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REMARKS ON THE PRECEDING PAPER BY CRESPO, IVORRA AND RAMOS ON THE STABILITY OF BIOREACTOR PROCESSES

JESÚS ILDEFONSO DÍAZ

Abstract. In this short note we indicate some improvement of an article published in this journal by Crespo, Ivorra and Ramos [6]. The techniques used are connected with several smoothing effects associated with linear partial differential operators which give rise to some accretive operators in $L^1(\Omega)$, as well as with some $H^2(\Omega)$ estimates independent on time.

1. Introduction and results

In the previous article in this journal Crespo, Ivorra and Ramos [6] obtained an interesting theorem on the stability of bioreactor processes under suitable conditions. The main goal of this short note is to present some improvement of their main result by using quite different techniques of proof.

Let us consider the non-dimensional form of the system considered in [6],

$$
\begin{gathered}
\frac{\partial S}{\partial t}=\frac{\sigma^2}{\mathrm{Th}_S}\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial S}{\partial r})+\frac{1}{\mathrm{Th}_S}\frac{\partial^2 S}{\partial z^2}+\frac{1}{\mathrm{Da}}\frac{\partial S}{\partial z}-\mu(S)B \quad \text{in } \Omega\times(0,T),\\
\frac{\partial B}{\partial t}=\frac{\sigma^2}{\mathrm{Th}_B}\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial B}{\partial r})+\frac{1}{\mathrm{Th}_B}\frac{\partial^2 B}{\partial z^2}+\frac{1}{\mathrm{Da}}\frac{\partial B}{\partial z}+\mu(S)B \quad \text{in } \Omega\times(0,T),\\
\frac{1}{\mathrm{Th}_S}\frac{\partial S}{\partial z}+\frac{1}{\mathrm{Da}}S=\frac{1}{\mathrm{Da}} \quad \text{in } \Gamma_{\mathrm{in}}\times(0,T),\\
\frac{1}{\mathrm{Th}_B}\frac{\partial B}{\partial z}+\frac{1}{\mathrm{Da}}B=0 \quad \text{in } \Gamma_{\mathrm{in}}\times(0,T),\\
\frac{\partial S}{\partial r}=\frac{\partial B}{\partial r}=0 \quad \text{in } \Gamma_{\mathrm{sym}}\times(0,T),\\
\frac{\partial S}{\partial r}=\frac{\partial B}{\partial r}=0 \quad \text{in } \Gamma_{\mathrm{wall}}\times(0,T),\\
\frac{\partial S}{\partial z}=\frac{\partial B}{\partial z}=0 \quad \text{in } \Gamma_{\mathrm{out}}\times(0,T),
\end{gathered}
\tag{1.1}
$$

jointly with by the initial conditions

$$
S(\cdot,\cdot,0)=S_{\mathrm{init}} \quad \text{and} \quad B(\cdot,\cdot,0)=B_{\mathrm{init}} \quad \text{in } \Omega, \tag{1.2}
$$

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where $\Omega = (0,1) \times (0,1)$ is the nondimensional domain, $\Gamma_{\rm in} = (0,1) \times \{1\}$, $\Gamma_{\rm out} = (0,1) \times \{0\}$, $\Gamma_{\rm wall} = \{1\} \times (0,1)$ and $\Gamma_{\rm sym} = \{0\} \times (0,1)$ are the non-dimensional boundary edges. This system can be more conceptually written as a special case of the system

$$\begin{gathered} u_t - L_1 u - f(u,v) = 0 \quad \text{in } \Omega \times (0,T), \\ v_t - L_2 v + f(u,v) = 0 \quad \text{in } \Omega \times (0,T), \\ \frac{\partial u}{\partial n} + b_1(x) u = g(x) \quad \text{on } \partial\Omega \times (0,T), \\ \frac{\partial v}{\partial n} + b_2(x) v = 0 \quad \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) \quad \text{on } \Omega \\ v(x,0) = v_0(x) \quad \text{on } \Omega, \end{gathered} \tag{1.3}$$

with obvious choices of the data. Which is relevant in our approach is that Ω is a convex bounded open set of R^3 with piece-wise smooth boundary and the elliptic operators are coercive and can be expressed, for $k = 1,2$, in terms of

$$L_k w = -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_j}(a_{ij}^k(x)\frac{\partial w}{\partial x_i}) + \sum_{i=1}^{N} \frac{\partial}{\partial x_j}(a_i^k(x) w)$$

with smooth coefficients $a_{ij}^k, a_i^k \in C^1(\overline{\Omega})$. We also may assume that $b_k(x)$ are smooth coefficients on each part of $\partial\Omega$ and

$$f(u,v) = \mu(u) v, \tag{1.4}$$

with $\mu(\cdot)$ satisfying the assumptions in [6]: i.e. $\mu \in C[0,+\infty)$, $\mu(0) = 0$,

$$0 < \mu(z) \le \overline{\mu} z + \overline{\mu} \quad \text{for any } z > 0, \tag{1.5}$$

and such that one of the following properties hold:

$$\mu \text{ is increasing and concave}, \tag{1.6}$$

or

$$\text{there exists } s > 0 \text{ such that } \mu \text{ is increasing on } (0,s) \text{ and decreasing on } (s,+\infty). \tag{1.7}$$

The L^1-framework associated to this system allows to get an improvement alternative to the main existence and uniqueness result of [5] (we do not try to get the more general assumptions on the data but only the ones which are relevant to our purposes).

Theorem 1.1. *Assume nonnegative the data $u_0, v_0 \in L^1(\Omega), g \in L^\infty(\partial\Omega)$. Then system (1.3) admits a weak solution $u, v \in C([0,T] : L^1(\Omega))^2$ with u, v being nonnegative functions. Moreover:*

(i) $u_t(\cdot,t), v_t(\cdot,t) \in L^\infty(\Omega)$ for $t \in (0,T]$ and $u, v \in L^2(\delta, T : H^2(\Omega))$ for any $\delta > 0$,

(ii) there exists a constant $C(T) > 0$ such that for $t \in (0,T]$ we have

$$\begin{gathered} \|u(\cdot,t)\|_{L^\infty(\Omega)} \le \max(t^{-3/2}\|u_0\|_{L^1(\Omega)}, \|g\|_{L^\infty(\partial\Omega)}) \\ \|v(\cdot,t)\|_{L^\infty(\Omega)} \le C(T) t^{-3/2} \|v_0\|_{L^1(\Omega)} e^{\overline{\mu} t}, \end{gathered}$$

(iii) if, in addition, $\mu(\cdot)$ is Lipschitz continuous then the weak solution is unique.

As a consequence of the above result, in order to define the notion of asymptotically stable solution (see [6, Definition 3.1]) the boundedness requirement on the perturbation of the stationary solution is not any important restriction since for any $t > 0$ the solution becomes bounded (even if the initial data are unbounded).

As a second remark, concerning the paper [6], we point out that in the special case of the constant (washout) stationary solution $(S_1^*, B_1^*) = (1, 0)$ the convergence, as $t \to +\infty$, proved in [6, Theorem 3.6], holds in sharper functional spaces (besides to hold in $L^2(\Omega)$).

Theorem 1.2. *Assume (as in* [6, Theorem 3.6]*) that*

$$\mu(1) < \frac{\mathrm{Th}_B}{(2\mathrm{Da})^2} + \frac{(\beta_1(\mathrm{Da}, \mathrm{Th}_B))^2}{\mathrm{Th}_B}, \tag{1.8}$$

with $\beta_1(\mathrm{Da}, \mathrm{Th}_B)$ *given in* [6, Theorem 3.6]. *Assume* $\|u_0 - 1\|_{L^1(\Omega)}$ *and* $\|v_0\|_{L^1(\Omega)}$ *small enough. Then* $(u(\cdot, t), v(\cdot, t))$ *converges to* $(1, 0)$, *strongly in* $H^1(\Omega) \cap L^\infty(\Omega)$ *and weakly in* $H^2(\Omega)$, *as* $t \to +\infty$.

Outline of the proof of Theorem 1.1. As in [5] the results hold by application of a fixed point argument. So, the qualitative properties mentioned in the statement follow from the correspondent properties established for the uncoupled systems

$$\begin{gathered} u_t - L_1 u = F(x,t) \quad \text{in } \Omega \times (0,T), \\ \frac{\partial u}{\partial n} + b_1(x) u = g(x) \quad \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) \quad \text{on } \Omega, \end{gathered} \tag{1.9}$$

and

$$\begin{gathered} v_t - L_2 v + a(x,t) v = 0 \quad \text{in } \Omega \times (0,T), \\ \frac{\partial v}{\partial n} + b_2(x) v = 0 \quad \text{on } \partial\Omega \times (0,T), \\ v(x,0) = v_0(x) \quad \text{on } \Omega. \end{gathered} \tag{1.10}$$

The existence, uniqueness and the regularity mentioned in (ii) for solutions of the uncoupled problem is a consequence of the Semigroup Theory in Banach Spaces applied to the space $L^1(\Omega)$. The smoothing effect mentioned in ii) was proved in [14] for the case of Dirichlet boundary conditions by using some properties of the associated Green function obtained in [16]. The adaptation to the case of Robin boundary conditions is a routine matter after the pioneering work by Brezis-Strauss [3] in which more general boundary conditions were considered (see also the treatment made in [1], [9], [2]). Notice that the convexity assumption on Ω allows the application of the $H^2(\Omega)$-regularity techniques (see, e.g. [13]). This assumption could be relaxed but we do not enter into details here.

The application of the Schauder Fixed Point Theorem is a mimetic application of the proof given in [5] for $L^2(\Omega)$ initial data (a $L^1(\Omega)$-compactness argument can be also applied as in [12]). The regularity $F \in W^{1,1}(0, T : L^\infty(\Omega))$ assumed in [14] can be obtained, for the regularity proof of the fixed point, by means of a previous Steklov average regularization process following passing to the limit as in [11].

The uniqueness of weak solution under the Lipschitz continuity condition on $\mu(\cdot)$ is an easy task which can be obtained, for instance, by a simple modification of the proof given in [5]. As a matter of fact the uniqueness of solution can be obtained (as in [9]) when $\mu(\cdot)$ is merely assumed to be Hölder continuous and increasing.

The non-negativeness (and even the existence) of solutions for the coupled system can be also obtained as in [5] (see also [8], [9], [15], [7] and [4] for some related works). □

Outline of the proof of Theorem 1.2. Once we know the $L^2(\Omega)$ convergence ([6, Theorem 3.6]), we get that, for any $\epsilon > 0$,

$$\max\Big(\int_\epsilon^t \frac{d}{d\tau}\|u(\tau)\|^2_{L^2(\Omega)}d\tau, \int_\epsilon^t \|\nabla u(\tau)\|^2_{L^2(\Omega)}d\tau\Big) \leq C,$$

for some $C > 0$ independent of t (see [6, formula (26)]). Then $u \in W^{1,1}(\epsilon, +\infty : L^2(\Omega)) \cap L^2(\epsilon, +\infty : H^1(\Omega))$. Moreover, by multiplying by $\frac{\partial}{\partial\tau}u(\tau)$, it is possible to show that $u \in W^{1,\infty}(\epsilon, +\infty : L^2(\Omega)) \cap L^\infty(\epsilon, +\infty : H^1(\Omega))$, as in [10, Theorem 6]. The coercivity of the linear operator L_1u allows to show that in fact $u(t) \to 1$ in $H^1(\Omega)$, as $t \to +\infty$ (see [10, Theorem 2]). Notice that since the ω-limit set for the system is formed by a discrete set of stationary solutions then we have the convergence when $t \to +\infty$ and not only for a subsequence (see [10, Remark 2]). Finally, since the regularity obtained in Theorem 1 and the $L^2(\Omega)$ convergence lead the universal estimate $\|u\|_{L^2(\delta,+\infty:H^2(\Omega))} \leq C$, for some $C > 0$ (here $\delta > 0$ is fixed), we get that $u(t) \rightharpoonup 1$ in $H^2(\Omega)$, as $t \to +\infty$, and since the inclusion $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$ is compact (remember the three-dimensional formulation of the problem) we get that $u(t) \to 1$ in $L^\infty(\Omega)$ as $t \to +\infty$. The proof of the convergence $v(\cdot, t) \to 0$, strongly in $H^1(\Omega) \cap L^\infty(\Omega)$ and weakly in $H^2(\Omega)$, as $t \to +\infty$, is entirely similar. □

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Jesús Ildefonso Díaz
Departamento de Matemática Aplicada, & Instituto de Matemática Interisciplinar, Universidad Complutense de Madrid, Plaza de Ciencias, 3, 28040 Madrid, Spain
E-mail address: jidiaz@ucm.es