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A BIDIMENSIONAL BI-LAYER SHALLOW-WATER MODEL

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ABSTRACT. The existence of global weak solutions in a periodic domain for a non-linear viscous bi-layer shallow-water model with capillarity effects and extra friction terms in a two-dimensional space has been proved in [21]. The main contribution of this article is to show the existence of global weak solutions without friction term or capillary effect following the ideas of [20] for the two dimensional case.

1. Introduction

The shallow-water equations are usually used to model some natural phenomena such as ocean circulation, coastal areas, rivers, lakes, avalanches, etc. However in many situations one layer of shallow-water cannot be used to model the system. In some cases such as the Strait of Gibraltar it is necessary to consider two layers of shallow-water system to model the flow. For this purpose many derivations of bi-layers and multi-layers shallow-water system have been done (see [1, 16, 18]).

In this article we study the existence of global weak solutions of the bi-layer shallow-water model derived in [16]. In [21], the authors obtained the existence of global weak solutions for a 2D viscous bi-layer shallow-water model derived in [16]. In their work they considered in a periodic domain Ω, a system composed of two layers of immiscible fluids with different and constant densities (ρ_1 and ρ_2, resp.) and viscosities (ν_1 and ν_2, resp.) and imposed $\nu_1 < \nu_2$. The system studied in [21] reads as follows:

$$\partial_t h_1 + \operatorname{div}(h_1 u_1) = 0; \tag{1.1}$$

$$\begin{aligned} &\rho_1\partial_t(h_1u_1) + \rho_1 \operatorname{div}(h_1u_1 \otimes u_1) - 2\nu_1 \operatorname{div}(h_1 D(u_1)) + \rho_1 g h_1 \nabla h_1 \\ &+ \rho_2 g h_1 \nabla h_2 - \Big(1 + \frac{c_0\beta(h_1)h_1}{6\nu_1}\Big) \operatorname{fric}(u_1, u_2) + c_0\beta(h_1)u_1 \\ &- \alpha_1 h_1 \nabla(\Delta h_1) - \alpha_2 h_1 \nabla(\Delta h_2) = 0; \end{aligned} \tag{1.2}$$

$$\partial_t h_2 + \operatorname{div}(h_2 u_2) = 0; \tag{1.3}$$

$$\begin{aligned} &\rho_2\partial_t(h_2u_2) + \rho_2 \operatorname{div}(h_2u_2 \otimes u_2) - 2\nu_2 \operatorname{div}(h_2 D(u_2)) + \rho_2 g h_2 \nabla h_2 \\ &+ \rho_2 g h_2 \nabla h_1 + \operatorname{fric}(u_1, u_2) - \alpha_2 h_2 \nabla(\Delta h) = 0. \end{aligned} \tag{1.4}$$

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where $\mathrm{fric}(u_1, u_2) = -c_1 B(h_1, h_2)(u_1 - u_2)$ with

$$B(h_1; h_2) = \frac{h_1 h_2}{\frac{\nu_1}{\nu_2} h_1 + \frac{\nu_2}{\nu_1} h_2}.$$

The drag coefficient B is introduced to control the friction terms at the interface. In their paper the special feature of their definition of weak solution is based on the test functions depending on the unknowns used for the momentum equations namely $h_i\varphi$. This one has been introduced in [5] and allowed them to get the compacticity when height limit vanishes. This particular definition of weak solutions with test functions $h\varphi$ was firstly introduced in [6]. With this particular definition of weak solutions the author in [12] proved the existence of global weak solutions of quantum Navier-Stokes equations in 3D. The main idea of his paper is to rewrite quantum Navier-Stokes equations as a viscous quantum Euler system by means of the effective velocity.

In [11], the authors proved the existence of global weak solutions for the compressible quantum Navier-Stokes equations by the use of a singular cold pressure. In [20] the authors, using the “BD-entropy” obtained the existence of usual global weak solutions for 3D compressible Navier-Stokes equations with degenerating viscosity. They derived “Mellet-Vasseur” type inequality which allows them to get global solutions in time.

For some fluids like electrorheological fluids which can change from a liquid like state to a solid like viscous state, another approach to study the existence of weak solution was developed in [15, 19].

Our analysis takes inspiration from the work developed in [20]. Our contribution compared with the work performed in [21] is that we obtain the existence of global usual weak solutions with test functions independent from the unknowns without friction term and without any condition on the two viscosities coefficients.

In [2, 3] the authors obtained global weak solutions for a 2D shallow-water system and Korteweg system with diffusion term of ttype $\nu\,\mathrm{div}(hD(u))$. They proved that the considered system is energetically consistent without any restriction on the data. The key point of this proof is based on an estimate of a new entropy, called “mathematical BD entropy”, which gives a bound of the term $\nabla\sqrt{h}$. We denote that to obtain this result in [2, 5, 7] it was necessary for the authors to add linear and quadratic terms of the form $r_0 u$, $r_1 h|u|u$ in the momentum equation. In [14] without additional regularizing terms, the authors obtained the existence of global weak solutions for the barotropic Navier-Stokes equations. They proved an inequality namely "Mellet-Vasseur” type inequality and obtained a control on $\int_\Omega h_i(1+|u_i|^2)\ln(1+|u_i|^2)dx$.

In [9] and [18], the authors proved the existence of global weak solutions of a bi-layers shallow-water model without any friction term but with a diffusion term of the form $\nu\Delta u$. This analysis used the method developed in [17] and the system is energetically consistent only for small enough initial data.

Following [20], we add friction terms $r_{i_0}u_i$, $r_{i_1}h_i|u_i|^2u_i$ and the Bohm potential term $\kappa_i h_i\nabla(\frac{\Delta\sqrt{h_i}}{\sqrt{h_i}})$ in the momentum equation. Following [2] and [20] the terms $r_{i_0}u_i$, $r_{i_1}h_i|u_i|^2u_i$ turn out to be essential to obtain the compactness of $\sqrt{h_i}u_i$ in $L^2(0,T;L^2(\Omega))$ and $h_iu_i\otimes u_i$ in $L^1(\Omega)$. The Bohm potential $\kappa_i h_i\nabla(\frac{\Delta\sqrt{h_i}}{\sqrt{h_i}})$ allows to deduce an estimate on $\nabla h_i^{1/4}$ in $L^4([0,T]\times\Omega)$ (see [20]). Another contribution in this paper is devoted to the convergence when the coefficients r_{i_0}, r_{i_1} and κ_i go to 0 for bi-layers shallow-water model. We consider in a periodic domain with periodic boundaries conditions the system

$$\partial_t h_1 + \operatorname{div}(h_1 u_1) = 0; \tag{1.5}$$

$$\begin{aligned} &\rho_1 \partial_t (h_1 u_1) + \rho_1 \operatorname{div}(h_1 u_1 \otimes u_1) - 2\nu_1 \operatorname{div}(h_1 D(u_1)) \\ &+ \rho_1 g h_1 \nabla h_1 + \rho_2 g h_1 \nabla h_2 = 0; \end{aligned} \tag{1.6}$$

$$\partial_t h_2 + \operatorname{div}(h_2 u_2) = 0; \tag{1.7}$$

$$\begin{aligned} &\rho_2 \partial_t (h_2 u_2) + \rho_2 \operatorname{div}(h_2 u_2 \otimes u_2) - 2\nu_2 \operatorname{div}(h_2 D(u_2)) \\ &+ \rho_2 g h_2 \nabla h_2 + \rho_2 g h_2 \nabla h_1 = 0 \end{aligned} \tag{1.8}$$

with initial conditions:

$$h_{i|t=0} = h_{i_0} \geq 0, \quad h_i u_{i|t=0} = m_{i_0}, \tag{1.9}$$

for which we assume the following regularities:

$$\begin{gathered} h_{i_0} \in L^2(\Omega), \quad \nabla h_{i_0} \in (L^2(\Omega))^2, \quad \nabla \sqrt{h_{i_0}} \in (L^2(\Omega))^2 \\ \frac{|m_{i_0}|^2}{h_{i_0}} \in L^1(\Omega), \quad \log_-(h_{i_0}) \in L^1(\Omega). \end{gathered} \tag{1.10}$$

for $i = 1, 2$. We denote by $D(u)$ the strain tensor, defined by $D(u) = \frac{\nabla u + \nabla^t u}{2}$, and by $A(u)$, the vorticity tensor such as $A(u) = \frac{\nabla u - \nabla^t u}{2}$.

The article is organized as follows: In Section 2 we give the definition of global weak solutions of the system (1.1)-(1.4) and we state the results of the existence of weak solutions for the system (1.1)-(1.4). And moreover we give some Theorems which are very useful in this current paper. Section 3 is devoted to the construction of approximate "Mellet-Vasseur" type inequality for any weak solutions. In this section we show that we can control (uniformly with respect to κ_i) this quantity, for any weak solutions of (3.1)-(3.2) with $\kappa_i > 0$. In Section 4, we study the limits as α_i defined in (3.16) approaches ∞. On the other hand, Section 5 is dedicated to the convergence of terms when r_{0_i}, r_{1_i} and κ_i go to zero. In Section 6 we give the proof of the "Mellet-Vasseur" type inequality. We denote that in this section we give also the proof of Theorem 3.6 by recovering the limit from Lemma 5.2.

2. Main results

We start this section with the definition of weak solutions.

Definition 2.1. We shall say that (h_1, h_2, u_1, u_2) is a weak solution of (1.1)-(1.4) if (1.1) and (1.3) hold in $(\mathcal{D}'(0,T) \times \Omega)^2$; (1.9) holds in $\mathcal{D}'(\Omega)$; the following assumptions are satisfied:

$$\begin{gathered} h_i \in L^\infty(0,T; L^2(\Omega)); \\ \nabla h_i \in L^2(0,T; (L^2(\Omega))^2) \text{ and } \sqrt{h_i} u_i \in L^\infty(0,T; (L^2(\Omega))^2); \\ \sqrt{h_i} D(u_i) \in L^2(0,T; (L^2(\Omega))^4); \\ \nabla \sqrt{h_i} \in L^2(0,T; L^2(\Omega)^2); \end{gathered} \tag{2.1}$$

for any $\varphi \in \mathcal{C}^\infty((0,T)\times\Omega)^2$ with $\varphi(T,\cdot)=0$, (φ with compact support), we have

$$\begin{aligned}
&-\rho_1 h_{1_0} u_{1_0}\varphi(0,\cdot) - \int_0^T\int_\Omega \rho_1 h_1 u_1 \partial_t\varphi - \rho_1\int_0^T\int_\Omega (h_1 u_1\otimes u_1) : D(\varphi)\\
&+ 2\nu_1\int_0^T\int_\Omega h_1(D(u_1):D(\varphi)) + \frac{1}{2}\rho_1 g\int_0^T\int_\Omega h_1^2 \operatorname{div}\varphi + \rho_1 g\int_0^T\int_\Omega h_1\nabla h_2\varphi\\
&= 0
\end{aligned} \tag{2.2}$$

and

$$\begin{aligned}
&-\rho_2 h_{2_0} u_{2_0}\varphi(0,\cdot) - \int_0^T\int_\Omega \rho_2 h_2 u_2 \partial_t\varphi - \rho_2\int_0^T\int_\Omega (h_2 u_2\otimes u_2) : D(\varphi)\\
&+ 2\nu_2\int_0^T\int_\Omega h_2(D(u_2):D(\varphi)) + \frac{1}{2}\rho_2 g\int_0^T\int_\Omega h_2^2 \operatorname{div}\varphi + \rho_2 g\int_0^T\int_\Omega h_2(\varphi\cdot\nabla h_1)\\
&= 0.
\end{aligned} \tag{2.3}$$

We will prove the following theorem.

Theorem 2.2. *There exists a global weak solution* (h_1,h_2,u_1,u_2) *of* (1.1)-(1.4) *satisfying the entropy inequalities* (2.4) *and* (2.8).

In this section, we give the classical energy estimate and the "mathematical BD entropy". These two inequalities will allow us to prove the main theorem.

Lemma 2.3. *Let* (h_1,h_2,u_1,u_2) *be a solution of the system* (1.1)-(1.4). *Then*

$$\begin{aligned}
&\frac{1}{2}\rho_1\frac{d}{dt}\int_\Omega h_1|u_1|^2 + \frac{1}{2}\rho_2\frac{d}{dt}\int_\Omega h_2|u_2|^2 + 2\nu_1\int_\Omega h_1(D(u_1):D(u_1))\\
&+ 2\nu_2\int_\Omega h_2(D(u_2):D(u_2)) + \frac{1}{2}g(\rho_1-\rho_2)\frac{d}{dt}\int_\Omega |h_1|^2 + \frac{1}{2}\rho_2 g\frac{d}{dt}\int_\Omega |h_1+h_2|^2\\
&\le 0.
\end{aligned} \tag{2.4}$$

Remark 2.4. From the energy estimate (2.4), we deduce the following:

$$\sqrt{h_1}u_1 \in L^\infty(0,T;(L^2(\Omega))^2); \quad \sqrt{h_2}u_2 \in L^\infty(0,T;(L^2(\Omega))^2); \tag{2.5}$$
$$h_1 \in L^\infty(0,T;L^2(\Omega)); \quad \sqrt{h_1}\,D(u_1) \in L^2(0,T;(L^2(\Omega))^4); \tag{2.6}$$
$$h_2 \in L^\infty(0,T;L^2(\Omega)); \quad \sqrt{h_2}\,D(u_2) \in L^2(0,T;(L^2(\Omega))^4). \tag{2.7}$$

However, it is well-known that these estimates are not enough to pass to the limit and get the stability of the system. So we are going to obtain further estimates from the BD entropy that we state in the following lemma, (see [5]).

Lemma 2.5. *If we assume that* (h_1, h_2, u_1, u_2) *is a smooth solution of system* (1.1)-(1.4), *then*

$$\begin{aligned}
&\frac{1}{2}\rho_2\frac{d}{dt}\int_\Omega h_1|\rho_1 u_1 + 2\nu_1\nabla\log h_1|^2 + \frac{1}{2}\rho_1\frac{d}{dt}\int_\Omega h_2|\rho_2 u_2 + 2\nu_2\nabla\log h_2|^2 \\
&+ \rho_1\rho_2\Big(\frac{1}{2}g(\rho_1-\rho_2)\frac{d}{dt}\int_\Omega |h_1|^2 + \frac{1}{2}\rho_2 g\frac{d}{dt}\int_\Omega |h_1+h_2|^2\Big) \\
&+ 2\nu_2\rho_1\rho_2\int_\Omega h_2(A(u_2):A(u_2)) + 2\nu_1\rho_1\rho_2\int_\Omega h_1(A(u_1):A(u_1)) \\
&+ 2\nu_1\rho_1\rho_2 g\int_\Omega|\nabla h_1|^2 + 2\nu_2\rho_1\rho_2 g\int_\Omega|\nabla h_2|^2 + 2\rho_2 g(\rho_2\nu_1+\rho_1\nu_2)\int_\Omega\nabla h_1\nabla h_2 \\
&\le 0
\end{aligned} \tag{2.8}$$

Remark 2.6. We would like to point out the boundedness of the 'non usual' terms appearing above.

(1) In the energy equality (2.8), it remains to control the four last terms on left-hand side.
(2) The proof of the previous two lemmas takes inspiration in [21].
(3) The classical energy and the BD entropy allow us to find the estimates:

$$\begin{gathered}
\nabla\sqrt{h_1}\in L^2(0,T;(L^2(\Omega))^2);\quad \nabla\sqrt{h_2}\in L^2(0,T;(L^2(\Omega))^2); \\
\nabla h_1\in L^2(0,T;(L^2(\Omega))^2);\quad \nabla h_2\in L^2(0,T;(L^2(\Omega))^2).
\end{gathered} \tag{2.9}$$

3. Construction of the "Mellet-Vasseur" type inequality

Following the idea proposed in [20] this section is devoted to the construction of an approximation of the "Mellet-Vasseur" type inequality for any weak solution for the system (3.1)-(3.4), with the initial conditions (1.9), verifying in additional $h_{i_0} \ge \frac{1}{\alpha_i}$ for $\alpha_i > 0$ and $\sqrt{h_{i_0}}u_{i_0} \in L^\infty(\Omega)$.

Proposition 3.1. *For any* $\kappa \ge 0$ *and* $\bar\kappa \ge 0$, *there exists a global weak solution to the system*

$$\partial_t h_1 + \operatorname{div}(h_1u_1) = 0; \tag{3.1}$$

$$\begin{aligned}
&\rho_1\partial_t(h_1u_1) + \rho_1\operatorname{div}(h_1u_1\otimes u_1) - 2\nu_1\operatorname{div}(h_1D(u_1)) + \rho_1 g h_1\nabla h_1 \\
&+ \rho_2 g h_1\nabla h_2 + r_0u_1 + r_1h_1|u_1|^2u_1 - \kappa h_1\nabla\Big(\frac{\Delta\sqrt{h_1}}{\sqrt{h_1}}\Big) = 0;
\end{aligned} \tag{3.2}$$

$$\partial_t h_2 + \operatorname{div}(h_2u_2) = 0; \tag{3.3}$$

$$\begin{aligned}
&\rho_2\partial_t(h_2u_2) + \rho_2\operatorname{div}(h_2u_2\otimes u_2) - 2\nu_2\operatorname{div}(h_2D(u_2)) \\
&+ \rho_2 g h_2\nabla h_2 + \rho_2 g h_2\nabla h_1 + \bar r_0u_2 + \bar r_1h_2|u_2|^2u_2 - \bar\kappa h_2\nabla\Big(\frac{\Delta\sqrt{h_2}}{\sqrt{h_2}}\Big) = 0
\end{aligned} \tag{3.4}$$

with the initial data (1.9) *satisfying* (1.10) *and* $-r_0\int_\Omega \log_- h_{i_0}dx<\infty$. *In particular, we have the energy inequality*

$$
\begin{aligned}
&\frac{d}{dt}E_1(t)+2\nu_1\int_\Omega h_1(D(u_1):D(u_1))+2\nu_2\int_\Omega h_2(D(u_2):D(u_2))+c_0\int_\Omega |u_1|^2\\
&+r_0\int_\Omega |u_1|^2+\bar r_0\int_\Omega |u_2|^2+r_1\int_\Omega h_1|u_1|^4+\bar r_1\int_\Omega h_2|u_2|^4=0,
\end{aligned}
\tag{3.5}
$$

where

$$
\begin{aligned}
E_1(t)=\int_\Omega\Big[&\frac12\rho_1h_1|u_1|^2+\frac12\rho_2h_2|u_2|^2+\frac12 g(\rho_1-\rho_2)|h_1|^2+\frac12\rho_2g|h_1+h_2|^2\\
&+\frac{\kappa}{2}|\nabla\sqrt{h_1}|^2+\frac{\bar\kappa}{2}|\nabla\sqrt{h_2}|^2\Big]
\end{aligned}
$$

and the BD-entropy

$$
\begin{aligned}
&\frac{d}{dt}E_2(t)+\rho_1\rho_2c_0\int_\Omega|u_1|^2+2\nu_2\rho_1\rho_2\int_\Omega h_2(A(u_2):A(u_2))\\
&+2\nu_1\rho_1\rho_2\int_\Omega h_1(A(u_1):A(u_1))+2\rho_2g(\rho_2\nu_1+\rho_1\nu_2)\int_\Omega\nabla h_1\nabla h_2\\
&+\kappa\int_\Omega h_1|\nabla^2\log h_1|^2+\bar\kappa\int_\Omega h_1|\nabla^2\log h_2|^2\\
&+2\nu_1\rho_1\rho_2g\int_\Omega|\nabla h_1|^2+2\nu_2\rho_1\rho_2g\int_\Omega|\nabla h_2|^2=0
\end{aligned}
\tag{3.6}
$$

where

$$
\begin{aligned}
E_2(t)=\int_\Omega\Big[&\frac12\rho_2h_1|\rho_1u_1+2\nu_1\nabla\log h_1|^2+\frac12\rho_1h_2|\rho_2v_2+2\nu_2\nabla\log h_2|^2\\
&-\bar r_0\log_- h_2-r_0\log_- h_1+\frac{\kappa}{2}|\nabla\sqrt{h_1}|^2+\frac{\bar\kappa}{2}|\nabla\sqrt{h_2}|^2\\
&+\rho_1\rho_2\Big(\frac12 g(\rho_1-\rho_2)|h_1|^2+\frac12\rho_2g|h_1+h_2|^2\Big)\Big].
\end{aligned}
$$

The proof of the above Proposition takes inspiration in [21]. It takes into account the additional terms.

Corollary 3.2. *The energy inequalities* (3.5)-(3.6) *yield the following new estimates*

$$\|\sqrt{\kappa}\nabla\sqrt{h_1}\|_{L^\infty(0,T,L^2(\Omega))}\le C,\quad \|\sqrt{\bar\kappa}\nabla\sqrt{h_2}\|_{L^\infty(0,T,L^2(\Omega))}\le C,\tag{3.7}$$
$$\|\sqrt{r_0}u_1\|_{L^2(0,T,L^2(\Omega))}\le C,\quad \|\sqrt{\bar r_0}u_2\|_{L^2(0,T,L^2(\Omega))}\le C,\tag{3.8}$$
$$\|\sqrt[4]{r_1}h_1u_1\|_{L^4(0,T,L^4(\Omega))}\le C,\quad \|\sqrt[4]{\bar r_1}h_2u_2\|_{L^4(0,T,L^4(\Omega))}\le C,\tag{3.9}$$
$$\|\sqrt{\kappa}\nabla^2\log h_1\|_{L^2(0,T,L^2(\Omega))}\le C,\quad \|\sqrt{\bar\kappa}\nabla^2\log h_2\|_{L^2(0,T,L^2(\Omega))}\le C,\tag{3.10}$$
$$\|\nabla\sqrt{h_1}\|_{L^\infty(0,T,L^2(\Omega))}\le C,\quad \|\nabla\sqrt{h_2}\|_{L^\infty(0,T,L^2(\Omega))}\le C,\tag{3.11}$$
$$\|\sqrt{h_1}A(u_1)\|_{L^2(0,T,L^2(\Omega))}\le C,\quad \sqrt{h_1}A(u_1)\|_{L^2(0,T,L^2(\Omega))}\le C.\tag{3.12}$$

where C is bounded by the initial data, uniformly on $r_0,\bar r_0,r_1,\bar r_1,\kappa$, and $\bar\kappa$.

Remark 3.3. (1) The following inequalities hold:

$$\sqrt{\kappa}\|\sqrt{h_1}\|_{L^2(0,T,H^2(\Omega))}+\kappa^{1/4}\|\nabla h_1^{1/4}\|_{L^4(0,T,L^4(\Omega))}\le C_1,$$

$$\sqrt{\bar\kappa}\|\sqrt{h_2}\|_{L^2(0,T,H^2(\Omega))}+\bar\kappa^{1/4}\|\nabla h_2^{1/4}\|_{L^4(0,T,L^4(\Omega))}\le C_2,$$

where C_1 and C_2 depend only on the initial data. These inequalities are a consequence of the bound on (3.10)

(2) The inequalities (3.11) and (2.6) yield

$$\sqrt{h_1}\in L^\infty(0,T;L^p(\Omega)),\quad \sqrt{h_2}\in L^\infty(0,T;L^p(\Omega))\quad \text{for } p\ge 1. \tag{3.13}$$

(3) The weak formulation reads as follows

$$\begin{aligned}
&-\rho_1 h_{1_0}u_{1_0}\varphi(0,\cdot)-\int_0^T\int_\Omega \rho_1 h_1 u_1\partial_t\varphi-\rho_1\int_0^T\int_\Omega (h_1u_1\otimes u_1):D(\varphi)\\
&+r_0\int_0^T\int_\Omega u_1+2\nu_1\int_0^T\int_\Omega h_1(D(u_1):D(\varphi))+r_1\int_0^T\int_\Omega h_1|u_1|^2u_1\varphi\\
&-\kappa\int_0^T\int_\Omega \Delta\sqrt{h_1}\sqrt{h_1}\operatorname{div}\varphi-2\kappa\int_0^T\int_\Omega\Delta\sqrt{h_1}\nabla\sqrt{h_1}\varphi\\
&+\frac12\rho_1 g\int_0^T\int_\Omega h_1^2\operatorname{div}\varphi+\rho_1 g\int_0^T\int_\Omega h_1\nabla h_2\varphi=0
\end{aligned}\tag{3.14}$$

and

$$\begin{aligned}
&-\rho_2 h_{2_0}u_{2_0}\varphi(0,\cdot)-\int_0^T\int_\Omega \rho_2 h_2 u_2\partial_t\varphi-\rho_2\int_0^T\int_\Omega (h_2u_2\otimes u_2):D(\varphi)\\
&+\bar r_0\int_0^T\int_\Omega u_2\varphi+2\nu_2\int_0^T\int_\Omega h_2(D(u_2):D(\varphi))+\bar r_1\int_0^T\int_\Omega h_2|u_2|^2u_2\varphi\\
&-\bar\kappa\int_0^T\int_\Omega \Delta\sqrt{h_2}\sqrt{h_2}\operatorname{div}\varphi-2\bar\kappa\int_0^T\int_\Omega\Delta\sqrt{h_2}\nabla\sqrt{h_2}\varphi\\
&+\frac12\rho_2 g\int_0^T\int_\Omega h_2^2\operatorname{div}\varphi+\rho_2 g\int_0^T\int_\Omega h_2(\varphi\cdot\nabla h_1)=0.
\end{aligned}\tag{3.15}$$

for any function test φ.

Next we consider $\varepsilon_1=r$, $\varepsilon_2=1$, $r_{1_0}=r_0$, $r_{2_0}=\bar r_0$, $r_{1_1}=r_1$, $r_{2_1}=\bar r_1$, $\kappa_1=\kappa$ and $\kappa_2=\bar\kappa$.

Our first main result is the next theorem which gives the "Mellet-Vasseur" type inequality (see [20]).

Theorem 3.4. *For any $\delta_i\in(0,2)$, there exists C_i depending only on δ_i, and the weak solutions (h_1,h_2,u_1,u_2) to (3.1)-(3.2) with $\kappa_i=0$ verify all the properties of Proposition 3.1, and satisfy the following "Mellet-Vasseur" type inequality for every $T>0$, and almost every $T>t$:*

$$\begin{aligned}
&\int_\Omega h_i(1+|u_i|^2)\ln(1+|u_i|^2)\\
&\le\int_\Omega h_{i_0}(1+|u_{i_0}|^2)\ln(1+|u_{i_0}|^2)+C_i\int_\Omega\Big(\frac{h_{i_0}|u_{i_0}|^2}{2}+\frac12|h_i|^2+|\nabla\sqrt{h_{i_0}}|^2\Big)\\
&+C_i\int_0^T\Big(\int_\Omega\big(h_i^{3-\frac{\delta_i}{2}}\big)^{\frac{2}{2-\delta_i}}\Big)^{\frac{2-\delta_i}{2}}\Big(\int_\Omega h_i\big(2+\ln(1+|u_i|^2)\big)\Big)^{\frac{\delta_i}{2}}\Big),
\end{aligned}$$

for $i=1,2$.

Remark 3.5. (1) The right hand side constant C_i of the above inequality does not depend on r_{i_0} and r_{i_1}. This theorem will be crucial to prove the strong convergence of $\sqrt{h_i}u_i$ in the space $L^2(0,T;L^2(\Omega))$ when r_{i_0} and r_{i_1} converge to 0.

(2) For a global weak solution of (3.1)-(3.4), under hypothesis of Definition 2.1, we need that (3.1)-(3.4) hold in $D'([0,T]\times\Omega)$ and the following be satisfied

$$
\begin{gathered}
h_i \geq 0, \quad h_i \in L^\infty(0,T;L^2(\Omega)),\\
h_i(1+|u_i|^2)\ln(1+|u_i|^2) \in L^\infty(0,T;L^1(\Omega)),\\
\nabla h_i \in L^2(0,T;L^2(\Omega)), \quad \nabla\sqrt{h_i} \in L^\infty(0,T;L^2(\Omega)),\\
\sqrt{h_i}u_i \in L^\infty(0,T;L^2(\Omega)), \sqrt{h_i}\nabla u_i \in L^2(0,T;L^2(\Omega)),
\end{gathered}
$$

As a sequence of Theorem 3.4, we have the same result as in [20]:

Theorem 3.6. *Let* (h_{i_0}, m_{i_0}) *satisfy* (1.9) *and*

$$
\int_\Omega h_{i_0}(1+|u_{i_0}|^2)\ln(1+|u_{i_0}|^2)dx < \infty.
$$

Then for $T>0$*, there exists a weak solution of* (3.1)-(3.4) *on* $(0,T)$.

The "Mellet-Vasseur" type inequality does not work for the solutions of (3.1)-(3.4) for $\kappa_i \geq 0$. The idea is to construct as in [20] an approximation of the "Mellet-Vasseur" type inequality. We define four $\mathcal{C}^\infty$ non-negative cut-off functions ϕ_{α_i} and ϕ_{β_i} as follows.

$$
\phi_{\alpha_i}(h_i) = 1 \text{ for any } h_i > \frac{1}{\alpha_i}, \quad \phi_{\alpha_i}(h_i) = 0 \text{ for any } h_i < \frac{1}{2\alpha_i}, \tag{3.16}
$$

where $\alpha_i > 0$ is any real number, and otherwise, $|\phi'_{\alpha_i}| \leq 2\alpha_i$; and $\phi_{\beta_i}(h_i) \in \mathcal{C}^\infty(\mathbb{R})$ is a non-negative function such that

$$
\phi_{\beta_i}(h_i) = 1 \text{ for any } h_i < \beta_i, \quad \phi_{\beta_i}(h_i) = 0 \text{ for any } h_i > 2\beta_i, \tag{3.17}
$$

where $\beta_i > 0$ is any real number, and $|\phi'_{\beta_i}| \leq \frac{2}{\beta_i}$.

We define $v_i = \phi_i(h_i)u_i$, and $\phi_i(h_i) = \phi_{\alpha_i}(h_i)\phi_{\beta_i}(h_i)$. The lemmas will be very useful to construct the approximation of the "Mellet-Vasseur" type inequality.

Lemma 3.7. *For any fixed* $\kappa_i > 0$*, we have*

$$
\|\nabla v_i\|_{\mathrm{L^2(0,T;L^2(\Omega))}} \leq \mathrm{C_i},
$$

where the constant C_i *depends on* $\kappa_i > 0$, r_{i_1}, β_i *and* α_i*; and*

$$
\partial_t h_i \in L^4(0,T;L^{\frac{6}{5}}(\Omega)) + L^2(0,T;L^{\frac{3}{2}}(\Omega)).
$$

For a proof of the above lemma, see [20]. Following the ideas in [20], we introduce a new $\mathcal{C}^\infty(\mathbb{R}^2)$, non-negative cut-off function $\varphi_{in}(h_i)$ which is given by

$$
\varphi_{in}(x) = \begin{cases} (1+|x|^2)\ln(1+|x|^2) & \text{if } 0 \leq |x| < n,\\ (1+8n^2)\ln(1+4n^2) & \text{if } |x| \geq 2n \end{cases} \tag{3.18}
$$

where $n > 0$ are large, and

$$
|\varphi'_{in}(x)| + |\varphi"_{in}(x)| \leq \frac{C_i}{n} \quad \text{for any } |x| \geq n.
$$

The first step of constructing the approximation of the Mellet-Vasseur type inequality is the following lemma.

Lemma 3.8. *For the weak solutions to* (3.1)-(3.4) *constructed in Proposition 3.1, and any* $\psi_i(t) \in \mathcal{D}(-1,+\infty)$*, we have*

$$\begin{aligned}
&\int_0^T \int_\Omega \partial_t \psi_i(t) h_i \varphi_{in}(v_i)\,dx\,dt - \int_0^T \int_\Omega \psi_i(t)\varphi_{in}'(v_i) F_i\,dx\,dt \\
&+ \int_0^T \int_\Omega \psi_i(t) S_i : \nabla(\varphi_{in}'(v_i))\,dx\,dt \\
&= \int_\Omega h_{i_0}\varphi_{in}(v_{i0})\psi_i(0)\,dx\,dt
\end{aligned} \tag{3.19}$$

where

$$S_i = h_i\phi_i(h_i)(D(u_i) + \kappa_i \frac{\Delta\sqrt{h_i}}{\sqrt{h_i}})\mathbb{I},$$

$$\begin{aligned}
F_i = h_i^2 u_i \phi_i'(h_i) \operatorname{div} \mathrm{u_i} + \mathrm{gh_i \nabla h_i \phi_i(h_i)} + \mathrm{h_i \nabla \phi_i(h_i) D(u_i)} + \mathrm{g\varepsilon_i h_i \nabla h_j \phi_i(h_i)} \\
+ r_{i_0} u_i \phi_i(h_i) + r_{i_1} h_i |u_i|^2 u_i \phi_i(h_i) + \kappa_i \sqrt{h_i} \nabla\phi_i(h_i) \Delta\sqrt{h_i} \\
+ 2\kappa_i \phi_i(h_i) \nabla\sqrt{h_i} \Delta\sqrt{h_i},
\end{aligned} \tag{3.20}$$

where $\mathbb{I}$ *is an identity matrix.*

Proof. To obtain the result it suffices to multiply equations (3.2) and (3.4) by $\phi_1(h_1)$ and $\phi_2(h_2)$ respectively; we have

$$\begin{aligned}
&\partial_t(h_i v_i) - h_i u_i \phi_i'(h_i)\partial_t h_i + \operatorname{div}(h_i u_i \otimes v_i) - h_i u_i \otimes u_i \nabla\phi_i(h_i) + h_i \nabla h_i \phi_i(h_i) \\
&- \operatorname{div}(\phi_i(h_i) h_i \mathbb{D} u_i) + h_i \nabla\phi_i(h_i) \mathbb{D} u_i + r_{i_0} h_i u_i \phi_i(h_i) + r_{i_1} h_i |u_i|^2 u_i \phi_i(h_i) \\
&+ g\varepsilon_i h_i \nabla h_j \phi_i(h_i) - \kappa_i \nabla(\sqrt{h_i}\phi_i(h_i)\Delta\sqrt{h_i}) \\
&+ \kappa_i \sqrt{h_i}\nabla\phi_i(h_i)\Delta\sqrt{h_i} + 2\kappa_i\phi_i(h_i)\nabla\sqrt{h_i}\Delta\sqrt{h_i} = 0.
\end{aligned}$$

Here we did a successive integration. □

Remark 3.9. Both $\nabla\sqrt{h_i}$ and $\partial_t h_i$ are functions, so the above equalities are justified by regularizing h_i and passing into the limit. We can rewrite the above equation as follows

$$\partial_t(h_i v_i) + \operatorname{div}(\mathrm{h_i u_i \otimes v_i}) - \operatorname{div} \mathrm{S_i} + \mathrm{F_i} = 0 \tag{3.21}$$

where S_i and F_i are as in (3.20), and we used

$$\begin{aligned}
h_i u_i \phi_i'(h_i)\partial_t(h_i) + h_i u_i \otimes u_i \phi_i'(h_i) \nabla h_i &= h_i u_i \phi_i'(h_i)(\partial_t h_i + \nabla h_i \cdot u_i) \\
&= -h_i^2 u_i \phi_i'(h_i) \operatorname{div} u_i.
\end{aligned}$$

We should remark that, thanks to Corollary 3.2 and Remark 3.3,

$$\|F_i\|_{L^{\frac{4}{3}}(0,T;L^1(\Omega))} \le C_i, \quad \|S_i\|_{L^2(0,T;L^2(\Omega))} \le C_i,$$

since $\sqrt{h_i}\phi_i(h_i)$ and $h_i\phi_i(h_i)$ are bounded. Those bounds depend on β_i and κ_i.

We first introduced a test function $\psi_i(t) \in \mathcal{D}(0,+\infty)$. Essentially this function vanishes for t close $t = 0$. We will later extend the result for $\psi_i(t) \in D(-1,+\infty)$. We define a new function $\Phi_i = \overline{\psi_i(t)\phi_{in}'(v_i)}$, where $\overline{f_i(t,x)} = f_i * \eta_{ik}(t,x)$, k is a small enough number. Note that, since $\phi_i(t)$ is compactly supported in $(0,\infty)$. Φ_i is well defined on $(0,\infty)$ for k small enough. We use it to test (3.21) to have

$$\int_0^T \int_\Omega \overline{\psi_i(t)\varphi_{in}'(\bar{v}_i)}[\partial_t(h_i v_i) + \operatorname{div}(h_i u_i \otimes v_i) - \operatorname{div} S_i + F_i]\,dx\,dt = 0$$

which in turn gives us

$$\int_0^T \int_\Omega \psi_i(t)\varphi'_{in}(\bar{v}_i)\overline{[\partial_t(h_i v_i) + \operatorname{div}(h_i u_i \otimes v_i) - \operatorname{div} S_i + F_i]}\,dx\,dt = 0 \tag{3.22}$$

The first term in (3.22) can be calculated as follows:

$$\begin{aligned}
&\int_\Omega \psi_i(t)\varphi'_{in}(\bar{v}_i)\overline{\partial_t(h_i v_i)}\,dx\,dt \\
&= \int_\Omega \psi_i(t)\varphi'_{in}(\bar{v}_i)\partial_t(h_i\bar{v}_i)\,dx\,dt + \int_\Omega \psi_i(t)\varphi'_{in}(\bar{v}_i)[\overline{\partial_t(h_i v_i)} - \partial_t(h_i\bar{v}_i)]\,dx\,dt \\
&= \int_\Omega \psi_i(t)\varphi'_{in}(\bar{v}_i)(\partial_t(h_i)\bar{v}_i + h_i\partial_t(\bar{v}_i))\,dx\,dt + R_1 \\
&= \int_\Omega \psi_i(t)\partial_t h_i\varphi'_{in}(\bar{v}_i)\bar{v}_i\,dx\,dt + \int_0^T \int_\Omega \psi_i(t)h_i\varphi_{in}(\partial_t\bar{v}_i)\,dx\,dt + R_1
\end{aligned}$$

where

$$R_1 = \int_0^T \int_\Omega \psi_i(t)\varphi'_{in}(\bar{v}_i)[\overline{\partial_t(h_i\bar{v}_i} - \partial_t(h_i\bar{v}_i)]\,dx\,dt.$$

Thanks to equation (3.1), we can rewrite the second term in (3.22) as follows

$$\begin{aligned}
&\int_0^T \int_\Omega \psi_i(t)\varphi'_{in}(\bar{v}_i)\overline{\operatorname{div}(h_i u_i \otimes v_i)}\,dx\,dt \\
&= \int_0^T \int_\Omega \psi_i(t)\partial_t h_i\varphi_{in}(\bar{v}_i)\,dx\,dt - \int_0^T \int_\Omega \psi_i(t)\partial_t h_i\varphi'_{in}(\bar{v}_i)\bar{v}_i\,dx\,dt + R_2,
\end{aligned} \tag{3.23}$$

and

$$R_2 = \int_0^T \int_\Omega \psi_i(t)\varphi'_{in}(\bar{v}_i)[\operatorname{div}(h_i u_i \otimes \bar{v}_i) - \overline{\operatorname{div}(h_i u_i \otimes v_i}]\,dx\,dt.$$

By (3.22)-(3.23), we have

$$\begin{aligned}
&\int_0^T \int_\Omega \psi_i(t)\partial_t(h_i\varphi_{in}(\bar{v}_i))\,dx\,dt + R_1 + R_2 - \int_0^T \int_\Omega \psi_i(t)\varphi'_i(\bar{v}_i)\overline{\operatorname{div} S_i}\,dx\,dt \\
&+ \int_0^T \int_\Omega \psi_i(t)\varphi'_{in}(\bar{v}_i)\bar{F}_i\,dx\,dt = 0.
\end{aligned}$$

Notice that $\bar{v}_i$ converges to v almost everywhere and

$$h_i\varphi_{in}(\bar{v}_i)\partial_t\psi_i \to h_i\varphi_{in}(v_i)\partial_t\psi_i \quad \text{in } L^1((0,T)\times\Omega).$$

So, up to a subsequence, we have

$$\int_0^T \int_\Omega h_i\varphi_{in}(\bar{v}_i)\partial_t\psi_i\,dx\,dt \to \int_0^T \int_\Omega h_i\varphi_{in}(v_i)\partial_t\psi_i\,dx\,dt \quad \text{as } k \to 0. \tag{3.24}$$

Since $\varphi'_{in}(\bar{v}_i)$ converges to $\varphi'_{in}(v_i)$ almost everywhere, and is uniformly bounded in $L^\infty((0,T)\times\Omega)$, we have

$$\int_0^T \int_\Omega \psi_i(t)\varphi'_{in}(\bar{v}_i)\bar{F}_i\,dx\,dt \to \int_0^T \int_\Omega \psi_i(t)\varphi'_{in}(v_i)F_i \quad \text{as } k \to 0. \tag{3.25}$$

Noticing that $\nabla v_i \in L^2(0,T;L^2(\Omega))$, we have

$$\overline{\nabla v_i} \to \nabla v_i \quad \text{strongly in } L^2(0,T;L^2(\Omega)).$$

Since $\bar{S}_i$ converges to S_i strongly in $L^2(0,T;L^2(\Omega))$, and $\varphi_i"(\bar{v}_i)$ converges to $\varphi_i"(v_i)$ almost everywhere and uniformly bounded in $L^\infty((0,T)\times\Omega))$, we obtain

$$\int_0^T \int_\Omega \psi_i(t)\varphi_{in}'(\bar{v}_i)\overline{\operatorname{div} S_i}\,dx\,dt = -\int_0^T \int_\Omega \psi_i(t)\bar{S}_i : \nabla(\varphi_{in}'(\bar{v}_i))\,dx\,dt, \tag{3.26}$$

which converges to

$$-\int_0^T \int_\Omega \psi_i(t) S_i : \nabla(\varphi_{in}'(v_i))\,dx\,dt \tag{3.27}$$

To handle R_1 and R_2, we use the following lemma due to Lions [13].

Lemma 3.10. *Let $f \in W^{1,p}(\mathbb{R}^N)$, $g \in L^q(\mathbb{R}^N)$ with $1 \le p, q \le \infty$, and $\frac{1}{P} + \frac{1}{q} \le 1$. Then we have*

$$\|\operatorname{div}(fg) * w_\varepsilon - \operatorname{div}(f(g * w_\varepsilon))\|_{L^r(\mathbb{R}^N)} \le C\|f\|_{W^{1,p}(\mathbb{R}^N)}\|g\|_{L^q(\mathbb{R}^N)}$$

for some $C \ge 0$ independent of ε, f and g, r is determined by $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. In addition,

$$\operatorname{div}(fg) * w_\varepsilon - \operatorname{div}(f(g * w_\varepsilon)) \to 0 \quad \text{in } L^r(\mathbb{R}^N)$$

as $\varepsilon \to 0$ if $r < \infty$.

This lemma includes the following statement.

Lemma 3.11. *Let $\partial_t f \in L^P(0,T)$, $g \in L^q(0,T)$ with $1 \le p, q \le \infty$, and $\frac{1}{P} + \frac{1}{q} \le 1$. Then we have*

$$\|\partial_t(fg) * w_\varepsilon - \partial_t(f(g * w_\varepsilon))\|_{L^r(0,T)} \le C\|f\|_{L^p(0,T)}\|g\|_{L^q(0,T)}$$

for some $C \ge 0$ independent of ε, f and g, r is determined by $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. In addition,

$$\partial_t(fg) * w_\varepsilon - \partial_t(f(g * w_\varepsilon)) \to 0 \quad \text{in } L^r(0,T)$$

as $\varepsilon \to 0$ if $r < \infty$.

With Lemmas 3.10 and 3.11 in hand, we are ready to handle the terms R_1 and R_2. For $\kappa_i > 0$, by Lemma 3.7 and Poincare inequality, we have $v_i \in L^2(0,T;L^6(\Omega))$. We also have, by Lemma 3.7,

$$\partial_t h_i \in L^4(0,T;L^{\frac{6}{5}}(\Omega)) + L^2(0,T;L^{\frac{3}{2}}(\Omega)).$$

Thus, applying Lemma 3.11,

$$\begin{aligned} |R_1| &\le \int_0^T \int_\Omega \Big|\psi_i(t)\varphi'(\bar{v}_i)[\overline{\partial_t(h_i v_i)} - \partial_t(h_i\bar{v}_i)]\Big|\,dx\,dt \\ &\le C(\psi_i)\int_0^T \int_\Omega \Big|\varphi'(\bar{v}_i)[\overline{\partial_t(h_i v_i)} - \partial_t(h_i\bar{v}_i)]\Big|\,dx\,dt \to 0 \quad \text{as } k \to 0. \end{aligned} \tag{3.28}$$

By a similar reasoning and using Lemma 3.10, We deduce that $R_2 \to 0$ as $k \to 0$. By (3.24)-(3.28), we have

$$\begin{aligned} &\int_0^T \int_\Omega \partial_t\psi_i(t) h_i\varphi_{in}(v_i)\,dx\,dt - \int_0^T \int_\Omega \psi_i(t)\varphi_{in}'(v_i) F\,dx\,dt \\ &+ \int_0^T \int_\Omega \psi_i(t) S_i : \nabla(\varphi_{in}'(v_i))\,dx\,dt = 0, \end{aligned} \tag{3.29}$$

for any test function $\psi_i \in \mathcal{D}(0,\infty)$.

We need to consider the test function $\psi_i \in \mathcal{D}(-1,\infty)$. For this, we need the continuity of $h_i(t)$ and $(\sqrt{h_i}u_i)(t)$ in the strong topology at $t=0$. In fact, thanks to Proposition 3.1, we have

$$\partial_t\sqrt{h_i} \in L^2(0,T;L^2(\Omega)), \quad \sqrt{h_i} \in L^2(0,T;H^2(\Omega)).$$

This gives us

$$\sqrt{h_i} \in C([0,T];L^2(\Omega)) \quad \text{and} \quad \nabla\sqrt{h_i} \in C([0,T];L^2(\Omega))$$

thanks to [8, Theorem 3 p. 287]. Similarly, we have

$$h_i \in C([0,T];L^2(\Omega)) \tag{3.30}$$

due to

$$\|\nabla h_i\|_{L^2(0,T;L^2(\Omega))} \le \|\nabla\sqrt{h_i}\|_{L^4(0,T;L^4(\Omega))}\|\sqrt{h_i}\|_{L^4(0,T;L^4(\Omega))}.$$

We have $\sqrt{h_i} \in L^\infty(0,T;L^p(\Omega))$ for any $p \ge 1$, and hence

$$\sqrt{h_i} \in C([0,T];L^p(\Omega)) \text{ for any } p \ge 1. \tag{3.31}$$

An analogous reasoning as in [20] gives us

$$\sqrt{h_i}u_i \in C([0,T];L^2(\Omega)). \tag{3.32}$$

Indeed: we have

$$\begin{aligned}
&\operatorname{ess}\limsup_{t\to 0}\int_\Omega \rho_1|\sqrt{h_1}u_1-\sqrt{h_{1_0}}u_{1_0}|^2dx + \operatorname{ess}\limsup_{t\to 0}\int_\Omega \rho_2|\sqrt{h_2}u_2-\sqrt{h_{2_0}}u_{2_0}|^2dx\\
&\le \varepsilon(t,x)-\varepsilon(0,x) + 2\operatorname{ess}\limsup_{t\to 0}\int_\Omega \rho_1\sqrt{h_{1_0}}u_{1_0}(\sqrt{h_{1_0}}u_{1_0}-\sqrt{h_1}u_1)dx\\
&\quad + \operatorname{ess}\limsup_{t\to 0}\int_\Omega \rho_2\sqrt{h_{2_0}}u_{2_0}(\sqrt{h_{2_0}}u_{2_0}-\sqrt{h_2}u_2)dx\\
&\quad + \operatorname{ess}\limsup_{t\to 0}\int_\Omega g(\rho_1-\rho_2)|h_1-h_{1_0}|^2dx\\
&\quad + 2\operatorname{ess}\limsup_{t\to 0}\int_\Omega g(\rho_1-\rho_2)h_1(h_{1_0}-h_1)dx\\
&\quad + \operatorname{ess}\limsup_{t\to 0}\int_\Omega g\rho_2|h_1+h_2-h_{1_0}-h_{2_0}|^2dx\\
&\quad + 2\operatorname{ess}\limsup_{t\to 0}\int_\Omega g\rho_2(h_1+h_2)(h_1+h_2-h_{1_0}-h_{2_0})dx\\
&\quad - \frac{3\kappa_1}{2}\operatorname{ess}\limsup_{t\to 0}\int_\Omega |\nabla\sqrt{h_1}-\nabla\sqrt{h_{1_0}}|^2dx\\
&\quad - \frac{3\kappa_2}{2}\operatorname{ess}\lim_{t\to 0}\sup\int_\Omega |\nabla\sqrt{h_2}-\nabla\sqrt{h_{2_0}}|^2dx\\
&\quad + 3\kappa_1\operatorname{ess}\limsup_{t\to 0}\int_\Omega \nabla\sqrt{h_{1_0}}(\nabla\sqrt{h_1}-\nabla\sqrt{h_{1_0}})dx\\
&\quad + 3\kappa_2\operatorname{ess}\limsup_{t\to 0}\int_\Omega \nabla\sqrt{h_{2_0}}(\nabla\sqrt{h_2}-\nabla\sqrt{h_{2_0}})dx,
\end{aligned}$$

where

$$\varepsilon(t,x) = \frac{1}{2}\int_\Omega \rho_1h_1|u_1|^2+\rho_2h_2|u_2|^2+g(\rho_1-\rho_2)|h_1|^2+\rho_2|h_1+h_2|^2$$

$$+ 2\kappa_1|\nabla\sqrt{h_1}|^2 + 2\kappa_2|\nabla\sqrt{h_2}|^2.$$

We have

$$3\kappa_i \operatorname{ess}\limsup_{t\to 0}\int_\Omega \nabla\sqrt{h_{i_0}}(\nabla\sqrt{h_i} - \nabla\sqrt{h_{i0}})dx = 0 \quad \text{for } i = 1,2. \tag{3.33}$$

So, using (3.5), (3.31) and the convexity of $h_i \longmapsto h_i^2$, we have

$$\begin{aligned}
&\operatorname{ess}\limsup_{t\to 0}\int_\Omega \rho_1|\sqrt{h_1}u_1 - \sqrt{h_{1_0}}u_{1_0}|^2dx + \operatorname{ess}\limsup_{t\to 0}\int_\Omega \rho_2|\sqrt{h_2}u_2 - \sqrt{h_{2_0}}u_{2_0}|^2dx\\
&\le 2\operatorname{ess}\limsup_{t\to 0}\int_\Omega \rho_1\sqrt{h_{1_0}}u_{1_0}(\sqrt{h_{1_0}}u_{1_0} - \sqrt{h_1}u_1)dx\\
&\quad + 2\operatorname{ess}\limsup_{t\to 0}\int_\Omega \rho_2\sqrt{h_{2_0}}u_{2_0}(\sqrt{h_{2_0}}u_{2_0} - \sqrt{h_2}u_2)dx
\end{aligned}$$

Following the line of [20] the right terms tend to 0 and we deduce that

$$\operatorname{ess}\limsup_{t\to 0}\int_\Omega |\sqrt{h_{i_0}}u_{i_0} - \sqrt{h_i}u_i|^2 = 0 \quad \text{for } i = 1,2,$$

which gives us $\sqrt{h_i}u_i \in C([0,T];L^2(\Omega))$. By (3.30) and (3.32), we obtain

$$\lim_{\tau\to 0}\frac{1}{\tau}\int_0^T\int_\Omega h_i\varphi_{in}(v_i)\,dx\,dt = \int_\Omega h_{i_0}\varphi_{in}(v_{i_0})dx$$

Considering (3.29) for the test function,

$$\psi_{\tau i}(t) = \psi_i(t) \text{ for } t \ge \tau, \quad \psi_{\tau i}(t) = \psi_i(\tau)\frac{t}{\tau} \text{ for } t \le \tau,$$

we obtain

$$\begin{aligned}
&\int_\tau^T\int_\Omega \partial_t\psi_i h_i\varphi_{in}(v_i)\,dx\,dt - \int_0^T\int_\Omega \psi_{\tau i}(t)\varphi'_{in}(v_i)F_i\,dx\,dt\\
&+ \int_0^T\int_\Omega \psi_{\tau i}(t)S_i : \nabla(\varphi'_{in}(v_i))\,dx\,dt\\
&= \frac{\psi_i(\tau)}{\tau}\int_0^\tau\int_\Omega h_i\varphi_{in}(v_i)\,dx\,dt.
\end{aligned}$$

Passing into the limit as $\tau\to 0$, this gives us

$$\begin{aligned}
&\int_\tau^T\int_\Omega \partial_t\psi_i h_i\varphi_{in}(v_i)\,dx\,dt - \int_0^T\int_\Omega \psi_i(t)\varphi'_{in}(v_i)F_i\,dx\,dt\\
&+ \int_0^T\int_\Omega \psi_i(t)S_i : \nabla(\varphi'_{in}(v_i))\,dx\,dt\\
&= \int_0^\tau\int_\Omega h_{i_0}\psi_i(0)\varphi_{in}(v_{i_0})\,dx\,dt
\end{aligned} \tag{3.34}$$

4. Recover the limits as $\alpha_i \to \infty$

In this section, we want to recover the limits in (3.19) as $\alpha_i \to \infty$. Here, we should remark that (h_1, h_2, u_1, u_2) is any fixed weak solution to (3.1)-(3.4) satisfying Proposition 3.1 with $\kappa_i > 0$. For any fixed weak solution (h_1, h_2, u_1, u_2), we have

$$\phi_{\alpha_i}(h_i) \to 1 \quad \text{almost everywhere for } (t,x),$$

and it is uniformly bounded in $L^\infty(\Omega)$; we also have
$$r_{0_i}\phi_{\beta_i}(h_i)u_i \in L^2(0,T;L^2(\Omega)),$$
and thus
$$v_{\alpha_i} = \phi_{\alpha_i}\phi_{\beta_i}u_i \to \phi_{\beta_i}u_i \quad \text{almost everywhere for } (t,x)$$
as $\alpha_i \to \infty$. By the Dominated Convergence Theorem, we have
$$v_{\alpha_i} \to \phi_{\beta_i}u_i \text{ in } L^2(0,T;L^2(\Omega)),$$
as $\alpha_i \to \infty$, and hence, we have
$$\varphi_{in}(v_{\alpha_i}) \to \varphi_{in}(\phi_{\beta_i}u_i) \quad \text{in } L^p((0,T)\times\Omega)$$
for any $1 \le p \le \infty$. For any fixed h_i, we have
$$\phi'_{\alpha_i}(h_i) \to 0 \quad \text{almost everywhere for } (t,x)$$
as $\alpha_i \to \infty$. for any fixed h_i. Since $|\phi'_{\alpha_i}(h_i)| \le 2\alpha_i$ as $\frac{1}{2\alpha_i} \le h_i \le \frac{1}{\alpha_i}$ and otherwise, $\phi'_{\alpha_i}(h_i) = 0$, we have
$$|h_i\phi'_{\alpha_i}(h_i)| \le 1 \quad \text{for all } h_i.$$
We find that
$$\int_0^T\int_\Omega \psi'_i(t)(h_i\varphi_{in}(v_{\alpha_i}))\,dx\,dt \to \int_0^T\int_\Omega \psi'_i(t)(h_i\varphi_{in}(\phi_{\beta_i}(h_i)u_i))\,dx\,dt$$
and
$$\int_\Omega h_{i_0}\varphi_{in}(v_{\alpha_i 0}) \to \int_\Omega h_{i_0}\varphi_{in}(\phi_{\beta_i}(h_{i_0})u_{i_0})$$
as $\alpha_i \to \infty$.

To pass into the limits in (3.34) as $\alpha_i \to \infty$, we rely on the following lemma.

Lemma 4.1. *If $\|a_{\alpha_i}\|_{L^\infty(0,T;\Omega)} \le C$, $a_{\alpha_i} \to a$ as $\alpha_i \to \infty$ a.e. for (t,x) in $L^p((0,T)\times\Omega)$ for any $1 \le p \le \infty$, $f \in L^1((0,T)\times\Omega)$, then we have*
$$\int_0^T\int_\Omega \phi_{\alpha_i}(h_i)a_{\alpha_i}f\,dx\,dt \to \int_0^T\int_\Omega af\,dx\,dt \quad \text{as } \alpha_i \to \infty,$$
and
$$\int_0^T\int_\Omega |h_i\phi'_{\alpha_i}(h_i)a_{\alpha_i}f|\,dx\,dt \to 0 \quad \text{as } \alpha_i \to \infty.$$

For a proof of the above lemma see [20]. Now we prove that
$$\int_0^T\int_\Omega \psi_i(t)S_{\alpha_i} : \nabla(\varphi_{in}(v_{\alpha_i}))\,dx\,dt \to \int_0^T\int_\Omega \psi_i(t)S : \nabla(\varphi'_{in}(\phi_{\beta_i}(h_i)u_i))\,dx\,dt \tag{4.1}$$
as $\alpha_i \to \infty$, where $S_i = \psi_{\beta_i}(h_i)h_i(D(u_i) + \kappa_i\frac{\Delta\sqrt{h_i}}{\sqrt{h_i}}\mathbb{I})$ and
$$\int_0^T\int_\Omega \psi_i(t)\varphi'_{in}(v_{\alpha_i})F_{\alpha_i}\,dx\,dt \to \int_0^T\int_\Omega \psi_i(t)\varphi'_{in}(\phi_{\beta_i}(h_i)u)F_i\,dx\,dt \tag{4.2}$$
where
$$\begin{aligned}F_i &= h_i^2u_i\phi'_{\beta_i}(h_i)\operatorname{div}u_i + gh_i\nabla h_i\phi_{\beta_i}(h_i) + h_i\nabla\phi_{\beta_i}(h_i)D(u_i)\\ &\quad + g\varepsilon_ih_i\nabla h_j\phi_{\beta_i}(h_i) + r_{i0}u_i\phi_{\beta_i}(h_i) + r_{i1}h_i|u_i|^2u_i\phi_{\beta_i}(h_i)\\ &\quad + \kappa_i\sqrt{h_i}\nabla\phi_{\beta_i}(h_i)\Delta\sqrt{h_i} + 2\kappa_i\phi_{\beta_i}(h_i)\nabla\sqrt{h_i}\Delta\sqrt{h_i},\end{aligned}$$

For the proof of (4.1) the reasoning is similarly as in [20]. Concerning (4.2) we just notice that $h_i\nabla h_j \in \mathrm{L}^1((0,\mathrm{T})\times\Omega)$. Letting $\alpha_i \to 0$ in (3.34), we have

$$\begin{aligned}
&\int_0^T\int_\Omega \psi_i'(h_i\varphi_{in}(\phi_{\beta_i}(h_i)u_i))\,dx\,dt - \int_0^T\int_\Omega \psi_i(t)\varphi_{in}'(\phi_{\beta_i}(h_i)u_i)F_i\,dx\,dt\\
&+\int_0^T\int_\Omega \psi_i(t)S_i:\nabla(\varphi_{in}'(\phi_{\beta_i}(h_i)u_i))\,dx\,dt\\
&=\int_\Omega \psi_i(0)h_{i_0}\varphi_{in}(\phi_{\beta_i}(h_{i_0})u_{i_0})\,dx\,dt
\end{aligned}$$

which in turn gives us the following lemma.

Lemma 4.2. *For any weak solutions to* (3.1)-(3.4) *satisfying Proposition 3.1, we have*

$$\begin{aligned}
&\int_0^T\int_\Omega \psi_i'(t)(h_i\varphi_{in}(\phi_{\beta_i}(h_i)u_i))\,dx\,dt - \int_0^T\int_\Omega \psi_i(t)\varphi_{in}'(\phi_{\beta_i}(h_i)u_i)F_i\,dx\,dt\\
&+\int_0^T\int_\Omega \psi_i(t)S_i:\nabla(\varphi_{in}'(\phi_{\beta_i}(h_i)u_i))\,dx\,dt\\
&=\int_\Omega \psi_i(0)h_{i_0}\varphi_{in}'(\phi_{\beta_i}(h_{i_0})u_{i_0})\,dx\,dt
\end{aligned}\tag{4.3}$$

where: $S_i=\psi_{\beta_i}(h_i)h_i(D(u_i)+\kappa_i\frac{\Delta\sqrt{h_i}}{\sqrt{h_i}}\mathbb{I})$ *and*

$$\begin{aligned}
F_i &= h_i^2u_i\phi_{\beta_i}'(h_i)\operatorname{div}u_i + gh_i\nabla h_i\phi_{\beta_i}(h_i) + h_i\nabla\phi_{\beta_i}(h_i)D(u_i)\\
&\quad+g\varepsilon_ih_i\nabla h_j\phi_{\beta_i}(h_i)+r_{i0}u_i\phi_{\beta_i}(h_i)+r_{i1}h_i|u_i|^2u_i\phi_{\beta_i}(h_i)\\
&\quad+\kappa_i\sqrt{h_i}\nabla\phi_{\beta_i}(h_i)\Delta\sqrt{h_i}+2\kappa_i\phi_{\beta_i}(h_i)\nabla\sqrt{h_i}\Delta\sqrt{h_i},
\end{aligned}$$

where $\mathbb{I}$ *is an identity matrix.*

5. Recover the limits as κ_i, r_{i_0} and r_{i_1} approach 0

The objective of this section is twofold. Firstly, to recover the limits in (4.3) as $\kappa_i\to 0$ and $\beta_i\to\infty$. Secondly, to apply Theorem 3.6 to prove Theorem 2.2 by letting as in [20] $r_{i_0}\to 0$ and $r_{i_1}\to 0$. We assume that $\beta_i=\kappa_i^{-3/4}$, thus $\beta_i\to\infty$ when $\kappa_i\to 0$. First, we state the following lemmas.

Lemma 5.1. *Let* $\kappa_i\to 0$ *and* $\beta_i\to\infty$*, we have*

$$\begin{gathered}
h_{i\kappa_i}\to h_i \quad \text{strongly in } L^2(0,T;L^2(\Omega)),\\
\nabla h_{i\kappa_i}\rightharpoonup \nabla h_i \quad \text{weakly in } L^2(0,T;H^{-1}(\Omega)),\\
h_{i\kappa_i}\varphi_{in}(\phi_{\beta_i}(h_{i\kappa_i})u_{i\kappa_i})\to h_i\varphi_{in}(u_i) \quad \text{strongly in } L^1((0,T)\times\Omega),\\
h_{i\kappa_i}\varphi_{in}'(\phi_{\beta}(h_{i\kappa_i})u_{i\kappa_i})\to h_i\varphi_{in}'(u_i) \quad \text{strongly in } L^2(0,T;L^2(\Omega)).
\end{gathered}$$

Lemma 5.2. *Let $\beta_i = \kappa_i^{-3/4}$, and $\kappa_i \to 0$, we have*

$$\begin{aligned}
&\int_0^T \int_\Omega |\psi_i'(t)| h_i \varphi_{in}(u_i)\,dx\,dt \\
&\le \frac{C}{n} + \Big| \int_0^T \int_\Omega \psi_i(t) \nabla h_i^2 \varphi_{in}'(u_i)\,dx\,dt \Big| + \Big| \int_0^T \int_\Omega \psi_i(t) h_i \nabla h_j \varphi_{in}'(u_i)\,dx\,dt \Big| \\
&\quad + C \int_0^T \int_\Omega (\frac{1}{2} h_{i_0} |u_{i_0}|^2 + \frac{1}{2} g h_{i_0}^2 + |\nabla \sqrt{h_{i_0}}|^2) dx + \psi_i(0) \int_\Omega h_{i0} \varphi_{in}(u_{i_0}) dx.
\end{aligned} \tag{5.1}$$

For a proof of the above lemma see [20]. With above two lemmas in hand, we are ready to recover the limits in (4.3) as $\kappa_i \to 0$ and $\beta_i \to \infty$.

Let $r_i = r_{i_0} = r_{i_1}$, we use $(h_1^{r_1}, h_2^{r_2}, u_1^{r_1}, u_2^{r_2})$ to denote the weak solutions to (3.1)-(3.4) verifying Proposition 3.1 with $\kappa_i = 0$. Here, we remark that the initial data should satisfy the following conditions, more precisely,

$$\begin{gathered}
h_{i0}^{r_i} \to h_{i_0} \quad \text{strongly in } L^2(\Omega), \\
\sqrt{h_{i_0}^{r_i}} u_{i0}^{r_i} \to \sqrt{h_{i_0}} u_{i_0} \quad \text{strongly in } L^2(\Omega)
\end{gathered}$$

as $r_i \to 0$ and

$$\begin{gathered}
h_{i_0} \text{ is bounded in } L^1(\Omega) \cap L^2(\Omega), \quad h_{i_0} \ge 0 \text{ a.e. in } \Omega, \\
h_{i0} |u_{i_0}|^2 = \frac{m_{i_0}^2}{h_{i_0}} \text{ is bounded in } L^1(\Omega), \\
\nabla \sqrt{h_{i_0}} \text{ is bounded in } L^2(\Omega), \\
\frac{1}{2} \int_\Omega h_{i_0} (1 + |u_{i_0}|^2) \ln(1 + |u_{i_0}|^2) dx \le C < \infty
\end{gathered} \tag{5.2}$$

By (3.5)-(3.6) one obtains the following estimates:

$$\begin{gathered}
\|\sqrt{h_i^{r_i}} u_i^{r_i}\|_{L^\infty(0,T;L^2(\Omega))} \le C, \quad \|h_i^{r_i}\|_{L^\infty(0,T;L^1 \cap L^2(\Omega))} \le C, \\
\|\nabla h_i^{r_i}\|_{L^2(0,T;L^2(\Omega))} \le C, \quad \|\nabla \sqrt{h_i^{r_i}} \\
|_{L^\infty(0,T;L^2(\Omega))} \le C, \\
\|\sqrt{h_i^{r_i}} \nabla u_i^{r_i}\|_{L^2(0,T;L^2(\Omega))} \le C;
\end{gathered} \tag{5.3}$$

and by Theorem 3.4, we have

$$\sup_{t \in [0,T]} \int_\Omega h_i^{r_i} |u_i^{r_i}|^2 \ln(1 + |u_i^{r_i}|^2) dx \le C. \tag{5.4}$$

In line with the ideas developed in [20], we have

$$\int_0^T \int_\Omega r_i |u_i^{r_i}|^2\,dx\,dt \le C, \quad \int_0^T \int_\Omega r_i |h_i^{r_i} u_i^{r_i}|^4\,dx\,dt \le C, \tag{5.5}$$

where the constant C only depends on the initial data and we can pass into the limits as $r_i \to 0$. In particular,

$$\sqrt{h_i^{r_i}} \to \sqrt{h_i} \quad \text{almost everywhere and strongly in } L^2_{\rm loc}((0,T) \times \Omega), \tag{5.6}$$

$$h_i^{r_i} \to h_i \quad \text{in } C^0(0,T; L^{\frac{3}{2}}_{\rm loc}(\Omega)), \tag{5.7}$$

$$h_i^{r_i\,2} \to h_i^2 \quad \text{strongly in } L^1_{\rm loc}((0,T) \times \Omega), \tag{5.8}$$

$$\sqrt{h_i^{r_i}} u_i^{r_i} \to \sqrt{h_i} u_i \quad \text{strongly in } L^2_{\rm loc}((0,T)\times\Omega)), \tag{5.9}$$

$$h_i^{r_i} u_i^{r_i} \to h_i u_i \quad \text{strongly in } L^2(0,T; L^p_{\rm loc}(\Omega)) \quad \text{for } p \in [1, \frac{3}{2}). \tag{5.10}$$

and the convergence of the diffusion terms

$$\begin{gathered} h_i^{r_i} \nabla u_i^{r_i} \to h_i \nabla u_i \quad \text{in } \mathcal{D}', \\ h_i^{r_i} (\nabla)^t u_i^{r_i} \to h_i (\nabla)^t u_i \quad \text{in } \mathcal{D}' \end{gathered} \tag{5.11}$$

For the proof of the convergence of the terms $r_i u_i^{r_i}$ and $r_i h_i^{r_i} |u_i^{r_i}|^2 u_i^{r_i}$ to zero when $r_i \to 0$ we refer the reader to [20].

6. Recover the limits as $n \to \infty$

We want in this section to recover the "Mellet-Vasseur" type inequality by letting $n \to \infty$. In particular, we prove Theorem 3.4 by recovering the limit from Lemma 5.2. In this section, (h_1, h_2, u_1, u_2) are the fixed weak solutions. Following the ideas proposed in [20] we only have to control the term

$$\Big| \int_0^T \int_\Omega \psi_i(t) h_i \nabla h_j \varphi'_{in}(u_i)\,dx\,dt \Big| \quad (i \neq j)$$

in the right term of (5.1) for the other terms the proof is the same as in [20]. We have

$$\begin{aligned} &\Big| \int_0^T \int_\Omega \psi_i(t) h_i \nabla h_j \varphi'_{in}(u_i)\,dx\,dt \Big| \\ &\leq \Big| \int_0^T \int_\Omega \psi_i(t) h_i \nabla h_j \varphi'_{in}(u_i) 1_{|u_i|\geq n}\,dx\,dt \Big| \\ &\quad + \Big| \int_0^T \int_\Omega \psi_i(t) h_i \nabla h_j \varphi'_{in}(u_i) 1_{|u_i|\leq n}\,dx\,dt \Big|, \end{aligned}$$

where 1_A is the indicator function that yields on A and zero outside A. So we have

$$\Big| \int_0^T \int_\Omega \psi_i(t) h_i \nabla h_j \varphi'_{in}(u_i) 1_{|u_i|\geq n}\,dx\,dt \Big| \leq \frac{C}{n} \|h_i\|_{L^2(0,T;L^2(\Omega))} \|\nabla h_j\|_{L^2(0,T;L^2(\Omega))}\,,$$

$$\begin{aligned} &\Big| \int_0^T \int_\Omega \psi_i(t) h_i \nabla h_j \varphi'_{in}(u_i) 1_{|u_i|\leq n}\,dx\,dt \Big| \\ &\leq \Big| \int_0^T \int_\Omega \psi_i(t) h_j \nabla h_i \varphi'_{in}(u_i) 1_{|u_i|\leq n}\,dx\,dt \Big| \\ &\quad + C \int_0^T \int_\Omega \psi_i(t) h_i h_j \frac{2u_{il} u_{ik}}{1+|u_i|^2} \partial_l u_{ik} 1_{|u_i|\leq n}\,dx\,dt \Big| \\ &\quad + C \Big| \int_0^T \int_\Omega \psi_i(t) h_i h_j (1 + \ln(1+|u_i|^2) \operatorname{div}(u_i) 1_{|u_i|\leq n}\,dx\,dt \Big| \\ &\leq \frac{C}{n} \|h_j\|_{L^2(0,T;L^2(\Omega))} \|\nabla h_i\|_{L^2(0,T;L^2(\Omega))} \\ &\quad + C \|\sqrt{h_i} \mathbb{D} u_i\|_{L^2(0,T;(L^2(\Omega))^4)} \|h_i\|_{L^2(0,T;L^2(\Omega))} \|h_j\|_{L^4(0,T;L^4(\Omega))} \\ &\quad + C \Big| \int_0^T \int_\Omega \psi_i(t) h_i h_j (1 + \ln(1+|u_i|^2) \operatorname{div}(u_i) 1_{|u_i|\leq n}\,dx\,dt \Big|, \end{aligned}$$

and

$$\begin{aligned}
&\Big|\int_0^T\int_\Omega \psi_i(t)h_ih_j(1+\ln(1+|u_i|^2)\operatorname{div}(u_i)1_{|u_i|\le n}\,dx\,dt\Big|\\
&\le C\Big|\int_0^T\int_\Omega (1+\ln(1+|u_i|^2)h_i|\mathbb{D}u_i|^2 1_{|u_i|\le n}\,dx\,dt\Big|\\
&\quad + C\Big|\int_0^T\int_\Omega h_ih_j^2(1+\ln(1+|u_i|^2)1_{|u_i|\le n}\,dx\,dt\Big|,
\end{aligned}$$

where

$$\begin{aligned}
&\Big|\int_0^T\int_\Omega h_ih_j^2(1+\ln(1+|u_i|^2)1_{|u_i|\le n}\,dx\,dt\Big|\\
&\le C\|\sqrt{h_i}u_i\|_{L^2(0,T;L^2(\Omega))}\|h_i\|_{L^2(0,T;L^2(\Omega))}\|h_j\|_{L^8(0,T;L^8(\Omega))}
\end{aligned}$$

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