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GENERAL FORM OF THE EULER-POISSON-DARBOUX EQUATION AND APPLICATION OF THE TRANSMUTATION METHOD

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Abstract. In this article, we find solution representations in the compact integral form to the Cauchy problem for a general form of the Euler-Poisson-Darboux equation with Bessel operators via generalized translation and spherical mean operators for all values of the parameter k, including also not studying before exceptional odd negative values. We use a Hankel transform method to prove results in a unified way. Under additional conditions we prove that a distributional solution is a classical one too. A transmutation property for connected generalized spherical mean is proved and importance of applying transmutation methods for differential equations with Bessel operators is emphasized. The paper also contains a short historical introduction on differential equations with Bessel operators and a rather detailed reference list of monographs and papers on mathematical theory and applications of this class of differential equations.

1. Introduction

The classical Euler-Poisson-Darboux (EPD) equation is defined by

$$\frac{\partial^2 u}{\partial t^2} + \frac{k}{t}\frac{\partial u}{\partial t} = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}, \quad u = u(x,t;k), \quad x \in \mathbb{R}^n,\ t > 0,\ -\infty < k < \infty. \tag{1.1}$$

The operator acting by variable t is the Bessel operator and we will be denoted by (see, for example, [36, p. 3])

$$(B_k)_t = \frac{\partial^2}{\partial t^2} + \frac{k}{t}\frac{\partial}{\partial t}.$$

When $n = 1$ equation (1.1) appeared in Leonard Euler's work (see [19, p. 227]) and later was studied by Siméon Denis Poisson [48], by Gaston Darboux [15] and by Bernhard Riemann [52].

For the Cauchy problem corresponding to (1.1) we add initial conditions

$$u(x,0;k) = f(x), \quad \frac{\partial u(x,t;k)}{\partial t}\Big|_{t=0} = 0. \tag{1.2}$$

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Interest in the multidimensional equation (1.1) has increased significantly after Alexander Weinstein's papers [70]–[74]. There the Cauchy problem for (1.1) is considered with $k \in \mathbb{R}$, the first initial condition being non-zero and the second initial condition equal zero. A solution to the Cauchy problem (1.1)–(1.2) in the classical sense was obtained in [70]–[75], and in the distributional sense in [6, 7, 11]. Tersenov [67] solved the Cauchy problem for (1.1) in the general form where the first and the second conditions are non-zero. Different problems for the equation (1.1) with many applications to gas dynamics, hydrodynamics, mechanics, elasticity and plasticity and so on were also studied in [2, 3, 5], [8]–[10], [13]–[18], [20]–[24], [27]–[35], [37], [41]–[43], [46, 47, 50, 51], [62]–[68], [76]. Of course this list of references is incomplete.

In this article we consider the singular hyperbolic differential equation, with respect to all variables, which is a generalization of multidimensional Euler-Poisson-Darboux (EPD) equation (1.1):

$$\frac{\partial^2 u}{\partial t^2} + \frac{k}{t}\frac{\partial u}{\partial t} = (\Delta_\gamma)_x\, u, \quad u = u(x,t;k), \quad k \in \mathbb{R}, \quad t > 0, \tag{1.3}$$

with the singular elliptic operator defined by

$$(\Delta_\gamma)_x = \sum_{i=1}^{n} (B_{\gamma_i})_{x_i} = \sum_{i=1}^{n}\Big(\frac{\partial^2}{\partial x_i^2} + \frac{\gamma_i}{x_i}\frac{\partial}{\partial x}\Big) = \sum_{i=1}^{n}\frac{1}{x_i^{\gamma_i}}\frac{\partial}{\partial x_i}x_i^{\gamma_i}\frac{\partial}{\partial x_i} \tag{1.4}$$

under the natural restrictions

$$\gamma_i > 0, \quad x = (x_1, \dots, x_n), \quad x_i > 0, \quad i = 1, 2, \dots, n,$$

together with initial conditions (1.2).

We call equation (1.3) the *Euler-Poisson-Darboux equation in general form*.

Let us emphasize that singular differential equations with the operator (1.4) including equations (1.1) and (1.3) were thoroughly studied in many papers by Kipriyanov's school, the results are partially systemized in his monograph [36]. In accordance with Kipriyanov's terminology the operator (1.4) is classified as B-elliptic operator (sometimes also the term Laplace-Bessel operator is used), and equations (1.1) and (1.3) are classified as B-hyperbolic equations. In connection with results of this scientific school let us mention papers of Ivanov [31, 32, 37] in which important problems for EPD equation were solved, such as generalizations to homogeneous symmetric Riemann spaces, energy equipartition property, equations with a product of EPD-type multipliers. Also note papers [3, 45, 46] on application of spherical mean and generalized translation operators, generalized mean value theorems. Differential equations of EPD type are applied in the study of fractional powers of EPD, generalized EPD operators and connected generalized Riesz–type potentials, cf. [53]–[55].

Another important approach to differential equations with Bessel operators is based on application of the transmutation theory. This method is essential in the study of singular problems with use of special classes of transmutations such as Sonine, Poisson, Buschman–Erdélyi ones and different forms of fractional integrodifferential operators, cf. [8]–[12], [33]–[35], [38]–[40], [57]–[61]. Abstract differential equations with Bessel operators were studied, and in fact were mostly initiated, in the famous monograph [11]. See also recent papers [26]–[28].

Considering the Cauchy problem (1.3)–(1.2) in more details, David Fox in [21] (cf. also [11, p. 243] and [66]) proved solution uniqueness for $k \geq 0$ and find a solution representation in the explicit form for all k except odd negative values. The explicit solution was found via Lauricella functions in fact as n-times series, which is not convenient for applications and numerical solving. In all the above references the case $k \neq -1, -3, -5, \dots$ was expelled and not studied. So in [3], [45]–[46] different approaches from those used in [21] to the solution of this Cauchy problem were considered.

In this paper we find solution representations to the above Cauchy problem in the compact integral form via generalized translation and spherical mean operators for all values of the parameter k, including also not studying before exceptional odd negative values. We use a Hankel transform method to prove results in a unified way. Under additional conditions we prove that a distributional solution is a classical one too.

2. Definitions and propositions

We use the subset of the Euclidean space

$$\mathbb{R}^n_+ = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : x_1 > 0, \dots, x_n > 0\}.$$

Let $|x| = \big(\sum_{i=1}^n x_i^2\big)^{1/2}$ and Ω be bounded or unbounded open set in $\mathbb{R}^n$ symmetric with respect to each hyperplane $x_i = 0$, $i = 1, \dots, n$, $\Omega_+ = \Omega \cap \mathbb{R}^n_+$ and $\overline{\Omega}_+ = \Omega \cap \overline{\mathbb{R}}^n_+$ where

$$\overline{\mathbb{R}}^n_+ = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : x_1 \geq 0, \dots, x_n \geq 0\}.$$

We consider the class $C^m(\Omega_+)$ consisting of m-times differentiable on Ω_+ functions and denote by $C^m(\overline{\Omega}_+)$ the subset of functions from $C^m(\Omega_+)$ such that all derivatives of these functions with respect to x_i for any $i = 1, \dots, n$ are continuous up to $x_i = 0$. Function $f \in C^m(\overline{\Omega}_+)$ we will call *even with respect to* x_i, $i = 1, \dots, n$ if $\frac{\partial^{2k+1} f}{\partial x_i^{2k+1}}\Big|_{x=0} = 0$ for all nonnegative integer $k \leq \frac{m-1}{2}$ (see [36, p. 21]). The class $C^m_{ev}(\overline{\Omega}_+)$ consists of functions from $C^m(\overline{\Omega}_+)$ even with respect to each variable x_i, $i = 1, \dots, n$. In the following we will denote $C^m_{ev}(\overline{\mathbb{R}}^n_+)$ by C^m_{ev}. We set

$$C^\infty_{ev}(\overline{\Omega}_+) = \cap C^m_{ev}(\overline{\Omega}_+)$$

with intersection taken for all finite m and $C^\infty_{ev}(\overline{\mathbb{R}}_+) = C^\infty_{ev}$. Let $\overset{\circ}{C}{}^\infty_{ev}(\overline{\Omega}_+)$ be the space of all functions $f \in C^\infty_{ev}(\overline{\Omega}_+)$ with a compact support. Elements of $\overset{\circ}{C}{}^\infty_{ev}(\overline{\Omega}_+)$ we will call *test functions* and use the notation $\overset{\circ}{C}{}^\infty_{ev}(\overline{\Omega}_+) = \mathcal{D}_+(\overline{\Omega}_+)$.

As the space of basic functions we will use the subspace of the space of rapidly decreasing functions:

$$S_{ev}(\mathbb{R}^n_+) = \Big\{f \in C^\infty_{ev} : \sup_{x \in \mathbb{R}^n_+} |x^\alpha D^\beta f(x)| < \infty, \forall \alpha, \beta \in \mathbb{Z}^n_+\Big\},$$

where $\alpha = (\alpha_1, \dots, \alpha_n)$, $\beta = (\beta_1, \dots, \beta_n)$, $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n$ are integer nonnegative numbers, $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n}$, $D^\beta = D^{\beta_1}_{x_1} \dots D^{\beta_n}_{x_n}$, $D_{x_j} = \frac{\partial}{\partial x_j}$.

We deal with multi-index $\gamma = (\gamma_1, \dots, \gamma_n)$ consists of positive fixed reals $\gamma_i > 0$, $i = 1, \dots, n$, $|\gamma| = \gamma_1 + \dots + \gamma_n$. Let $L^\gamma_p(\Omega_+)$, $1 \leq p < \infty$, be the space of all measurable in Ω_+ functions even with respect to each variable x_i, $i = 1, \dots, n$ such that

$$\int_{\Omega_+} |f(x)|^p x^\gamma dx < \infty,$$

where

$$x^\gamma=\prod_{i=1}^n x_i^{\gamma_i}.$$

For a real number $p\geq 1$, the $L_p{}^\gamma(\Omega_+)$-norm of f is defined by

$$\|f\|_{L_p{}^\gamma(\Omega_+)}=\Big(\int_{\Omega_+}|f(x)|^p x^\gamma dx\Big)^{1/p}.$$

The weighted measure of Ω_+ is denoted by $\operatorname{meas}_\gamma(\Omega)$ and is defined by

$$\operatorname{meas}_\gamma(\Omega_+)=\int_{\Omega_+}x^\gamma dx.$$

For every measurable function $f(x)$ defined on $\mathbb{R}^n_+$ we consider

$$\mu_\gamma(f,t)=\operatorname{meas}_\gamma\{x\in\mathbb{R}^n_+:|f(x)|>t\}=\int_{\{x:\,|f(x)|>t\}^+}x^\gamma dx$$

where $\{x:|f(x)|>t\}^+=\{x{\in}\mathbb{R}^n_+:|f(x)|>t\}$. We will call the function $\mu_\gamma=\mu_\gamma(f,t)$ a *weighted distribution function* $|f(x)|$.

The space $L_\infty{}^\gamma(\Omega_+)$ is defined as a set of measurable on Ω_+ and even with respect to each variable functions $f(x)$ such as

$$\|f\|_{L_\infty{}^\gamma(\Omega_+)}=\operatorname{ess\,sup}_{x\in\Omega_+,\gamma}|f(x)|=\inf_{a\in\Omega_+}\{\mu_\gamma(f,a)=0\}<\infty.$$

For $1\leq p\leq\infty$ the $L_{p,\mathrm{loc}}{}^\gamma(\Omega_+)$ is the set of functions $u(x)$ defined almost everywhere in Ω_+ such that $uf\in L_p{}^\gamma(\Omega_+)$ for any $f\in\overset{\circ}{C}{}^\infty_{ev}(\overline{\Omega}_+)$. Each function $u(x)\in L_{1,loc}{}^\gamma(\Omega_+)$ will be identified with the functional $u\in\mathcal{D}'_+(\overline{\Omega}_+)$ acting according to the formula

$$(u,f)_\gamma=\int_{\mathbb{R}^n_+}u(x)\,f(x)\,x^\gamma\,dx,\quad f\in\overset{\circ}{C}{}^\infty_{ev}(\overline{\mathbb{R}}^n_+). \tag{2.1}$$

Functionals $u\in\mathcal{D}'_+(\overline{\Omega}_+)$ acting by the formula (2.1) will be called *regular weighted functionals*. All other functionals $u\in\mathcal{D}'_+(\overline{\Omega}_+)$ will be called *singular weighted functionals*.

We will use the regular weighted functional $(t^2-|x|^2)^\lambda_{+,\gamma}$ defined by

$$((t^2-|x|^2)^\lambda_{+,\gamma},\varphi)_\gamma=\int_{\{x\in\mathbb{R}^n_+:|x|<t\}}(t^2-|x|^2)^\lambda\varphi(x)x^\gamma dx,\quad \varphi\in S_{ev},\quad \lambda{\in}\mathbb{C}. \tag{2.2}$$

The symbol j_ν is used for the normalized Bessel function:

$$j_\nu(t)=\frac{2^\nu\Gamma(\nu+1)}{t^\nu}J_\nu(t),$$

where $J_\nu(t)$ is the Bessel function of the first kind of order ν (see [69]). The function $j_\nu(t)$ is even by t.

We use the multidimensional Hankel (Fourier-Bessel) transform. The multidimensional Hankel transform of a function $f(x)$ is given by (see [4]):

$$F_B[f](\xi)=(F_B)_x[f(x)](\xi)=\widehat{f}(\xi)=\int_{\mathbb{R}^n_+}f(x)\,\mathbf{j}_\gamma(x;\xi)x^\gamma dx,$$

where

$$\mathbf{j}_\gamma(x;\xi)=\prod_{i=1}^n j_{\frac{\gamma_i-1}{2}}(x_i\xi_i),\quad \gamma_1>0,\dots,\gamma_n>0.$$

For $f\in S_{ev}$ inverse multidimensional Hankel transform is defined by

$$F_B^{-1}[\widehat{f}(\xi)](x)=f(x)=\frac{2^{n-|\gamma|}}{\prod_{j=1}^n\Gamma^2\left(\frac{\gamma_j+1}{2}\right)}\int_{\mathbb{R}^n_+}\mathbf{j}_\gamma(x,\xi)\widehat{f}(\xi)\xi^\gamma\,d\xi.$$

We will deal with the singular Bessel differential operator B_ν (see, for example, [36, p. 5]):

$$(B_\nu)_t=\frac{\partial^2}{\partial t^2}+\frac{\nu}{t}\frac{\partial}{\partial t}=\frac{1}{t^\nu}\frac{\partial}{\partial t}t^\nu\frac{\partial}{\partial t},\quad t>0.$$

and the elliptical singular operator or the Laplace-Bessel operator Δ_γ:

$$\Delta_\gamma=(\Delta_\gamma)_x=\sum_{i=1}^n(B_{\gamma_i})_{x_i}=\sum_{i=1}^n\Big(\frac{\partial^2}{\partial x_i^2}+\frac{\gamma_i}{x_i}\frac{\partial}{\partial x}\Big)=\sum_{i=1}^n\frac{1}{x_i^{\gamma_i}}\frac{\partial}{\partial x_i}x_i^{\gamma_i}\frac{\partial}{\partial x_i}. \tag{2.3}$$

The operator (2.3) belongs to the class of B-elliptic operators by Kipriyanovs' classification (see [36]).

The *B–polyharmonic of order* p function $f=f(x)$ is the function $f\in C^{2p}_{ev}(\overline{\mathbb{R}}{}^+_n)$ such that

$$\Delta_\gamma^p f=0, \tag{2.4}$$

where Δ_γ is operator (2.3). The operator (2.4) was considered in [36]. The B-polyharmonic of order 1 function we will call B-harmonic.

Using [1, formulas 9.1.27] we obtain

$$(B_\nu)_t j_{\frac{\nu-1}{2}}(\tau t)=-\tau^2 j_{\frac{\nu-1}{2}}(\tau t). \tag{2.5}$$

We will use the generalized convolution operator defined by

$$(f*g)_\gamma=\int_{\mathbb{R}^n_+}f(y)({}^\gamma T^y g)(x)y^\gamma dy,$$

where ${}^\gamma T^y$ is multidimensional generalized translation

$${}^\gamma T^y={}^{\gamma_1}T^{y_1}_{x_1}\dots{}^{\gamma_n}T^{y_n}_{x_n},$$

each one-dimensional operator ${}^{\gamma_i}T^{y_i}_{x_i}$, $i=1,\dots,n$ acts according to (see [42])

$$\begin{aligned}&{}^{\gamma_i}T^{y_i}_{x_i}f(x)\\&=\frac{\Gamma(\frac{\gamma_i+1}{2})}{\Gamma(\frac{\gamma_i}{2})\Gamma(\frac{1}{2})}\int_0^\pi f(x_1,\dots,x_{i-1},\sqrt{x_i^2+y_i^2-2x_iy_i\cos\alpha_i},x_{i+1},\dots,x_n)\\&\quad\times\sin^{\gamma_i-1}\alpha_i\,d\alpha_i\,.\end{aligned}$$

Based on the multidimensional generalized translation ${}^\gamma T^y$ the weighted spherical mean $M_t^\gamma[f(x)]$ of a suitable function is defined by the formula

$$M_t^\gamma[f(x)]=\frac{1}{|S_1^+(n)|_\gamma}\int_{S_1^+(n)}{}^\gamma T_x^{t\theta}f(x)\theta^\gamma dS, \tag{2.6}$$

where

$$\theta^\gamma=\prod_{i=1}^n\theta_i^{\gamma_i},\quad S_1^+(n)=\{\theta:|\theta|=1,\theta\in\mathbb{R}^n_+\},\quad |S_1^+(n)|_\gamma=\frac{\prod_{i=1}^n\Gamma\left(\frac{\gamma_i+1}{2}\right)}{2^{n-1}\Gamma\left(\frac{n+|\gamma|}{2}\right)}.$$

It is easy to see that

$$M_0^\gamma[f(x)]=f(x),\quad \frac{\partial}{\partial t}M_t^\gamma[f(x)]\Big|_{t=0}=0. \tag{2.7}$$

Lemma 2.1. *Let $u\in S_{ev}$ then*
$$F_B[\Delta_\gamma f](\xi)=-|\xi|^2F_B[f](\xi). \tag{2.8}$$

Proof. We have
$$\begin{aligned}F_B[\Delta_\gamma f](\xi)&=\int_{\mathbb{R}^n_+}[\Delta_\gamma f(x)]\,\mathbf{j}_\gamma(x;\xi)x^\gamma dx\\&=\sum_{i=1}^n\int_{\mathbb{R}^n_+}\Big[\frac{1}{x_i^{\gamma_i}}\frac{\partial}{\partial x_i}x_i^{\gamma_i}\frac{\partial}{\partial x_i}f(x)\Big]\mathbf{j}_\gamma(x;\xi)x^\gamma dx.\end{aligned}$$
Integrating by parts by variable x_i and using formula (2.5), we obtain
$$\begin{aligned}F_B[\Delta_\gamma f](\xi)&=\sum_{i=1}^n\int_{\mathbb{R}^n_+}f(x)\Big[\frac{1}{x_i^{\gamma_i}}\frac{\partial}{\partial x_i}x_i^{\gamma_i}\frac{\partial}{\partial x_i}\mathbf{j}_\gamma(x;\xi)\Big]x^\gamma dx\\&=\sum_{i=1}^n(-\xi_i^2)\int_{\mathbb{R}^n_+}f(x)\mathbf{j}_\gamma(x;\xi)x^\gamma dx\\&=-|\xi|^2\int_{\mathbb{R}^n_+}f(x)\,\mathbf{j}_\gamma(x;\xi)x^\gamma dx=-|\xi|^2F_B[f](\xi).\end{aligned}$$
□

Lemma 2.2. *We have the formula*
$$\frac{(F_B)_x(t^2-|x|^2)_{+,\gamma}^{\frac{k-n-|\gamma|-1}{2}}}{\Gamma\big(\frac{k-n-|\gamma|+1}{2}\big)}=\frac{t^{k-1}\prod_{i=1}^n\Gamma\left(\frac{\gamma_i+1}{2}\right)}{2^n\Gamma\left(\frac{k+1}{2}\right)}j_{\frac{k-1}{2}}(t|x|), \tag{2.9}$$
where $(t^2-|x|^2)_{+,\gamma}^{\frac{k-n-|\gamma|-1}{2}}$ is defined by (2.2).

Formula (2.9) is obtained similarly as [25, (5) p. 291].

Lemma 2.3. *Let $u=u(x,t;k)$ denote the solution to* (1.3)*. Then the solution satisfies the next two important recursion formulas:*
$$u(x,t;k)=t^{1-k}u(x,t;2-k), \tag{2.10}$$
$$u_t(x,t;k)=tu(x,t;2+k). \tag{2.11}$$

This is particular cases of Weinstein's formulas which state that for any equation of the form $u_{tt}+\frac{k}{t}u_t=X(u)$, in which X is an operator does not depend on t the relations (2.10) and (2.11) hold (see [21]).

Lemma 2.4. *The weighted spherical mean $M_t^\gamma[f(x)]$ is the transmutation operator (see the definition of transmutation operators in* [8, 9, 10] *or* [58, 59]*) intertwining $(\Delta_\gamma)_x$ and $(B_{n+|\gamma|-1})_t$ for the $f\in C^2_{ev}$:*
$$(B_{n+|\gamma|-1})_tM_t^\gamma[f(x)]=M_t^\gamma[(\Delta_\gamma)_xf(x)]. \tag{2.12}$$

Proof. First of all we note that the function $f\in C^2_{ev}$ satisfies the equation
$$\int_{B_t^+(n)}f(x)\,x^\gamma\,dx=\int_0^t\lambda^{n+|\gamma|-1}\,d\lambda\int_{S_1^+(n)}f(\lambda\theta)\theta^\gamma dS_\theta, \tag{2.13}$$

which can be easy obtained by passing to spherical coordinates $x=\lambda\theta$, $|\theta|=1$ in the left hand of (2.13). From (2.13) we obtain

$$\begin{aligned} &|S_1^+(n)|_\gamma \int_0^t \lambda^{n+|\gamma|-1} M_\lambda^\gamma[f(x)]d\lambda \\ &= \int_0^t \lambda^{n+|\gamma|-1} d\lambda \int_{S_1^+(n)} (T^{\lambda y} f)(x)\ y^\gamma dS_y \\ &= \int_{B_r^+(n)} (T^z f)(x) z^\gamma dz. \end{aligned} \tag{2.14}$$

Let us apply the operator Δ_γ to both sides of (2.14) with respect to x, then we obtain

$$\begin{aligned} &|S_1^+(n)|_\gamma \sum_{i=1}^n \int_0^t \lambda^{n+|\gamma|-1} B_{\gamma_i} M_\lambda^\gamma[f(x)]d\lambda \\ &= \sum_{i=1}^n \int_{B_t^+(n)} (B_{\gamma_i})_{x_i} (T^z f)(x) z^\gamma dz \\ &= \sum_{i=1}^n \int_{B_t^+(n)} (B_{\gamma_i})_{z_i} T_x^z f(x) z^\gamma dz. \end{aligned} \tag{2.15}$$

We have the Green formula

$$\int_{\overline{\Omega}^+} (v\Delta_\gamma w - w\Delta_\gamma v) x^\gamma dx = \int_{\Gamma=\partial\overline{\Omega}^+} \Big(v\frac{\partial w}{\partial \vec{\nu}} - w\frac{\partial v}{\partial \vec{\nu}}\Big) x^\gamma\, d\Gamma_x\,, \tag{2.16}$$

where $w, v \in C^2_{ev}(\overline{\Omega}^+)$, $\vec{\nu}$ is the outward normal to the boundary $\Gamma = \partial\overline{\Omega}^+$ of the $\overline{\Omega}^+$. This formula was presented in [45].

By applying formula (2.16) to the right-hand side of (2.15), we obtain

$$\sum_{i=1}^n \int_{B_t^+(n)} (B_{\gamma_i})_{z_i} (T^z f)(x) z^\gamma dz = \sum_{i=1}^n \int_{S_t^+(n)} \frac{\partial}{\partial z_i} (T^z f)(x) \cos(\vec{\nu}, \vec{e}_i) z^\gamma\, dS_z,$$

where $\vec{e}_i$ is the direction of the axis Oz_i, $i=1,\dots,n$.

Now, by using the fact that the direction of the outward normal to the boundary of a ball with center the origin coincides with the direction of the position vector of the point on the ball, we obtain the relation

$$\begin{aligned} \sum_{i=1}^n \int_{B_t^+(n)} (B_{\gamma_i})_{z_i} (T^z f)(x) z^\gamma dz &= t^{n+|\gamma|-1} \int_{S_1^+(n)} \frac{\partial}{\partial t} (T^{t\theta} f)(x) \theta^\gamma dS_\theta \\ &= |S_1^+(n)|_\gamma t^{n+|\gamma|-1} \frac{\partial}{\partial t} M_t^\gamma[f(x)]. \end{aligned}$$

Returning to (2.15), we obtain

$$\sum_{i=1}^n \int_0^t \lambda^{n+|\gamma|-1} B_{\gamma_i} M_\lambda^\gamma[f(x)]d\lambda = t^{n+|\gamma|-1} \frac{\partial}{\partial t} M_t^\gamma[f(x)]. \tag{2.17}$$

By differentiating relation (2.17) with respect to t, we obtain

$$\sum_{i=1}^n t^{n+|\gamma|-1} B_{\gamma_i} M_t^\gamma[f(x)] = (n+|\gamma|-1) t^{n+|\gamma|-2} \frac{\partial}{\partial t} M_t^\gamma[f(x)] + t^{n+|\gamma|-1} \frac{\partial^2}{\partial t^2} M_t^\gamma[f(x)]$$

or

$$\sum_{i=1}^{n} B_{\gamma_i} M_t^\gamma[f(x)] = \frac{n+|\gamma|-1}{t}\frac{\partial}{\partial t}M_t^\gamma[f(x)] + \frac{\partial^2}{\partial t^2}M_t^\gamma[f(x)],$$

and so

$$(\Delta_\gamma)_x M_t^\gamma[f(x)] = \frac{n+|\gamma|-1}{t}\frac{\partial}{\partial t}M_t^\gamma[f(x)] + \frac{\partial^2}{\partial t^2}M_t^\gamma[f(x)]. \tag{2.18}$$

Now let us consider $(\Delta_\gamma)_x M_t^\gamma[f(x)]$. Using the commutativity of B_{γ_i} and $T_{x_i}^{t\theta_i}$ (see [36]) we obtain

$$\begin{aligned}(\Delta_\gamma)_x M_t^\gamma[f(x)] &= \frac{1}{|S_1^+(n)|_\gamma}(\Delta_\gamma)_x \int_{S_1^+(n)} {}^\gamma T_x^{t\theta} f(x)\theta^\gamma dS_\theta \\ &= \frac{1}{|S_1^+(n)|_\gamma}\int_{S_1^+(n)} {}^\gamma T_x^{t\theta}[(\Delta_\gamma)_x f(x)]\theta^\gamma dS_\theta = M_t^\gamma[(\Delta_\gamma)_x f(x)].\end{aligned}$$

which with (2.18) gives (2.12). □

A similarly proof can be found in [45].

3. Transmutation method

An important and powerful approach to differential equations with Bessel operators is based on application of the transmutation theory. This method is essential in the study of singular problems with use of special classes of transmutations such as Sonine, Poisson, Buschman-Erdélyi ones and different forms of fractional integrodifferential operators, cf. [8]–[12], [33]–[35], [38]–[40], [57]–[61].

In this section we show how the transmutation method can be used to find the solution of the Cauchy problem for the general Euler-Poisson-Darboux equation

$$(B_k)_t u = (\Delta_\gamma)_x u, \quad u = u(x,t;k), \quad x \in \mathbb{R}_+^n,\ t>0,\ k \in \mathbb{R}, \tag{3.1}$$

$$u(x,0;k) = f(x), \quad u_t(x,0;k) = 0. \tag{3.2}$$

Theorem 3.1. *Let $f = f(x)$, $x \in \mathbb{R}_+^n$ be twice continuous differentiable function even with respect of each variable. Then for the case $k > n+|\gamma|-1$ the function*

$$u(x,t;k) = \frac{2^n\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{k-n-|\gamma|+1}{2}\right)\prod_{i=1}^n \Gamma\left(\frac{\gamma_i+1}{2}\right)} \int_{B_1^+(n)} [{}^\gamma T^{ty} f(x)](1-|y|^2)^{\frac{k-n-|\gamma|-1}{2}} y^\gamma \, dy \tag{3.3}$$

is the solution to problem (3.1)–(3.2)*. The solution to* (3.1)–(3.2) *for the case $k = n+|\gamma|-1$ is the weighted spherical mean $M_t^\gamma[f(x)]$.*

Proof. Using Lemma 2.4 we obtain that the weighted spherical mean of any twice continuously differentiable function $f = f(x)$ even with respect to each of the independent variables $x_1, \dots, x_n$ on $\mathbb{R}_+^n$ satisfies the general Euler-Poisson-Darboux equation

$$(B_k)_t M_t^\gamma[f(x)] = (\Delta_\gamma)_x M_t^\gamma[f(x)], \quad k = n+|\gamma|-1$$

and initial conditions (see (2.7))

$$M_0^\gamma[f(x)] = f(x), \quad M_t^\gamma[f(x)]\Big|_{t=0} = 0.$$

It means the the weighted spherical mean $M_t^\gamma[f(x)]$ is the solution to the problem (3.1)-(3.2) for the $k = n+|\gamma|-1$.

To obtain the solution to (3.1)–(3.2) for $k > n + |\gamma| - 1$ we will use the method of descent. First, we will seek solution to the Cauchy problem (3.1)–(3.2) for the case $k > n + |\gamma|$.

Let $\gamma' = (\gamma_1, \dots, \gamma_n, \gamma'_{n+1})$, $\gamma'_{n+1} > 0$, $x' = (x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1}_+$ and

$$(\Delta_{\gamma'})_{x'} = (B_{\gamma_1})_{x_1} + \dots + (B_{\gamma_n})_{x_n} + (B_{\gamma'_{n+1}})_{x_{n+1}}.$$

Consider the equation of type (3.1),

$$(B_k)_t u = (\Delta_{\gamma'})_{x'} u, \quad u = u(x', t; k), \quad x' \in \mathbb{R}^{n+1}_+,\ t > 0$$

with the initial conditions

$$u(x', 0; k) = f_1(x'), \quad u_t(x', 0; k) = 0.$$

When $k = n + |\gamma'| = n + |\gamma| + \gamma'_{n+1}$ the weighted spherical mean $M_t^\gamma[f_1(x')]$ is a solution to this Cauchy problem:

$$\begin{gathered} u(x', t; k) \\ = \frac{1}{|S_1^+(n+1)|_{\gamma'}} \int_{S_1^+(n+1)} [{}^{\gamma_1}T^{ty_1} \dots {}^{\gamma_n}T^{ty_n}\, {}^{\gamma'_{n+1}}T^{ty_{n+1}} f_1(x)](y')^{\gamma'} dS_{y'}, \qquad (3.4) \\ y' = (y_1, \dots, y_n, y'_{n+1}) \in \mathbb{R}^{n+1}_+, \\ |S_1^+(n+1)|_{\gamma'} = \frac{\prod_{i=1}^n \Gamma\big(\frac{\gamma_i+1}{2}\big)\Gamma\big(\frac{\gamma'_{n+1}+1}{2}\big)}{2^n \Gamma\big(\frac{n+1+|\gamma|+\gamma'_{n+1}}{2}\big)} = \frac{\prod_{i=1}^n \Gamma\big(\frac{\gamma_i+1}{2}\big)\Gamma\big(\frac{k-n-|\gamma|+1}{2}\big)}{2^n \Gamma\left(\frac{k+1}{2}\right)}. \end{gathered}$$

Let us put $f_1(x_1, \dots, x_n, 0) = f(x_1, \dots, x_n)$, where f is the function which appears in initial conditions (3.2). In this way the u defined by (3.4) becomes a function only of $x_1, \dots, x_n$ which satisfies equation (3.1) and initial conditions (3.2). We have

$$u(x, t; k) = \frac{1}{|S_1^+(n+1)|_{\gamma'}} \int_{S_1^+(n+1)} [{}^\gamma T^{ty} f(x)](y')^{\gamma'} dS_{y'}, \quad \gamma'_{n+1} = k - n - |\gamma|.$$

Now we rewrite the integral over the part of the sphere $S_1^+(n+1)$ as an integral over the part of ball $B_1^+(n) = \{y \in \mathbb{R}^n_+ : \sum_{i=1}^n y_i^2 \leq 1\}$. We write the surface integral over multiple integral:

$$\begin{aligned} \int_{S_1^+(n+1)} [{}^\gamma T^{ty} f(x)](y')^{\gamma'} dS_{y'} &= \int_{B_1^+(n)} [{}^\gamma T^{ty} f(x)](1 - y_1^2 - \dots - y_n^2)^{\frac{\gamma'_{n+1}-1}{2}} y^\gamma dy \\ &= \int_{B_1^+(n)} [{}^\gamma T^{ty} f(x)](1 - |y|^2)^{\frac{k-n-|\gamma|-1}{2}} y^\gamma dy, \end{aligned}$$

where $B_1^+(n)$ is a projection of the $S_1^+(n+1)$ on the equatorial plane $x_{n+1} = 0$. We have

$$u(x, t; k) = \frac{2^n \Gamma\left(\frac{k+1}{2}\right)}{\prod_{i=1}^n \Gamma\left(\frac{\gamma_i+1}{2}\right)\Gamma\big(\frac{k-n-|\gamma|+1}{2}\big)} \int_{B_1^+(n)} [{}^\gamma T^{ty} f(x)](1-|y|^2)^{\frac{k-n-|\gamma|-1}{2}} y^\gamma dy.$$

Although (3.3) was obtained as the solution to (3.1)–(3.2) for the case $k > n+|\gamma|$ the integral on its right-hand side converges and for $k > n+|\gamma|-1$. We can verify by direct substitution (3.3) in (3.1)–(3.2) that (3.3) satisfies to the differential equation (3.1) and to the initial conditions (3.2) for all values of k which are greater than

$n+|\gamma|-1$. Let show it. Changing coordinates from y to y/t and using that $(B_{\gamma_i})^{\gamma_i}_{x_i} T^{y_i}_{x_i} = (B_{\gamma_i})^{\gamma_i}_{y_i} T^{y_i}_{x_i}$ (see [42]) we obtain

$$\begin{aligned} I &= (\Delta_\gamma)_x \int_{B^+_1(n)} [{}^\gamma T^{ty} f(x)](1-|y|^2)^{\frac{k-n-|\gamma|-1}{2}} y^\gamma dy \\ &= \sum_{i=1}^n (B_{\gamma_i})_{x_i} \int_{B^+_1(n)} [{}^\gamma T^{ty} f(x)](1-|y|^2)^{\frac{k-n-|\gamma|-1}{2}} y^\gamma dy \\ &= t^{1-k} \sum_{i=1}^n \int_{B^+_t(n)} [(B_{\gamma_i})^\gamma_{x_i} T^y f(x)](t^2-|y|^2)^{\frac{k-n-|\gamma|-1}{2}} y^\gamma dy \\ &= t^{1-k} \sum_{i=1}^n \int_{B^+_t(n)} [(B_{\gamma_i})^\gamma_{y_i} T^y f(x)](t^2-|y|^2)^{\frac{k-n-|\gamma|-1}{2}} y^\gamma dy, \end{aligned} \tag{3.5}$$

where $B^+_t(n) = \{y \in \mathbb{R}^n_+ : \sum_{i=1}^n y_i^2 \leq t\}$.

For integration over $\overline{\Omega}^+$ the functions $w, v \in C^2_{ev}(\overline{\Omega}^+)$ we have the Green formula (2.16). By applying (2.16) to the right-hand side of (3.5), we obtain

$$I = t^{1-k} \sum_{i=1}^n \int_{S^+_t(n)} \big[\frac{\partial}{\partial y_i}^\gamma T^y f(x)\big](t^2-|y|^2)^{\frac{k-n-|\gamma|-1}{2}} \cos(\vec{\nu}, \vec{e}_i)\, y^\gamma\, dS,$$

where $\vec{e}_i$ is the direction of the axis Oy_i, $i=1,\dots,n$, and thus $\cos(\vec{\nu}, \vec{e}_i) = \frac{y_i}{t}$. Now, by using the fact that the direction of the outward normal to the boundary of a ball with center the origin coincides with the direction of the position vector of the point on the ball, we obtain the relation

$$I = \frac{1}{t^k} \frac{\partial}{\partial t} t^k \frac{\partial}{\partial t} \int_{B^+_1(n)} \big[{}^\gamma T^{ty} f(x)\big]\, (1-|y|^2)^{\frac{k-n-|\gamma|-1}{2}}\, y^\gamma\, dy.$$

Given (3.5) and that $\frac{1}{t^k}\frac{\partial}{\partial t} t^k \frac{\partial}{\partial t} = (B_k)_t$, we have

$$\begin{aligned} &(\Delta_\gamma)_x \int_{B^+_1(n)} [{}^\gamma T^{ty} f(x)](1-|y|^2)^{\frac{k-n-|\gamma|-1}{2}} y^\gamma dy \\ &= (B_k)_t \int_{B^+_1(n)} \big[{}^\gamma T^{ty} f(x)\big]\, (1-|y|^2)^{\frac{k-n-|\gamma|-1}{2}}\, y^\gamma\, dy. \end{aligned}$$

It means that $u(x,t;k)$ defined by the formula (3.3) indeed satisfies to equation (3.1) for $k > n+|\gamma|-1$. Validity of the first and the second initial conditions follows from [42, (5.20) and (5.21)] respectively. □

Theorem 3.2. *Let $f = f(x)$, $f \in C_{ev}^{[\frac{n+|\gamma|-k}{2}]+2}$. Then for $k < n+|\gamma|-1$, $k \neq -1, -3, -5, \dots$ the function*

$$u(x,t;k) = t^{1-k} \Big(\frac{\partial}{t\partial t}\Big)^m (t^{k+2m-1} u(x,t;k+2m)), \tag{3.6}$$

is the solution to (3.1)–(3.2), where m is a minimum integer such that $m \geq \frac{n+|\gamma|-k-1}{2}$ and $u(x,t;k+2m)$ is the solution to the Cauchy problem

$$(B_{k+2m})_t u = (\Delta_\gamma)_x u, \tag{3.7}$$

$$u(x,0;k+2m) = \frac{f(x)}{(k+1)(k+3)\dots(k+2m-1)}, \quad u_t(x,0;k+2m) = 0. \tag{3.8}$$

Proof. To proof that (3.6) is a solution of (3.1)–(3.2) when $k<n+|\gamma|-1$, $k\neq -1$, -3, -5, …, we use the recursion formulas (2.10) and (2.11). Let choose minimum integer m such that $k+2m\geq n+|\gamma|-1$. Now we can write the solution of the Cauchy problem

$$(B_{k+2m})_t u=(\Delta_\gamma)_x u,$$
$$u(x,0;k+2m)=g(x),\quad u_t(x,0;k+2m)=0,\quad g\in C^2_{ev}$$

by (3.3). We have

$$\begin{aligned} &u(x,t;k+2m)\\ &=\frac{2^n\Gamma\left(\frac{k+2m+1}{2}\right)}{\prod_{i=1}^n\Gamma\left(\frac{\gamma_i+1}{2}\right)\Gamma\left(\frac{k+2m-n-|\gamma|+1}{2}\right)}\int_{B_1^+(n)}[{}^\gamma T^{ty}g(x)](1-|y|^2)^{\frac{k+2m-n-|\gamma|-1}{2}}y^\gamma dy, \end{aligned}$$

and, using (2.10) we obtain

$$t^{k+2m-1}u(x,t;k+2m)=u(x,t;2-k-2m).$$

Applying (2.11) to the last formula m times we obtain

$$\left(\frac{\partial}{t\partial t}\right)^m(t^{k+2m-1}u(x,t;k+2m)=u(x,t;2-k).$$

Applying again (2.10) we can write

$$u(x,t;k)=t^{1-k}\left(\frac{\partial}{t\partial t}\right)^m(t^{k+2m-1}u(x,t;k+2m)), \tag{3.9}$$

which gives the solution to (3.7). Now we obtain the function g such that the (3.8) is true. From (3.6) we have asymptotic relation

$$u(x,t;k)=(k+1)(k+3)\dots(k+2m-1)u(x,t;k+2m)+Ctu(x,t;k+2m)+O(t^2),$$

as $t\to 0$, where C is a constant. Therefore, if

$$g(x)=\frac{f(x)}{(k+1)(k+3)\dots(k+2m-1)}$$

then $u(x,t;k)$ defined by (3.6) satisfies to the initial conditions (3.2).

Let us recall that for $u(x,t;k+2m)$ to be a solution to (3.7)–(3.8) it is sufficient that $f\in C^2_{ev}$. To be able to carry out the construction (3.6), it is sufficient to require that $f\in C_{ev}^{[\frac{n+|\gamma|-k}{2}]+2}$. □

Theorem 3.3. *If f is B-polyharmonic of order $\frac{1-k}{2}$ and even with respect to each variable then one of the solutions to the Cauchy problem* (3.7)–(3.8) *for the $k=-1,-3,-5,\dots$ is given by*

$$u(x,t;k)=f(x),\quad k=-1, \tag{3.10}$$

$$u(x,t;k)=f(x)+\sum_{h=1}^{-\frac{k+1}{2}}\frac{\Delta_\gamma^h f}{(k+1)\dots(k+2h-1)}\frac{t^{2h}}{2\cdot 4\cdots 2h},\quad k=-3,-5,\dots \tag{3.11}$$

Proof. Let us first take $k=-1$ and assume that $\lim_{t\to 0}\frac{\partial^2 u(x,t;-1)}{\partial t^2}$ exists. Let $t\to 0$ in

$$(\Delta_\gamma)_x u^{-1}(x,t;k)=\frac{\partial^2 u(x,t;-1)}{\partial t^2}-\frac{1}{t}\frac{\partial u(x,t;-1)}{\partial t},$$

i.e.,

$$(\Delta_\gamma)_x u(x,0;-1) = \lim_{t\to 0}\frac{\partial^2 u(x,t;-1)}{\partial t^2} - \lim_{t\to 0}\frac{1}{t}\frac{\partial u(x,t;-1)}{\partial t} = 0.$$

We find that $(\Delta_\gamma)_x u(x,0;-1)=0$ which shows that f must be B-harmonic. So the function f satisfies (3.7)–(3.8) for the $k=-1$.

When $k=-3$ we have

$$\lim_{t\to 0}\frac{\partial^2 u(x,t;-3)}{\partial t^2} = \lim_{t\to 0}\frac{1}{t}\frac{\partial u(x,t;-3)}{\partial t}.$$

From the general form of the Euler-Poisson-Darboux equation for $k=-3$ we obtain

$$\begin{aligned}\lim_{t\to 0}(\Delta_\gamma)_x u(x,t;-3) &= \lim_{t\to 0}\frac{\partial^2 u(x,t;-3)}{\partial t^2} - 3\lim_{t\to 0}\frac{1}{t}\frac{\partial u(x,t;-3)}{\partial t}\\ &= -2\lim_{t\to 0}\frac{1}{t}\frac{\partial u(x,t;-3)}{\partial t}.\end{aligned}$$

It is follows from (2.11) that

$$\frac{1}{t}\frac{\partial u(x,t;-3)}{\partial t} = u(x,t;-1)$$

hence

$$\lim_{t\to 0}(\Delta_\gamma)_x u(x,t;-3) = -2u(x,0;-1). \tag{3.12}$$

If $\lim_{t\to 0}\frac{\partial^4 u(x,t;-3)}{\partial t^4}$ exists and all odd derivatives of $u(x,t;-3)$ tend to zero when $t\to 0$, then $\lim_{t\to 0}\frac{\partial^2 u(x,t;-1)}{\partial t^2}$ also exists. Therefore, $\lim_{t\to 0}(\Delta_\gamma)_x u(x,t;-1)=0$ and by (3.12) we have $\lim_{t\to 0}(\Delta_\gamma)_x^2 u(x,t;-3)=0$. This remark can be easily generalized to include all exceptional values. So, in this case a solution to the Cauchy problem for the general form of the Euler-Poisson-Darboux equation for the case $k=-3,-5,\dots$ is given by the formula

$$u(x,t;k) = f(x) + \sum_{h=1}^{-\frac{k+1}{2}} \frac{\Delta_\gamma^h f}{(k+1)\dots(k+2h-1)}\,\frac{t^{2h}}{2\cdot 4\cdots 2h},\quad k=-3,-5,\dots$$

and as we proved earlier $u(x,t;-1)=f(x)$. □

4. Solution of the singular Cauchy problem using the Hankel transform

In this section we look for the solution $u\in S'_{ev}(\mathbb{R}^n_+)\times C^2(0,\infty)$ to the problem

$$(B_k)_t u = (\Delta_\gamma)_x u,\quad u=u(x,t;k),\quad x\in\mathbb{R}^n_+,\ t>0, \tag{4.1}$$

$$u(x,0;k)=f(x),\quad u_t(x,0;k)=0. \tag{4.2}$$

when $f(x)\in S'_{ev}(\mathbb{R}^n_+)$, $k\in\mathbb{R}\setminus\{-1,-3,-5,\dots\}$.

The notation $u\in S'_{ev}(\mathbb{R}^n_+)\times C^2(0,\infty)$ means that $u(x,t;k)$ belongs to $S'_{ev}(\mathbb{R}^n_+)$ by variable x and belongs to $C^2(0,\infty)$ by variable t.

Theorem 4.1. *There exists the solution from the class $u\in S'_{ev}(\mathbb{R}^n_+)\times C^2(0,\infty)$ to the problem (4.1)–(4.2) when $k\neq -1,-3,-5,\dots$ and it is defined by the formula*

$$u(x,t;k) = \frac{2^n t^{1-k}\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{k-n-|\gamma|+1}{2}\right)\prod_{i=1}^n\Gamma\left(\frac{\gamma_i+1}{2}\right)}\left((t^2-|x|^2)_{+,\gamma}^{\frac{k-n-|\gamma|-1}{2}} * f(x)\right)_\gamma. \tag{4.3}$$

The solution (4.3) is unique for $k\geq 0$, and not unique for $k<0$.

Proof. Applying multidimensional Hankel transform to (4.1) with respect to the variables $x_1,\dots,x_n$ only and using (2.8) we obtain

$$\Big(|\xi|^2+\frac{\partial^2}{\partial t^2}+\frac{k}{t}\frac{\partial}{\partial t}\Big)\widehat{u}(\xi,t)=0, \tag{4.4}$$

$$\lim_{t\to 0}\widehat{u}(\xi,t;k)=\widehat{f}(\xi),\quad \lim_{t\to 0}\frac{\partial\widehat{u}(\xi,t;k)}{\partial t}=0, \tag{4.5}$$

where $\xi=(\xi_l,\xi_2,\dots,\xi_n)\in\mathbb{R}^n_+$ corresponds to $x=(x_1,\dots,x_n)\in\mathbb{R}^n_+$, $|\xi|^2=\xi_1^2+\xi_2^2+\dots+\xi_n^2$,

$$\widehat{u}(\xi,t;k)=(F_B)_x[u(x,t;k)](\xi)=\int_{\mathbb{R}^n_+}u(x,t;k)\,\mathbf{j}_\gamma(x;\xi)x^\gamma dx$$

and $\widehat{f}(\xi)=F_B[f](\xi)$.

In [6] the solution $\widehat{G}^k(\xi,t)$ to the Cauchy problem

$$\Big(|\xi|^2+\frac{\partial^2}{\partial t^2}+\frac{k}{t}\frac{\partial}{\partial t}\Big)\widehat{G}^k(\xi,t)=0,$$

$$\lim_{t\to 0}\widehat{G}^k(\xi,t)=1,\quad \lim_{t\to 0}\frac{\partial\widehat{G}^k(\xi,t)}{\partial t}=0$$

was obtained and it has the form

$$\widehat{G}^k(\xi,t)=j_{\frac{k-1}{2}}(|\xi|t), \tag{4.6}$$

for $k\geq 0$,

$$\widehat{G}^k(\xi,t)=j_{\frac{k-1}{2}}(|\xi|t)+At^{\frac{1-k}{2}}J_{\frac{1-k}{2}}(|\xi|t), \tag{4.7}$$

for $k<0$, $k\neq -1,-3,-5,\dots$,

$$\widehat{G}^k(\xi,t)=Bt^{\frac{1-k}{2}}J_{\frac{1-k}{2}}(|\xi|t)-\frac{\pi 2^{\frac{k-1}{2}}}{\Gamma\left(\frac{1-k}{2}\right)}(|\xi|t)^{\frac{1-k}{2}}Y_{\frac{1-k}{2}}(|\xi|t), \tag{4.8}$$

for $k=-1,-3,-5,\dots$.

In (4.6)–(4.8) A and B are arbitrary complex numbers and $Y_\nu(z)$ is a Bessel functions of the the second kind. The solutions (4.7), (4.8) depend on the constants A and B since they are is not unique (see [6]). When $\widehat{G}^k(\xi,t)$ is found, the solution of (4.4)–(4.5) is

$$\widehat{u}(\xi,t;k)=\widehat{G}^k(\xi,t)\cdot\widehat{f}(\xi)$$

and the solution to (4.1)–(4.2) is then given by

$$u(x,t;k)=((F_B^{-1})_\xi[\widehat{G}^k(\xi,t)]*f(x))_\gamma=(G^k(x,t)*f(x))_\gamma.$$

We are looking for the solution when $k\neq -1,-3,-5,\dots$ and when $A=0$. The obtained solution will be unique for $k\geq 0$ and will be one of the possible solutions for $k<0$, $k\neq -1,-3,-5,\dots$. So we are interested in case when $\widehat{G}^k(\xi,t)=j_{\frac{k-1}{2}}(|\xi|t)$ Using (2.9) we can find $(F_B^{-1})_\xi[j_{\frac{k-1}{2}}(|\xi|t)](x)$:

$$\begin{aligned} G^k(x,t)&=(F_B^{-1})_\xi[j_{\frac{k-1}{2}}(|\xi|\cdot t)](x)\\ &=\frac{2^n t^{1-k}\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\big(\frac{k-n-|\gamma|+1}{2}\big)\prod_{i=1}^n\Gamma\left(\frac{\gamma_i+1}{2}\right)}(t^2-|x|^2)_{+,\gamma}^{\frac{k-n-|\gamma|-1}{2}}. \end{aligned}$$

Then the solution to (4.1)–(4.2) has the form

$$u(x,t;k)=\frac{2^n t^{1-k}\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{k-n-|\gamma|+1}{2}\right)\prod_{i=1}^n\Gamma\left(\frac{\gamma_i+1}{2}\right)}\Big((t^2-|x|^2)_{+,\gamma}^{\frac{k-n-|\gamma|-1}{2}}*f(x)\Big)_\gamma, \tag{4.9}$$

for $k\neq -1,-3,-5,\dots$.

Since $(t^2-|x|^2)^{\lambda}_{+,\gamma}$ has its support in the interior of the part of the sphere $S_1^+(n)$ when $x_1\geq 0,\dots,x_n\geq 0$, we may conclude that the convolution exists for arbitrary $\varphi(x)\in S'_+$. In [21] it was shown that the solution to the singular Cauchy problem (4.1)–(4.2) is unique when k is nonnegative and not unique when k is negative. □

Corollary 4.2. *For $k>n+|\gamma|-1$ when $f\in C^2_{ev}$ the solution to* (4.1)–(4.2) *exists in the classical sense and is defined by*

$$\begin{aligned} &u(x,t;k)\\ &=\frac{2^n\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{k-n-|\gamma|+1}{2}\right)\prod_{i=1}^n\Gamma\left(\frac{\gamma_i+1}{2}\right)}\int_{B_1^+(n)}(1-|y|^2)^{\frac{k-n-|\gamma|-1}{2}}\,{}^\gamma T^{ty}f(x)y^\gamma dy, \end{aligned}\tag{4.10}$$

which coincides with formula (3.3).

Proof. In the case when $k>n+|\gamma|-1$ and $f(x)$ is continuous and even with respect to all variables the integral in (4.9) exists in the classical sense. So, taking in (4.9) usual function $(t^2-|x|^2)^\lambda$ instead of the weighted generalized function $(t^2-|x|^2)^\lambda_{+,\gamma}$, passing to the integral over the part of the ball $B_t^+=\{x{\in}\mathbb{R}^n_+ : |x|{<}t\}$ and changing the variables by formula $x=ty$ we obtain (4.10). □

5. Case x is one-dimensional

In this section we concentrate on the case when x is one-dimensional. Then problems and constructed above solutions are simplified. For these problems we consider below some illustrative examples with explicit solution representations and some visual graphs using the Wolfram Alpha. In this case we have the Cauchy problem

$$\frac{\partial^2 u}{\partial x^2}+\frac{\gamma}{x}\frac{\partial u}{\partial x}=\frac{\partial^2 u}{\partial t^2}+\frac{k}{t}\frac{\partial u}{\partial t}, \tag{5.1}$$

$$u(x,0;k)=f(x),\quad \frac{\partial u(x,t;k)}{\partial t}\Big|_{t=0}=0,\quad f(x)\in C^2_{ev}(\overline{\mathbb{R}}^1_+). \tag{5.2}$$

When $k>\gamma>0$ the solution to (5.1)–(5.2) is given by (see (3.3))

$$u(x,t;k)=\frac{2\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{k-\gamma}{2}\right)\Gamma\left(\frac{\gamma+1}{2}\right)}\int_0^1(1-y^2)^{\frac{k-\gamma-2}{2}}\,{}^\gamma T^{ty}f(x)y^\gamma dy. \tag{5.3}$$

When $k<\gamma$ the solution to (5.1)–(5.2) is found by the formulas (3.6), (3.10) or (3.11).

Example 5.1. We are looking for the solution to

$$\frac{\partial^2 u}{\partial x^2}+\frac{\gamma}{x}\frac{\partial u}{\partial x}=\frac{\partial^2 u}{\partial t^2}+\frac{k}{t}\frac{\partial u}{\partial t},\quad k>\gamma>0,$$

$$u(x,0;k)=j_{\frac{\gamma-1}{2}}(x),\quad \frac{\partial u(x,t;k)}{\partial t}\Big|_{t=0}=0,\quad f(x)\in C^2_{ev}(\overline{\mathbb{R}}^1_+).$$

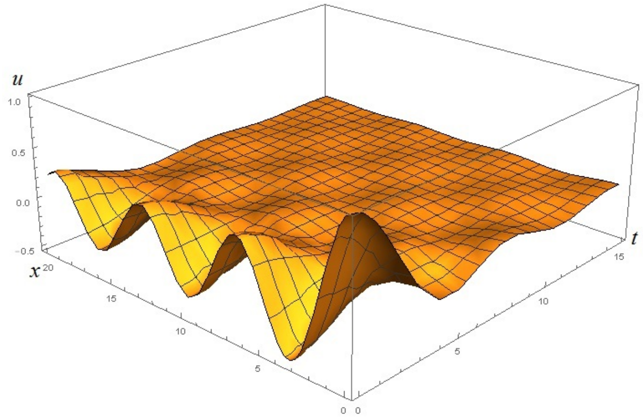

FIGURE 1. $u(x,t;k) = j_{-\frac{1}{6}}(x) j_{\frac{3}{4}}(t)$.

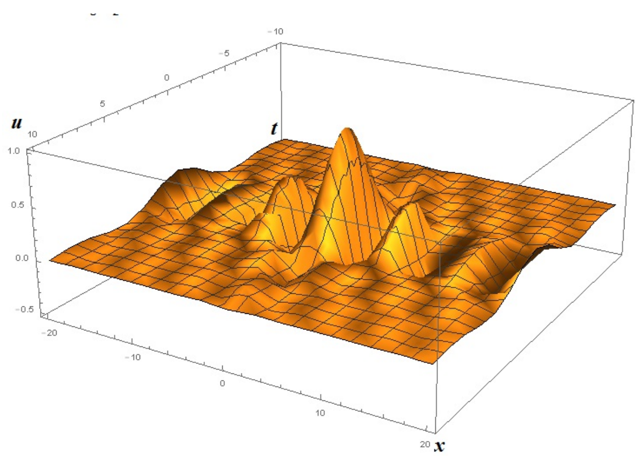

FIGURE 2. $u(x,t;k) = j_{-\frac{1}{6}}(x) j_{\frac{3}{4}}(t)$.

By (5.3) we obtain

$$u(x,t;k) = \frac{2\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{k-\gamma}{2}\right)\Gamma\left(\frac{\gamma+1}{2}\right)} \int_0^1 (1-y^2)^{\frac{k-\gamma-2}{2}} \, {}^{\gamma}T^{ty} j_{\frac{\gamma-1}{2}}(x) y^{\gamma} dy.$$

The next formula is valid

$$T^{ty} j_{\frac{\gamma-1}{2}}(x) = j_{\frac{\gamma-1}{2}}(x) j_{\frac{\gamma-1}{2}}(ty)$$

and so

$$u(x,t;k) = j_{\frac{\gamma-1}{2}}(x)\, t^{\frac{1-\gamma}{2}} \frac{2^{\frac{\gamma+1}{2}} \Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{k-\gamma}{2}\right)} \int_0^1 (1-y^2)^{\frac{k-\gamma-2}{2}} J_{\frac{\gamma-1}{2}}(ty)\, y^{\frac{\gamma+1}{2}} dy.$$

Using [49, formula 2.12.4.6],

$$\int_0^a x^{\nu+1}(a^2-x^2)^{\beta-1}J_\nu(cx)dx=\frac{2^{\beta-1}a^{\beta+\nu}}{c^\beta}\Gamma(\beta)J_{\beta+\nu}(ac), \tag{5.4}$$

for $a>0$, $\operatorname{Re}\beta>0$, $\operatorname{Re}\nu>-1$, we obtain

$$u(x,t;k)=j_{\frac{\gamma-1}{2}}(x)j_{\frac{k-1}{2}}(t). \tag{5.5}$$

The plot of (5.5) when $k=\frac{5}{2}$ and $\gamma=\frac{2}{3}$ is presented on Figure 1, and obtained through the Wolfram-Alpha. We can continue the solution to negative values of x and t as an even function. The plot of such continuation is presented on Figure 2.

If we denote

$$\begin{gathered} {}^{k,\gamma}T_x^t f(x)=C(\gamma,k)\int_0^1(1-y^2)^{\frac{k-\gamma-2}{2}}T^{ty}f(x)y^\gamma dy,\\ C(\gamma,k)=\frac{2\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{k-\gamma}{2}\right)\Gamma\left(\frac{\gamma+1}{2}\right)}. \end{gathered} \tag{5.6}$$

We can consider the operator (5.6) as a generalized translation operator (see [43]). For this operator the next property holds

$${}^{k,\gamma}T_x^t j_{\frac{\gamma-1}{2}}(x)=j_{\frac{\gamma-1}{2}}(x)j_{\frac{k-1}{2}}(t)\,.$$

Example 5.2. The solution to

$$\begin{gathered} \frac{\partial^2u}{\partial x^2}+\frac{\gamma}{x}\frac{\partial u}{\partial x}=\frac{\partial^2u}{\partial t^2}+\frac{k}{t}\frac{\partial u}{\partial t},\quad 1-\gamma\le k<\gamma,\ k\ne-1,-3,-5,\dots,\quad \gamma>\frac{1}{2},\\ u(x,0;k)=j_{\frac{\gamma-1}{2}}(x),\quad \frac{\partial u(x,t;k)}{\partial t}\Big|_{t=0}=0. \end{gathered}$$

is given by (5.1) where $m=1$,

$$u(x,t;k)=\frac{1}{t^k}\frac{\partial}{\partial t}(t^{k+1}u(x,t;k+2)),$$

and $u(x,t;k+2)$ is the solution to the Cauchy problem

$$\begin{gathered} (B_{k+2})_t u=(\Delta_\gamma)_x u,\\ u(x,0;k+2)=\frac{j_{\frac{\gamma-1}{2}}(x)}{k+1},\quad u_t(x,0;k+2)=0. \end{gathered}$$

Using the previous example we obtain

$$\begin{gathered} u(x,t;k+2)=\frac{1}{k+1}j_{\frac{\gamma-1}{2}}(x)j_{\frac{k+1}{2}}(t),\\ u(x,t;k)={}_0F_1\Big(\frac{\gamma+1}{2};-\frac{x^2}{4}\Big)\,{}_0F_1\Big(;\frac{k+1}{2};-\frac{t^2}{4}\Big). \end{gathered}$$

The plot of (5.5) when $k=\frac{1}{3}$ and $\gamma=\frac{3}{2}$ is presented on Figure 3 obtained through the Wolfram-Alpha.

We can continue the solution to negative values of x and t as an even function. The plot of such continuation is presented on Figure 4.

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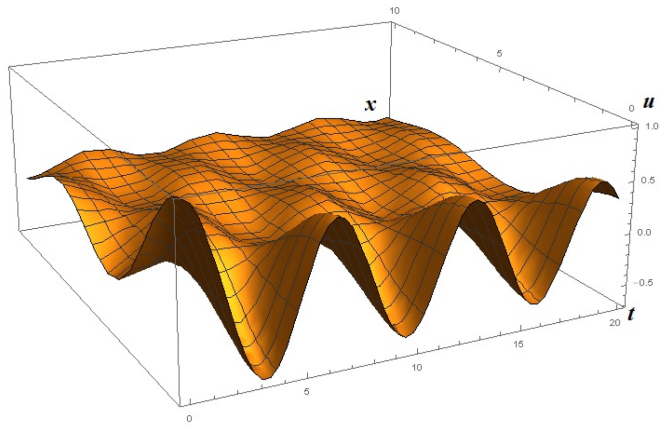

FIGURE 3. $u(x,t;k) = {}_0F_1\left(;\frac{2}{3};-\frac{t^2}{4}\right)\,{}_0F_1\left(;\frac{5}{4};-\frac{x^2}{4}\right)$.

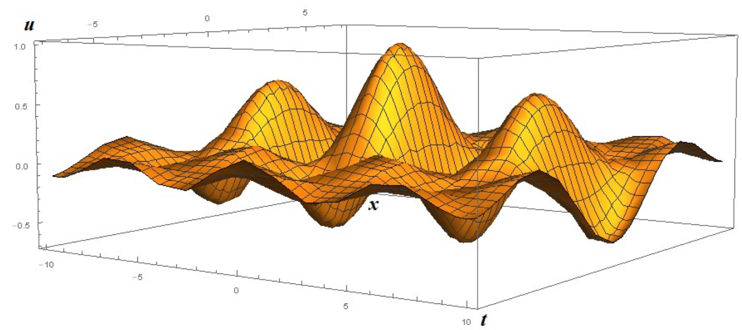

FIGURE 4. $u(x,t;k) = {}_0F_1\left(;\frac{2}{3};-\frac{t^2}{4}\right)\,{}_0F_1\left(;\frac{5}{4};-\frac{x^2}{4}\right)$.

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