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ASYMPTOTIC BEHAVIOR OF SOLUTIONS TO A SYSTEM OF SCHRÖDINGER EQUATIONS

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Abstract. This article concerns the behaviour of solutions to a coupled system of Schrödinger equations that has applications in many physical problems, especially in nonlinear optics. In particular, when the solution exists globally, we obtain the growth of the solutions in the energy space. Finally, some conditions are also obtained for having blow-up in this space.

1. Introduction

In this work, we consider the following initial value problem (IVP) for two coupled nonlinear Schrödinger equations (NLS):

$$\begin{gathered} iu_t + \Delta u + (\alpha |u|^{2p} + \beta |u|^q |v|^{q+2})u = 0, \\ iv_t + \Delta v + (\alpha |v|^{2p} + \beta |v|^q |u|^{q+2})v = 0, \\ u(x,0) = u_0(x), \qquad v(x,0) = v_0(x), \end{gathered} \tag{1.1}$$

where $x \in \mathbb{R}^n$, α, $\beta \in \mathbb{R}$, $p > 0$ and $q > 0$.

For β a real positive constant, $\alpha = 1$ and $q = p - 1$, system (1.1) leads to the model

$$\begin{gathered} iu_t + \Delta u + (|u|^{2p} + \beta |u|^{p-1} |v|^{p+1})u = 0, \\ iv_t + \Delta v + (|v|^{2p} + \beta |v|^{p-1} |u|^{p+1})v = 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x). \end{gathered} \tag{1.2}$$

This problem arises as a model for propagation of polarized laser beams in birefringent Kerr medium in nonlinear optics (see, for example, [4, 16, 24, 27, 35, 36] and the references therein for a complete discussion about the physical standpoint of the problem). The two functions u and v are the components of the slowly varying envelope of the electrical field, t is the distance in the direction of propagation, x are orthogonal variables and Δ is the diffraction operator. The case $n = 1$ corresponds to propagation in a planar geometry, the case $n = 2$ describes the propagation in a bulk medium and the case $n = 3$ represents the propagation of pulses in a bulk medium with time dispersion. The focusing nonlinear terms in (1.2) describes the

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dependence of the refraction index of material on the electric field intensity and the birefringence effects. The parameter $\beta > 0$ has to be interpreted as the birefringence intensity and describes the coupling between the two components of the electric-field envelope.

If α and β are real constants and $u = v$, system (1.1) reduces to the nonlinear Schrödinger with double power nonlinearity.

$$\begin{gathered} iu_t + \Delta u + (\alpha|u|^{2p} + \beta|u|^{2(q+1)})u = 0, \\ u(x,0) = u_0(x). \end{gathered} \tag{1.3}$$

Special case of (1.3) is the cubic-quintic nonlinear Schrödinger equation ($p = q = 1$)

$$iu_t + \Delta u + (\alpha|u|^2 + \beta|u|^4)u = 0. \tag{1.4}$$

This equation arises in a number of independent physics field: nuclear hydrodynamic with Skyrme [20], the optical pulse propagations in dielectrical media of non-Kerr type [23]. Also, it is used to describe the boson gas with two and three body interaction [2, 3].

The equation (1.3) is just one of many models of Schrödinger equations. Many of different aspects of this model were investigated by various techniques by any authors [10, 14, 18, 17, 19, 28, 21, 33] and references therein. In [33] was consider

$$\begin{gathered} iu_t + \Delta u + (\alpha|u|^{p_1} + \beta|u|^{p_2})u = 0, \\ u(x,0) = u_0(x), \end{gathered} . \tag{1.5}$$

with $(x,t) \in \mathbb{R}^n \times \mathbb{R}$, $n \geq 3$ and $0 < p_1 < p_2 \leq \frac{4}{n-2}$ and they proved local and global well-posedness, they also addresses issues related to finite time blow-up, assymptotic behaviour and scattering in the energy space $H^1(\mathbb{R}^n)$.

System (1.1), admits the mass and the energy conservation in the spaces $L^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$ and $H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n)$ respectively. More precisely, the mass (L^2 norm):

$$M(u(t), v(t)) := \|u(t)\|^2_{L^2(\mathbb{R}^n)} + \|v(t)\|^2_{L^2(\mathbb{R}^n)} = M(u_0, v_0), \tag{1.6}$$

and the energy

$$\begin{aligned} E(t) :=& E(u(t), v(t)) := \|\nabla u(t)\|^2_{L^2(\mathbb{R}^n)} + \|\nabla v(t)\|^2_{L^2(\mathbb{R}^n)} - \mathcal{X}(t) \\ =& E(0) := E(u_0, v_0), \end{aligned} \tag{1.7}$$

are conserved by the flow of (1.1), where

$$\mathcal{X}(t) = \frac{\alpha}{p+1}\big[\|u(t)\|^{2p+2}_{L^{2p+2}(\mathbb{R}^n)} + \|v(t)\|^{2p+2}_{L^{2p+2}(\mathbb{R}^n)}\big] + \frac{2\beta}{q+2}\|u(t)v(t)\|^{q+2}_{L^{q+2}(\mathbb{R}^n)}. \tag{1.8}$$

For some remarks and proofs of conservation laws for nonlinear Schrödinger equations, we refer to [29].

Well-posedness issues and the blow-up phenomenon for the IVP (1.1) has been studied in the literature, see for example in [11, 13, 16, 26, 27, 30, 35] and references therein. The system (1.2) has scaling, this is if u and v are two solutions from (1.2) and $\lambda > 0$ then

$$\eta(x,t) = \lambda^{1/p}u(\lambda x, \lambda^2 t), \quad \omega(x,t) = \lambda^{1/p}v(\lambda x, \lambda^2 t), \tag{1.9}$$

are also solutions of (1.2). Hence, putting

$$p = \frac{2}{n - 2s_0},$$

the Sobolev space $\dot{H}^{s_0}$ is invariant under the scaling (1.9). In what follows we list some important results that are relevant in our work.

(1) Local solution: Under assumptions $s \geq \max\{s_0, 0\}$ and $p > [s]/2$, if $p \notin \mathbb{Z}$ then the solution of the Cauchy problem (1.2), exists locally in time.

(2) Global solution: Assuming that $0 < p < 2/n$, the solution of the Cauchy problem (1.2), exists globally in time (see [16], see also Theorem 1.2 and Section 4 in this work).

(3) When $p \geq 2/n$, the solution of the Cauchy problem (1.2), blows-up in a finite time for some initial data, especially for a class of sufficiently large data (see [13, 16, 26, 30] and Theorem 1.4 in this work). On the other hand, the solution of the Cauchy problem (1.2) *exists globally for other initial data*, especially for a class of sufficiently small data (see [11, 16, 27]).

In [35], Xiaoguang et al. obtained a sharp threshold of blow-up solution for (1.2). To study the blow-up threshold, they considered the stationary system

$$\begin{aligned} &\Delta Q - \frac{(2-n)p+2}{2}Q + (|Q|^{2p} + \beta|Q|^{p-1}|R|^{p+1})Q = 0, \\ &\Delta R - \frac{(2-n)p+2}{2}R + (|R|^{2p} + \beta|R|^{p-1}|Q|^{p+1})R = 0, \end{aligned} \tag{1.10}$$

associated with (1.2).

Let, $s_c = n/2 - 1/p$,

$$\begin{gathered} \sigma_{p,n,\beta} := (\frac{pn}{2})^{1/4(1-1/p)}\sqrt{\|Q\|^2_{L^2(\mathbb{R}^n)} + \|R\|^2_{L^2(\mathbb{R}^n)}}, \\ \Gamma(u,v) := E^{s_c}(u,v)M^{1-s_c}(u,v), \\ \vartheta(u,v) := (\|\nabla u\|^2_{L^2(\mathbb{R}^n)} + \|\nabla v\|^2_{L^2(\mathbb{R}^n)})^{s_c/2}(\|u\|^2_{L^2(\mathbb{R}^n)} + \|v\|^2_{L^2(\mathbb{R}^n)})^{(1-s_c)/2}. \end{gathered}$$

The following is the result proved by Xiaoguang et al. [35].

Theorem 1.1 ([35]). *Let $2/n \leq p < A_n$, where $A_n = \infty$ if $n = 1, 2$, and $A_n = 2/(n-2)$ if $n \geq 3$, and let $(|x|u_0, |x|v_0) \in L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$. Assume that*

$$\Gamma(u_0, v_0) < \Gamma(Q,R) \equiv \frac{s_c}{n}^{s_c}(\sigma_{p,n,\beta})^2,$$

then the following two conclusions are valid.

1. *If $\vartheta(u_0, v_0) < \vartheta(Q,R)$, then the solution of the Cauchy problem* (1.2) *exists globally in time.*
2. *If $\vartheta(u_0, v_0) > \vartheta(Q,R)$, then the solution of the Cauchy problem* (1.2) *blows-up in finite time.*

In [7], they considered the initial value problem (IVP) associated with the coupled system of supercritical nonlinear Schrödinger equations

$$\begin{aligned} iu_t + \Delta u + \theta_1(\omega t)(|u|^{2p} + \beta|u|^{p-1}|v|^{p+1})u = 0, \\ iv_t + \Delta v + \theta_2(\omega t)(|v|^{2p} + \beta|v|^{p-1}|u|^{p+1})v = 0, \end{aligned} \tag{1.11}$$

where θ_1 and θ_2 are periodic functions. They proved that, for given initial data $\varphi, \psi \in H^1(\mathbb{R}^n)$, as $|\omega| \to \infty$, the solution (u_ω, v_ω) of IVP (1.11) converges to the solution (U,V) of the IVP associated with

$$\begin{aligned} iU_t + \Delta U + I(\theta_1)(|U|^{2p} + \beta|U|^{p-1}|V|^{p+1})U = 0, \\ iV_t + \Delta V + I(\theta_2)(|V|^{2p} + \beta|V|^{p-1}|U|^{p+1})V = 0, \end{aligned} \tag{1.12}$$

with the same initial data, where $I(g)$ is the average of the periodic function g. Moreover, if the solution (U, V) is global and bounded, then they also proved that the solution (u_ω, v_ω) is also global provided $|\omega| \gg 1$.

Our main result characterize the asymptotic properties of solutions of (1.1) and gives the growth of the Sobolev norm in H^1.

Theorem 1.2. *Let $u_0, v_0 \in L^2(|x|^2dx) \cap H^1(\mathbb{R}^n)$ and $u(t), v(t)$ be solutions of* (1.1) *with $t \geq 1$, we have*

(1) If $0 < p \leq \frac{2}{n}$ and $p \geq q+1$ if $\beta > 0$ or $p \leq q+1$ if $\beta < 0$ then

$$E(0) - \frac{b_0}{4\, t^{np}} \leq \int \left(|\nabla u(x,t)|^2 + |\nabla v(x,t)|^2\right) dx.$$

And if moreover $\mathcal{X} \leq 0$ (see (1.8)*, e.g., $\alpha \leq 0$ and $\beta \leq 0$), we also have*

$$\begin{aligned} &\|\nabla u(t)\|_{L^2_x(\mathbb{R}^n)} + \|\nabla v(t)\|_{L^2_x(\mathbb{R}^n)} \\ &\leq \min\Big\{\Big(c_0 + \frac{2b_0^{1/2}}{np}\Big) - \frac{b_0^{1/2}(2-np)}{np} t^{-np/2}, \quad E(0)\Big\}, \end{aligned} \tag{1.13}$$

$$\|xu(t)\|_{L^2_x} + \|x\, v(t)\|_{L^2_x} \leq 2t\Big(c_0 + \frac{2b_0^{1/2}}{np}\Big) + \frac{4b_0^{1/2}(np-1)}{np} t^{1-np/2}, \tag{1.14}$$

$$\lim_{t\to+\infty} \int \left(|\nabla u(x,t)|^2 + |\nabla v(x,t)|^2\right) dx = E(0), \tag{1.15}$$

where $b_0 := b_0(n,p)$ and $c_0 = c_0(u_0, v_0)$ are defined in (5.7) *and* (5.16) *respectively.*

(2) If $0 < q \leq \frac{2}{n} - 1$ and $p \leq q+1$ if $\alpha > 0$ or $p \geq q+1$ if $\alpha < 0$ then

$$E(0) - \frac{b_1}{4\, t^{n(q+1)}} \leq \int \left(|\nabla u(x,t)|^2 + |\nabla v(x,t)|^2\right) dx.$$

And if moreover $\mathcal{X} \leq 0$ (e.g., $\alpha \leq 0$ and $\beta \leq 0$), we also have

$$\begin{aligned} &\|\nabla u(t)\|_{L^2_x(\mathbb{R}^n)} + \|\nabla v(t)\|_{L^2_x(\mathbb{R}^n)} \\ &\leq \min\Big\{\Big(c_0 + \frac{2b_1^{1/2}}{n(q+1)}\Big) - \frac{b_1^{1/2}(2-n(q+1))}{n(q+1)} t^{-n(q+1)/2}, \quad E(0)\Big\}, \\ &\|xu(t)\|_{L^2_x} + \|x\, v(t)\|_{L^2_x} \leq 2t\Big(c_0 + \frac{2b_1^{1/2}}{n(q+1)}\Big) + \frac{4b_1^{1/2}(n(q+1)-1)}{n(q+1)} t^{1-n(q+1)/2}, \end{aligned} \tag{1.16}$$

$$\lim_{t\to+\infty} \int \left(|\nabla u(x,t)|^2 + |\nabla v(x,t)|^2\right) dx = E(0),$$

where $b_1 := b_1(n,q) \geq 0$ and $c_0 = c_0(u_0, v_0) \geq 0$ are defined in (5.20) *and* (5.16) *respectively.*

Remark 1.3. (i) The restriction $t \geq 1$ in Theorem 1.2 can be replaced by $t \geq c_0$, where $c_0 > 0$ is any arbitrarily small constant.

(ii) Observe also that using interpolation

$$\|u\|_{H^\theta} \leq \|u\|_{L^2}^{1-\theta} \|u\|_{H^1}^{\theta}, \quad \theta \in [0,1],$$

the theorem above also gives the growth of the Sobolev norm in $H^\theta(\mathbb{R}^n)$, $\theta \in [0,1]$. The growth of Sobolev norms, in the Schrödinger equation was studied by Bourgain [6]. See also [31, 9] and references there.

(iii) If $np = 2$ and $n(q+1) = 2$ then

$$\frac{\partial}{\partial t}\Big[\int \left(|J(u)|^2 + |J(v)|^2\right)\, dx - tf(t)\Big] = 0$$

(see equality (5.1)) and therefore if $\alpha < 0$, $\beta < 0$ and $u_0, v_0 \in L^2(|x|^2dx)$ then

$$\|v\|_{L^{2p+2}}^{2p+2} + \|u\|_{L^{2p+2}}^{2p+2} \le \frac{(p+1)(\|xu_0\|_{L^2}^2 + \|xv_0\|_{L^2}^2)}{4|\alpha|t^2},$$

$$\|uv\|_{L^{q+2}}^{q+2} \le \frac{(q+2)(\|xu_0\|_{L^2}^2 + \|xv_0\|_{L^2}^2)}{8|\beta|t^2}$$

Our blow-up result is as follows.

Theorem 1.4. *Let $u_0, v_0 \in L^2(|x|^2dx) \cap H^1(\mathbb{R}^n)$ and $u(t), v(t)$ be solutions of (1.1), we have*

(1) If $np \ge 2$ and $p \le q+1$ if $\beta > 0$ or $p \ge q+1$ if $\beta < 0$, then there exists $0 < T^* < \infty$ such that

$$\lim_{t\to T^*} \|\nabla u(t)\|_{L^2} = \infty, \quad \lim_{t\to T^*} \|\nabla v(t)\|_{L^2} = \infty,$$

in the following three cases:

(1) *$E(0) = 0$ and*

$$\operatorname{Im}\int (\overline{u}_0 x\cdot\nabla u_0 + \overline{v}_0 x\cdot\nabla v_0)\, dx < 0,$$

(2) *$E(0) < 0$,*

(3) *$E(0) > 0$ and*

$$\Big(\operatorname{Im}\int (\overline{u}_0 x\cdot\nabla u_0 + \overline{v}_0 x\cdot\nabla v_0)\, dx\Big)^2 > \frac{npE(0)}{2}\int |x|^2(|u_0|^2 + |v_0|^2)\, dx.$$

(2) If $n(q+1) \ge 2$ and $p \ge q+1$ if $\alpha > 0$ or $p \le q+1$ if $\alpha < 0$, then there exists $0 < T^* < \infty$ such that

$$\lim_{t\to T^*} \|\nabla u(t)\|_{L^2} = \infty, \quad \lim_{t\to T^*} \|\nabla v(t)\|_{L^2} = \infty,$$

in the following three cases:

(1) *$E(0) = 0$ and*

$$\operatorname{Im}\int (\overline{u}_0 x\cdot\nabla u_0 + \overline{v}_0 x\cdot\nabla v_0)\, dx < 0,$$

(2) *$E(0) < 0$,*

(3) *$E(0) > 0$ and*

$$\Big(\operatorname{Im}\int (\overline{u}_0 x\cdot\nabla u_0 + \overline{v}_0 x\cdot\nabla v_0)\, dx\Big)^2 > \frac{n(q+1)E(0)}{2}\int |x|^2(|u_0|^2 + |v_0|^2) dx.$$

Remark 1.5. If

$$\lim_{t\to T^*} \|\nabla u(t)\|_{L^2} = \infty, \quad \text{and} \quad \lim_{t\to T^*} \|\nabla v(t)\|_{L^2} = \infty,$$

then by the energy conservation (1.7) we have that $\lim_{t\to T^*}\mathcal{X}(t) = \infty$, and this limit implies

$$\lim_{t\to T^*} \|u(t)\|_{L^\infty} = \infty, \quad \lim_{t\to T^*} \|v(t)\|_{L^\infty} = \infty.$$

2. Notation

Let $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, we denote the partial derivative of u with respect to spatial variable x_j as: u_{x_j}, $\partial_{x_j} u$ or $\frac{\partial u}{\partial x_j}$. Similarly we denote the partial derivative of u with respect to time variable $t \in \mathbb{R}$ as: u_t, $\partial_t u$ or $\frac{\partial u}{\partial t}$. All the integrals in our work are defined on $\mathbb{R}^n$, in this way $\int f(x)\,dx := \int_{\mathbb{R}^n} f(x)\,dx$. If $f(x)$ is a function of $x \in \mathbb{R}^n$, the laplacian of f is denoted by

$$\Delta f(x) = \sum_{j=1}^{n} \partial_{x_j}^2 f(x), \quad x = (x_1, \dots, x_n).$$

The gradient of f is denoted by

$$\nabla f(x) = (\partial_{x_1} f, \dots, \partial_{x_n} f).$$

The product of two vectors $x = (x_1, \dots, x_n) \in \mathbb{C}^n$ $y = (y_1, \dots, y_n) \in \mathbb{C}^n$ is denoted by

$$x \cdot y = \sum_{j=1}^{n} x_j y_j,$$

and this manner $|x|^2 = x \cdot \overline{x}$.

3. Preliminary results

In this section we present important results that will be useful in the following sections.

Lemma 3.1 (Gronwall Inequality)**.** *Let u and β be continuous and α and δ Riemann integrable functions on $J = [a, b]$ with δ and β nonnegative on J.*

If u satisfies the integral inequality

$$u(t) \le \alpha(t) + \delta(t) \int_a^t \beta(s) u(s) ds, \quad \forall t \in J,$$

then

$$u(t) \le \alpha(t) + \delta(t) \int_a^t \alpha(s)\beta(s) \exp\Big(\int_s^t \delta(r)\beta(r) dr \Big).$$

For a proof of the above lemma, see [15, Theorem 11]. Observe that there are no assumptions on the signs of the functions α and u.

Theorem 3.2 (Existence of solutions in the energy space)**.** *Assume $0 \le \max\{p, q+1\} < 2/(n-2)$ if $\alpha < 0$ and $\beta < 0$ (focusing case), otherwise $0 \le \max\{p, q+1\} < 2/n$. Then for any $(u_0, v_0) \in H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n)$, there are $T_{\max} > 0$ and a unique solution $(u, v) \in C([0, T_{\max}); H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n))$ of (1.1) satisfying $(u(0), v(0)) = (u_0, v_0)$. Moreover, it holds the blow up alternatives: (i) $T_{\max} = \infty$, or (ii) $T_{\max} < \infty$ with*

$$\lim_{t \to T_{\max}} (\|\nabla u(t)\|_{L^2(\mathbb{R}^n)} + \|\nabla v(t)\|_{L^2(\mathbb{R}^n)}) = \infty.$$

When (i) occurs, we say that the solution is global. When (ii) occurs, we say that the solution blows up in finite time. The proof of this theorem is similar to that for the Schrödinger equation and it combines Strichartz estimates with the contraction mapping principle.

Lemma 3.3. *Let u and v be solutions of* (1.1), *then*

$$\begin{aligned} &\frac{\partial}{\partial t}\Big\{\int \operatorname{Im}\left(\overline{u}x\cdot\nabla u+\overline{v}x\cdot\nabla v\right)dx\Big\} \\ &= 2E(0)+\frac{\alpha(2-np)}{p+1}\int\left(|u|^{2p+2}+|v|^{2p+2}\right)dx+\frac{2\beta\left(2-n(1+q)\right)}{q+2}\int|u\,v|^{q+2}dx. \end{aligned} \tag{3.1}$$

Proof. Differentiating with respect to t and integrating by parts we obtain

$$\frac{\partial}{\partial t}\Big\{\int \operatorname{Im}\left(\overline{u}x\cdot\nabla u\right)dx\Big\}=2\operatorname{Im}\int\overline{u}_t x\cdot\nabla u dx-n\int\operatorname{Im}\left(\overline{u}u_t\right)dx, \tag{3.2}$$

using the first equation in (1.1) we have

$$\int\operatorname{Im}\left(\overline{u}u_t\right)dx=-\int|\nabla u|^2dx+\alpha\int|u|^{2p+2}dx+\beta\int|u|^{2+q}\,|v|^{2+q}dx, \tag{3.3}$$

similarly

$$\begin{aligned} \operatorname{Im}\int u_t x\cdot\nabla\overline{u}dx &= \operatorname{Re}\int\Delta u x\cdot\nabla\overline{u}\,dx+\alpha\operatorname{Re}\int|u|^{2p}ux\cdot\nabla\overline{u}\,dx \\ &\quad+\beta\operatorname{Re}\int|u|^{q}\,|v|^{2+q}ux\cdot\nabla\overline{u}\,dx. \end{aligned} \tag{3.4}$$

Using integration by parts twice, it is easy to see that

$$\int\Delta u x\cdot\nabla\overline{u}\,dx=(n-2)\int|\nabla u|^2dx-\int\Delta\overline{u}x\cdot\nabla u\,dx$$

and therefore

$$\operatorname{Re}\int\Delta u x\cdot\nabla\overline{u}\,dx=\frac{(n-2)}{2}\int|\nabla u|^2dx. \tag{3.5}$$

Integrating by parts again gives

$$\begin{aligned} 2\operatorname{Re}\int|u|^{2p}ux\cdot\nabla\overline{u}\,dx &= -n\int|u|^{2p+2}dx-\int|u|^2x\cdot\nabla(|u|^{2p})dx \\ &= -n\int|u|^{2p+2}dx-\frac{2p}{2p+2}\int x\cdot\nabla(|u|^{2p+2})dx \\ &= -n\int|u|^{2p+2}dx+\frac{2pn}{2p+2}\int|u|^{2p+2}dx \\ &= \frac{-2n}{2p+2}\int|u|^{2p+2}dx. \end{aligned} \tag{3.6}$$

Similarly,

$$\begin{aligned} &2\operatorname{Re}\int\,|u|^{q}\,|v|^{2+q}ux\cdot\nabla\overline{u}\,dx \\ &= -n\int|uv|^{q+2}dx-\frac{q}{q+2}\int|v|^{q+2}x\cdot\nabla(|u|^{q+2})dx \\ &\quad-\int|u|^{q+2}x\cdot\nabla(|v|^{q+2})dx. \end{aligned} \tag{3.7}$$

Combining (3.4)-(3.7) it follows that

$$
\begin{aligned}
&\operatorname{Im}\int u_t x\cdot\nabla\overline{u}\\
&=\frac{(n-2)}{2}\int|\nabla u|^2\,dx-\frac{n\,\alpha}{2p+2}\int|u|^{2p+2}dx-\frac{n\beta}{2}\int|uv|^{q+2}dx\\
&\quad-\frac{q\beta}{2(q+2)}\int|v|^{q+2}x\cdot\nabla(|u|^{q+2})dx-\frac{\beta}{2}\int|u|^{q+2}x\cdot\nabla(|v|^{q+2})dx.
\end{aligned}
\tag{3.8}
$$

The symmetry of (1.1) in u and v and one integration by parts gives

$$
\begin{aligned}
&\operatorname{Im}\int u_t x\cdot\nabla\overline{u}+v_t x\cdot\nabla\overline{v}\,dx\\
&=\frac{(n-2)}{2}\int\left(|\nabla u|^2+|\nabla v|^2\right)\,dx-\frac{n\,\alpha}{2p+2}\int\left(|u|^{2p+2}+|v|^{2p+2}\right)\,dx\\
&\quad-n\beta\int|uv|^{q+2}dx-\frac{q\beta\,n}{2(q+2)}\int|v|^{q+2}|u|^{q+2}dx+\frac{\beta\,n}{2}\int|u|^{q+2}\,|v|^{q+2}dx.
\end{aligned}
\tag{3.9}
$$

Now from (3.2), (3.3) and (3.9) is not hard to see that

$$
\begin{aligned}
&\frac{\partial}{\partial t}\left\{\int\operatorname{Im}\left(\overline{u}x\cdot\nabla u+\overline{v}x\cdot\nabla v\right)dx\right\}\\
&=2\int\left(|\nabla u|^2+|\nabla v|^2\right)\,dx\\
&\quad-\frac{n\alpha p}{p+1}\int\left(|u|^{2p+2}+|v|^{2p+2}\right)dx-\frac{n\beta(q+1)}{q+2}\int|uv|^{q+2}dx.
\end{aligned}
\tag{3.10}
$$

We conclude the proof of Lemma by using the conservation law (1.7). □

The following Lemma is an obvious result.

Lemma 3.4. *Let u and v be solutions of the coupled system* (1.1)*, we have*

$$\frac{\partial}{\partial t}|u|^2=2\operatorname{Im}(\Delta\overline{u}u)\quad\text{and}\quad\frac{\partial}{\partial t}|v|^2=2\operatorname{Im}(\Delta\overline{v}\,v).\tag{3.11}$$

The following lemma will be useful to prove the asymptotic behaviour of solutions of (1.1).

Lemma 3.5. *Let $u_0,v_0\in L^2(|x|^2dx)\cap H^1(\mathbb{R}^n)$ and $u(t),v(t)$ solutions of* (1.1)*, then if $0\le t\le T$, we have*

$$\left(\int|x|^2|u(x,t)|^2dx\right)^{1/2}\le\left(\int|x|^2|u_0|^2dx\right)^{1/2}+2\int_0^t\|\nabla u(t')\|_{L^2}dt',\tag{3.12}$$

$$\left(\int|x|^2|v(x,t)|^2dx\right)^{1/2}\le\left(\int|x|^2|v_0|^2dx\right)^{1/2}+2\int_0^t\|\nabla v(t')\|_{L^2}dt'.\tag{3.13}$$

Proof. Using Lemma 3.4 we obtain

$$\frac{\partial}{\partial t}\int|x|^2|u(t)|^2\,dx=\int|x|^2\frac{\partial\,|u(t)|^2}{\partial t}\,dx=2\int|x|^2\operatorname{Im}\left(u\Delta\overline{u}\right)\,dx,\tag{3.14}$$

integrating by parts once, we have

$$\int|x|^2u\Delta\overline{u}\,dx=-2\int ux\cdot\nabla\overline{u}\,dx-\int|x|^2|\nabla u|^2\,dx,\tag{3.15}$$

inserting (3.15) in (3.14) we arrive at

$$\frac{\partial}{\partial t}\int |x|^2|u(t)|^2\,dx = -4\operatorname{Im}\int ux\cdot\nabla\overline{u}\,dx = 4\operatorname{Im}\int \overline{u}x\cdot\nabla u\,dx. \tag{3.16}$$

Let $\Omega(t)=\|xu\|_{L^2}$, then using Cauchy-Schwartz, the inequality (3.16) implies

$$\frac{d\Omega(t)^2}{dt} = 2\Omega(t)\frac{d\Omega(t)}{dt} \le 4\Omega(t)\|\nabla u\|_{L^2}, \tag{3.17}$$

and from (3.17) integrating, we have

$$\Omega(t) \le \Omega(0) + 2\int_0^t \|\nabla u\|_{L^2}\,dt'.$$

Similarly we obtain the inequality (3.13). □

In this article we use the operators J and L defined by

$$Jw = e^{i|x|^2/4t}(2it)\nabla(e^{-i|x|^2/4t}w) = (x+2it\nabla)w, \quad Lw = (i\partial_t + \Delta)w.$$

With this notation the system (1.1) is

$$\begin{gathered} Lu = -F(u,v) = -(\alpha|u|^{2p} + \beta|u|^q|v|^{q+2})u,\\ Lv = -F(v,u). \end{gathered} \tag{3.18}$$

We note that (see Remark after proof Theorem 3.8).

$$J(Lu) = L(Ju) \tag{3.19}$$

Lemma 3.6. *Let u and v be solutions of coupled system* (1.1)*, then we have*

$$\begin{aligned} \operatorname{Im}\Big(\int J(|u|^{2p}u)\cdot\overline{Ju}\,dx\Big) &= -\frac{2(np-2)}{(p+1)}t\int |u|^{2p+2}\,dx\\ &\quad - \frac{2}{(p+1)}\frac{\partial}{\partial t}\Big\{t^2\int |u|^{2p+2}\,dx\Big\},\\ \operatorname{Im}\Big(\int J(|v|^{2p}v)\cdot\overline{Jv}\,dx\Big) &= -\frac{2(np-2)}{(p+1)}t\int |v|^{2p+2}\,dx\\ &\quad - \frac{2}{(p+1)}\frac{\partial}{\partial t}\big\{t^2\int |v|^{2p+2}\,dx\big\}. \end{aligned}$$

Proof. Using the definition of J, the scalar product of vectors and differentiating gives

$$\begin{aligned} &J(|u|^{2p}u)\cdot\overline{Ju}\\ &= |x|^2|u|^{2p+2} - 2it|u|^{2p}ux\cdot\nabla\overline{u} + 2it\overline{u}\nabla(|u|^{2p}u)\cdot x + 4t^2\nabla(|u|^{2p}u)\cdot\nabla\overline{u}\\ &= |x|^2|u|^{2p+2} + 2it|u|^{2p}x\cdot(\overline{u}\nabla u - u\nabla\overline{u}) + 2it|u|^2\nabla(|u|^{2p})\cdot x\\ &\quad + 4t^2|u|^{2p}|\nabla u|^2 + 4t^2u\nabla(|u|^{2p})\cdot\nabla\overline{u}, \end{aligned}$$

taking the imaginary part we have

$$\begin{aligned} \operatorname{Im}\big(J(|u|^{2p}u)\cdot\overline{Ju}\big) &= 2t|u|^2\nabla(|u|^{2p})\cdot x + 4t^2\operatorname{Im}\big(u\nabla(|u|^{2p}).\nabla\overline{u}\big)\\ &= 2t\frac{p}{p+1}\nabla(|u|^{2p+2})\cdot x + 4t^2\operatorname{Im}\big(u\nabla(|u|^{2p}).\nabla\overline{u}\big), \end{aligned} \tag{3.20}$$

and after integration over $\mathbb{R}^n$, we obtain

$$\begin{aligned}&\operatorname{Im}\int J(|u|^{2p}u)\overline{Ju}\,dx\\&=\frac{2tp}{p+1}\int\nabla(|u|^{2p+2})\cdot x\,dx+4t^2\operatorname{Im}\int u\nabla(|u|^{2p}).\nabla\overline{u})\,dx.\end{aligned}\tag{3.21}$$

Integrating by parts, we have

$$\begin{gathered}\int\nabla(|u|^{2p+2})\cdot x\,dx=-n\int|u|^{2p+2}\,dx,\\\int u\nabla(|u|^{2p}).\nabla\overline{u}=-\int|u|^{2p}|\nabla u|^2\,dx-\int|u|^{2p}u\Delta\overline{u}\,dx.\end{gathered}$$

Substituting into the equation (3.21) and applying Lemma 3.4, we arrive at

$$\begin{aligned}\operatorname{Im}\int J(|u|^{2p}u)\overline{Ju}\,dx&=-\frac{2tpn}{p+1}\int|u|^{2p+2}\,dx-4t^2\int|u|^{2p}\operatorname{Im}(\Delta\overline{u}\,u)\,dx\\&=-\frac{2tpn}{p+1}\int|u|^{2p+2}\,dx-2t^2\int|u|^{2p}\frac{\partial}{\partial t}|u|^2\,dx\\&=-\frac{2tpn}{p+1}\int|u|^{2p+2}\,dx-\frac{2t^2}{p+1}\int\frac{\partial}{\partial t}|u|^{2p+2}\,dx,\end{aligned}$$

we conclude the proof by observing that

$$t^2\frac{\partial}{\partial t}\left(|u|^{2p+2}\right)=\frac{\partial}{\partial t}\left(t^2|u|^{2p+2}\right)-2t\,|u|^{2p+2}.$$

□

Lemma 3.7. *Let u and v be solutions of coupled system* (1.1)*, then we have*

$$\begin{aligned}&\operatorname{Im}\Big(\int J(|u|^q|v|^{q+2}u)\cdot\overline{Ju}\,dx\Big)+\operatorname{Im}\Big(\int J(|v|^q|u|^{q+2}v)\cdot\overline{Jv}\,dx\Big)\\&=-\frac{4t\,(n(q+1)-2)}{q+2}\int|u\,v|^{q+2}\,dx-\frac{4}{q+2}\frac{\partial}{\partial t}\Big\{t^2\int\left(|u\,v|^{q+2}\right)dx\Big\}.\end{aligned}\tag{3.22}$$

Proof. From the definition of J we have

$$J(|u|^q|v|^{q+2}u)=|u|^q|v|^{q+2}ux+2it\nabla(|u|^q|v|^{q+2}u),\tag{3.23}$$

making the scalar product of (3.23) with $\overline{Ju}=x\overline{u}-2it\nabla\overline{u}$ and differentiating gives

$$\begin{aligned}&J(|u|^q|v|^{q+2}u)\cdot\overline{Ju}\\&=|x|^2|u|^q|v|^{q+2}|u|^2-2it|u|^q|v|^{q+2}ux\cdot\nabla\overline{u}+2it\overline{u}x\cdot\nabla(|u|^q|v|^{q+2}u)\\&\quad+4t^2\nabla(|u|^q|v|^{q+2}u)\cdot\nabla\overline{u}\\&=|x|^2|u|^q|v|^{q+2}|u|^2+2it|u|^q\,|v|^{q+2}x\cdot(\overline{u}\nabla u-u\nabla\overline{u})+2it|u|^2x\cdot\nabla(|u|^q|v|^{q+2})\\&\quad+4t^2|u|^q|v|^{q+2}|\nabla u|^2+4t^2u\,\nabla(|u|^q|v|^{q+2})\cdot\nabla\overline{u}.\end{aligned}\tag{3.24}$$

Taking the imaginary part of (3.24) and differentiating again, we obtain

$$\begin{aligned}
&\operatorname{Im}(J(|u|^q|v|^{q+2}u)\cdot\overline{Ju})\\
&=2t|u|^2x\cdot\nabla(|u|^q|v|^{q+2})+4t^2\operatorname{Im}\big(u\,\nabla(|u|^q|v|^{q+2})\cdot\nabla\overline{u}\big)\\
&=2t|v|^{q+2}x\cdot|u|^2\nabla(|u|^q)+2t|u|^{q+2}x\cdot\nabla(|v|^{q+2})\\
&\quad+4t^2\operatorname{Im}\big(u\nabla(|u|^q|v|^{q+2})\cdot\nabla\overline{u}\big)\\
&=\frac{2tq}{2+q}|v|^{q+2}x\cdot\nabla(|u|^{q+2})+2t|u|^{q+2}x\cdot\nabla(|v|^{q+2})\\
&\quad+4t^2\operatorname{Im}\big(u\,\nabla(|u|^q|v|^{q+2})\cdot\nabla\overline{u}\big)\,.
\end{aligned}\tag{3.25}$$

Observe that

$$\int u\nabla(|u|^q|v|^{q+2})\cdot\nabla\overline{u}dx=-\int|u|^q|v|^{q+2}\,|\nabla u|^2dx-\int|u|^q|v|^{q+2}\Delta\overline{u}u\,dx,$$

using the Lemma 3.4 it follows that

$$\begin{aligned}
4t^2\operatorname{Im}\int u\,\nabla(|u|^q|v|^{q+2})\cdot\nabla\overline{u}dx&=-4t^2\int|u|^q|v|^{q+2}\operatorname{Im}\left(\Delta\overline{v}\,v\right)\,dx\\
&=-2t^2\int|v|^{q+2}|u|^q\frac{\partial}{\partial t}|u|^2\,dx\\
&=-\frac{4t^2}{q+2}\int|v|^{q+2}\frac{\partial}{\partial t}|u|^{q+2}\,dx.
\end{aligned}\tag{3.26}$$

Combining (3.25), (3.26) and integrating by parts in $\mathbb{R}^n$, it is not difficult to see that

$$\begin{aligned}
&\int\operatorname{Im}(J(|u|^q|v|^{q+2}u)\cdot\overline{Ju})dx+\int\operatorname{Im}(J(|v|^q|u|^{q+2})\cdot\overline{Jv})dx\\
&=\frac{2tq}{q+2}\int x\cdot\nabla(|u\,v|^{q+2})dx+2t\int x\cdot\nabla(|u\,v|^{q+2})dx-\frac{4t^2}{q+2}\int\frac{\partial}{\partial t}\left(|u\,v|^{q+2}\right)dx\\
&=-\frac{4tn(q+1)}{q+2}\int|u\,v|^{q+2}\,dx-\frac{4t^2}{q+2}\int\frac{\partial}{\partial t}\left(|u\,v|^{q+2}\right)dx,
\end{aligned}$$

the proof of lemma follows using the following identity

$$t^2\frac{\partial}{\partial t}\left(|u\,v|^{q+2}\right)=\frac{\partial}{\partial t}\left(t^2\,|u\,v|^{q+2}\right)-2t\,|u\,v|^{q+2}.$$

□

Theorem 3.8 (Pseudo-Conformal Law). *Let u and v be solutions of the coupled system* (1.1)*, then*

$$\begin{aligned}
&\frac{\partial}{\partial t}\Big\{\int|Ju|^2+|Jv|^2-\frac{4\alpha t^2}{(p+1)}\int\Big[|u|^{2(p+1)}+|v|^{2(q+1)}\Big]\,dx\\
&-\frac{8\beta t^2}{(q+2)}\int|u\,v|^{q+2}\,dx\Big\}\\
&=\frac{4\alpha(np-2)\,t}{(p+1)}\int\Big[|u|^{2(p+1)}+|v|^{2(q+1)}\Big]\,dx\\
&\quad+\frac{8\beta t}{(q+2)}[(q+1)n-2]\int|u\,v|^{q+2}\,dx.
\end{aligned}\tag{3.27}$$

Proof. From (3.18) and (3.19), we obtain

$$L(Ju) = J(Lu) = -\alpha J(|u|^{2p}u) - \beta J(|u|^q|v|^{q+2}u) \tag{3.28}$$

and by the definition of L, we have

$$i\frac{\partial}{\partial t}(Ju) + \Delta(Ju) = -\alpha J(|u|^{2p}u) - \beta J(|u|^q|v|^{q+2}u). \tag{3.29}$$

Computing the scalar product of (3.29) with $\overline{Ju}$, taking two times the imaginary part, after integration in $\mathbb{R}^n$, we obtain

$$\begin{aligned}&\frac{\partial}{\partial t}\int |Ju(x)|^2\,dx - 2\,\mathrm{Im}\int |\nabla(Ju(x))|^2\,dx\\ &= -2\alpha\,\mathrm{Im}\int J(|u|^{2p}u)\cdot\overline{Ju}\,dx - 2\beta\,\mathrm{Im}\int J(|u|^q|v|^{q+2}u)\cdot\overline{Ju}\,dx.\end{aligned} \tag{3.30}$$

Therefore,

$$\begin{aligned}&\frac{\partial}{\partial t}\int |Ju(x)|^2\,dx\\ &= -2\alpha\,\mathrm{Im}(\int J(|u|^{2p}u)\cdot\overline{Ju}\,dx) - 2\beta\,\mathrm{Im}(\int J(|u|^q|v|^{q+2}u)\cdot\overline{Ju}\,dx).\end{aligned} \tag{3.31}$$

Similarly,

$$\begin{aligned}&\frac{\partial}{\partial t}\int |Jv(x)|^2\,dx\\ &= -2\alpha\,\mathrm{Im}(\int J(|v|^{2p}v)\cdot\overline{Jv}\,dx) - 2\beta\,\mathrm{Im}(\int J(|v|^q|u|^{q+2}v)\cdot\overline{Jv}\,dx).\end{aligned} \tag{3.32}$$

Adding (3.31) and (3.32) and applying the lemmas 3.6 and 3.7 we completes the proof. □

Remark 3.9. Let $u \in \mathcal{S}(\mathbb{R}^n)$, we consider the multiplication differential operator

$$\widehat{Pu}(\xi) = \sum_{l=1}^{n}\zeta_l\xi^{\theta_l}\widehat{u}(\xi), \quad \xi \in \mathbb{R}^n, \tag{3.33}$$

where $\zeta_l \in \mathbb{R}$ and the multi-index $\theta_l = (\theta_l^j)_{j=1,\dots,n} \in (\mathbb{Z}^+)^n$. In order for the differential operators

$$L = \partial_t - iP, \quad J = x + tQ, \quad x \in \mathbb{R}^n,$$

to commute, where Q is also a multiplication differential operator, it is easy to see that we need

$$\begin{gathered}Q(u) = i(P(xu) - xP(u)) = i(P(x_ju) - x_jP(u))_{j=1,\dots,n},\\ x = (x_j)_{j=1,\dots,n} \in \mathbb{R}^n\end{gathered} \tag{3.34}$$

and using properties of Fourier transform we have

$$\widehat{Qu}(\xi) = \Big(\sum_{l=1}^{n}\zeta_l\theta_l^j\xi^{\theta_l - e_j}\widehat{u}(\xi)\Big)_{j=1,\dots,n}, \quad \xi \in \mathbb{R}^n, \tag{3.35}$$

where the canonical unit vector is $e_j = \overbrace{(0,\dots,0,1,0,\dots,0)}^{\text{j}}$. Observe that in this case J also commutes with cL for any constant $c \in \mathbb{C}$ and reciprocally L commutes with cJ for any constant $c \in \mathbb{C}$.

In our case, if we consider

$$Pu = \Delta u \Rightarrow \widehat{Pu}(\xi) = -\sum_{l=1}^{n} \xi^{2e_l}\widehat{u}(\xi),$$

and by definition of Q (see (3.35)) we obtain

$$\widehat{Qu}(\xi) = -\left(2\xi^{2e_j - e_j}\widehat{u}(\xi)\right)_{j=1,\dots,n} = -2\xi\widehat{u}(\xi),$$

and therefore

$$Qu = 2i\nabla u.$$

In the case $n = 1$, considering the operator $\partial_t + \partial_x^{2k+1}$, $x \in \mathbb{R}$, then

$$\widehat{Pu}(\xi) = (-1)^{k+1}\xi^{2k+1}\widehat{u}(\xi), \quad \xi \in \mathbb{R},$$

and $\widehat{Qu}(\xi) = (-1)^{k+1}(2k+1)\xi^{2k}\widehat{u}(\xi)$, thus

$$Qu = (-1)^k(2k+1)\partial_x^{2k}u,$$

in the particular case $k = 1$ (KdV equation), we obtain $J = x - 3t\partial_x^2$.

4. A priori estimates in $H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n)$

Here we will give conditions about of the global existence. We begin with the well-known result: The Gagliardo-Nirenberg inequality.

Lemma 4.1. *Let $f : \mathbb{R}^n \mapsto \mathbb{R}$. Fix $1 \le q, r \le \infty$ and a natural number m. Suppose also that a real number λ and a natural number j are such that*

$$\frac{1}{p} = \frac{j}{n} + \big(\frac{1}{r} - \frac{m}{n}\big)\lambda + \frac{1-\lambda}{q} \quad \text{and} \quad \frac{j}{m} \le \lambda \le 1.$$

Then

(1) *every function $f : \mathbb{R}^n \mapsto \mathbb{R}$ that lies in $L^q(\mathbb{R}^n)$ with mth derivative in $L^r(\mathbb{R}^n)$ also has jth derivative in $L^p(\mathbb{R}^n)$;*
(2) *furthermore, there exists a constant C depending only on m, n, j, q, r and λ such that*

$$\|D^j f\|_{L^p} \le C\|D^m f\|_{L^r}^{\lambda}\|f\|_{L^q}^{1-\lambda}. \tag{4.1}$$

In the particular case $j = 0$, $r = q = 2$ and $m = 1$, we have

$$\|f\|_{L^p} \le C\|Df\|_{L^2}^{\lambda}\|f\|_{L^2}^{1-\lambda}, \tag{4.2}$$

where

$$0 \le \lambda := \lambda(r) = \frac{(r-2)n}{2r} \le 1.$$

Considering the energy equation (1.7), we can to obtain an "a priori" estimate for

$$\|\nabla u(t)\|_{L^2(\mathbb{R}^n)}^2 + \|\nabla v(t)\|_{L^2(\mathbb{R}^n)}^2 \tag{4.3}$$

if $(2p+2)\lambda(2p+2) \le 2$ and $(4+2q)\lambda(4+2q) \le 2$, i.e. if

$$0 < p \le \frac{2}{n}, \quad 0 < q \le \frac{2}{n} - 1, \tag{4.4}$$

or if

$$0 < p \le \frac{2}{n}, \quad \text{and} \quad \beta \le 0,$$

or if

$$0 < q \le \frac{2}{n} - 1, \quad \text{and} \quad \alpha \le 0,$$

where in the equality, we obtain "a priori" estimate only to $\|u_0\|_{L^2} \leq C$ and $\|v_0\|_{L^2} \leq C$ (small data).

We observe that if $\mathcal{X} \leq 0$, then from (1.7) it follows that

$$\int \left(|\nabla u(x,t)|^2 + |\nabla v(x,t)|^2\right) dx \leq E(u_0, v_0), \quad \forall t \geq 0. \tag{4.5}$$

In the next section we will see that in some cases when $\mathcal{X} \leq 0$, we can also get us a better asymptotic growth to (4.3).

5. Asymptotic growth in the energy space

In this section we prove Theorem 1.2. From Theorem 3.8 we obtain

$$\begin{aligned} &\frac{\partial}{\partial t}\Big[\int \left(|J(u)|^2 + |J(v)|^2\right) dx - tf(t)\Big] \\ &= \frac{4t\alpha(np-2)}{p+1}\int |u|^{2p+2} + |v|^{2p+2}\, dx + \frac{8t\beta[n(q+1)-2]}{q+2}\int |uv|^{q+2}\, dx, \end{aligned} \tag{5.1}$$

where the function

$$f(t) = 4t\mathcal{X}(t) = \frac{4\alpha t}{(p+1)}\int \left[|u|^{2(p+1)} + |v|^{2(p+1)}\right] dx + \frac{8\beta t}{(q+2)}\int |uv|^{q+2}\, dx. \tag{5.2}$$

We consider two cases.

Case I: If

$$\beta\, n(q+1) \leq \beta\, np \Longleftrightarrow \begin{cases} p \geq q+1 & \text{if } \beta > 0, \\ \text{or} \\ p \leq q+1 & \text{if } \beta < 0. \end{cases}$$

In this case

$$8t\beta[n(q+1)-2] \leq 8t\beta(np-2),$$

and (5.1) implies

$$\begin{aligned} &\frac{\partial}{\partial t}\Big[\int \left(|J(u)|^2 + |J(v)|^2\right) dx - tf(t)\Big] \\ &\leq \frac{4t\alpha(np-2)}{p+1}\int |u|^{2p+2} + |v|^{2p+2}\, dx + \frac{8t\beta(np-2)}{q+2}\int |uv|^{q+2}\, dx \\ &= (np-2)f(t). \end{aligned} \tag{5.3}$$

Integrating the inequality above,

$$\begin{aligned} &\int (|J(u)|^2 + |J(v)|^2)\, dx - tf(t) \\ &\leq a_0 + (np-2)\int_0^t f(t')dt' \\ &\leq a_0 + (np-2)\int_0^1 f(t')dt' + (np-2)\int_1^t f(t')dt', \end{aligned} \tag{5.4}$$

where

$$a_0 = \int |x|^2 \left(|u_0(x)|^2 + |v_0(x)|^2\right) dx, \tag{5.5}$$

which gives

$$F(t) := -tf(t) \leq b_0 + \int_1^t \Big(\frac{2-np}{t'}\Big)F(t')dt', \tag{5.6}$$

where

$$b_0 = b_0(n,p) := a_0 + (np-2)\int_0^1 f(t')dt'. \tag{5.7}$$

The Gronwall inequality in (5.6) with $np \le 2$, implies

$$F(t) \le b_0 e^{-\int_1^t (np-2)/t' dt'} = b_0 t^{2-np}, \quad t \ge 1. \tag{5.8}$$

From the conservation of energy (1.7) we deduce

$$\int \left(|\nabla u|^2 + |\nabla v|^2\right) dx = E(0) + \frac{f(t)}{4\,t}, \tag{5.9}$$

and from (5.8) and (5.9) it follows that

$$\int \left(|\nabla u|^2 + |\nabla v|^2\right) dx \ge E(0) - \frac{b_0}{4\,t^{np}}, \quad t \ge 1.$$

On the other hand, if $f(t) = 4t\mathcal{X}(t) \le 0$ (e.g. $\alpha \le 0$ and $\beta \le 0$) the above inequality and (4.5) imply (1.15). and from inequalities (5.4)-(5.8) we obtain

$$\begin{aligned}\int (|J(u)|^2 + |J(v)|^2)\,dx + |tf(t)| &\le b_0 + (2-np)\int_1^t \frac{b_0 t'^{2-np}}{t'}dt' \\ &= b_0 t^{2-np} \quad \text{if } np \le 2 \text{ and } t \ge 1.\end{aligned} \tag{5.10}$$

By the definition of J it follows that

$$|J(u)|^2 = |x|^2|u|^2 + 4t^2|\nabla u|^2 - 4t\,\mathrm{Im}\,\overline{u}x\cdot\nabla u.$$

Hence if $np \le 2$, using Cauchy-Schwartz we obtain

$$\begin{aligned}&\int |x|^2\left(|u|^2+|v|^2\right)dx + 4t^2\int\left(|\nabla u|^2+|\nabla v|^2\right)dx \\ &\le b_0 t^{2-np} + 4t\int \mathrm{Im}\,\overline{u}x\cdot\nabla u dx + 4t\int \mathrm{Im}\,\overline{v}x\cdot\nabla v dx \\ &\le b_0 t^{2-np} + 4t\|x\,u\|_{L^2}\|\nabla u\|_{L^2} + 4t\|x\,v\|_{L^2}\|\nabla v\|_{L^2},\end{aligned} \tag{5.11}$$

and from (5.11) we have

$$\left(\|x\,u\|_{L^2} - 2t\|\nabla u\|_{L^2}\right)^2 + \left(\|x\,v\|_{L^2} - 2t\|\nabla v\|_{L^2}\right)^2 \le b_0\,t^{2-np}, \tag{5.12}$$

and consequently

$$2t\left(\|\nabla u\|_{L^2} + \|\nabla v\|_{L^2}\right) \le \|x\,u\|_{L^2} + \|x\,v\|_{L^2} + 2b_0^{1/2}t^{1-np/2}; \tag{5.13}$$

therefore, using Lemma 3.5, we obtain

$$2t\left(\|\nabla u\|_{L^2} + \|\nabla v\|_{L^2}\right) \le 2b_0^{1/2}t^{1-np/2} + a_0 + 2\int_0^t \left(\|\nabla u\|_{L^2} + \|\nabla v\|_{L^2}\right)dt'. \tag{5.14}$$

Let $\mathcal{W}(t) = \|\nabla u(t)\|_{L^2} + \|\nabla v(t)\|_{L^2}$, the above inequality gives

$$\begin{aligned}t\mathcal{W}(t) &\le b_0^{1/2}t^{1-np/2} + \frac{a_0}{2} + \int_0^t \mathcal{W}(t')dt' \\ &= b_0^{1/2}t^{1-np/2} + \frac{a_0}{2} + \int_0^1 \mathcal{W}(t')dt' + \int_1^t \mathcal{W}(t')dt' \\ &:= b_0^{1/2}t^{1-np/2} + c_0 + \int_1^t \left(\frac{1}{t'}\right)t'\mathcal{W}(t')dt',\end{aligned} \tag{5.15}$$

where

$$c_0 = \frac{a_0}{2} + \int_0^1 \mathcal{W}(t')dt', \tag{5.16}$$

and a_0 as defined in (5.5), and by Gronwall's inequality (see Lema 3.1), we concludes that if $np \le 2$ and $t \ge 1$, then

$$\begin{aligned} t\mathcal{W}(t) &\le b_0^{1/2} t^{1-np/2} + c_0 + \int_1^t \Big(b_0^{1/2} t'^{1-np/2} + c_0 \Big) \frac{1}{t'} \exp\Big\{ \int_{t'}^t \frac{1}{r} dr \Big\} dt' \\ &\le b_0^{1/2} t^{1-np/2} + c_0 + t \int_1^t \Big(b_0^{1/2} t'^{1-np/2} + c_0 \Big) \frac{1}{t'^2} dt'. \end{aligned} \tag{5.17}$$

Consequently, if $np \le 2$ and $t \ge 1$ we estimate $\mathcal{W}(t)$ by

$$\mathcal{W}(t) \le \Big(\frac{2b_0^{1/2}}{np} + c_0 \Big) - \frac{b_0^{1/2}(2-np)}{np} t^{-np/2}.$$

Using this inequality and (5.12) is easy to verify the estimate (1.14).

Case II: If

$$\alpha\, n(q+1) \ge \alpha np \Longleftrightarrow \begin{cases} p \le q+1 & \text{if } \alpha > 0, \\ p \ge q+1 & \text{if } \alpha < 0. \end{cases}$$

In this case

$$4t\alpha[n(q+1)-2] \ge 4t\alpha(np-2),$$

and (5.1) implies

$$\begin{aligned} &\frac{\partial}{\partial_t} \Big[\int \left(|J(u)|^2 + |J(v)|^2 \right) dx - tf(t) \Big] \\ &\le \frac{4t\alpha[n(q+1)-2]}{p+1} \int |u|^{2p+2} + |v|^{2p+2}\, dx \\ &\quad + \frac{8t\beta[n(q+1)-2]}{q+2} \int |uv|^{q+2}\, dx \\ &= -[2-n(q+1)] f(t), \end{aligned} \tag{5.18}$$

and similarly as the above case we can show that if $n(q+1) \le 2$, then

$$\begin{aligned} \int \left(|\nabla u|^2 + |\nabla v|^2 \right) dx &= E(0) + \frac{f(t)}{4\,t} \\ &\ge E(0) - \frac{b_1}{4t^{n(q+1)}}, \quad t \ge 1, \end{aligned} \tag{5.19}$$

where

$$b_1 = b_1(n,q) := a_0 - [2-n(q+1)] \int_0^1 f(t')dt'. \tag{5.20}$$

Similarly as in Case I, if $f(t) = 4t\mathcal{X}(t) \le 0$, from the inequalities above we obtain

$$\begin{aligned} \int (|J(u)|^2 + |J(v)|^2)\, dx + |tf(t)| &\le b_1 + (2-n(q+1)) \int_1^t \frac{b_1 t'^{2-n(q+1)}}{t'} dt' \\ &= b_1 t^{2-n(q+1)} \quad \text{if } n(q+1) \le 2 \text{ and } t \ge 1. \end{aligned} \tag{5.21}$$

Let $\mathcal{W}(t) = \|\nabla u(t)\|_{L^2} + \|\nabla v(t)\|_{L^2}$, as in Case I, we obtain

$$\begin{aligned} &t\mathcal{W}(t)\\ &\le b_1^{1/2} t^{1-n(q+1)/2} + c_0 + \int_1^t \Big(b_1^{1/2} t'^{1-n(q+1)/2} + c_0\Big) \frac{1}{t'} \exp\Big\{\int_{t'}^t \frac{1}{r} dr\Big\} dt' \\ &\le b_1^{1/2} t^{1-n(q+1)/2} + c_0 + t\int_1^t \Big(b_1^{1/2} t'^{1-n(q+1)/2} + c_0\Big) \frac{1}{t'^2} dt'. \end{aligned} \tag{5.22}$$

Consequently, if $n(q+1) \le 2$ and $t \ge 1$ we estimate $W(t)$ by

$$\mathcal{W}(t) \le \Big(\frac{2b_1^{1/2}}{n(q+1)} + c_0\Big) - \frac{b_1^{1/2}(2 - n(q+1))}{n(q+1)} t^{-n(q+1)/2}.$$

Finally using this inequality and (5.12) is easy to verify the estimate (1.16).

Corollary 5.1. *Let*

$$P(t) = \|x\,u(t)\|_{L_x^2}^2 + \|x\,v(t)\|_{L_x^2}^2, \quad W(t) = \|\nabla u(t)\|_{L^2}^2 + \|\nabla v(t)\|_{L^2}^2.$$

Then: (i) If $E(0) \gg 1$ is large and $P(0) \ll 1$ is very small, then in the right side of (1.13) *we have*

$$\Big(c_0 + \frac{2b_0^{1/2}}{np}\Big) - \frac{b_0^{1/2}(2-np)}{np} t^{-np/2} < E(0).$$

(ii) With the conditions of Theorem 1.2, i.e. if $np \le 2$ and $p \ge q+1$ if $\beta > 0$ or $p \le q+1$ if $\beta < 0$ and $\mathcal{X} \le 0$ (see (1.8)*, e.g., $\alpha \le 0$ and $\beta \le 0$) we have*

$$t^2 W(t) - 2b_0 t^{2-np} \le P(t) \le 2a_0 + 8t \int_0^t W(t')\, dt', \tag{5.23}$$

and similarly if $n(q+1) \le 2$ and $p \ge q+1$ if $\alpha > 0$ or $p \le q+1$ if $\alpha < 0$ and $\mathcal{X} \le 0$, then

$$t^2 W(t) - 2b_0 t^{2-n(q+1)} \le P(t) \le 2a_0 + 8t \int_0^t W(t')\, dt'. \tag{5.24}$$

(iii) With the hypotheses of Theorem 1.2 item (1) we have

$$4W(t) - \frac{b_0}{t^{np}} \le \frac{d}{dt}\Big(\frac{P(t)}{t}\Big). \tag{5.25}$$

Proof. First we prove item (i): we consider $\mathcal{X} \le 0$ and $np \le 2$. From energy equation (1.7) we have

$$E(0) = W(t) + |\mathcal{X}(t)|.$$

and therefore

$$W(t) \le E(0) \quad \text{and} \quad |\mathcal{X}(t)| \le E(0). \tag{5.26}$$

consequently from definition of c_0 in (5.16) and Cauchy-Schwarz we obtain

$$c_0 \le \frac{a_0}{2} + \sqrt{2}\Big(\int_0^1 W(t) dt\Big)^{1/2} \le \frac{a_0}{2} + \sqrt{2} E(0)^{1/2} < \frac{E(0)}{2}, \tag{5.27}$$

if $P(0) = a_0 \ll 1$ and $E(0) \gg 1$. Similarly from definition of b_0 in (5.7) we have

$$b_0 \le a_0 + 4(2-np)\int_0^1 t|\mathcal{X}(t)| dt \le a_0 + 2E(0)(2-np) \le 5E(0), \tag{5.28}$$

if $P(0) = a_0 \ll 1$ and $E(0) \gg 1$, thus

$$\frac{2b_0^{1/2}}{np} \le \frac{2\sqrt{5}E(0)^{1/2}}{np} < \frac{E(0)}{2}, \tag{5.29}$$

finally combining (5.27) and (5.29) we have

$$\Big(c_0 + \frac{2b_0^{1/2}}{np}\Big) - \frac{b_0^{1/2}(2-np)}{np} t^{-np/2} \le c_0 + \frac{2b_0^{1/2}}{np} < E(0).$$

To prove (ii), using Lemma 3.5 ,it follows that

$$P(t) \le 2P(0) + 8\Big(\int_0^t \|\nabla u(t')\|_{L^2} dt'\Big)^2 + 8\Big(\int_0^t \|\nabla v(t')\|_{L^2} dt'\Big)^2, \tag{5.30}$$

and Cauchy-Schwarz inequality gives

$$P(t) \le 2P(0) + 8t \int_0^t W(t')dt', \tag{5.31}$$

and this proves the side right of (5.23). On the other hand, from (5.13) we obtain

$$4t^2 \left(\|\nabla u\|_{L^2}^2 + \|\nabla v\|_{L^2}^2\right) \le 4\|x\, u\|_{L^2}^2 + 4\|x\, v\|_{L^2}^2 + 8b_0 t^{2-np},$$

and this inequality proves the side left of (5.23). In a similar way we prove (5.24).

Now we prove (iii): using equality (3.16) in the first inequality from (5.11), we obtain

$$\begin{aligned} P(t) + 4t^2 W(t) &\le b_0 t^{2-np} + 4t \int \operatorname{Im} \overline{u} x \cdot \nabla u dx + 4t \int \operatorname{Im} \overline{v} x \cdot \nabla v dx \\ &\le b_0 t^{2-np} + tP'(t), \end{aligned}$$

hence

$$P(t) + 4t^2 W(t) \le b_0 t^{2-np} + tP'(t);$$

this inequality proves (5.25). □

6. Blow-up in $H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n)$

In this section we prove Theorem 1.4. Using Lemma 3.3 and equality (3.16) we obtain

$$\begin{aligned} &\frac{\partial^2}{\partial t^2} \int |x|^2 (|u(t)|^2 + |v(t)|^2)\, dx \\ &= 4\frac{\partial}{\partial t}\Big\{ \operatorname{Im} \int (\overline{u} x \cdot \nabla u + \overline{v} x \cdot \nabla v)\, dx \Big\} \\ &= 8E(0) + \frac{4\alpha(2-np)}{p+1} \int \left(|u|^{2p+2} + |v|^{2p+2}\right) dx \\ &\quad + \frac{8\beta\ (2 - n(1+q))}{q+2} \int |u\, v|^{q+2} dx. \end{aligned} \tag{6.1}$$

We consider two cases.

Case I: If

$$\beta\, p \le \beta\, (q+1) \Longleftrightarrow \begin{cases} p - q \le 1 & \text{if } \beta > 0, \\ p - q \ge 1 & \text{if } \beta < 0. \end{cases}$$

In this case

$$8\beta[2 - n(q+1)] \le 8\beta(2-np),$$

and (6.1) gives

$$
\begin{aligned}
&\frac{\partial^2}{\partial t^2}\int |x|^2(|u(t)|^2+|v(t)|^2)\,dx\\
&\le 8E(0)+\frac{4\alpha(2-np)}{p+1}\int\big(|u|^{2p+2}+|v|^{2p+2}\big)\,dx+\frac{8\beta\,(2-np)}{q+2}\int |u\,v|^{q+2}dx \qquad (6.2)\\
&\le 8E(0)-\frac{(np-2)f(t)}{t}.
\end{aligned}
$$

From the conservation of energy (1.7) we deduce

$$
-\frac{f(t)}{4\,t}=E(0)-\int\big(|\nabla u|^2+|\nabla v|^2\big)\ dx; \qquad (6.3)
$$

therefore,

$$
-\frac{f(t)}{t}\le 4E(0). \qquad (6.4)
$$

Combining (6.2), (6.4) and that $np\ge 2$, we have

$$
\frac{\partial^2}{\partial t^2}\int |x|^2(|u(t)|^2+|v(t)|^2)\,dx\le 4npE(0). \qquad (6.5)
$$

Integrating and using (3.16) we can show that

$$
\begin{aligned}
&\frac{\partial}{\partial t}\int |x|^2(|u(t)|^2+|v(t)|^2)\,dx\\
&\le 4\,\mathrm{Im}\int\left(\overline{u}_0x\cdot\nabla u_0+\overline{v}_0x\cdot\nabla v_0\right)\,dx+4npE(0)t,
\end{aligned} \qquad (6.6)
$$

integrating again we obtain

$$
\begin{aligned}
&\int |x|^2(|u(t)|^2+|v(t)|^2)dx\\
&\le\int |x|^2(|u_0|^2+|v_0|^2)dx+4t\,\mathrm{Im}\int\left(\overline{u}_0x\cdot\nabla u_0+\overline{v}_0x\cdot\nabla v_0\right)\,dx+2npE(0)t^2\\
&:=A_0+B_0t+C_0t^2:=P_0(t).
\end{aligned} \qquad (6.7)
$$

It is not difficult to see that there exists a $T>0$ such that $\int |x|^2(|u(T)|^2+|v(T)|^2)dx=0$ in the following three cases:

(1) $E(0)=0$ and

$$
\mathrm{Im}\int\left(\overline{u}_0x\cdot\nabla u_0+\overline{v}_0x\cdot\nabla v_0\right)\,dx<0,
$$

(2) $E(0)<0$,

(3) $E(0)>0$ and

$$
\Big(\mathrm{Im}\int\left(\overline{u}_0x\cdot\nabla u_0+\overline{v}_0x\cdot\nabla v_0\right)\,dx\Big)^2>\frac{npE(0)}{2}\int |x|^2(|u_0|^2+|v_0|^2)dx.
$$

Figures 1, 2 and 3 correspond to the cases (1), (2) and (3) above.

Now the Heisenberg inequality (Uncertainty inequality)

$$
\|f\|_{L^2}^2\le\frac{2}{n}\|xf\|_{L^2}\|\,\nabla f\|_{L^2}, \qquad (6.8)
$$

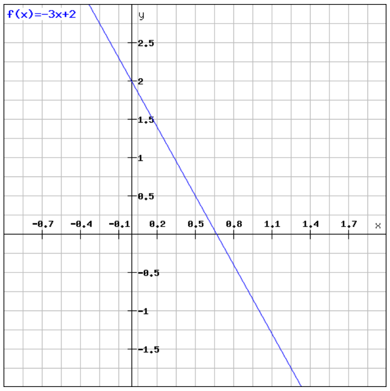

FIGURE 1. Graph of $P_0(t)$ corresponding to case (1).

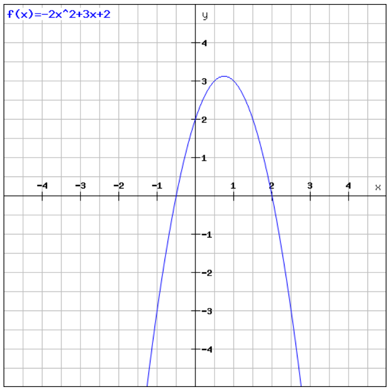

FIGURE 2. Graph of $P_0(t)$ corresponding to case (2).

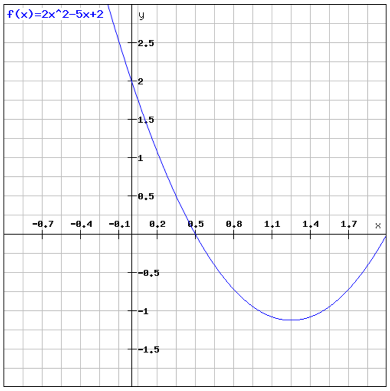

FIGURE 3. Graph of $P_0(t)$ corresponding to case (3).

implies that if the initial data u_0 and v_0 satisfies (1), (2) or (3) then, there exists $0 < T^* \leq T$ such that

$$\lim_{t\to T^*} \|\nabla u(t)\|_{L^2} = \infty, \quad \lim_{t\to T^*} \|\nabla v(t)\|_{L^2} = \infty.$$

Case II: If

$$\alpha\,(q+1) < \alpha p \Longleftrightarrow \begin{cases} p-q>1 & \text{if } \alpha>0, \\ p-q<1 & \text{if } \alpha<0. \end{cases}$$

In this case

$$4\alpha(2-np) \le 4\alpha[2-n(q+1)],$$

and (6.1) gives

$$\frac{\partial^2}{\partial t^2}\int |x|^2(|u(t)|^2+|v(t)|^2)\,dx \le 8E(0) - \frac{(n(q+1)-2)f(t)}{t}.$$

As in Case I, using (6.4) and $n(q+1)\ge 2$, we have

$$\frac{\partial^2}{\partial t^2}\int |x|^2(|u(t)|^2+|v(t)|^2)\,dx \le 4n(1+q)E(0).$$

Integrating two times and using (3.16) we obtain

$$\begin{aligned} &\int |x|^2(|u(t)|^2+|v(t)|^2)dx \\ &\le \int |x|^2(|u_0|^2+|v_0|^2)dx + 4t\,\mathrm{Im}\int \left(\overline{u}_0 x\cdot\nabla u_0 + \overline{v}_0 x\cdot\nabla v_0\right)\,dx + 2n(1+q)E(0)t^2 \\ &:= A_0 + B_0 t + C_1 t^2. \end{aligned}$$

It is not difficult to see that there exists a $T>0$ such that $\int |x|^2(|u(T)|^2 + |v(T)|^2)dx = 0$ in the following three cases:

(1) $E(0)=0$ and

$$\mathrm{Im}\int \left(\overline{u}_0 x\cdot\nabla u_0 + \overline{v}_0 x\cdot\nabla v_0\right)\,dx < 0,$$

(2) $E(0)<0$,
(3) $E(0)>0$ and

$$\Big(\mathrm{Im}\int \big(\overline{u}_0 x\cdot\nabla u_0 + \overline{v}_0 x\cdot\nabla v_0\big)\,dx\Big)^2 > \frac{n(q+1)E(0)}{2}\int |x|^2(|u_0|^2+|v_0|^2)dx.$$

Using the Heinseberg inequality (6.8) again we concludes in this case that if the initial data u_0 and v_0 satisfies (1)–(3) then, there exists $0<T^*\le T$ such that

$$\lim_{t\to T^*}\|\nabla u(t)\|_{L^2} = \infty, \quad \lim_{t\to T^*}\|\nabla v(t)\|_{L^2} = \infty.$$

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