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INVERSE SOURCE CAUCHY PROBLEM FOR A TIME FRACTIONAL DIFFUSION-WAVE EQUATION WITH DISTRIBUTIONS

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Abstract. We study the inverse Cauchy problem for a time fractional diffusion-wave equation with distributions in right-hand sides. The problem is to find a solution (continuous in time in generalized sense) of the direct problem and an unknown continuous time-dependent part of a source. The unique solvability of the problem is established.

1. Introduction

Elliptic and parabolic initial and boundary value problems for differential and pseudo-differential equations having distributions in right-hand sides are investigated by many authors; see [3, 9, 16, 20] and references therein.

Equations with fractional derivatives and inverse problems to them appear in different branches of science and engineering. The conditions of classical solvability of the Cauchy and boundary value problems to equations with the regularized time fractional derivative were obtained in [2, 5, 6, 7, 15, 17, 18, 19, 25, 26] and other works. The inverse boundary value problems to a time fractional diffusion equation with different unknown functions or parameters were investigated, for example, in [1, 4, 8, 10, 11, 12, 13, 21, 22, 27]. Most papers are devoted to the inverse problems with an unknown right-hand side (see, for example, [1, 8, 11, 21, 27]). Mainly such problems were studied under regular data.

In this article, for the equation

$$u_t^{(\beta)} - \Delta u = g(t)F_0(x), \quad (x,t) \in \mathbb{R}^n \times (0,T] := Q \tag{1.1}$$

with the Riemann-Liouville fractional derivative of order $\beta \in (m-1,m)$, $m,n \in \mathbb{N}$, we study the inverse Cauchy problem

$$\frac{\partial^{j-1}}{\partial t^{j-1}} u(x,0) = F_j(x), \quad x \in \mathbb{R}^n,\ j = 1,2,\dots,m, \tag{1.2}$$

$$(u(\cdot,t), \varphi_0(\cdot)) = F(t), \quad t \in [0,T] \tag{1.3}$$

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where F_j $(j=0,1,\dots,m)$ are given distributions, F is given continuous function, the symbol $(u(\cdot,t),\varphi_0(\cdot))$ stands for the value of an unknown distribution u on given test function φ_0 for every $t\in[0,T]$, $g(t)$ is an unknown continuous function on $[0,T]$. We prove the existence and uniqueness of a solution (u,g) of the problem in the cases $m=1,2$.

Note that the inverse boundary value problems of finding a pair (u,g) under smooth given data in right-hand sides and similar (integral) over-determination conditions were studied, for example, in [1, 11]. The over-determination condition of kind (1.3), but with the scalar product (u,φ_0) in abstract Hilbert space was used in [8].

Conditions of the unique classical solvability of the Cauchy problem (1.2) for the diffusion-wave equation with the Caputo partial derivative of order $\beta\in(0,2)$ and the Cauchy type problem for the diffusion-wave equation with the Riemann-Liouville partial derivative of order $\beta\in(0,2)$ where obtained in [25, 26], respectively. The method of the Green function was used to prove the solvability of these problems. The representations of components of the Green vector-functions for mentioned problems by means of the H-functions of Fox [14, 24], were obtained also in [6].

2. Notation, definitions and auxiliary results

We use the following: $\mathcal{D}(\mathbb{R}^n)$ is the space of indefinitely differentiable functions with compact supports in $\mathbb{R}^n$, $C^{\infty,(0)}(\bar{Q})=\{v\in C^\infty(\overline{Q}):(\frac{\partial}{\partial t})^k v|_{t=T}=0,\quad k\in\mathbb{Z}_+\}$, $\mathcal{D}(\overline{Q})$ is the space of functions from $C^{\infty,(0)}(\bar{Q})$ having compact supports with respect to the space variables, $\mathcal{D}'(\mathbb{R}^n)$ and $\mathcal{D}'(\overline{Q})$ are the spaces of linear continuous functionals (distributions [23, p. 13-15]) on $\mathcal{D}(\mathbb{R}^n)$ and $\mathcal{D}(\overline{Q})$, respectively, $\mathcal{E}'(\mathbb{R}^n)=[C^\infty(\mathbb{R}^n)]'$ is the space of distributions with compact supports, the symbol (f,φ) stands for the value of the distribution f on the test function φ,

$$\mathcal{D}'_C(\bar{Q})=\{v\in\mathcal{D}'(\bar{Q}):(v(\cdot,t),\varphi(\cdot))\in C[0,T]\quad\forall\varphi\in\mathcal{D}(\mathbb{R}^n)\}.$$

We denote $(g\hat{*}\varphi)(x)=\big(g(\xi),\varphi(x+\xi)\big)$, by $f*g$ the convolution of the distributions f and g:

$$(f*g,\varphi)=(f,g\hat{*}\varphi)\quad\text{for any test function }\varphi,$$

by $f\times g$ the direct product of the distributions f and g:

$$(f\times g,\varphi)=(f(x),(g(t),\varphi(x,t))\quad\text{for any test function }\varphi(x,t).$$

We shall use the function

$$f_\lambda(t)=\frac{\theta(t)t^{\lambda-1}}{\Gamma(\lambda)}\text{ for }\lambda>0\quad\text{and}\quad f_\lambda(t)=f'_{1+\lambda}(t)\text{ for }\lambda\le 0,$$

where $\Gamma(\lambda)$ is the Gamma-function, $\theta(t)$ is the Heaviside function. Note that

$$f_\lambda*f_\mu=f_{\lambda+\mu},\quad f_\lambda\hat{*}f_\mu=f_{\lambda+\mu}.$$

The Riemann-Liouville derivative $v^{(\beta)}(t)$ of order $\beta>0$ is defined by the formula

$$v^{(\beta)}(t)=f_{-\beta}(t)*v(t),$$

the Djrbashian-Caputo fractional derivative (regularized fractional derivative)

$$D^\beta v(t)=\frac{1}{\Gamma(m-\beta)}\int_0^t(t-\tau)^{m-\beta-1}\frac{d^m}{d\tau^m}v(\tau)d\tau\quad\text{for }m-1<\beta<m,\ m\in\mathbb{N},$$

and therefore,

$$D^\beta v(t) = v^{(\beta)}(t) - \sum_{j=0}^{m-1} f_{j+1-\beta}(t) v^{(j)}(0) \quad \text{for } \beta \in (m-1, m).$$

Assume that

$$C_{2,\beta}(Q) = \{v \in C(Q) \mid \Delta v, D_t^\beta v \in C(Q)\},$$

and for $\beta \in (m-1, m)$,

$$C_{2,\beta}(\bar{Q}) = \{v \in C_{2,\beta}(Q) : \frac{\partial^j v}{\partial t^j} \in C(\bar{Q}),\ j = \overline{0, m-1}\},$$
$$(Lv)(x,t) \equiv v_t^{(\beta)}(x,t) - \Delta v(x,t),$$
$$(L^{reg} v)(x,t) \equiv D_t^\beta v(x,t) - \Delta v(x,t),$$
$$(\widehat{L}v)(x,t) \equiv f_{-\beta}(t) \hat{*} v(x,t) - \Delta v(x,t), \quad (x,t) \in Q,$$
$$\mathcal{X}(\overline{Q}) = \{v \in C^{\infty,(0)}(\overline{Q}) : \widehat{L}v \in \mathcal{D}(\bar{Q})\}.$$

Lemma 2.1. *The Green formula holds:*

$$\begin{aligned}
&\int_Q v(x,\tau)(\widehat{L}\psi)(x,\tau) dx\, d\tau \\
&= \int_Q (L^{reg} v)(x,\tau)\psi(x,\tau) dx\, d\tau + \sum_{j=1}^{m} \int_{\mathbb{R}^n} \frac{\partial^{j-1}}{\partial \tau^{j-1}} v(x,0) \big(f_{j-\beta}(\tau), \psi(x,\tau)\big) dx,
\end{aligned}$$
$$\beta \in (m-1, m), \quad m \in \mathbb{N},\ v \in C_{2,\beta}(\bar{Q}),\ \psi \in \mathcal{X}(\bar{Q})\,.$$

Proof. Integrating by parts, for any $v \in C_{2,\beta}(\bar{Q})$, $\psi \in \mathcal{X}(\bar{Q})$ we have

$$\begin{aligned}
&\int_Q D_t^\beta v(x,t)\psi(x,t)\, dx\, dt \\
&= \frac{1}{\Gamma(m-\beta)} \int_Q \Big(\int_0^t (t-\tau)^{m-1-\beta} \frac{\partial^m}{\partial \tau^m} v(x,\tau)\big) d\tau\Big) \psi(x,t)\, dx\, dt \\
&= \frac{1}{\Gamma(m-\beta)} \int_{\mathbb{R}^n} dx \int_0^T \Big(\int_\tau^T (t-\tau)^{m-1-\beta} \psi(x,t) dt\Big) \frac{\partial^m}{\partial \tau^m} v(x,\tau) d\tau \\
&= \frac{1}{\Gamma(m-\beta)} \int_{\mathbb{R}^n} dx \int_0^T \Big(\int_0^{T-\tau} \eta^{m-1-\beta} \psi(x,\eta+\tau) d\eta\Big) \frac{\partial^m}{\partial \tau^m} v(x,\tau) d\tau \\
&= \int_{\mathbb{R}^n} dx \int_0^T \big(f_{m-\beta}(\tau) \hat{*} \psi(x,\tau)\big) \frac{\partial^m}{\partial \tau^m} v(x,\tau) d\tau \\
&= \int_{\mathbb{R}^n} \Big((f_{m-\beta}(\tau) \hat{*} \psi(x,\tau)) \frac{\partial^{m-1}}{\partial \tau^{m-1}} v(x,\tau)\Big)\Big|_{\tau=0}^{\tau=T} dx \\
&\quad - \int_{\mathbb{R}^n} dx \int_0^T (f_{m-\beta}(\tau) \hat{*} \psi(x,\tau))_\tau \frac{\partial^{m-1}}{\partial \tau^{m-1}} v(x,\tau) d\tau \\
&= - \int_{\mathbb{R}^n} (f_{m-\beta} \hat{*} \psi)(x,0) \frac{\partial^{m-1}}{\partial \tau^{m-1}} v(x,0) dx \\
&\quad + \int_{\mathbb{R}^n} dx \int_0^T (f_{m-1-\beta}(\tau) \hat{*} \psi(x,\tau)) \frac{\partial^{m-1}}{\partial \tau^{m-1}} v(x,\tau) d\tau
\end{aligned}$$

$$
\begin{aligned}
&= -\sum_{j=0}^{m-1} \int_{\mathbb{R}^n} \big(f_{m-j-\beta} \hat{*} \psi\big)(x,0) \frac{\partial^{m-j-1}}{\partial \tau^{m-j-1}} v(x,0) dx \\
&\quad + \int_{\mathbb{R}^n} dx \int_0^T \big(f_{-\beta}(\tau) \hat{*} \psi(x,\tau)\big) v(x,\tau) d\tau \\
&= -\sum_{j=1}^{m} \int_{\mathbb{R}^n} \frac{\partial^{j-1}}{\partial \tau^{j-1}} v(x,0) \big(f_{j-\beta}(t), \psi(x,t)\big) dx + \int_Q \big(f_{-\beta}(t) \hat{*} \psi(x,t)\big) v(x,t)\, dx\, dt.
\end{aligned}
$$

□

We use the following assumptions:

(A1) $F_j \in \mathcal{E}'(\mathbb{R}^n), \quad j = \overline{0,m}$,

(A2) $F, F^{(\beta)} \in C[0,T]$, $\varphi_0 \in \mathcal{D}(\mathbb{R}^n)$, $(F_0, \varphi_0) \neq 0$.

Definition 2.2. The pair $(u,g) \in \mathcal{D}'_C(\bar{Q}) \times C[0,T]$ is called a solution of the problem (1.1)-(1.3) if

$$
\begin{aligned}
&\int_0^T \big(u(\cdot,t), (\hat{L}\psi)(\cdot,t)\big) dt \\
&= \int_0^T g(t) \big(F_0(\cdot), \psi(\cdot,t)\big) dt + \sum_{j=1}^{m} \big(F_j(y) \times f_{j-\beta}(t), \psi(y,t)\big) \quad \forall \psi \in X(\bar{Q})
\end{aligned}
\tag{2.1}
$$

and condition (1.3) holds.

Note that (2.1) is obtained as the generalization of the above Green formula. Then from (1.2) and (1.3) it follows the compatibility conditions

$$
(F_j, \varphi_0) = F^{(j-1)}(0), \quad j = \overline{1,m}. \tag{2.2}
$$

Definition 2.3. The vector-function $(G_0(x,t), G_1(x,t), \dots, G_m(x,t))$ is called a Green vector-function of the Cauchy problem (1.2) to the equation

$$
(Lu)(x,t) = F(x,t), \quad (x,t) \in Q,
$$

and also of such problem to the equation

$$
(L^{\rm reg} u)(x,t) = F(x,t), \quad (x,t) \in Q, \tag{2.3}
$$

if under rather regular F, $F_1, \dots, F_m$ the function

$$
\begin{aligned}
u(x,t) = &\int_0^t d\tau \int_{\mathbb{R}^n} G_0(x-y, t-\tau) F(y,\tau) dy \\
&+ \sum_{j=1}^{m} \int_{\mathbb{R}^n} G_j(x-y,t) F_j(y) dy, \quad (x,t) \in \overline{Q}
\end{aligned}
\tag{2.4}
$$

is a classical (from $C_{2,\beta}(\bar{Q})$) solution of the problem (2.3), (1.2).

It follows from the definition 2.3 that

$$
(LG_0)(x,t) = \delta(x,t), \quad (x,t) \in Q
$$

where δ is the Dirac delta-function,

$$
(L^{\rm reg} G_j)(x,t) = 0, \quad (x,t) \in Q, \quad \frac{\partial^{j-1}}{\partial t^{j-1}} G_j(x,0) = \delta(x), \quad x \in \mathbb{R}^n,\ j = \overline{1,m}.
$$

Let

$$(\widehat{\mathcal{G}}_0\varphi)(y,\tau) = \int_\tau^T dt \int_{\mathbb{R}^n} \varphi(x,t) G_0(x-y,t-\tau)dx, \quad (y,\tau) \in \bar{Q},$$

$$(\widehat{\mathcal{G}}_j\varphi)(y) = \int_0^T dt \int_{\mathbb{R}^n} \varphi(x,t) G_j(x-y,t)dx, \quad y \in \mathbb{R}^n.$$

Lemma 2.4. *For any $\psi \in \mathcal{X}(\bar{Q})$ the following equalities hold:*

$$(\widehat{\mathcal{G}}_0(\widehat{L}\psi))(y,\tau) = \psi(y,\tau), \quad (y,\tau) \in \bar{Q}, \tag{2.5}$$

$$(\widehat{\mathcal{G}}_j(\widehat{L}\psi))(y) = \big(f_{j-\beta}(\tau), \psi(y,\tau)\big), \quad y \in \mathrm{R}^n, \quad j = \overline{1,m}. \tag{2.6}$$

Proof. Substituting the classical solution (2.4) of the Cauchy problem (2.3), (1.2) in the Green formula (instead of v) one obtains

$$\begin{aligned}
&\int_Q \Big(\int_0^t d\tau \int_{\mathbb{R}^n} G_0(x-y,t-\tau)F(y,\tau)dy\Big)(\hat{L}\psi)(x,t)\,dx\,dt \\
&+ \sum_{j=1}^m \int_Q \Big(\int_{\mathbb{R}^n} G_j(x-y,t)F_j(y)dy\Big)(\hat{L}\psi)(x,t)\,dx\,dt \\
&= \int_Q F(x,t)\psi(x,t)\,dx\,dt + \sum_{j=1}^m \int_{\mathbb{R}^n} F_j(x)\big(f_{j-\beta}(t), \psi(x,t)\big)dx;
\end{aligned}$$

that is

$$\begin{aligned}
&\int_Q \Big(\int_\tau^T dt \int_{\mathbb{R}^n} G_0(x-y,t-\tau)(\hat{L}\psi)(x,t)dx\Big)F(y,\tau)dyd\tau \\
&+ \sum_{j=1}^m \int_{\mathbb{R}^n} \Big(\int_Q G_j(x-y,t)(\hat{L}\psi)(x,t)\,dx\,dt\Big)F_j(y)dy \\
&= \int_Q \psi(y,\tau)F(y,\tau)dyd\tau + \sum_{j=1}^m \int_{\mathbb{R}^n} \big(f_{j-\beta}(t), \psi(y,t)\big)F_j(y)dy.
\end{aligned}$$

We obtain the formulas (2.5) and (2.6) after an arbitrariness of $F, F_1, \dots, F_m$. □

Lemma 2.5. *For $m = 1, 2$, any $\varphi \in \mathcal{D}(\bar{Q})$ there exists $\psi \in \mathcal{X}(\bar{Q})$ such that*

$$(\widehat{L}\psi)(x,t) = \varphi(x,t), \qquad (x,t) \in \bar{Q}.$$

Proof. We show that

$$\psi(y,\tau) = \int_\tau^T dt \int_{\mathbb{R}^n} G_0(x-y,t-\tau)\varphi(x,t)dx, \quad (y,\tau) \in \bar{Q}$$

is the unknown function. Indeed, it will follow from Lemma 2.9 (relatively G_0) that $\psi \in C^{\infty,(0)}(\bar{Q})$ for any $\varphi \in \mathcal{D}(\bar{Q})$ in the cases $m = 1, 2$ and, in addition, we have

$$\begin{aligned}
(\widehat{L}\psi)(y,\tau) &= \widehat{L}\big(G_0(x,t), \varphi(x+y,t+\tau)\big) \\
&= \big(G_0(x,t), (\widehat{L}\varphi)(x+y,t+\tau)\big) \\
&= \big((LG_0)(x,t), \varphi(x+y,t+\tau)\big) \\
&= \big(\delta(x,t), \varphi(x+y,t+\tau)\big) \\
&= \varphi(y,\tau), \quad (y,\tau) \in \bar{Q}.
\end{aligned}$$

□

Corollary 2.6. *The following hold:*

$$G_j(x,t)=f_{j-\beta}(t)*G_0(x,t),\quad (x,t)\in Q,\quad j=\overline{1,m},\quad m=1,2. \tag{2.7}$$

Proof. Using (2.5) and the analogue of the Fubini theorem, we obtain

$$\begin{aligned}\big(f_{j-\beta}(\tau),\psi(y,\tau)\big)&=\big(f_{j-\beta}(\tau),(\widehat{\mathcal{G}}_0(\widehat{L}\psi))(y,\tau)\big)\\&=\Big(f_{j-\beta}(\tau),\int_\tau^T\int_{\mathbb{R}^n}G_0(x-y,t-\tau)(\hat{L}\psi)(x,t)\,dx\,dt\Big)\\&=\int_Q\big(f_{j-\beta}(t)*G_0(x-y,t)\big)(\hat{L}\psi)(x,t)\,dx\,dt.\end{aligned}$$

From (2.6) we obtain

$$\big(f_{j-\beta}(\tau),\psi(y,\tau)\big)=(\widehat{\mathcal{G}}_j(\widehat{L}\psi))(y)=\int_Q G_j(x-y,t)(\hat{L}\psi)(x,t)\,dx\,dt,\quad j=\overline{1,m}.$$

Therefore, for any $\psi\in\mathcal{X}(\bar{Q})$, $j=\overline{1,m}$ we have

$$\int_Q\Big(G_j(x-y,t)-f_{j-\beta}(t)*G_0(x-y,t)\Big)(\widehat{L}\psi)(x,t)\,dx\,dt=0,\quad y\in\mathbb{R}^n.$$

By Lemma 2.5, for all $\varphi\in\mathcal{D}(\bar{Q})$, $j=\overline{1,m}$, $m=1,2$ we obtain

$$\int_Q\Big(G_j(x-y,t)-f_{j-\beta}(t)*G_0(x-y,t)\Big)\varphi(x,t)\,dx\,dt=0,\quad y\in\mathbb{R}^n,$$

and the desirable formula (2.7) follows from the Du Bois-Reymond lemma. □

Lemma 2.7. *There exists a Green function $G_0(x,t)$ of the Cauchy problem* (1.1), (1.2).

Proof. From [6, 7] we have the representation

$$G_0(x,t)=\frac{\pi^{-n/2}t^{\beta-1}}{|x|^n}H^{2,0}_{1,2}\Big(\frac{|x|^2}{4t^\beta};(\beta,\beta);(1,1),(n/2,1)\Big). \tag{2.8}$$

The function $H^{m,n}_{p,q}(\cdot)$ is defined in Fox [14, 24] (with the same arguments in a different format)

$$H^{m,n}_{p,q}\Big(z;(a_1,\gamma_1),\dots,(a_p,\gamma_p);(b_1,\beta_1),\dots,(b_q,\beta_q)\Big)=\int_{\mathbb{C}}\mathcal{H}(s)z^{-s}\,ds\,,$$

where

$$\mathcal{H}(s)=\frac{\prod_{j=1}^m(b_j+\beta_j s)\prod_{i=1}^n\Gamma(1-a_i-\gamma_i s)}{\prod_{i=n+1}^q\Gamma(a_i+\gamma_i s)\prod_{j=m+1}^p\Gamma(1-b_j-\beta_j s)}\,,$$

$z^{-s}=\exp[s(\log|z|+i\arg z)]$, $z\neq 0$, $i^2=1$, $\Gamma(x)$ is the usual Gamma function, $\mathbb{C}$ is the boundless contour that separates the poles $b_{jl}=\frac{-b_j-l}{\beta_j}$, $1\le j\le m$, $l=0,1,\dots$, of the function $\Gamma(b_j+\beta_j s)$ to the left and the poles $a_{ik}=\frac{1-a_i-k}{\gamma_i}$, $1\le i\le m$, $k=0,1,\dots$, of the function $\Gamma(1-a_i-\gamma_i s)$ to the right (it is assumed that they does not coincide).

Let

$$a^*=\sum_{i=1}^n\gamma_i-\sum_{i=n+1}^p\gamma_i+\sum_{i=1}^m\beta_i-\sum_{i=m+1}^q\beta_i,$$

$$\Delta^* = \sum_{i=1}^{q} \beta_i - \sum_{i=1}^{p} \gamma_i.$$

For the function (2.8) we have $a^* = 2 - \beta \neq 0$, $\Delta^* = 2 - \beta \neq 0$, and by [14, Thm. 1.1] the function $G_0(x,t)$ exists for all $x \neq 0$, $t > 0$. □

Corollary 2.8. *The Green vector-function of the Cauchy problem* (1.1), (1.2) *with* $m = 1, 2$ *exists.*

Proof. By Corollary 2.6

$$G_j(x,t) = (f_{m-\beta}(t) * G_0(x,t))_t^{(m-j)}, \quad j = \overline{1,m}, \ m = 1,2.$$

Using (2.8), the formula of fractional differentiation of the H-function [14, Thm. 2.7]

$$\begin{aligned}
&f_\varrho(z) * \Big[z^\omega H_{p,q}^{m,n}\Big(cz^\sigma; (a_1,\gamma_1),\dots,(a_p,\gamma_p); (b_1,\beta_1),\dots,(b_q,\beta_q)\Big)\Big] \\
&= z^{\omega+\varrho} H_{p+1,q+1}^{m,n+1}\Big(cz^\sigma; (-\omega,\sigma),(a_1,\gamma_1),\dots,(a_p,\gamma_p); (b_1,\beta_1),\dots, \\
&\quad (b_q,\beta_q),(-\omega-\varrho,\sigma)\Big)
\end{aligned}$$

for $a^* > 0$, $\varrho > 0$, $\sigma \min_{1\le j\le m}[\frac{Reb_j}{\beta_j}] + Re\omega > -1$ and properties of this function (see, for example, [6, 14]), in the case $a^* = 2 - \beta > 0$ (that is for $m = 1, 2$) we obtain

$$\begin{aligned}
&f_{m-\beta}(t) * G_0(x,t) \\
&= f_{m-\beta}(t) * \Big[\frac{\pi^{-n/2} t^{\beta-1}}{|x|^n} H_{1,2}^{2,0}\Big(\frac{|x|^2}{4t^\beta}; (\beta,\beta); (1,1),(n/2,1)\Big)\Big] \\
&= f_{m-\beta}(t) * \Big[\frac{\pi^{-n/2} t^{\beta-1}}{|x|^n} H_{2,1}^{0,2}\Big(\frac{4t^\beta}{|x|^2}; (0,1),(1-n/2,1); (1-\beta,\beta)\Big)\Big] \\
&= \frac{\pi^{-n/2} t^{m-1}}{|x|^n} H_{3,2}^{0,3}\Big(\frac{4t^\beta}{|x|^2}; \Big((1-\beta,\beta),(0,1),(1-n/2,1); (1-\beta,\beta),(1-m,\beta)\Big) \\
&= \frac{\pi^{-n/2} t^{m-1}}{|x|^n} H_{2,1}^{0,2}\Big(\frac{4t^\beta}{|x|^2}; (0,1),(1-n/2,1); (1-m,\beta))\Big).
\end{aligned}$$

By the formula of differentiation [14, Prop. 2.8] of the H-function

$$\begin{aligned}
&(\frac{d}{dz})^k \Big[z^\omega H_{p,q}^{m,n}\Big(cz^\sigma; (a_1,\alpha_1),\dots,(a_p,\alpha_p); (b_1,\beta_1),\dots,(b_q,\beta_q)\Big)\Big] \\
&= z^{\omega-k} H_{p+1,q+1}^{m,n+1}\Big(cz^\sigma; (-\omega,\sigma),(a_1,\alpha_1),\dots,(a_p,\alpha_p); (b_1,\beta_1),\dots, \\
&\quad (b_q,\beta_q),(k-\omega,\sigma)\Big)
\end{aligned}$$

for $\omega, c \in \mathbb{C}$, $\sigma > 0$, $k \in \mathbb{Z}_+$, we have

$$\begin{aligned}
&(f_{m-\beta}(t) * G_0(x,t))_t^{(m-j)} \\
&= \frac{\pi^{-n/2} t^{m-1}}{|x|^n} H_{3,2}^{0,3}\Big(\frac{4t^\beta}{|x|^2}; (1-m,\beta),(0,1),(1-n/2,1); (1-m,\beta),(1-j,\beta)\Big) \\
&= \frac{\pi^{-n/2} t^{j-1}}{|x|^n} H_{2,1}^{0,2}\Big(\frac{4t^\beta}{|x|^2}; (0,1),(1-n/2,1); (1-j,\beta)\Big)
\end{aligned}$$

$$= \frac{\pi^{-n/2} t^{j-1}}{|x|^n} H^{2,0}_{1,2}\Big(\frac{|x|^2}{4t^\beta}; (j,\beta); (1,1), (n/2,1)\Big).$$

So, we find the representations

$$G_j(x,t) = \frac{\pi^{-n/2} t^{j-1}}{|x|^n} H^{2,0}_{1,2}\Big(\frac{|x|^2}{4t^\beta}; (j,\beta); (1,1), (n/2,1)\Big), \tag{2.9}$$

for $j = \overline{1,m}$, $m = 1,2$.

For every function G_j, $j = \overline{1,m}$, $m = 1,2$ we have $a^* = \Delta^* = 2-\beta > 0$. So, by [14, Thm. 1.1] these functions exist for all $x \neq 0$, $t > 0$. $\square$

Let $\mathcal{D}^k(\mathrm{R}^n)$ be the space of functions from $C^k(\mathrm{R}^n)$ having compact supports, $\|\varphi\|_{D^k(\mathrm{R}^n)} = \max_{|\kappa|\le k} \max_{x\in supp\varphi} |D^\kappa \varphi(x)|$ where $\kappa = (\kappa_1,\dots,\kappa_n)$, $\kappa_j \in \mathrm{Z}_+$, $j \in \{1,\dots,n\}$, $|\kappa| = \kappa_1 + \dots + \kappa_n$, $D^\kappa\varphi(x) = \frac{\partial^{|\kappa|}}{\partial x_1^{\kappa_1}\dots\partial x_n^{\kappa_n}}\varphi(x)$,

$$(\widehat{G_j}\varphi)(y,t) = \int_{\mathrm{R}^n} G_j(x-y,t)\varphi(x)\,dx, \quad (y,t) \in \bar{Q},\ j = \overline{0,m}.$$

Lemma 2.9. *For $\beta \in (m-1,m)$, $m = 1,2$, all $k \in \mathrm{Z}_+$, multi-index κ, $|\kappa| = k$, $\varphi \in \mathcal{D}(\mathrm{R}^n)$ the following bounds hold:*

$$\big|D^\kappa_y(\widehat{G_0}\varphi)(y,t)\big| \le c_k t^{\beta-1}\|\varphi\|_{\mathcal{D}^k(\mathrm{R}^n)}, \quad (y,t) \in Q,$$
$$\big|D^\kappa_y(\widehat{G_j}\varphi)(y,t)\big| \le c_k t^{j-1}\|\varphi\|_{\mathcal{D}^k(\mathrm{R}^n)}, \quad (y,t) \in \bar{Q},\ j = \overline{1,m}.$$

Hereinafter $c_k, \widehat{c}_k, d_k, \widehat{d}_k, C_k, C$ ($k \in \mathrm{Z}_+$) are positive constants.

Proof. We use the bounds of components of the Green vector-function. We obtain them from the properties of the H-functions. It is known [14] that

$$H^{q,0}_{p,q}\Big(z; (a_1,\gamma_1),\dots,(a_p,\gamma_p); (b_1,\alpha_1),\dots,(b_q,\alpha_q)\Big) \le C|z|^{\frac{\mu^*+1/2}{\Delta^*}} e^{-c|z|^{\frac{1}{\Delta^*}}},$$

as $|z| \to \infty$ in the case $a^* > 0$ where $\mu^* = \sum_{i=1}^q b_i - \sum_{i=1}^p a_i + \frac{p-q}{2}$, $c = (2-\beta)\beta^{\frac{\beta}{2-\beta}}$.

Using the representations (2.8) and (2.9) we obtain

$$\mu_0^* = \frac{n+1}{2} - \beta, \quad \mu_j^* = \frac{n+1}{2} - j, \quad j = \overline{1,m},\ m = 1,2.$$

So, in the case $|x| > t^{\beta/2}$ we obtain

$$|G_0(x,t)| \le C\frac{t^{\beta-1}}{|x|^n}\Big(\frac{|x|^2}{t^\beta}\Big)^{1+\frac{n-\beta}{2(2-\beta)}} e^{-c\big(\frac{|x|^2}{t^\beta}\big)^{\frac{1}{2-\beta}}},$$
$$|G_j(x,t)| \le C\frac{t^{j-1}}{|x|^n}\Big(\frac{|x|^2}{t^\beta}\Big)^{\frac{n+2-2j}{2(2-\beta)}} e^{-c\big(\frac{|x|^2}{t^\beta}\big)^{c12-\beta}}, \quad j = \overline{1,m}.$$

Using [14, Thm 1.11], we obtain the following bounds in the case $|x| < t^{\beta/2}$:

$$|G_0(x,t)| \le C\frac{t^{\beta-1}}{|x|^n}\Big(\frac{|x|^2}{t^\beta}\Big)^{\min\{1,\frac{n}{2}\}} = C\begin{cases} \frac{|x|^{2-n}}{t}, & n > 2 \\ \frac{1}{t}\big(1 + \big|ln\frac{|x|^2}{t^\beta}\big|\big), & n = 2 \\ t^{\frac{\beta}{2}-1}, & n = 1 \end{cases},$$

$$|G_j(x,t)| \le C\frac{t^{j-1}}{|x|^n}\Big(\frac{|x|^2}{t^\beta}\Big)^{\min\{1,\frac{n}{2}\}} = C\begin{cases} |x|^{2-n}t^{j-1-\beta}, & n > 2 \\ t^{j-1-\beta}\big(1 + \big|ln\frac{|x|^2}{t^\beta}\big|\big), & n = 2 \\ t^{j-1-\frac{\beta}{2}}, & n = 1, \end{cases}$$

for $j=\overline{1,m}$. Then in the case $n>2$ for all multi-index α, $|\alpha|=k$, $\varphi\in\mathcal{D}(\mathbb{R}^n)$ we have

$$
\begin{aligned}
&\Big|\int_{\mathbb{R}^n} G_0(x-y,t-\tau)D^\alpha\varphi(x)dx\Big|\\
&\le \int_{\{x\in\mathbb{R}^n:|x-y|^2<(t-\tau)^\beta\}} G_0(x-y,t-\tau)|D^\alpha\varphi(x)|dx\\
&\quad+\int_{\{x\in\mathbb{R}^n:|x-y|^2>(t-\tau)^\beta\}} G_0(x-y,t-\tau)|D^\alpha\varphi(x)|dx\\
&\le C\int_{\{x\in\mathbb{R}^n:|x-y|^2<(t-\tau)^\beta\}} \frac{|D^\alpha\varphi(x)|}{(t-\tau)|x-y|^{n-2}}dx\\
&\quad+C\int_{\{x\in\mathbb{R}^n:|x-y|^2>(t-\tau)^\beta\}} \frac{(t-\tau)^{\beta-1}}{|x-y|^n}\cdot\Big(\frac{|x-y|^2}{(t-\tau)^\beta}\Big)^{1+\frac{n-\beta}{2(2-\beta)}}\\
&\quad\times e^{-c\big(\frac{|x-y|^2}{(t-\tau)^\beta}\big)^{\frac{1}{2-\beta}}}|D^\alpha\varphi(x)|dx\\
&\le C_1\Big[\frac{1}{(t-\tau)}\int_0^{(t-\tau)^{\beta/2}} r dr+\int_{t^{\beta/2}}^d r^{1+\frac{n-\beta}{2-\beta}}(t-\tau)^{-1-\frac{(n-\beta)\beta}{2(2-\beta)}}\\
&\quad\times e^{-c\big(\frac{r^2}{(t-\tau)^\beta}\big)^{\frac{1}{2-\beta}}}dr\Big]\|\varphi\|_{\mathcal{D}^k(\mathbb{R}^n)}\\
&\le C_2\Big[(t-\tau)^{\beta-1}+(t-\tau)^{\beta-1}\int_1^{+\infty} z^{\frac{n}{2}-\beta}e^{-cz}\,dz\Big]\|\varphi\|_{\mathcal{D}^k(\mathbb{R}^n)}\\
&\le c_0(t-\tau)^{\beta-1}\|\varphi\|_{\mathcal{D}^k(\mathbb{R}^n)},\quad y\in\mathbb{R}^n,\quad 0\le\tau<t\le T,
\end{aligned}
$$

where $d=\operatorname{diam}\operatorname{supp}\varphi$,

$$
\begin{aligned}
&\Big|\int_{\mathbb{R}^n} G_j(x-y,t)D^\alpha\varphi(x)dx\Big|\\
&\le C\Big[\int_{\{x\in\mathbb{R}^n:|x-y|^2<t^\beta\}} \frac{t^{j-1-\beta}}{|x-y|^{n-2}}dx\\
&\quad+\int_{\{x\in\mathbb{R}^n:|x-y|^2>t^\beta\}} \frac{t^{j-1}}{|x-y|^n}\cdot\Big(\frac{|x-y|^2}{4t^\beta}\Big)^{\frac{n+2-2j}{2(2-\beta)}} e^{-c\Big(\frac{|x-y|^2}{4t^\beta}\Big)^{\frac{1}{2-\beta}}}dx\Big]\|\varphi\|_{\mathcal{D}^k(\mathbb{R}^n)}\\
&\le C_3\Big[t^{j-1-\beta}\int_0^{t^{\beta/2}} r\,dr+\int_{t^{\beta/2}}^{d_0} r^{-1+\frac{n+2-2j}{2-\beta}}t^{j-1-\frac{(n+2-2j)\beta}{2(2-\beta)}}e^{-c\big(\frac{r^2}{t^\beta}\big)^{\frac{1}{2-\beta}}}dr\Big]\|\varphi\|_{\mathcal{D}^k(\mathbb{R}^n)}\\
&\le C_4\Big[t^{j-1}+t^{j-1}\int_1^{+\infty} z^{\frac{n}{2}-j}e^{-cz}\,dz\Big]\|\varphi\|_{\mathcal{D}^k(\mathbb{R}^n)}\\
&\le c_jt^{j-1}\|\varphi\|_{\mathcal{D}^k(\mathbb{R}^n)},\quad (y,t)\in\bar{Q},
\end{aligned}
$$

$j=\overline{1,m}$, and similarly for $n=1,2$. Integrating by parts we finish the proof. □

Theorem 2.10. *Assume that* (A1) *with* $m=1$ *(*$m=2$*) holds,* $g\in C[0,T]$*. Then there exists a unique solution* $u\in\mathcal{D}'_C(\bar{Q})$ *of the Cauchy problem* (1.1), (1.2) *with* $m=1$ *(*$m=2$*, respectively). It is defined by*

$$
\Big(u(\cdot,t),\varphi(\cdot)\Big)=h_\varphi(t)\quad\forall\varphi\in\mathcal{D}(\mathrm{R}^n),\ t\in[0,T] \tag{2.10}
$$

where

$$h_\varphi(t) = \int_0^t g(\tau)\Big(F_0(\cdot), (\widehat{G}_0\varphi)(\cdot, t-\tau)\Big)d\tau + \sum_{j=1}^m \Big(F_j(\cdot), (\widehat{G}_j\varphi)(\cdot, t)\Big), \quad t \in [0,T].$$

Proof. We say that the distribution F has the order of the singularity $s(F) \le k$, $k \in \mathbb{Z}_+$ if there exist the functions $g_\kappa \in L_{1,loc}(\mathrm{R}^n)$, $|\kappa| \le k$ such that

$$\big(F, \varphi\big) = \sum_{|\kappa|\le k} \int_{\mathrm{R}^n} g_\kappa(y) D^\kappa \varphi(y) dy \quad \forall \varphi \in \mathcal{D}(\mathrm{R}^n). \tag{2.11}$$

A distribution from $\mathcal{E}'(\mathrm{R}^n)$ has a finite order of the singularity. So, $s(F_j) \le k_j$ with some $k_j \in \mathbb{Z}_+$, $j = \overline{0,m}$. Using this fact and Lemma 2.9, we show that the function (2.10) belongs to $\mathcal{D}'_C(\bar{Q})$. Namely, it follows from (2.11) for F_j, $j = \overline{0,m}$ that there exist positive constants B_j such that

$$\big|\big(F_j, \varphi\big)\big| \le B_j \|\varphi\|_{\mathcal{D}^{k_j}(\mathbb{R}^n)} \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^n), \quad j = \overline{0,m}.$$

Then by Lemma 2.9,

$$\begin{aligned}
\big|\big(F_0(y), (\widehat{G_0}\varphi)(y, t-\tau)\big)\big| &\le B_0 \|(\widehat{G_0}\varphi)(\cdot, t-\tau)\|_{\mathcal{D}^{k_0}(\mathbb{R}^n)} \\
&\le \widehat{c}_0 \|\varphi\|_{\mathcal{D}^{k_0}(\mathbb{R}^n)} (t-\tau)^{\beta-1}, \quad 0 \le \tau < t \le T, \\
\int_0^t |g(\tau)| \, \big|\big(F_0(y), (\widehat{G_0}\varphi)(y, t-\tau)\big)\big| d\tau &\le \widehat{d}_0 \|\varphi\|_{\mathcal{D}^{k_0}(\mathbb{R}^n)}\, t^\beta, \\
\big|\big(F_j(\cdot), (\widehat{G}_j\varphi)(\cdot, t)\big)\big| &\le B_j \|(\widehat{G}_j\varphi)(\cdot, t)\|_{\mathcal{D}^{k_j}(\mathbb{R}^n)} \\
&\le \widehat{d}_j \|\varphi\|_{\mathcal{D}^{k_j}(\mathbb{R}^n)} t^{j-1}, \quad t \in [0,T], \ j = \overline{1,m}.
\end{aligned}$$

So, $h_\varphi \in C[0,T]$ for any $\varphi \in \mathcal{D}(\mathbb{R}^n)$.

Using Lemma 2.4, we show that the function (2.10) satisfies (2.1). Indeed, for all $\psi \in \mathcal{X}(\overline{Q})$,

$$\begin{aligned}
\big(u, \widehat{L}\psi\big) &= \int_0^T \Big(u(\cdot, t), (\widehat{L}\psi)(\cdot, t)\Big) dt \\
&= \int_0^T \Big(\int_0^t g(\tau)\Big(F_0(y), (\widehat{G}_0(\widehat{L}\psi))(y, t-\tau)\Big) d\tau\Big) dt \\
&\quad + \sum_{j=1}^m \int_0^T \Big(F_j(y), (\widehat{G}_j(\widehat{L}\psi))(y, t)\Big) dt \\
&= \Big(F_0(y), \int_0^T dt \int_0^t g(\tau)(\widehat{G}_0(\widehat{L}\psi))(y, t-\tau) d\tau\Big) \\
&\quad + \sum_{j=1}^m \Big(F_j(y), \int_0^T (\widehat{G}_j(\widehat{L}\psi))(y, t) dt\Big) \\
&= \Big(F_0(y), \int_0^T g(\tau) d\tau \int_\tau^T (\widehat{G}_0(\widehat{L}\psi))(y, t-\tau) dt\Big) \\
&\quad + \sum_{j=1}^m \Big(F_j(y), \int_0^T (\widehat{G}_j(\widehat{L}\psi))(y, t) dt\Big)
\end{aligned}$$

$$= \Big(F_0(y)\cdot g(\tau), \big(\widehat{\mathcal{G}}_0(\widehat{L}\psi)\big)(y,\tau)\Big) + \sum_{j=1}^{m}\Big(F_j, \widehat{\mathcal{G}}_j(\widehat{L}\psi)\Big).$$

From Lemma 2.4 we obtain (2.1). By Definition 2.2 the function (2.10) is the solution of (1.1), (1.2).

If u_1, u_2 are two solutions of the problem (1.1), (1.2), then for $u = u_1 - u_2$ from (2.1) we obtain

$$(u, \widehat{L}\psi) = 0 \quad \forall \psi \in \mathcal{X}(\overline{Q}).$$

By using Lemma 2.5 we obtain $(u,\varphi) = 0$ for all $\varphi \in \mathcal{D}(\bar{Q})$, and also $(u(\cdot,t),\varphi(\cdot)) = 0$ for all $\varphi \in \mathcal{D}(\mathbb{R}^n)$, $t \in [0,T]$, that is $u = 0$ in $\mathcal{D}'_C(\overline{Q})$. □

3. Existence and uniqueness for the inverse problem

We pass to the problem (1.1)–(1.3) with $m = 1$ and $m = 2$. It follows from (1.1) that

$$\big(u_t^{(\beta)}(\cdot,t), \varphi(\cdot)\big) = \big(u(\cdot,t), \Delta\varphi(\cdot)\big) + (F_0,\varphi)g(t) \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^n),$$

in particular,

$$\big(u_t^{(\beta)}(\cdot,t), \varphi_0(\cdot)\big) = \big(u(\cdot,t), \Delta\varphi_0(\cdot)\big) + (F_0,\varphi_0)g(t).$$

By the over-determination condition (1.3) we obtain

$$F^{(\beta)}(t) = \big(u(\cdot,t), \Delta\varphi_0(\cdot)\big) + (F_0,\varphi_0)g(t).$$

Using the assumption (A2) we find the expression for $g(t)$ through u

$$g(t) = \big[F^{(\beta)}(t) - \big(u(\cdot,t), \Delta\varphi_0(\cdot)\big)\big]\big[(F_0,\varphi_0)\big]^{-1}, \quad t \in [0,T]. \tag{3.1}$$

It follows from Theorem 2.10 and the assumption (A2) that the right-hand side of (3.1) is a continuous function on $[0,T]$. By substituting it in (2.10) (instead of $g(t)$) one obtains

$$\begin{aligned}\big(u(\cdot,t),\varphi(\cdot)\big) &= \frac{1}{(F_0,\varphi_0)}\int_0^t \big[F^{(\beta)}(\tau) - \big(u(\cdot,\tau),\Delta\varphi_0(\cdot)\big)\big]\big(F_0(\cdot), (\widehat{G}_0\varphi)(\cdot,t-\tau)\big)d\tau \\ &\quad + \sum_{j=1}^{m}\big(F_j(\cdot), (\widehat{G}_j\varphi)(\cdot,t)\big) \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^n),\ t \in [0,T],\end{aligned}$$

in particular,

$$\begin{aligned}&\big(u(\cdot,t),\Delta\varphi_0(\cdot)\big) \\ &= \frac{1}{(F_0,\varphi_0)}\int_0^t \big[F^{(\beta)}(\tau) - \big(u(\cdot,\tau),\Delta\varphi_0(\cdot)\big)\big]\big(F_0(\cdot), (\widehat{G}_0\Delta\varphi_0)(\cdot,t-\tau)\big)d\tau \\ &\quad + \sum_{j=1}^{m}\big(F_j(\cdot), (\widehat{G}_j\Delta\varphi_0)(\cdot,t)\big), \quad t \in [0,T].\end{aligned}$$

Denote $r(u,t) = \big(u(\cdot,t),\Delta\varphi_0(\cdot)\big)$. Then we have

$$r(u,t) = -\int_0^t K(t,\tau)r(u,\tau)d\tau + v(t), \quad t \in [0,T]$$

where

$$K(t,\tau) = \frac{\big(F_0(\cdot), (\widehat{G}_0\Delta\varphi_0)(\cdot,t-\tau)\big)}{(F_0,\varphi_0)}, \tag{3.2}$$

$$v(t)=\int_0^t K(t,\tau)F^{(\beta)}(\tau)d\tau+\sum_{j=1}^{m}\big(F_j(\cdot),(\widehat{G}_j\Delta\varphi_0)(\cdot,t)\big),\quad t\in[0,T]. \tag{3.3}$$

Theorem 3.1. *Assume that* (A1), (A2) *and* (2.2) *with* $m=1$ *(*$m=2$*) hold. Then there exists a unique solution* $(u,g)\in\mathcal{D}'_C(\bar{Q})\times C[0,T]$ *of the problem* (1.1)–(1.3) *with* $m=1$ *(*$m=2$*, respectively): u is defined by* (2.10),

$$g(t)=\big[F^{(\beta)}(t)-r(t)\big][(F_0,\varphi_0)]^{-1},\quad t\in[0,T] \tag{3.4}$$

where $r(t)$ *is the solution of the integral equation*

$$r(t)=-\int_0^t K(t,\tau)r(\tau)d\tau+v(t),\quad t\in[0,T] \tag{3.5}$$

with the integrable kernel (3.2)*, and the function* v *is defined by* (3.3)*.*

Proof. As in the proof of Theorem 2.10 we obtain

$$\begin{aligned}
\big|\big(F_0(\cdot),(\widehat{G}_0\Delta\varphi_0)(\cdot,t-\tau)\big)\big| &\le B_0\|(\widehat{G}_0\Delta\varphi_0)(\cdot,t-\tau)\|_{\mathcal{D}^{k_0+2}(\mathbb{R}^n)}\\
&\le \widehat{d}_{0,2}\|\varphi_0\|_{\mathcal{D}^{k_j+2}(\mathbb{R}^n)}(t-\tau)^{\beta-1},\\
\big|\big(F_j(\cdot),(\widehat{G}_j\Delta\varphi_0)(\cdot,t)\big)\big| &\le B_j\|(\widehat{G}_j\Delta\varphi_0)(\cdot,t)\|_{\mathcal{D}^{k_j}(\mathbb{R}^n)}\\
&\le \widehat{d}_{j,2}\|\varphi_0\|_{\mathcal{D}^{k_j+2}(\mathbb{R}^n)}t^{j-1},\quad j=\overline{1,m},
\end{aligned}$$

where $\widehat{d}_{j,2}$, $j=\overline{0,m}$ are positive constants. So, the kernel (3.2) is integrable, the function (3.3) is continuous on $[0,T]$, and the equation (3.5) has the unique solution $r\in C[0,T]$.

Let r,g be defined by (3.5), (3.4), respectively. Then on previous considerations the function (2.10) is the solution of the Cauchy problem (1.1)–(1.2) with the known $g(t)$, $m=1$ ($m=2$) and satisfies the conditions (2.2). Define $F^*(t)=\big(u(\cdot,t),\varphi_0(\cdot)\big)$. It satisfies the conditions

$$(F_j,\varphi_0)=F^{*(j-1)}(0),\quad j=\overline{1,m}. \tag{3.6}$$

From the over-determination condition (1.3) we obtain

$$g(t)=\big[F^{*(\beta)}(t)-\big(u(\cdot,t),\Delta\varphi_0(\cdot)\big)\big][(F_0,\varphi_0)]^{-1},\quad t\in[0,T]. \tag{3.7}$$

As in the previous reasoning we obtain that the function $\big(u(\cdot,t),\Delta\varphi_0(\cdot)\big)$ satisfies the equation (3.5), and by uniqueness of a solution of this equation we obtain $\big(u(\cdot,t),\Delta\varphi_0(\cdot)\big)=r(t)$ for all $t\in[0,T]$. Then it follows from (3.7) and (3.4) that $F^{*(\beta)}(t)=F^{(\beta)}(t)$, $t\in[0,T]$. From the conditions (2.2) and (3.6) we obtain $F^{*(j-1)}(0)=F^{(j-1)}(0)$, $j=\overline{0,m}$. Then $F^*(t)=F(t)$, $t\in[0,T]$. So, the function (2.10), where r,g are defined by (3.5), (3.4), respectively, is the solution of the problem (1.1)–(1.3) with $m=1$ ($m=2$).

If (u_1,g_1), (u_2,g_2) are two solutions of the problem (1.1)–(1.3) then for $u=u_1-u_2$, $g=g_1-g_2$ we obtain the problem

$$\begin{gathered}
Lu(x,t)=F_0(x)g(t),\quad (x,t)\in Q,\\
u(x,0)=0,\quad x\in\mathbb{R}^n,\\
\big(u(\cdot,t),\varphi_0(\cdot)\big)=0,\quad t\in[0,T].
\end{gathered}$$

As before, we find

$$\big(u(\cdot,t),\varphi(\cdot)\big) = -\int_0^t r(\tau)\big(F_0(\cdot),(\widehat{G}_0\varphi)(\cdot,t-\tau)\big)d\tau \quad \forall\varphi\in\mathcal{D}(\mathbb{R}^n),$$

$$g(t) = -\frac{r(t)}{(F_0,\varphi_0)}, \quad t\in[0,T],$$

where $r(t)$ is a solution of the Volterra integral equation

$$r(t) = -\int_0^t K(t,\tau)r(\tau)d\tau, \quad t\in[0,T].$$

By uniqueness of a solution of this equation we obtain $r(t)=0$ for all $t\in[0,T]$. Then, from the previous equalities, $g(t)=0$ for all $t\in[0,T]$ and $u=0$ in $\mathcal{D}'_C(\bar{Q})$. □

In the same way as above, we can prove the existence and uniqueness of a solution $(u,g)\in\mathcal{D}'_C(Q)\times C(0,T]$ of the inverse source Cauchy problem to equation

$$u_t^{(\alpha)} + a^2(-\Delta)^{\gamma/2}u = F_0(x)g(t), \quad (x,t)\in Q$$

where $\alpha\in(0,2)$, $\min\{n,2,\gamma\} > (n-1)/2$, $\gamma>\alpha$, $(-\Delta)^{\gamma/2}u$ is defined with the use of the Fourier transform as follows $\mathcal{F}[(-\Delta)^{\gamma/2}u] = |\lambda|^{\gamma/2}\mathcal{F}[u]$ and

$$\mathcal{D}'_C(Q) = \{v\in\mathcal{D}'(\bar{Q}) : (v(\cdot,t),\varphi(\cdot))\in C(0,T] \quad \forall\varphi\in\mathcal{D}(\mathbb{R}^n)\}.$$

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