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CRITICAL CASE FOR THE VISCOUS CAHN-HILLIARD EQUATION

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Abstract. We prove the existence of solutions of the viscous Cahn-Hilliard equation in whole domain when the nonlinear term in the second order diffusion grows as u^q for the critical case when $N \geq 3$. Our results improve the ones in [9, 12].

1. Introduction

In this article, we study the initial-value problem

$$\begin{gathered} u_t = \Delta[\varphi(u) - \alpha\Delta u + \beta u_t] \quad \text{in } \mathbb{R}^N \times (0,T) := Q, \\ u(x,0) = u_0 \quad \text{in } \mathbb{R}^N \times \{0\}\,, \end{gathered} \tag{1.1}$$

where the nonlinearity φ satisfies the following assumptions:

- (H1) $\varphi \in W^{1,\infty}_{loc}(\mathbb{R})$, $\varphi(0) = 0$, and $\varphi(s)s \geq 0$, for any $s \in \mathbb{R}$.
- (H2) There exists $K > 0$ such that
$$|\varphi(u)| \leq K(|u| + |u|^q), \tag{1.2}$$
for some $q \in (1,\infty)$ if $N = 1, 2$; or $q \in \left(1, \frac{N+2}{N-2}\right]$ if $N \geq 3$.
- (H3) There exists $s_0 > 0$ such that $\varphi'(s) \geq 0$, if $|s| \geq s_0$.

Forward-backward parabolic equations arise in a variety of applications, such as edge detection in image processing [25], aggregation models in population dynamics [24], and stratified turbulent shear flow [1], theory of phase transitions [4, 5, 21], control theory in [11], etc. A different well-known equation of this type is the Perona-Malik equation,

$$w_t = \operatorname{div}\Big(\frac{\nabla w}{1 + |\nabla w|^2}\Big)\,, \tag{1.3}$$

which is parabolic if $|\nabla w| < 1$ and backward parabolic if $|\nabla w| > 1$. Similarly, the equation

$$u_t = \Delta\Big(\frac{u}{1+u^2}\Big) \tag{1.4}$$

is parabolic if $|u| < 1$ and backward parabolic if $|u| > 1$. Observe that in one space dimension the above equations are formally related setting $u = w_x$. A different

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well-known equation of application in theory of phase transitions is

$$u_t = \Delta\varphi(u) \tag{1.5}$$

where the famous choice of nonlinearity $\varphi(u) = u^3 - u$.

Clearly, forward-backward parabolic equations lead to ill-posed problems. Often a higher order term is added to the right-hand side to regularize the equation. Two main classes of additional terms are encountered in the mathematical literature, which, e.g. in case of equation (1.4), (1.5), reduce to:

(i) $\epsilon\Delta[\psi(u)]_t$, with $\psi' > 0$, leading to third-order pseudo-parabolic equations ($\epsilon > 0$ being a small parameter; for example, see [2, 8, 14, 20, 23, 26, 27, 32, 33, 34]);

(ii) $-\epsilon\Delta^2 u$, leading to fourth-order Cahn-Hilliard type equations (for example, see [3, 4, 28, 31] and references therein).

Remarkably, when $\psi(u) = u$ either of the above regularizations can be regarded as a particular case of the viscous Cahn-Hilliard equation,

$$\nu u_t = \Delta[\varphi(u) - \alpha\Delta u + \beta u_t] \quad (\alpha, \beta, \nu > 0)\,, \tag{1.6}$$

choosing either $\alpha = \epsilon$ or $\beta = \epsilon$; here $\varphi(u) = u^3 - u$ or $\varphi(u) = \frac{u}{1+u^2}$ for equation (1.5), whereas in general it involves a non-monotonic function.

Equation (1.6) has been derived by several authors using different physical considerations (in particular, see [16, 18, 22]). It is worth mentioning the wide literature concerning both the relationship between the viscous Cahn-Hilliard equation and *phase field models*, and generalized versions of the equation suggested in [16] (and references therein). Besides, the existence results were obtained under suitable nonlinearity φ in bounded smooth domain of $\mathbb{R}^N$ (see [9, 10, 13]). Moreover, in the latter reference authors give us the rigorous proof of convergence to solutions of either the Cahn-Hilliard equation, or of the Allen-Cahn equation, or of the Sobolev equation, depending on the choice of the parameter α, β. Recently, in [12] the authors gave the analysis of equation (1.6) in $\mathbb{R}^N$ under some assumptions on the growth of nonlinearity φ satisfying (H2), but not including the critical case $q = \frac{N+2}{N-2}$.

In light of the above considerations, by using some sharp a priori estimates for a suitable auxiliary approximation problem, we will prove the existence of solutions of problem (1.1) for a class of nonlinear functions φ satisfying the growth condition (H2) including the critical case $q = \frac{N+2}{N-2}$. Thus, our existence results enhance a part of the ones of Dlotko, et al. [12]. Our existence theorem is as follows:

Theorem 1.1. *Let $u_0 \in H^1(\mathbb{R}^N)$, and $q = \frac{N+2}{N-2}$. Let φ satisfy* (H1)–(H3). *Then, there exists a weak solution of problem* (1.1).

Remark 1.2. Note that we do not assume the boundedness on φ', see [9, (1.1)]. Thus, our results also improve the ones of Bui, et al. [9].

Before proving Theorem 1.1, we give a definition of weak solutions of (1.1).

Definition 1.3. Let $\alpha, \beta > 0$, and let $u_0 \in H^1(\mathbb{R}^N)$. By a weak solution of problem (1.1) we mean any function $u \in C([0,T]; H^2(\mathbb{R}^N)) \cap C^1([0,T]; L^2(\mathbb{R}^N))$ such that $\varphi(u) \in C([0,T]; L^2(\mathbb{R}^N))$, and

$$\begin{gathered} u_t = \Delta v \quad \text{in } Q \\ u = u_0 \quad \text{in } \mathbb{R}^N \times \{0\} \end{gathered} \tag{1.7}$$

in the sense of distribution. Here $v \in C([0,T]; H^2(\mathbb{R}^N) \cap H^1(\mathbb{R}^N))$ and for every $t \in [0,T]$ the function $v(\cdot,t)$ is the unique solution of the elliptic problem

$$\begin{gathered} -\beta\Delta v(\cdot,t) + v(\cdot,t) = \varphi(u)(\cdot,t) - \alpha\Delta u(\cdot,t) \quad \text{in } \mathbb{R}^N, \\ \lim_{|x|\to\infty} v(x,t) = 0. \end{gathered} \tag{1.8}$$

The function v is called a *chemical potential*.

2. Proof of Theorem 1.1

We first mention the description of our method. We start by considering the existence of weak solutions of the viscous Cahn-Hilliard problem with Dirichlet boundary conditions in the ball B_n, which has center at the origin and radius $n \geq 1$:

$$\begin{gathered} u_t = \Delta[\varphi_n(u) - \alpha\Delta u + \beta u_t] \quad \text{in } B_n \times (0,T) =: Q_n \\ u = \Delta u = 0 \quad \text{on } \partial B_n \times (0,T) \\ u = u_{0n} = u_0\phi_n \quad \text{in } B_n \times \{0\}\,, \end{gathered} \tag{2.1}$$

where $\phi_n(x) = \phi(x/n)$, and $\phi \in \mathcal{C}^\infty(\mathbb{R}^N)$ such that $\phi(x) = 1$ if $|x| < 1/2$, and $\phi(x) = 0$ if $|x| > 1$. And φ_n is just a truncated function of φ as in [9]:

$$\varphi_n(u) = \begin{cases} \varphi(u), & \text{if } |u| \leq n, \\ \varphi(n) + (u-n), & \text{if } u > n, \\ \varphi(-n) + (u+n), & \text{if } u < n. \end{cases}$$

Secondly, we establish a priori estimates for those solutions of problem (2.1) being independent of n. Finally, we shall pass to the limit as $n \to \infty$ (in a suitable way) to get a desired result.

It is not difficult to verify that φ_n is a globally Lipschitz function, and $\varphi_n(u)u \geq 0$. A well-posed result for problem (2.1) is proved in [9, Theorem 2.1]. Thus, there exists a unique weak solution u_n of problem (2.1) in $B_n \times (0,T)$. Remind that

$$v_n = \varphi_n(u_n) - \alpha\Delta u_n + \beta u_{nt}.$$

Then, multiplying both sides of this equation with $\partial_t u_n$ and integrating over $B_n \times (0,t)$ yields

$$\begin{aligned} &\int_{B_n} \Phi_n(u_n)(x,t)\,dx + \frac{\alpha}{2}\int_{B_n} |\nabla u_n|^2(x,t)\,dx + \beta\int_0^t\int_{B_n} u_{nt}^2\,dx\,ds \\ &= \int_0^t\int_{B_n} v_n\partial_t u_n\,dx\,ds + \int_{B_n} \Phi_n(u_{0n})(x)\,dx + \frac{\alpha}{2}\int_{B_n} |\nabla u_{0n}|^2\,dx, \quad \text{for } t \in (0,T), \end{aligned}$$

with $\Phi_n(u) = \int_0^u \varphi_n(s)ds$. Note that $\Delta v_n = u_{nt}$. Then, we obtain

$$\begin{aligned} &\int_\Omega \Phi_n(u_n)(x,t)\,dx + \frac{\alpha}{2}\int_\Omega |\nabla u_n|^2(x,t)\,dx + \beta\int_0^t\int_\Omega u_{nt}^2\,dx\,ds \\ &+ \int_0^t\int_\Omega |\nabla v_n|^2\,dx\,ds \\ &= \int_\Omega \Phi_n(u_{0n})(x)\,dx + \frac{\alpha}{2}\int_\Omega |\nabla u_{0n}|^2\,dx \end{aligned} \tag{2.2}$$

Since $\varphi_n(s)s \ge 0$ and assumption (H2), we have

$$0 \le \Phi_n(u) = \int_0^u \varphi_n(s)ds \le \frac{K}{2}u^2 + \frac{K(N-2)}{2N} u^{\frac{2N}{N-2}},$$

so there is a positive constant $C = C(K,N)$ such that

$$\int_{B_n} \Phi_n(u_{0n})dx \le C\Big(\int_{B_n} u_0^2 dx + \int_{B_n} u_0^{\frac{2N}{N-2}} dx\Big). \tag{2.3}$$

From Sobolev's embedding theorem, we obtain

$$\|u_{0n}\|_{L^{\frac{2N}{N-2}}(B_n)} \le C(N)\|\nabla u_{0n}\|_{L^2(B_n)}. \tag{2.4}$$

A combination of (2.4) and (2.3) implies that $\int_{B_n} \Phi(u_{0n})\,dx$ is bounded by a constant depending only on $\|u_0\|_{H^1(\mathbb{R}^N)}$. Therefore, there is a positive constant $C = C(N, \|u_0\|_{H^1(\mathbb{R}^N)})$ such that

$$\begin{aligned} &\int_{B_n} \Phi_n(u_n)(x,t)\,dx + \frac{\alpha}{2}\int_{B_n} |\nabla u_n|^2(x,t)\,dx \\ &+ \beta\int_0^t\!\!\int_{B_n} u_{nt}^2\,dx\,ds + \int_0^t\!\!\int_{B_n} |\nabla v_n|^2\,dx\,ds \le C. \end{aligned} \tag{2.5}$$

Next, using u_n as a test function to the first equation of (2.1) yields

$$\begin{aligned} &\frac{1}{2}\int_{B_n} u_n^2(x,t)\,dx + \frac{\beta}{2}\int_{B_n} |\nabla u_n|^2(x,t)\,dx + \alpha\int_0^t\!\!\int_{B_n} (\Delta u_n)^2\,dx\,ds \\ &+ \int_0^t \int_{B_n} \varphi_n'(u)|\nabla u_n|^2 dx\,ds \\ &\le \int_{B_n} u_{0n}^2(x,t)\,dx + \frac{\beta}{2}\int_{B_n} |\nabla u_{0n}|^2(x,t)\,dx \end{aligned}$$

Using (H3) yields

$$\begin{aligned} &\frac{1}{2}\int_{B_n} u_n^2(x,t)\,dx + \frac{\beta}{2}\int_{B_n} |\nabla u_n|^2(x,t)\,dx + \alpha\int_0^t\!\!\int_{B_n} (\Delta u_n)^2\,dx\,ds \\ &\le \int_{B_n\times(0,t)\cap\{|u_n|\le s_0\}} -\varphi_n'(u)|\nabla u_n|^2 dx\,ds + \int_{B_n} u_{0n}^2(x,t)\,dx \\ &\quad + \frac{\beta}{2}\int_{B_n} |\nabla u_{0n}|^2(x,t)\,dx \end{aligned}$$

By (H1), there is a positive constant C_0 such that $|\varphi_n'(s)| < C_0$, for any $|s| \le s_0$. Then,

$$\begin{aligned} &\frac{1}{2}\int_{B_n} u_n^2(x,t)\,dx + \frac{\beta}{2}\int_{B_n} |\nabla u_n|^2(x,t)\,dx + \alpha\int_0^t\!\!\int_{B_n} (\Delta u_n)^2\,dx\,ds \\ &\le C_0\int_{B_n\times(0,t)\cap\{|u_n|\le s_0\}} |\nabla u_n|^2 dx\,ds + \int_{B_n} u_{0n}^2(x,t)\,dx + \frac{\beta}{2}\int_{B_n} |\nabla u_{0n}|^2(x,t)\,dx \end{aligned}$$

By (2.5), $\int_{B_n} |\nabla u_n|^2 dx$ is bounded by a constant depending only on $\|u_0\|_{H^1(\mathbb{R}^N)}$. This fact and the last inequality imply that there is a positive constant, still denoted by $C = C(\|u_0\|_{H^1(\mathbb{R}^N)})$ such that

$$\frac{1}{2}\int_{B_n} u_n^2(x,t)\,dx + \frac{\beta}{2}\int_{B_n} |\nabla u_n|^2(x,t)\,dx + \alpha\int_0^t\!\!\int_{B_n} (\Delta u_n)^2\,dx\,ds \le C. \tag{2.6}$$

Next, we show that $\|\varphi_n(u_n)\|_{L^2(B_n\times(0,T))}$ is uniformly bounded for any $n \ge 1$. By (H2), it suffices to show that $u_n \in L^{\frac{2(N+2)}{N-2}}(B_n\times(0,T))$ is bounded by a constant not depending on n. Indeed, from Sobolev's embedding theorem we have for $N \ge 3$,

$$\|u_n(\cdot,t)\|_{L^{\frac{2N}{N-2}}(B_n)} \le C_1(N)\|\nabla u_n(\cdot,t)\|_{L^2(B_n)}.$$

From (2.6) or (2.5), there is a positive constant $C = C\left(\|u_0\|_{H^1(\mathbb{R}^N)}\right)$ such that

$$\|u_n(\cdot,t)\|_{L^{\frac{2N}{N-2}}(B_n)} \le C. \tag{2.7}$$

Thanks to Gagliardo-Nirenberg inequality, we obtain

$$\|u_n(\cdot,t)\|_{L^{\frac{2N}{N-4}}(B_n)} \le C_2(N)\|\nabla u_n(\cdot,t)\|_{L^{\frac{2N}{N-2}}(B_n)}\|u_n(\cdot,t)\|_{L^{\frac{2N}{N-2}}(B_n)}. \tag{2.8}$$

Combining (2.7) and (2.8) yields

$$\|u_n(\cdot,t)\|_{L^{\frac{2N}{N-4}}(B_n)} \le C_2'(N)\|\nabla u_n(\cdot,t)\|_{L^{\frac{2N}{N-2}}(B_n)}. \tag{2.9}$$

Using Sobolev's embedding theorem again yields

$$\|\nabla u_n(\cdot,t)\|_{L^{\frac{2N}{N-2}}(B_n)} \le C_3(N)\|D^2u_n(\cdot,t)\|_{L^2(B_n)}.$$

By the boundary condition, we can use the integration by parts formula to get $\|D^2u_n(\cdot,t)\|^2_{L^2(B_n)} = \|\Delta u_n(\cdot,t)\|^2_{L^2(B_n)}$. Thus,

$$\|\nabla u_n(\cdot,t)\|^2_{L^{\frac{2N}{N-2}}(B_n)} \le C_3(N)\|\Delta u_n(\cdot,t)\|^2_{L^2(B_n)}. \tag{2.10}$$

By (2.10) and (2.9), there exists a constant $C > 0$ not depending on n such that

$$\|u_n(\cdot,t)\|^2_{L^{\frac{2N}{N-4}}(B_n)} \le C\|\Delta u_n(\cdot,t)\|^2_{L^2(B_n)}. \tag{2.11}$$

Now, it follows from the interpolation theorem that

$$\|u_n(\cdot,t)\|_{L^{\frac{2(N+2)}{N-2}}(B_n)} \le \|u_n(\cdot,t)\|^{\theta}_{L^{\frac{2N}{N-4}}(B_n)}\|u_n(\cdot,t)\|^{1-\theta}_{L^{\frac{2N}{N-2}}(B_n)},$$

with $\theta = \frac{N-2}{N+2}$.

By (2.7), from the last inequality, we obtain

$$\|u_n(\cdot,t)\|^{\frac{2(N+2)}{N-2}}_{L^{\frac{2(N+2)}{N-2}}(B_n)} \le C\|u_n(\cdot,t)\|^2_{L^{\frac{2N}{N-4}}(B_n)} \tag{2.12}$$

A combination of (2.12), (2.11), and (2.6) yields

$$\int_0^T \|u_n(\cdot,t)\|^{\frac{2(N+2)}{N-2}}_{L^{\frac{2(N+2)}{N-2}}(B_n)}dt \le C\int_0^T \|\Delta u_n(\cdot,t)\|^2_{L^2(B_n)}dt \le C(T,N,u_0). \tag{2.13}$$

Therefore, we obtain the above claim.

It remains to pass to the limit as $n \to \infty$ in the equation satisfied by u_n. Thanks to the uniform estimates in (2.2), (2.5), (2.6), and (2.13), we can mimic the proof of [9, Theorem 2.4] to get

$$u_n \overset{*}{\rightharpoonup} u \quad \text{in } L^\infty((0,T);H^1(\mathbb{R}^N))\,, \tag{2.14}$$

$$u_{nt} \rightharpoonup u_t \quad \text{in } L^2(Q)\,, \tag{2.15}$$

$$\Delta u_n \rightharpoonup \Delta u \quad \text{in } L^2(Q), \tag{2.16}$$

$$u_n \to u \quad \text{a.e. in } Q, \tag{2.17}$$

up to a subsequence.

Next, we prove that $\varphi_n(u_n)$ converges weakly to $\varphi(u)$ in $L^2(Q)$. In fact, we observe that $\varphi_n(u_n) \to \varphi(u)$ as $n \to \infty$ a.e. in Q by (2.17). Moreover, the sequence $\{\varphi_n(u_n)\}_{n\geq 1}$ is uniformly bounded in $L^2(B_n \times (0,T))$ for any $n \geq 1$. Thus, there is a subsequence (still denoted by $\{\varphi_n(u_n)\}_{n\geq 1}$) such that $\varphi_n(u_n)$ converges weakly to $\varphi(u)$ in $L^2(Q)$, see [17, Theorem 13.44].

Now, it suffices to show that u is a weak solution of (1.1). We write the equation satisfied by u_n in the weak sense:

For any $\psi \in \mathcal{C}^1([0,T];\mathcal{C}_c^2(\mathbb{R}^N))$ such that $\psi(.,T) = 0$, we have

$$\begin{aligned} &\int_{Q(\psi)} -u_n\psi_t \,dx\,ds - \int_{\operatorname{supp}(\psi)} u_{0n}(x)\psi(x,0)\,dx \\ &= \int_{Q(\psi)} (\varphi(u_n)\Delta\psi - \alpha\Delta u_n\Delta\psi + \beta u_n\Delta\psi_t)\,dx\,ds, \end{aligned}$$

for any $n \geq 1$ such that $\operatorname{supp}(\psi) \subset B_n$, and $Q(\psi) = \operatorname{supp}(\psi) \times (0,T)$. Passing to the limit as $n \to \infty$ in the above equation yields

$$\int_{Q(\psi)} -u\psi_t \,dx\,ds - \int_{\operatorname{supp}(\psi)} u_0(x)\psi(x,0)dx \quad = \int_{Q(\psi)} (\varphi(u)\Delta\psi - \alpha\Delta u\Delta\psi + \beta u\Delta\psi_t)\,dx\,ds.$$

Or, u is a weak solution of problem (1.1). This completes the proof.

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