F(x, t) \geq c_1|t|^\theta - c_2, \quad t \in \mathbb{R}, \ x \in \Omega \tag{1.4}$$

holds for some $c_1, c_2 > 0$. However, there are superlinear functions f that (1.4) is not satisfied. For example the function $f(t) = t\ln(1 + |t|)$ which does not satisfy the estimate (1.4) proving that (AR) condition does not work anymore. At the same time, the function f satisfies the nonquadraticity condition at infinity given by hypotheses (NQ) ahead.

The main feature in this work is to find existence and multiplicity of solutions for fourth order elliptic problems given by problem (1.1) where the nonlinear term is nonquadratic at the infinity. As was mentioned before in this work is not required that f satisfy the (AR) condition. For further results on elliptic problems without the (AR) condition we infer the reader to [5, 10, 18, 19, 20, 27, 29, 30, 31, 34] and references therein.

At this moment we shall consider the nonquadraticity condition at infinity introduced by Costa and Magalhães [8] stated in the form

(NQ) setting $H(x, t) := f(x, t)t - 2F(x, t)$, we have that

$$\lim_{|t|\to\infty} H(x, t) = +\infty, \text{ uniformly for } x \in \Omega.$$

Now we shall consider the Mountain Pass Theorem, under the Cerami condition, writing our main first result in the form:

Theorem 1.1. *Suppose that f satisfies* (H1)–(H3) *and* (NQ). *Then problem* (1.1) *admits at least one nontrivial solution.*

Remark 1.2. Notice that Theorem 1.1 holds for Dirichlet boundary conditions. More specifically, we consider the elliptic problem

$$\alpha\Delta^2 u + \beta\Delta u = f(x, u) \quad \text{in } \Omega,$$

$$u = 0, \quad \frac{\partial u}{\partial \eta} = 0 \quad \text{on } \partial\Omega,$$

where $\frac{\partial u}{\partial \eta}$ denotes the normal derivative on the boundary. For this problem the energy functional $I : H_0^2(\Omega) \to \mathbb{R}$ is given by

$$I(u) = \frac{1}{2} \int_\Omega \left(\alpha|\nabla u|^2 - \beta|\nabla u|^2\right) dx - \int_\Omega F(x,u)dx, \quad u \in H_0^2(\Omega).$$

Here the first eigenvalue $\mu_1 > 0$ can be characterized by

$$\mu_1 = \inf \Big\{ \int_\Omega \left(\alpha|\nabla u|^2 - \beta|\nabla u|^2\right) dx, \|u\|_2 = 1, u \in H_0^2(\Omega) \Big\}.$$

For more details on this subject we refer the reader to [12, 13].

Remark 1.3. Here we assume that $N = 1, 2, 3$ and $p \in (1, \infty)$ where the function f is subcritical. Using assumptions (H1)–(H3) the energy functional I admits also the mountain pass geometry. Besides that, using hypothesis (NQ), we also mention that I verifies the Cerami condition. Hence the Theorem 1.1 remains true in this setting.

Now, using a truncation technique for fourth order elliptic problems, taking into account the Strong Maximum Principle for elliptic equations we can write our second result in the following form.

Theorem 1.4. *Suppose that f satisfies* (H1)–(H3) *and* (NQ). *Then problem* (1.1) *admits at least two nontrivial solutions $u, w \in \mathcal{H}$ satisfying $u > 0$ and $w < 0$ in Ω.*

Now, using some symmetric conditions and the Symmetric Mountain Pass Theorem, our third result can be stated in the following form

Theorem 1.5. *Suppose that f satisfies* (H1)–(H3) *and* (NQ). *Assume also that $t \to f(x,t)$ is an odd function for any $x \in \Omega$ fixed. Then the problem* (1.1) *admits infinitely many nontrivial solutions.*

Fourth-order elliptic problems involving biharmonic operator have been widely studied during the previous years, see [3, 6, 11, 26, 28]. In most of them was employed variational methods such as the Mountain Pass Theorem where $\alpha = 1$ and $\beta = 0$ and $f_0 \equiv 0$. For example in [26] the authors considered problem (1.1) under those conditions and the nonlinear f appears as simple power or it satisfies the (AR) condition. In [28] was considered problem (1.1) with $\alpha = 1$, $\beta < \lambda_1$ and f superlinear at infinity. Furthermore, in [28] the authors considered the hypothesis

$$\liminf_{|t| \to \infty} \frac{tf(x,t) - 2F(x,t)}{|t|^\sigma} \geq a, \quad t \in \mathbb{R},\ x \in \Omega \tag{1.5}$$

holding uniformly in $x \in \Omega$ for some $\sigma > \frac{N}{4}(p-1)$ where $a > 0$ is a suitable constant. Here we mention also that hypotheses in the spirt of (1.5) was introduced by Costa-Magalhães in [8]. In other works the authors have considered the assumption

$$t \to \frac{f(t)}{t} \text{ is an increasing function for each } t > 0. \tag{1.6}$$

The assumption (1.6) is crucial in order to ensure that any Palais-Smale sequence is bounded, see [7, 27]. For further results on this subject we refer the reader to [4, 19, 36]. As an example for our setting we consider the following functions

(a) $f_1(t) = t \ln(1 + |t|) + \sin t$, $t \in \mathbb{R}$ where $\mu_1 > 1$,

(b) $f_2(t) = t\ln(1+|t|) + at\sin t$, $t \in \mathbb{R}$ where $a \in (0, 1/2)$.

Here we mention that f_1, f_2 do not satisfy neither the (AR) condition nor assumption (1.6). Indeed, these functions do not satisfy the estimate given in (1.4). Furthermore, due the periodic term we observe that

$$\frac{d}{dt}\Big[\frac{f_1(t)}{t}\Big] \quad \text{and} \quad \frac{d}{dt}\Big[\frac{f_2(t)}{t}\Big]$$

are sign changing functions. The main point in this work is to consider the extremal case putting $\sigma = 0$ in assertion (1.5). Moreover, we consider the case where assumption (1.6) is not verified. Hence our results extend and complement the early results above-mentioned.

To ensure multiplicity of solutions for the elliptic problem (1.1) we have a difficulty because u^+ does not belong to $\mathcal{H}$ in general. This problem is inherent to elliptic operators of higher order such as problem (1.1). To overcome this difficulty we shall use a truncation technique together with strong maximum principle proving that problem (1.1) admits at least a weak solutions with constant sign. Here the main point is to consider an auxiliary elliptic problem and an elliptic system in order to use Stampacchia's result. On the other hand, assuming that f is an odd function, the functional I is even which allow us to apply the Symmetric Mountain Pass Theorem under the Cerami condition given by Ambrosetti-Rabinowitz. The main strategy here is to prove that I is anti-coercive in an appropriate sense, see Section 3 ahead. Hence our results extend and complement early results above-mentioned proving existence and multiplicity of solutions which are defined in sign.

The paper is organized as follows: In Section 2 we give the variational framework to the elliptic problem (1.1) proving the compactness condition, i.e, the Cerami condition. Section 3 is devoted to the proof of our main results. In Appendix we prove that any critical point $u \in \mathcal{H}$ for the functional I satisfies $\Delta u = 0$ on $\partial\Omega$.

Throughout this work $C, C_1, C_2, \dots$ denote positive constants. The norm in $L^p(\Omega)$ is denoted by $\|\cdot\|_p$ for each $p \in [1,\infty]$. For any function $u \in L^p(\Omega)$ we write $u = u^+ + u^-$ where $u^+ = \max(u, 0)$ and $u^- = \min(u, 0)$, $p \in [1,\infty]$. The norm in $\mathcal{H}$ is denoted by $\|\cdot\|$.

2. Variational framework

In this section we shall prove some properties related to the elliptic problem (1.1) given a variational framework for our problem.

Let $\mathcal{H}$ be a Banach space endowed with the norm $\|\|$. Consider $I : \mathcal{H} \to \mathbb{R}$ a functional of C^1 class. A sequence $(u_n) \in \mathcal{H}$ is said to be a Palais-Smale sequence at level $c \in \mathbb{R}$, in short $(PS)_c$, when $I(u_n) \to c$ and $\|I'(u_n)\|_{\mathcal{H}'} \to 0$ as $n \to \infty$. Recall that I satisfies the Palais-Smale condition at the level c, in short $(PS)_c$ condition, when any $(PS)_c$ sequence admits a convergent subsequence. We say simply that I verifies the Palais-Smale condition when $(PS)_c$ condition holds true for any $c \in \mathbb{R}$. Similarly, a sequence $(u_n) \in \mathcal{H}$ is said to be a Cerami sequence at the level $c \in \mathbb{R}$, in short $(Ce)_c$ sequence, whenever $I(u_n) \to c$ and $(1 + \|u_n\|)\|I'(u_n)\|_{\mathcal{H}'} \to 0$ as $n \to \infty$. The functional I satisfies the Cerami condition at the level $c \in \mathbb{R}$, in short $(Ce)_c$ condition, whenever any Cerami sequence at the level c possesses a convergent subsequence. When I satisfies the Cerami condition at any level $c \in \mathbb{R}$ we say purely that I satisfies the Cerami condition, in short (Ce) condition. Here we refer the reader to [4, 7, 36].

Now we recall the classical mountain pass theorem given in [1].

Theorem 2.1. *Let $\mathcal{H}$ be a Banach space and $I : \mathcal{H} \to \mathbb{R}$ a functional of C^1 class. Suppose that I admits the following mountain pass geometry*

(i) *There exist $r > 0$, $\rho > 0$ such that $I(u) \geq \rho > 0$, for any $u \in \mathcal{H}$ with $\|u\| = r$.*
(ii) *There exists $e \in \mathcal{H}$ with $\|e\| > r$ such that $I(e) \leq 0$.*

Suppose also that I satisfies the $(PS)_c$ condition or $(Ce)_c$ condition where we define

$$c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I(\gamma(t)) \tag{2.1}$$

where $\Gamma := \{\gamma \in C^0([0, 1], \mathcal{H}) : \gamma(0) = 0, \gamma(1) = e\}$. Then c is a critical value which satisfies $c \geq \rho$.

Now we shall prove the Cerami condition using the nonquadraticity condition at infinity. The main difficulty here is to ensure that any Cerami sequence is bounded.

Proposition 2.2. *Suppose that f satisfies* (H1)–(H3) *and* (NQ). *Then the functional I satisfies the Cerami condition at any level $c \in \mathbb{R}$.*

Proof. Let $(u_n) \subset \mathcal{H}$ be a sequence in such way that

$$I(u_n) \to c, \quad \|I'(u_n)\|_{\mathcal{H}'}(1 + \|u_n\|) \to 0,$$

where $c \in \mathbb{R}$. Since f has subcritical growth it suffices to prove that (u_n) is bounded sequence.

The proof follows arguing by contradiction. Suppose that, up to a subsequence, $\|u_n\| \to +\infty$ as $n \to +\infty$. Setting $v_n := u_n/\|u_n\|$ we obtain $\|v_n\| = 1$ and there exits $v \in \mathcal{H}$ in such way that $v_n \rightharpoonup v$ in $\mathcal{H}$. In this case, along a subsequence, we obtain

$$\begin{gathered} v_n \rightharpoonup v \quad \text{weakly in } \mathcal{H}, \\ v_n \to v \quad \text{strongly in } L^q(\Omega), \\ v_n(x) \to v(x), \quad \text{a. e. in } \Omega, \\ |v_n(x)| \leq h_q(x), \quad h \in L^q(\mathbb{R}^N), \end{gathered} \tag{2.2}$$

holds for any $2 \leq q < 2_*$.

At this stage we claim that $v \neq 0$. This fact can be proved arguing by contradiction. In fact, assuming that $v \equiv 0$, it follows from (H1) and (H3) that

$$|F(x, u)| \leq C|u|^2 + C|u|^p, \quad \forall (x, u) \in \Omega \times \mathbb{R}. \tag{2.3}$$

holds for some $C > 0$. Fix $m > 0$ a constant. Taking into account (2.3) we have

$$\int_\Omega |F(x, \sqrt{4m}v_n)|\,dx \leq C \int_\Omega v_n^2\,dx + C \int_\Omega |v_n|^p\,dx$$

holds for some $C = C(\Omega, f, m, p) > 0$. Using the strong convergence in (2.2) and the compact embedding we know that

$$\int_\Omega F(x, \sqrt{4m}v_n) \to 0$$

as $n \to +\infty$ for any fixed $m > 0$.

By the generality of constant m, without any loss of generality, we suppose that $\sqrt{4m} < \|u_n\|$. Let $t_n \in [0, 1]$ be in such way that

$$I(t_n u_n) = \max_{t\in[0,1]} I(tu_n). \tag{2.4}$$

Using the definition of t_n in (2.4), considering $t = \sqrt{4m} < \|u_n\|$ we deduce that

$$\begin{aligned} I(t_n u_n) &\geq I\Big(\frac{\sqrt{4m}}{\|u_n\|} u_n\Big) \\ &= 2m - \int_\Omega F(x, \sqrt{4m} v_n)\, dx \ \geq m > 0, \end{aligned} \tag{2.5}$$

for any $n \geq n_0$, where $n_0 \in \mathbb{N}$ depends only on m. It is important to emphasize that if $t_n = 0$ holds for any $n \in \mathbb{N}$ we obtain $0 = I(t_n u_n) \geq m > 0$ which is a contradiction. Furthermore, assuming that $t_n = 1$ holds for any $n \in \mathbb{N}$ we also obtain $m \leq I(t_n u_n) = I(u_n) \to c$ which does not make sense for any $m > c$. Hence, up to a subsequence, we can assume that $t_n \in (0, 1)$. This assertion together with (2.4) imply that $I'(t_n u_n)(t_n u_n) = 0$ for any $n \in \mathbb{N}$.

Now we shall divide the proof in two cases:

Case 1: Along a subsequence we suppose that $t_n < (2/\|u_n\|)$. In this case we use hypotheses (H1)–(H3) and the Sobolev embedding to get $c_1, c_2 > 0$, satisfying the estimates

$$\Big| \int_\Omega H(x, t_n u_n\,) dx \Big| \leq c_1 (t_n \|u_n\|)^2 + c_2 (t_n \|u_n\|)^p \leq 4c_1 + c_2 2^p < \infty.$$

Using the fact that $t_n \in (0, 1)$ it follows from the identity $I'(t_n u_n)(t_n u_n) = 0$ that

$$0 = t_n^2 \|u_n\|^2 - \int_\Omega f(x, t_n u_n)(t_n u_n)\, dx = 2I(t_n u_n) - \int_\Omega H(x, t_n u_n)\, dx.$$

As a consequence we obtain

$$I(t_n u_n) = \frac{1}{2} \int_\Omega H(x, t_n u_n)\, dx \leq c_3$$

where $c_3 > 0$. This give us an absurd with (2.5) due the fact that $m > 0$ in that expression is arbitrary. Hence Case 1 is not possible. It remains to focus in the following case

Case 2: Along a subsequence we have $t_n \geq (2/\|u_n\|)$. In this case we consider a new sequence $s_n := \frac{1}{\|u_n\|} < t_n$. Analyzing the energy functional I we obtain the identity

$$\begin{aligned} \frac{I(t_n u_n)}{t_n^2 \|u_n\|^2} - \frac{I(s_n u_n)}{s_n^2 \|u_n\|^2} &= \int_\Omega \frac{F(x, t_n u_n)}{s_n^2 \|u_n\|^2} \mathrm{d}x - \int_\Omega \frac{F(x, s_n u_n)}{t_n^2 \|u_n\|^2} \mathrm{d}x \\ &= -\frac{1}{\|u_n\|^2} \int_\Omega \Big(\frac{F(x, t_n u_n)}{t_n^2} - \frac{F(x, s_n u_n)}{s_n^2} \Big)\, \mathrm{d}x \end{aligned}$$

At this moment, using Calculus Fundamental Theorem, we infer that

$$\begin{aligned} \frac{I(t_n u_n)}{t_n^2 \|u_n\|^2} - \frac{I(s_n u_n)}{s_n^2 \|u_n\|^2} &= -\int_\Omega \int_{s_n}^{t_n} \frac{d}{d\tau} \Big(\frac{F(x, \tau u_n)}{\tau^2 \|u_n\|^2} \Big) \mathrm{d}\tau\, \mathrm{d}x \\ &= -\int_\Omega \int_{s_n}^{t_n} \frac{H(x, \tau u_n)}{\tau^3 \|u_n\|^2}\, \mathrm{d}\tau\, \mathrm{d}x. \end{aligned}$$

Now we shall define the function $\phi : \mathbb{R} \to \mathbb{R}$ given by $\phi(t) = e^{-1/t^2}$, $t \neq 0$ and $\phi(0) = 0$. It follows from elementary calculus that $\phi \in C^\infty(\mathbb{R}, \mathbb{R})$ and $0 \leq \phi(t) \leq 1$ for any $t \in \mathbb{R}$. Furthermore, we observe that $\phi^{(j)}(0) = 0$ for any $j \in \mathbb{N}$. Using the condition (NQ) and the function ϕ given just above we infer that

$$H(x, t) \geq R\phi(t), \quad \forall |t| \geq M,\ x \in \Omega \tag{2.6}$$

holds for any $R > 0$ where $M = M(R) > 0$.

On the other hand, by hypothesis (H1) and continuity of the function H, we have that

$$H(x,t) \geq -C|t|, \quad \forall |t| \leq M,\ x \in \Omega \tag{2.7}$$

for some positive constant $C = C(R) < \infty$. Using estimates (2.6) and (2.7) we ensure the global inequality

$$H(x,t) \geq R\phi(t) - C|t|, \quad \forall (x,t) \in \Omega \times \mathbb{R}.$$

Therefore, we have

$$\frac{I(t_n u_n)}{t_n^2\|u_n\|^2} - \frac{I(s_n u_n)}{s_n^2\|u_n\|^2} \leq I_1 + I_2,$$

where I_1 and I_2 are defined by

$$I_1 = -\frac{R}{\|u_n\|^2}\int_\Omega \Big[\int_{s_n}^{t_n} \frac{\phi(\tau u_n)}{\tau^3} d\tau\Big]\, dx, \tag{2.8}$$

$$I_2 = \frac{C}{\|u_n\|^2}\int_\Omega \Big[\int_{s_n}^{t_n} \frac{|\tau u_n|}{\tau^3} d\tau\Big]\, dx. \tag{2.9}$$

Firstly, analyzing the integral (2.9) we easily see that

$$I_2 = C\int_\Omega \Big(\frac{1}{s_n} - \frac{1}{t_n}\Big)\frac{|u_n|}{\|u_n\|^2}\, dx.$$

At this moment using the fact that $t_n > \frac{2}{\|u_n\|}$ the integral I_2 can be estimated as follows

$$I_2 \leq \frac{C}{2}\int_\Omega |v_n|\, dx \to 0.$$

Now we shall analyze the integral in (2.8). According to mean value theorem for integrals there exists $c_n \in (s_n, t_n)$ in such way that

$$\int_{s_n}^{t_n} \frac{\phi(\tau u_n)}{\tau^3}\, d\tau = \phi(c_n u_n)\int_{s_n}^{t_n} \frac{1}{\tau^3}\, d\tau = -\phi(c_n u_n)(1/2t_n^2 - 1/2s_n^2).$$

As a consequence we get the identity

$$I_1 = \frac{R}{\|u_n\|^2}\Big(\frac{1}{t_n^2} - \frac{1}{s_n^2}\Big)\int_\Omega \phi(c_n u_n)\, dx.$$

The above identity implies that

$$I_1 \leq -\frac{3R}{8}\int_\Omega \phi(c_n u_n)\, dx.$$

Now we define the set

$$\Omega_n = \cup_{M>0}\{x \in \Omega : |(c_n u_n)(x)| \geq \frac{1}{M}\}.$$

The strategy here is to find a subsequence $(u_{n_k}) \in \mathcal{H}$ in such way that

$$|\Omega_{n_k}| \geq \delta_0 > 0, \quad \forall k \in \mathbb{N} \tag{2.10}$$

holds for some numbers $\delta_0 > 0, M_0 > 0$ fixed. If this is true, then

$$I_1 \leq -\frac{3R}{8}\int_{\Omega_{n_k}} e^{-M_0^2}\, dx \leq -\frac{3R}{8}\delta_0 e^{-M_0^2} < 0.$$

Now we proceed with the proof of inequality (2.10). Suppose by contradiction that for any subsequence $(u_{n_k}) \in \mathcal{H}$ and for each $\delta > 0$ and $M > 0$ we have

$$0 < |\Omega_{n_k}| \le \delta.$$

These inequalities show that $|\Omega_{n_k}| = 0$ for any $M > 0$. Consider the measurable set $\Omega_0 = \cup_{k=1}^{\infty} \Omega_{n_k}$. It is easy to verify that

$$|\Omega_0| = \big| \cup_{k=1}^{\infty} \Omega_{n_k} \big| \le \sum_{k=1}^{\infty} |\Omega_{n_k}| = 0. \tag{2.11}$$

Note that we can rewrite the set $\Omega = \Omega_0 \dot\cup (\Omega \setminus \Omega_0)$. It is important to emphasize that

$$I(t_{n_k} u_{n_k}) = \frac{1}{2} \int_\Omega H(x, t_{n_k} u_{n_k})\, dx + o_k(1)$$

Using the previous identity and (2.11) we easily see that

$$\begin{aligned} I(t_{n_k} u_{n_k}) &= \frac{1}{2}\Big(\int_{\Omega_0} H(x, t_{n_k} u_{n_k})\, dx \int_{\Omega \setminus \Omega_0} H(x, t_{n_k} u_{n_k})\, dx \Big) + o_k(1) \\ &= \frac{1}{2} \int_{\Omega \setminus \Omega_0} H(x, t_{n_k} u_{n_k})\, dx + o_k(1) \end{aligned} \tag{2.12}$$

Analyzing the elements in the set $\Omega \setminus \Omega_0$ and using the Morgan's Law we get

$$\Omega \setminus \Omega_0 = \big\{ x \in \Omega : |(c_{n_k} u_{n_k})(x)| < \frac{1}{M},\ \forall k \in \mathbb{N},\ \forall M > 0 \big\}.$$

Now we consider the set $\Omega_1 = \{x \in \Omega; |(c_{n_k} u_{n_k})(x)| = 0\ \forall k \in \mathbb{N}\}$. Clearly, we see that $\Omega_1 \subset \Omega \setminus \Omega_0$. Additionally, given any $x \in \Omega \setminus \Omega_0$, we observe that

$$|(c_{n_k} u_{n_k})(x)| < \frac{1}{M}, \quad \forall k \in \mathbb{N}$$

holds for any $M > 0$. As the inequality just above is verified for any $M > 0$ and for any integer k we obtain that $|(c_{n_k} u_{n_k})(x)| = 0$ for all $k \in \mathbb{N}$. As a consequence $x \in \Omega_1$ and $\Omega \setminus \Omega_0 = \Omega_1$. Furthermore, using the fact that $0 < s_{n_k} \le c_{n_k} \le t_{n_k}$, we mention also that

$$u_{n_k}(x) = 0, x \in \Omega \setminus \Omega_0,\ \forall k \in \mathbb{N}.$$

Thus, we can be rewritten the set $\Omega \setminus \Omega_0$ in the form

$$\Omega \setminus \Omega_0 = \{x \in \Omega; |u_{n_k}(x)| = 0\ \forall k \in \mathbb{N}\}.$$

From now on, using the estimate given in (2.12) and the above assertion, we assume that

$$I(t_{n_k} u_{n_k}) = \frac{1}{2} \int_{\Omega \setminus \Omega_0} H(x, t_{n_k} u_{n_k})\, dx + o_k(1) = o_k(1),$$

which implies the convergence

$$\lim_{k \to +\infty} I(t_{n_k} u_{n_k}) = 0.$$

This is a contradiction with (2.5) for each $m > 0$. Therefore the assertion given in (2.10) is true. In conclusion, we have been proved that $v \ne 0$, i.e, the set

$$\widehat{\Omega} = \{x \in \Omega : v(x) \ne 0\}$$

has positive Lebesgue measure. Additionally, we know that $|u_n| \to \infty$ a.e. in $\widehat{\Omega}$. Taking into account hypothesis (NQ) we observe that $H(x,t) \ge -C$ holds for any

$(x,t) \in \Omega \times \mathbb{R}$ and for some $C > 0$. Under these conditions it follows from Fatou's Lemma that

$$\begin{aligned} c &= \liminf_{n\to\infty} \left\{ I(u_n) - \frac{1}{2} I'(u_n)u_n \right\} \\ &= \frac{1}{2} \liminf_{n\to\infty} \int_\Omega H(x,u_n)dx \\ &\geq \frac{1}{2} \int_\Omega \liminf_{n\to\infty} H(x,u_n)dx = \infty. \end{aligned}$$

Hence we have a contradiction proving that $(u_n) \in \mathcal{H}$ is now a bounded sequence. This finishes the proof of Proposition 2.2. $\square$

3. Proof of main results

Proof of Theorem 1.1. Initially, using hypotheses (H1)–(H3) and the fact that $f \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$, we have estimate

$$|F(x,t)| \leq \frac{(\varepsilon + f_0)}{2}|t|^2 + C|t|^p, \quad \forall (x,t) \in \Omega \times \mathbb{R}, \tag{3.1}$$

for any $\varepsilon > 0$ and for some $C = C(\varepsilon) > 0$. Now we mention the following variational inequality

$$\|u\|_2^2 \leq \frac{1}{\mu_1}\|u\|^2, \quad u \in \mathcal{H}. \tag{3.2}$$

Noticing the definition of I given in (1.2), it follows from (3.1), (3.2) and Sobolev inequality that

$$I(u) \geq \Big(\frac{1}{2} - \frac{(\varepsilon + f_0)}{2\mu_1}\Big)\|u\|^2 - C\|u\|^p.$$

for some positive constant $C > 0$. Now we define

$$r = \left(\frac{\varepsilon_0}{4\mu_1 C}\right)^{1/(p-2)}, \quad \rho = r^2\left(\frac{\varepsilon_0}{4\mu_1}\right)$$

where $\varepsilon_0 = \frac{\mu_1 - f_0}{2} > 0$. Under these conditions, for any $\varepsilon \in (0, \varepsilon_0)$, we infer that

$$I(u) \geq \rho > 0, \quad \forall u \in \mathcal{H},\ \|u\| = r.$$

This shows the first statement in the mountain pass geometry given in Theorem 2.1.

Now using condition (H2) and the continuity of F, given any $R > \mu_1$ there exists $C_R < \infty$ in such way that

$$F(x,t) \geq \frac{Rt^2}{2} - C_R, \quad \forall (x,t) \in \Omega \times \mathbb{R}. \tag{3.3}$$

Let $\varphi_1 > 0$ be the eigenfunction associated to μ_1 which satisfies $\int_\Omega \varphi_1^2 dx = 1$. Consider $e = t\varphi_1$ with $t > 0$. Using (3.3) we get the estimate

$$I(e) = I(t\varphi_1) \leq \frac{t^2}{2}(\mu_1 - R) + C_R|\Omega|.$$

As a consequence there exists $t_0 > 0$ large enough in such way that, considering $R > \mu_1$, we obtain $\|e\| = \|t_0\varphi_1\| > r$ and $I(e) < 0$. These facts shows that I admits the mountain pass geometry proving the existence of a Cerami condition (u_n) at the mountain pass level given by (2.1). According to Proposition 2.2 there exists $u \in \mathcal{H}$ in such way that $u_n \to u$ in $\mathcal{H}$. As a consequence u is a critical point for I and $I(u) \geq \rho > 0$. Hence u is a weak solution to the elliptic problem (1.1). Here was used the fact that $u = 0$ and $\Delta u = 0$ on $\partial\Omega$ which we shall prove in the Appendix. This completes the proof. $\square$

Proof of theorem 1.4. In this section, via Strong Maximum Principle, we shall prove that problem (1.1) admits at least two solutions which one solution is positive and another one is negative. To do that we reduce the fourth-order problem (1.1) to a second-order elliptic problem. Hence, using a truncation argument, we analyze the positive part and negative part of the nonlinearity f obtaining a multiplicity result.

As a first step we shall consider $v = -\Delta u$ in problem (1.1), i.e, we reduce problem (1.1) into the elliptic system

$$\begin{gathered} -\Delta u = v \quad \text{in } \Omega, \\ -\alpha\Delta v - \beta v = f(x,u) \quad \text{in } \Omega, \\ u = v = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{3.4}$$

Here we emphasize that $u, v \in H_0^1(\Omega)$. Putting $v^- \in H_0^1(\Omega)$ as test function in the problem (3.4) we observe that

$$\int_\Omega \alpha \nabla v \nabla v^- \, dx - \beta \int_\Omega v v^- \, dx = \int_\Omega f(x,u)\, v^- \, dx. \tag{3.5}$$

It is worthwhile to mention that

$$\nabla v^- = \begin{cases} \nabla v, & \text{if } v < 0 \\ 0, & \text{if } v \geq 0, \end{cases}$$

Using the variational inequality for λ_1 and the identity (3.5) we obtain

$$\Big(\alpha - \frac{\beta}{\lambda_1}\Big) \int_\Omega |\nabla v^-|^2 \, dx \leq \int_\Omega \alpha |\nabla v^-|^2 \, dx - \beta \int_\Omega |v^-|^2 \, dx = \int_\Omega f(x,u)\, v^- \, dx.$$

At this moment we shall consider the function f^+, i.e, we define the truncation

$$f^+(x,t) = \begin{cases} f(x,t), & \text{if } t \geq 0 \\ 0, & \text{if } t < 0. \end{cases}$$

Now we define the functional $I^+ : \mathcal{H} \to \mathbb{R}$ of C^1 class given by

$$I^+(u) = \frac{\alpha}{2} \int_\Omega |\Delta u|^2 dx - \frac{\beta}{2} \int_\Omega |\nabla u|^2 dx - \int_\Omega F^+(x,u)\, dx$$

where $F^+(x,t) = \int_0^t f^+(x,s) ds$ for any $x \in \Omega$, $t \in \mathbb{R}$. Moreover, $u \in \mathcal{H}$ is a critical point for I^+ if and only if we have

$$\alpha \int_\Omega \Delta u \Delta \phi dx - \beta \int_\Omega \nabla u \nabla \phi dx - \int_\Omega f^+(x,u) \phi \, dx = 0$$

for any $\phi \in \mathcal{H}$. Then we can find weak solution to the elliptic problem (1.1) finding positive critical points to the functional I^+. It is no hard to verify that I^+ admits the mountain pass geometry using the same ideas discussed in the proof of Theorem 1.1. Furthermore, the functional I^+ satisfies the $(Ce)_c$ condition at any energy level $c \in \mathbb{R}$. This is ensured using the ideas discussed in the proof of Proposition 2.2. As a consequence, using the Mountain Pass Theorem, there exists a critical point $u \in \mathcal{H}$ for the functional I^+ verifying $I^+(u) > 0$.

Now we shall consider the auxiliary elliptic problem

$$
\begin{gathered}
-\Delta u = v \quad \text{in } \Omega \\
-\alpha\Delta v - \beta v = f^{+}(x,u) \quad \text{in } \Omega \\
u = v = 0 \quad \text{on } \partial\Omega.
\end{gathered} \tag{3.6}
$$

Using the same ideas discussed above and changing the function f by f^{+} we get the estimate

$$
0 \leq \Big(\alpha - \frac{\beta}{\lambda_1}\Big) \int_\Omega |\nabla v^-|^2\,dx \leq \int_{[v<0]} f^{+}(x,u)\,v^-\,dx. \tag{3.7}
$$

It follows from problem (3.6) that

$$
\begin{gathered}
-\Delta u = v < 0 \quad \text{in } [v<0] \\
u \leq 0 \quad \text{on } \partial[v<0],
\end{gathered} \tag{3.8}
$$

where we define $[v<0] = \{x \in \Omega : v(x) < 0\}$. This set is an open set due the fact that v is in $C^{0,\alpha}(\overline{\Omega})$. This fact can be ensured using regularity arguments on elliptic equations involving operators of fourth-order and the fact that f^{+} is a continuous function. For further results on regularity for elliptic equations we infer the reader to Agmon, Douglis and Niremberg [2] or Gupta and Kwong [17].

Now, using the strong maximum principle [14], we note that (3.8) implies $u<0$ in $[v<0]$. As a consequence, using the last assertion, we know that

$$
\begin{aligned}
\int_\Omega f^{+}(x,u)\,v^-\,dx &= \int_{[v\geq 0]} f^{+}(x,u)\,v^-\,dx + \int_{[v<0]} f^{+}(x,u)\,v^-\,dx \\
&= \int_{[v<0]} f^{+}(x,u)\,v^-\,dx = 0
\end{aligned}
$$

Therefore, estimate (3.7) and the variational inequality for λ_1 imply

$$
0 \leq \Big(\alpha - \frac{\beta}{\lambda_1}\Big) \int_\Omega |\nabla v^-|^2\,dx \leq 0.
$$

Hence $v^- \equiv 0$ and $v = v^+ \geq 0$ in Ω. As a consequence,

$$
-\Delta u = v \geq 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega.
$$

Using one more time the strong maximum principle we easily seen that $u>0$ in Ω, i.e, we guarantee that problem (1.1) admits at least one positive solution.

Analogously, we define the function f^- by

$$
f^-(x,t) = \begin{cases} f(x,t), & \text{if } t \leq 0 \\ 0, & \text{if } t > 0. \end{cases}
$$

Now we define the functional $I^- : \mathcal{H} \to \mathbb{R}$ of C^1 class given by

$$
I^-(u) = \frac{\alpha}{2}\int_\Omega |\Delta u|^2 dx - \frac{\beta}{2}\int_\Omega |\nabla u|^2 dx - \int_\Omega F^-(x,u)\,dx
$$

where $F^-(x,t) = \int_0^t f^-(x,s)ds$ for any $x \in \Omega, t \in \mathbb{R}$. Moreover, $u \in \mathcal{H}$ is a critical point for I^- if only if we have

$$
\alpha\int_\Omega \Delta u \Delta\phi dx - \beta\int_\Omega \nabla u\nabla\phi dx - \int_\Omega f^-(x,u)\phi\,dx = 0
$$

for any $\phi \in \mathcal{H}$. Then we can find weak solution to the elliptic problem (1.1) finding negative critical points to the functional I^-. It is no hard to verify that I^- admits the mountain pass geometry using the same ideas discussed in the proof of Theorem 1.1. Furthermore, the functional I^- satisfies the $(Ce)_c$ condition at any energy level $c \in \mathbb{R}$. This is verified using the ideas discussed in the proof of Proposition 2.2. As a consequence, using the Mountain Pass Theorem, there exists a critical point $w \in \mathcal{H}$ for the functional I^- satisfying $I^-(w) > 0$. At the same time, using the strong principle strong maximum twice we obtain a second solution $w \in \mathcal{H}$ to the elliptic problem (1.1) satisfying $w < 0$ in Ω. Here was used the fact that f^- is also a continuous function. This completes the proof. □

To prove Theorem 1.5, we shall apply the following version of the symmetric Mountain Pass Theorem.

Theorem 3.1. *Let X be an infinite dimensional Banach space and $\mathcal{I} \in C^1(X, \mathbb{R})$ be even, satisfy $(Ce)_c$ for any $c \in \mathbb{R}$, and $\mathcal{I}(0) = 0$. If $X = V \oplus W$, where V is finite dimensional, and $\mathcal{I}$ satisfies*

(1) *there exists $r, \rho > 0$ such that*
$$\mathcal{I}(u) \geq r > 0, \quad \text{for any } u \in \partial B_\rho(0) \cap W;$$
(2) *for any finite dimensional subspace $\overline{X} \subset X$ there exists $\xi = \xi(\overline{X})$ such that*
$$\mathcal{I}(u) \leq 0 \quad \text{for any } u \in \overline{X} \setminus B_\xi(0),$$

then $\mathcal{I}$ possesses an unbounded sequence of critical values.

Proof of Theorem 1.5. Firstly, we would like to apply Theorem 3.1 choosing $X = \mathcal{H}$ and $\mathcal{I} = I$. Hence, using hypotheses $(f_0) - (f_2)$, we deduce the estimate
$$|F(x,t)| \leq \frac{(\varepsilon + f_0)}{2}|t|^2 + C|t|^p, \quad \forall (x,t) \in \Omega \times \mathbb{R},$$
which holds for any $\varepsilon > 0$ and for some $C = C(\varepsilon) > 0$. This implies that $I(u) \geq \rho$ for any $u \in \mathcal{H}$ satisfying $\|u\| = r$. Here we choose $r, \rho > 0$ small, see the proof of Theorem 1.1, and therefore (1) of Theorem 3.1 holds.

Now consider any subspace $E \subset \mathcal{H}$ which is finite dimensional. Let $B_r(0) \subset \mathcal{H}$ be the open ball centered at the origin and radius $r > 0$. Here we is need to guarantee that $I(v) \leq 0$ for any $v \in E \backslash B_r(0)$ where $r > 0$ is chosen large enough. To do that, we use the estimate (3.3) and the fact that all the norms in space E are equivalent obtaining constants $C_0 > 0$ and $C_R > 0$ in such way that
$$I(v) \leq \frac{1}{2}\left(1 - C_0 R\right)\|u\|^2 + C_R|\Omega|, \quad v \in E$$
holds for any $R > 0$. Since $R > 0$ is arbitrary we conclude that $I(v) \to -\infty$ as $\|v\| \to \infty$, $v \in E$, and therefore (1) of Theorem 3.1 holds. Furthermore, the functional I satisfies the Cerami condition, see Proposition 2.2. Then applying the symmetric mountain pass theorem we obtain a sequence of critical values $c_n > 0$ such that $c_n \to +\infty$ as $n \to \infty$. In particular, we obtain a sequence of critical points $u_n \in \mathcal{H}$ for the functional I satisfying $I(u_n) \to +\infty$ as $n \to \infty$. This completes the proof. □

4. Appendix

The main objective in this section is to ensure that any critical point $u \in \mathcal{H}$ for I satisfies $u = 0$ and $\Delta u = 0$ on $\partial\Omega$. In other words, we shall prove that any critical point of I give us a weak solution for the problem (1.1) satisfying the Navier boundary conditions.

Let $u \in \mathcal{H} = H_0^1(\Omega) \cap H^2(\Omega)$ be a critical point for I, i.e, we have that

$$\alpha \int_\Omega \Delta u \Delta \phi \, dx + \beta \int_\Omega \phi \Delta u \, dx = \int_\Omega f(x,u)\phi \, dx.$$

holds for any function $\phi \in \mathcal{H}$. From standard trace theorem we know that $u = 0$ on $\partial\Omega$. It remains to show that $\Delta u = 0$ on $\partial\Omega$. To do that we consider $v = -\Delta u$ and $h(x) = f(x,u)$. It is easy to verify that $v \in L^2(\Omega)$ and $h \in L^2(\Omega)$. Furthermore, we have

$$\int_\Omega v[-\alpha\Delta\phi - \beta\phi] \, dx = \int_\Omega h(x)\phi \, dx, \quad \forall \phi \in \mathcal{H}. \tag{4.1}$$

Let $w \in H_0^1(\Omega) \cap H^2(\Omega)$ be the unique weak solution for the elliptic problem

$$\begin{gathered} -\alpha\Delta w - \beta w = h(x) \quad \text{in } \Omega \\ w = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{4.2}$$

To ensure the existence of solutions to the elliptic problem (4.2) for each $h \in L^2(\Omega)$ we apply standard minimization procedures. To get the uniqueness of solutions to the elliptic problem (4.2) we consider any weak solutions w_1 and w_2 proving that

$$\begin{gathered} -\alpha\Delta(w_1 - w_2) - \beta(w_1 - w_2) = 0 \quad \text{in } \Omega \\ (w_1 - w_2) = 0 \quad \text{on } \partial\Omega. \end{gathered}$$

Choosing $\overline{w} = w_1 - w_2$ and taking $\overline{w}$ as testing function we get

$$\int_\Omega \alpha|\nabla\overline{w}|^2 - \beta\overline{w}^2 \, dx = 0.$$

On the other hand, using the hypothesis $\mu_1 > 0$, we deduce that $-\infty < \beta < \alpha\lambda_1$. In particular, putting $\beta \le 0$, it follows from the above estimate that

$$\int_\Omega \alpha|\nabla\overline{w}|^2 \, dx \le 0.$$

This implies that $\overline{w} \equiv 0$. Now, taking $0 < \beta < \alpha\lambda_1$, the variational inequality says

$$0 \le \int_\Omega \Big(\alpha - \frac{\beta}{\lambda_1}\Big)|\nabla\overline{w}|^2 \, dx \le 0.$$

Hence the last estimate implies that $\overline{w} \equiv 0$. To sum up, we have been shown that $w_1 \equiv w_2$ in Ω proving that problem (4.2) admits exactly one weak solution for each $h \in L^2(\Omega)$.

At this moment, using the weak formulation for (4.2) we obtain

$$\int_\Omega -\alpha w\Delta\phi - \beta w\phi \, dx = \int_\Omega h(x)\phi \, dx \tag{4.3}$$

holds for any $\phi \in \mathcal{H}$. Putting together the identities (4.1) and (4.3) we deduce that

$$\int_\Omega (v - w)(\alpha\Delta\phi - \beta\phi) \, dx = 0, \quad \forall \phi \in C_0^\infty(\Omega).$$

At this stage, using Du Bois Raymond's Lemma [14], we infer that $v = w$ a.e. in Ω. As a consequence $v = w$ in $L^2(\Omega)$ which says that $v = -\Delta u = 0$ on $\partial\Omega$.

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