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EXISTENCE OF SOLUTIONS TO $(2,p)$-LAPLACIAN EQUATIONS BY MORSE THEORY

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Abstract. In this article, we use Morse theory to investigate a type of Dirichlet boundary value problem related to the $(2,p)$-Laplacian operator, where the nonlinear term is characterized by the first eigenvalue of the Laplace operator. The investigation is heavily based on a new decomposition about the Banach space $W_0^{1,p}(\Omega)$, where $\Omega \subset \mathbb{R}^N$ $(N \geqslant 1)$ is a bounded domain with smooth enough boundary.

1. Introduction and statement of main results

Recently, much attention has been paid to the existence of solutions to the quasilinear elliptic problems of (q,p)-Laplacian type

$$\begin{gathered} -\Delta_q u - \Delta_p u = h(x,u), \quad x \in \Omega, \\ u = 0, \quad x \in \partial\Omega, \end{gathered} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N (N \geqslant 1)$ is a bounded domain with smooth boundary $\partial\Omega$, $\Delta_q u = \operatorname{div}(|\nabla u|^{q-2}\nabla u)$ and $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ are respectively the q-Laplacian and p-Laplacian of u. Solutions to (1.1) are the steady state solutions of the general reaction-diffusion equation

$$u_t = \operatorname{div}(H(u)\nabla u) + h(x,u), \tag{1.2}$$

where $H(u) = |\nabla u|^{q-2} + |\nabla u|^{p-2}$. Equation (1.2) has a wide range of applications in physics and related sciences such as biophysics [6], plasma physics [16] and chemical reaction design [1]. The stationary solutions to (1.2) have been studied by many authors using variational methods; see [2, 9, 11, 15, 17, 18].

In this article, we use Morse theory to show the existence of solutions to the (2,p)-Laplacian equation

$$\begin{gathered} -\Delta u - \Delta_p u = f(x,u), \quad x \in \Omega, \\ u = 0, \quad x \in \partial\Omega, \end{gathered} \tag{1.3}$$

where $p > 2$. This work is motivated by our previous research of (1.3); see [9, 18]. Assume that the nonlinear term f satisfies the following hypotheses.

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(H1) $f \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$, $f(x,t) \geqslant 0$ for all $x \in \overline{\Omega}$, $t > 0$ and $f(x,t) = 0$ for all $x \in \overline{\Omega}$, $t \leqslant 0$.

(H2) For f_0, $f_\infty < \infty$, the limits

$$\lim_{t \to 0^+} \frac{f(x,t)}{t} = f_0, \quad \lim_{t \to \infty} \frac{f(x,t)}{t^{p-1}} = f_\infty$$

exist uniformly for $x \in \overline{\Omega}$.

In [18], we show that if f satisfies (H1) and (H2) with $f_0 < \lambda_1$ and $f_\infty > \mu_1$, then (1.3) has a positive solution, where

$$\lambda_1 = \inf\Big\{ \int_\Omega |\nabla u|^2 : u \in H_0^1(\Omega), \int_\Omega |u|^2 = 1 \Big\},$$
$$\mu_1 = \inf\Big\{ \int_\Omega |\nabla u|^p : u \in W_0^{1,p}(\Omega), \int_\Omega |u|^p = 1 \Big\}.$$

What about the situation that $f_0 > \lambda_1$ or $f_\infty < \mu_1$? In [9], we have studied the case $f_\infty < \mu_1$ and obtain the existence result of non-negative solutions to (1.3). But we can not guarantee the existence of nontrivial solutions to (1.3); see [9, Proposition 1.3]. In this article, we pour our attention into the situation that $f_0 > \lambda_1$ and give some spirit for the existence of nontrivial solutions to (1.3). Now, we give some assumptions for the nonlinearity f of the present article.

(H3) $f \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$ and there exist a constant $C > 0$ and $q \in (p, p^*)$ such that for all $x \in \overline{\Omega}$, $t \in \mathbb{R}$,

$$|f(x,t)| \leqslant C(1 + |t|^{q-1}),$$

where $p^* = Np/(N-p)$ if $N > p$ and $p^* = \infty$ if $N \leqslant p$.

(H4) The following limit holds uniformly for $x \in \overline{\Omega}$,

$$\lim_{|t| \to \infty} \frac{F(x,t)}{|t|^p} = +\infty,$$

where $F(x,t) = \int_0^t f(x,s) \mathrm{d}s$.

(H5) There exists $R > 0$ such that for all $x \in \overline{\Omega}$, $\frac{f(x,t)}{|t|^{p-2}t}$ is increasing for $t \geqslant R$ and decreasing for $t \leqslant -R$.

(H6) There exists $l > 0$ such that

$$\lim_{t \to 0} \frac{f(x,t)}{t} = l$$

uniformly for $x \in \overline{\Omega}$.

Our main result is the following theorem.

Theorem 1.1. *Suppose that* (H3)–(H6) *are satisfied with* $l > \lambda_1$ *and* $l \notin \sigma(\Delta)$, *where* $\sigma(\Delta)$ *is the spectral set of* $(-\Delta, H_0^1(\Omega))$. *Then* (1.3) *has at least one nontrivial solution.*

Let X be a real Banach space and $I \in C^1(X, \mathbb{R})$. We say that $\{u_n\} \subset X$ is a Cerami sequence if $\{I(u_n)\}$ is bounded and $(1 + \|u_n\|)\|I'(u_n)\| \to 0$ as $n \to \infty$ and I satisfies the Cerami condition if any Cerami sequence has a convergent subsequence. Denote by $C_k(I, u)$ and $C_k(I, \infty)$ the kth critical group of I at an isolated critical point u and the kth critical group of I at infinity respectively.

To prove Theorem 1.1, we use the following theorem.

Theorem 1.2 ([3, 12])**.** *Suppose that $I \in C^1(X,\mathbb{R})$ satisfies the Cerami condition and I has only finitely many critical points. Let θ, the zero element of X, be a critical point of I. If for some $k \in \mathbb{N}$ we have $C_k(I,\theta) \neq C_k(I,\infty)$, then I has a nonzero critical point.*

2. Proof of main results

First, we introduce some notation. Let

$$X = W_0^{1,p}(\Omega), \quad \|u\| = \Big(\int_\Omega |\nabla u|^p\Big)^{1/p}, \quad \|u\|_H = \Big(\int_\Omega |\nabla u|^2\Big)^{1/2}.$$

X^* is the dual space of X and $c_1, c_2, \dots$ denote various positive constants whose exact values are not essential to the analysis of the relevant problems. $|D|$ means the Lebesgue measure of Lebesgue measurable set D. Define the energy functional $I : X \to \mathbb{R}$ by

$$I(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{1}{p}\int_\Omega |\nabla u|^p - \int_\Omega F(x,u), \quad u \in X.$$

It is obvious that the functional I is well defined and belongs to $C^1(X,\mathbb{R})$; see [9, 14]. Furthermore,

$$\langle I'(u), v\rangle = \int_\Omega \nabla u \cdot \nabla v + \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v - \int_\Omega f(x,u)v, \quad u, v \in X,$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between X^* and X. Clearly, critical points of I are the weak solutions to (1.3).

Lemma 2.1. *Assume that* (H3)–(H6) *are satisfied. Then I satisfies the Cerami condition.*

Proof. By (H3), (H4) and (H6), we know that there exists $\mu > 0$ such that for all $(x,t) \in \overline{\Omega} \times \mathbb{R}$,

$$F(x,t) \geqslant -\mu |t|^p. \tag{2.1}$$

Letting $\{u_n\}$ be a Cerami sequence of I, we first prove that $\{u_n\}$ is bounded in X. If $\{u_n\}$ is unbounded, up to a subsequence, then we may assume that for some $c \in \mathbb{R}$,

$$I(u_n) \to c, \quad \|u_n\| \to \infty, \quad (1 + \|u_n\|)\|I'(u_n)\| \to 0, \quad n \to \infty.$$

In particular,

$$\begin{aligned} c &= \lim_{n\to\infty} (I(u_n) - \frac{1}{p}\langle I'(u_n), u_n\rangle) \\ &= \lim_{n\to\infty} \Big(\Big(\frac{1}{2} - \frac{1}{p}\Big)\int_\Omega |\nabla u_n|^2 + \int_\Omega \Big(\frac{1}{p} f(x,u_n)u_n - F(x,u_n)\Big)\Big). \end{aligned} \tag{2.2}$$

Let $w_n = u_n/\|u_n\|$. Since $\{w_n\}$ is bounded in X, up to a subsequence, we have as $n \to \infty$,

$$\begin{gathered} w_n \rightharpoonup w \quad \text{in } X, \\ w_n \to w \quad \text{in } L^s(\Omega), \quad s \in [1, p^*), \\ w_n(x) \to w(x) \quad \text{a.e. } x \in \Omega. \end{gathered} \tag{2.3}$$

If $w = 0$, as in [8], we choose a sequence $\{t_n\} \subset [0,1]$ such that

$$I(t_n u_n) = \max_{t\in[0,1]} I(t u_n).$$

For any $\alpha > 0$, let $v_n = (2\alpha p)^{1/p} w_n$. Since $v_n \to 0, n \to \infty$ in $L^q(\Omega)$ and

$$|F(x,t)| \leqslant C(1+|t|^q),$$

the continuity of the Nemitskii operator implies that $F(\cdot, v_n) \to 0, n \to \infty$ in $L^1(\Omega)$. For n large enough, since $(2\alpha p)^{1/p}\|u_n\|^{-1} \in (0,1)$,

$$I(t_n u_n) \geqslant I(v_n) \geqslant 2\alpha - \int_\Omega |F(x, v_n)| \geqslant \alpha.$$

We have shown that $I(t_n u_n) \to \infty$. Since $I(0) = 0$, $I(u_n) \to c$, we know that $t_n \in (0,1)$ and

$$\begin{aligned} &\int_\Omega |\nabla(t_n u_n)|^2 + \int_\Omega |\nabla(t_n u_n)|^p - \int_\Omega f(x, t_n u_n) t_n u_n \\ &= \langle I'(t_n u_n), t_n u_n \rangle = t_n \frac{d}{dt}\Big|_{t=t_n} I(t u_n) = 0 \end{aligned} \tag{2.4}$$

for n large enough. Let $G(x,t) = f(x,t)t - pF(x,t)$ and

$$c_1 = 1 + \sup_{\overline{\Omega}\times[-R,R]} G(x,t) - \inf_{\overline{\Omega}\times[-R,R]} G(x,t).$$

By (H5), for any $x \in \overline{\Omega}$, $0 \leqslant s \leqslant t$ or $t \leqslant s \leqslant 0$,

$$G(x,s) \leqslant G(x,t) + c_1. \tag{2.5}$$

It follows from (2.4) that

$$\begin{aligned} &\big(\frac{1}{2} - \frac{1}{p}\big) \int_\Omega |\nabla u_n|^2 + \int_\Omega \Big(\frac{1}{p} f(x, u_n) u_n - F(x, u_n)\Big) \\ &\geqslant \big(\frac{1}{2} - \frac{1}{p}\big) \int_\Omega |\nabla t_n u_n|^2 + \int_\Omega \Big(\frac{1}{p} f(x, t_n u_n) t_n u_n - F(x, t_n u_n)\Big) - \frac{c_1}{p}|\Omega| \\ &= \frac{1}{2} \int_\Omega |\nabla t_n u_n|^2 + \frac{1}{p} \int_\Omega |\nabla t_n u_n|^p - \int_\Omega F(x, t_n u_n) - \frac{c_1}{p}|\Omega| \\ &= I(t_n u_n) - \frac{c_1}{p}|\Omega| \to \infty, \quad n \to \infty. \end{aligned}$$

This contradicts (2.2).

If $w \neq 0$, then the set $\Theta := \{x \in \Omega : w(x) \neq 0\}$ has positive Lebesuge measure. For a.e. $x \in \Theta$, we have $|u_n(x)| \to \infty$ as $n \to \infty$. By (f_2), as $n \to \infty$,

$$\frac{F(x, u_n(x))}{|u_n(x)|^p} |w_n(x)|^p \to \infty, \quad \text{a.e. } x \in \Theta.$$

Using (2.1) and the Fatou lemma, we get

$$\int_\Theta \frac{F(x, u_n(x))}{|u_n(x)|^p} |w_n(x)|^p \to \infty, \quad n \to \infty. \tag{2.6}$$

The Lebegue convergence theorem implies that $\int_{w=0} |w_n|^p \to 0$ as $n \to \infty$. Combining this with (2.1), we have

$$\int_{\Omega\setminus\Theta} \frac{F(x, u_n(x))}{|u_n(x)|^p} |w_n(x)|^p \geqslant -\mu \int_{\Omega\setminus\Theta} |w_n|^p \geqslant -\mu \int_\Omega |w_n|^p. \tag{2.7}$$

Since $X \hookrightarrow H_0^1(\Omega)$, there exists $c_2 > 0$ such that

$$\|u_n\|_H^2 \leqslant c_2 \|u_n\|^2.$$

It follows that

$$\frac{c_2}{2}\|u_n\|^2 + \frac{1}{p}\|u_n\|^p - c \geqslant \frac{1}{2}\|u_n\|_H^2 + \frac{1}{p}\|u_n\|^p - c = \int_\Omega F(x, u_n) + o(1).$$

By (2.6), (2.7) and (2.3), we have

$$\begin{aligned}\frac{c_2}{2}\|u_n\|^{2-p} + \frac{1}{p} - \frac{c}{\|u_n\|^p} &\geqslant \int_\Omega \frac{F(x, u_n)}{\|u_n\|^p} + o(1) \\ &= \Big(\int_\Theta + \int_{\Omega\setminus\Theta}\Big) \frac{F(x, u_n(x))}{|u_n(x)|^p} |w_n(x)|^p + o(1) \to \infty,\end{aligned}$$

as $n \to \infty$. This is impossible.

Secondly, up to a subsequence, we may assume that

$$\begin{gathered} u_n \rightharpoonup u_0 \quad \text{in } X, \\ u_n \to u_0 \quad \text{in } L^s(\Omega), \quad s \in [1, p^*), \\ u_n(x) \to u_0(x) \quad \text{a.e. } x \in \Omega. \end{gathered}$$

We will prove $u_n \to u_0$ in X. Defined $A : X \to X^*$ by

$$\langle Au, v\rangle = \int_\Omega \nabla u \cdot \nabla v + \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v, \quad u, v \in X.$$

For any $u, v \in X$, there exists $c_3 > 0$ such that (see [5])

$$\begin{aligned} &\langle Au_n - Au_0, u_n - u_0\rangle \\ &= \int_\Omega |\nabla(u_n - u_0)|^2 + \int_\Omega (|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_0|^{p-2}\nabla u_0) \cdot \nabla(u_n - u_0) \\ &\geqslant \int_\Omega |\nabla(u_n - u_0)|^2 + c_3 \int_\Omega |\nabla(u_n - u_0)|^p \\ &\geqslant c_3\|u_n - u_0\|^p. \end{aligned} \tag{2.8}$$

By (H3) we know that

$$\int_\Omega f(x, u_n)(u_n - u_0) \to 0, \quad n \to \infty.$$

According to $\langle I'(u_n), u_n - u_0\rangle \to 0$ as $n \to \infty$, we have $\langle Au_n, u_n - u_0\rangle \to 0$ as $n \to \infty$. Then

$$\langle Au_n - Au_0, u_n - u_0\rangle \to 0, \quad n \to \infty.$$

It follows form (2.8) that $u_n \to u_0$ as $n \to \infty$. The proof is complete. ☐

For the proof of Theorem thm1.1, we assume that I has only finitely many critical points. Since I satisfies the Cerami condition, the critical group $C_k(I, \infty), k \in \mathbb{N}$ makes sense.

Lemma 2.2. *Assume that* (H3)-(H6) *hold. Then for any* $k \in \mathbb{N}$, *we have* $C_k(I, \infty) \cong 0$.

Proof. Let $S = \{u \in X : \|u\| = 1\}$. By (H4), we see that for any $u \in S$,

$$I(tu) \to -\infty, \quad t \to \infty.$$

Choose $a < \min\{\inf_{\|u\|\leqslant 2} I(u), -\frac{1}{p}c_1|\Omega|\}$. Then for any $u \in S$, there exists $t > 1$ such that $I(tu) \leqslant a$, that is,

$$I(tu) = \frac{t^2}{2}\int_\Omega |\nabla u|^2 + \frac{t^p}{p} - \int_\Omega F(x, tu) \leqslant a.$$

By (2.5), we can find that $G(x,t) \geqslant -c_1$ when $s=0$. Therefore,

$$\begin{aligned}\frac{d}{dt}I(tu) &= t\int_\Omega |\nabla u|^2 + t^{p-1} - \int_\Omega uf(x,tu)\\ &\leqslant \frac{1}{t}\Big(pa + \int_\Omega pF(x,tu) - \int_\Omega tuf(x,tu)\Big)\\ &= \frac{1}{t}\Big(pa - \int_\Omega G(x,tu)\Big)\\ &\leqslant \frac{1}{t}(pa + c_1|\Omega|) < 0.\end{aligned}$$

By the implicit function theorem, there exists an unique $T \in C(S,\mathbb{R})$ such that for any $u \in S$, $I(T(u)u) = a$.

For $u \neq 0$, set $\widetilde{T}(u) = \frac{1}{\|u\|}T(\frac{u}{\|u\|})$. Then $\widetilde{T} \in C(X\backslash\{\theta\},\mathbb{R})$ and for all $u \in X\backslash\{\theta\}$, $I(\widetilde{T}(u)u) = a$. Moreover, if $I(u) = a$, then $\widetilde{T}(u) = 1$.

We define a function $\widehat{T} : X \setminus \{\theta\} \to \mathbb{R}$ as

$$\widehat{T}(u) = \begin{cases}\widetilde{T}(u), & I(u) > a,\\ 1, & I(u) \leqslant a.\end{cases}$$

Since $I(u) = a$ implies $\widetilde{T}(u) = 1$, we know that $\widehat{T} \in C(X\backslash\{\theta\},\mathbb{R})$.

Finally, we set $\eta : [0,1] \times (X \setminus \{\theta\}) \to X \setminus \{\theta\}$ as

$$\eta(s,u) = (1-s)u + s\widehat{T}(u)u.$$

It is easy to see that η is a strong deformation retract from $X \setminus \{\theta\}$ to I^a. Thus,

$$C_k(I,\infty) = H_k(X,I^a) \cong H_k(X, X\setminus\{\theta\}) = 0, \quad k \in \mathbb{N}.$$

Here, $H_k(A,B), k \in \mathbb{N}$ denotes the kth singular relative homology group of the topological pair (A,B) with coefficients in a field $\mathbb{F}$. The proof is complete. □

Since $l > \lambda_1$ and $l \notin \sigma(\Delta)$, there exists $n_0 \in \mathbb{N}$ such that $\lambda_1 < \lambda_2 \leqslant \dots \leqslant \lambda_{n_0} < l < \lambda_{n_0+1} \leqslant \dots$. Let φ_i be the corresponding eigenfunction of λ_i with $\|\varphi_i\| = 1$ and $V = \operatorname{span}\{\varphi_1,\varphi_2\dots\varphi_{n_0}\}$. Assume W is the complementary space of V in X. For the details on the term of complementary space we refer the readers to [4, p.94]. Then $X = V\bigoplus W$. We may assume that

$$\int_\Omega |\nabla u|^2 \geqslant \lambda_{n_0+1}\int_\Omega u^2, \quad u \in W. \tag{2.9}$$

In fact, since $\partial\Omega$ is smooth enough, $\operatorname{span}\{\varphi_i : i = 1,2,\dots\}$ is dense in $H^m_\vartheta(\Omega)$ with some $m \in \mathbb{N}$ and $2m-2 \geqslant N$, where

$$H^m_\vartheta(\Omega) = \big\{u \in H^m(\Omega) : \Delta^j u = 0 \quad \text{on } \partial\Omega \text{ for } j < \frac{m}{2}\big\}$$

denotes the Hilbert space with the scalar product

$$(u,v) := \begin{cases}\int_\Omega \Delta^k u\Delta^k v, & m = 2k,\\ \int_\Omega \nabla(\Delta^k u)\cdot\nabla(\Delta^k v), & m = 2k+1.\end{cases}$$

$H^m_\vartheta(\Omega)$ is a closed subspace of $H^m(\Omega)$ which satisfies $H^m_\vartheta(\Omega) \subset H^m(\Omega)$ with continuous embedding; see [7] for the details. Since the embedding $H^m(\Omega) \hookrightarrow W^{1,p}(\Omega)$ is continuous, we have that $\operatorname{span}\{\varphi_i : i = 1,2,\dots\}$ is dense in $W^{1,p}_0(\Omega)$. Let $W = \overline{\operatorname{span}\{\varphi_i : i = n_0+1, n_0+2,\dots\}}$, the closure of $\operatorname{span}\{\varphi_i : i = n_0+1, n_0+2,\dots\}$ in

X. Then the continuous embedding $W_0^{1,p}(\Omega) \hookrightarrow H_0^1(\Omega)$ implies the desired result (2.9).

Lemma 2.3. *Assume that* (H3)–(H6) *are satisfied. Then there exists $\rho > 0$ such that*

$$\begin{gathered} I(u) \leqslant 0, \quad u \in V, \quad \|u\| \leqslant \rho, \\ I(u) > 0, \quad u \in W, \quad 0 < \|u\| \leqslant \rho. \end{gathered} \tag{2.10}$$

Proof. Let $c_4 = \min_{1\leqslant i\leqslant n_0} \int_\Omega \varphi_i^2$. For $u = \sum_{i=1}^{n_0} a_i\varphi_i \in V$, define

$$|u|_1^2 = \sum_{i=1}^{n_0} a_i^2, \quad |u|_p^p = \sum_{i=1}^{n_0} |a_i|^p, \quad |u|_\infty = \max_{x\in\Omega} |u(x)|.$$

We know that there exist $c_5, c_6, c_7 > 0$ such that

$$\|u\|^2 \leqslant c_5|u|_1^2, \quad |u|_p^p \leqslant c_6\|u\|^p, \quad |u|_\infty \leqslant c_7\|u\|.$$

It follows from (H6) that there exist $\varepsilon, r > 0$ such that $\lambda_{n_0} + 2\varepsilon < l < \lambda_{n_0+1} - \varepsilon$ and

$$\frac{(l-\varepsilon)t^2}{2} \leqslant F(x,t) \leqslant \frac{(l+\varepsilon)t^2}{2}, \quad x \in \Omega, \ |t| \leqslant r. \tag{2.11}$$

Fix $0 < \rho_1 \leqslant \min\{r/c_7, (\varepsilon pc_4/(2c_5c_6))^{1/(p-2)}\}$. For $u = \sum_{i=1}^{n_0} a_i\varphi_i \in V$ with $\|u\| \leqslant \rho_1$, we have

$$\begin{aligned} I(u) &= \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{1}{p}\int_\Omega |\nabla u|^p - \int_\Omega F(x,u) \\ &\leqslant \frac{1}{2}\sum_{i=1}^{n_0} a_i^2\lambda_i \int_\Omega \varphi_i^2 + \frac{1}{p}\sum_{i=1}^{n_0} |a_i|^p - \frac{\lambda_{n_0}+\varepsilon}{2}\sum_{i=1}^{n_0} a_i^2 \int_\Omega \varphi_i^2 \\ &\leqslant -\frac{\varepsilon}{2}c_4\sum_{i=1}^{n_0} a_i^2 + \frac{1}{p}\sum_{i=1}^{n_0} |a_i|^p \\ &\leqslant -\frac{\varepsilon}{2}\frac{c_4}{c_5}\|u\|^2 + \frac{1}{p}c_6\|u\|^p \leqslant 0. \end{aligned}$$

By (H3), (2.9) and (2.11), we have the following estimates for $u \in W$.

$$\begin{aligned} I(u) &= \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{1}{p}\int_\Omega |\nabla u|^p - \int_\Omega F(x,u) \\ &= \frac{1}{2}\int_\Omega \big(|\nabla u|^2 - \lambda_{n_0+1}|u|^2\big) + \frac{1}{p}\int_\Omega |\nabla u|^p \\ &\quad - \int_{|u|\leqslant r} \Big(F(x,u) - \frac{\lambda_{n_0+1}}{2}|u|^2\Big) - \int_{|u|\geqslant r} \Big(F(x,u) - \frac{\lambda_{n_0+1}}{2}|u|^2\Big) \\ &\geqslant \frac{1}{p}\int_\Omega |\nabla u|^p - c_8\int_\Omega |u|^q \\ &\geqslant \frac{1}{p}\|u\|^p - c_9\|u\|^q. \end{aligned}$$

Then, there exists $\rho_2 > 0$ such that

$$I(u) > 0, \quad u \in W, \quad 0 < \|u\| \leqslant \rho_2.$$

Taking $\rho = \min\{\rho_1, \rho_2\}$, the proof is complete. □

Combining Lemma 2.3 with the linking theorem at zero [10], we can get the following lemma.

Lemma 2.4. *Assume that* (H3)–(H6) *holds. Then* θ*, the zero function of* X*, is a critical point of* I *and* $C_{n_0}(I,\theta) \ncong 0$.

Proof of Theorem 1.1. Since $f(x,0) = 0$, the zero function θ is a trivial critical point of I. According to Lemma 2.4, we have $C_{n_0}(I,\theta) \ncong 0$ while by Lemma 2.2 $C_k(I,\infty) \cong 0$ for all $k \in \mathbb{N}$. Now the desired result of Theorem 1.1 follows from Theorem 1.2. □

We conclude this article with some remarks. We obtained the existence of nontrivial solutions for a type of (2,p)-Laplacian equations with the nonlinearity f having $(p-1)$-superlinear and subcritical growth by critical point methods and Morse theory. In connection with the study, a natural question is what happens if the $(2,p)$-Laplacian is replaced by the general (q,p)-Laplacian with $1 < q < p$. In our proof of the main result, the main ingredient is the decomposition $X = V \oplus W$ with the property

$$\int_\Omega |\nabla u|^2 \geqslant \lambda_{n_0+1} \int_\Omega u^2, \quad u \in W,$$

which is deduced from the properties of the $-\Delta$'s eigenfunctions and some progresses for the polyharmonic equations with Navier boundary conditions. Recently, some researches have been obtained for the p-Laplacian type equation by Morse theory; see [13] for example. We think the question mentioned above deserves a further investigation with the developments of the spectrum theory of $-\Delta_p$ and some ideas form the researches of the p-Laplacian type equation by Morse theory.

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