Electronic Journal of Differential Equations, Vol. 2017 (2017), No. 196, pp. 1–18.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

BLOW-UP OF SOLUTIONS FOR WEAKLY COUPLED SYSTEMS OF COMPLEX GINZBURG-LANDAU EQUATIONS

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Communicated by Mokhtar Kirane

Abstract. Blow-up phenomena of weakly coupled systems of several evolution equations, especially complex Ginzburg-Landau equations is shown by a straightforward ODE approach, not by the so-called test-function method used in [38] which gives the natural blow-up rate. The difficulty of the proof is that, unlike the single case, terms which come from the Laplacian cannot be absorbed into the weakly coupled nonlinearities. A similar ODE approach is applied to heat systems by Mochizuki [32] to obtain the lower estimate of lifespan.

1. Introduction

In this article, we study a blow-up phenomena (blow-up rate and estimates of lifespan) for the following Cauchy problem for the Ginzburg-Landau systems with the weakly coupled nonlinearity by developing an ODE approach used in [32]:

$$\begin{gathered}
\partial_t u + \alpha_1 \Delta u = \beta_1 |v|^p, \quad t \in [0,T), \quad x \in \mathbb{X}, \\
\partial_t v + \alpha_2 \Delta v = \beta_2 |u|^q, \quad t \in [0,T), \quad x \in \mathbb{X}, \\
u(0,x) = u_0(x), \quad v(0,x) = v_0(x), \quad x \in \mathbb{X},
\end{gathered} \tag{1.1}$$

where $u = u(t,x)$ and $v = v(t,x)$ are unknown complex-valued functions of (t,x), $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{C}\backslash\{0\}$ and $p, q \geq 1$ are constants, $\mathbb{X}$ denotes the n-dimensional Euclidean space $\mathbb{R}^n$ or the Torus $\mathbb{T}^n$ with $n \in \mathbb{N}$, and $\Delta := \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ denotes the Laplacian on $\mathbb{X}$. T denotes the maximal existence time of the function (u,v) and is called lifespan.

Our aim in the present paper is to understand a blow-up mechanism of the weakly-coupled system (1.1) by an ODE argument which is more direct than the test-function method used in [38].

When $\alpha_1 = \alpha_2 = -1$, $\beta_1 = \beta_2 = 1$, u and v are real-valued and u_0 and v_0 are non-trivial non-negative, the problem (1.1) becomes the Cauchy problem for the

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2010 *Mathematics Subject Classification.* 35B44, 35Q55.
Key words and phrases. Weakly coupled; complex Ginzburg-Landau equation; blow-up.

Submitted July 22, 2017. Published August 8, 2017.

following heat systems with the weakly coupled nonlinearities:

$$\begin{gathered} \partial_t u - \Delta u = v^p, \quad t \in [0,T), \quad x \in \mathbb{X}, \\ \partial_t v - \Delta v = u^q, \quad t \in [0,T), \quad x \in \mathbb{X}, \\ u(0,x) = u_0(x) \geq 0, \quad v(0,x) = v_0(x) \geq 0, \quad x \in \mathbb{X}. \end{gathered} \tag{1.2}$$

We extend the blow-up result for (1.2) into the complex setting.

We, at first, recall several previous results about blow-up of single heat equations. The following Cauchy problem for the single heat equation with a power-type nonlinearity has been extensively studied:

$$\begin{gathered} \partial_t u - \Delta u = u^p, \quad t \in [0,T), \quad x \in \mathbb{R}^n, \\ u(0,x) = u_0(x), \quad x \in \mathbb{R}^n, \end{gathered} \tag{1.3}$$

where u is unknown non-negative function, u_0 is non-negative functions which is not identically 0, and $p \geq 1$. As a pioneering work, Fujita [10] showed that if $p < p_F := 1 + 2/n$, where p_F is called the Fujita exponent, then any non-trivial non-negative solutions blow up in a finite time by using the contradiction argument. See also [11]. He used the following ODE coming from (1.3) without the Laplacian to show the blow-up result:

$$\frac{d}{dt} f(t) = f(t)^p, \tag{1.4}$$

where f is a positive C^1-function. If $f(0) > 0$, then the solution of (1.4) is given by

$$f(t) = (f(0)^{-p+1} - (p-1)t)^{-\frac{1}{p-1}}$$

and therefore f blows up at $t = \frac{f(0)^{-p+1}}{p-1}$ and the blow-up rate is $-\frac{1}{p-1}$. Later, Hayakawa [15] and Kobayashi, Sirao, and Tanaka [24] independently showed that in the critical case where $p = p_F$, any non-trivial non-negative solutions blow up in a finite time. Moreover, Giga and Kohn [14] proved that if $p > 1$ when $n = 1, 2$ and $1 < p < \frac{n+2}{n-2}$ when $n \geq 3$, then the positive solution u of (1.3) blows up in type I rate, i.e.

$$\|u(t)\|_{L^\infty(\mathbb{R}^n)} \leq C(T-t)^{-\frac{1}{p-1}}, \quad t \in [0,T) \tag{1.5}$$

for some constant C independent of t and T. The estimate (1.5) implies that behavior of blow-up solutions of (1.3) and that of (1.4) have similar blow-up rate and lifespan. Later, Matano and Merle [28] extended the result of [14] to $\frac{n+2}{n-2} < p < p_{JL}$, where p_{JL} is the Joseph-Lundgren exponent given by $p_{JL} := \infty$ $(3 \leq n \leq 10)$, $p_{JL} := 1 + \frac{4}{n-4-\sqrt{n-1}}$ $(n \geq 11)$. We remark that Herrero and Velázquez [17] showed that for $n \geq 11$ and $p_{JL} < p$, there exists a blow-up solution of (1.3) which violates the estimate (1.5). Thus the analogy from ODE (1.4) does not work to the heat equation (1.3) in the case where the power and spacial dimension are sufficiently high.

In the subcritical case where $p < p_F$, Lee and Ni [27] obtained the sharp estimate of lifespan to (1.3)

$$c\epsilon^{-\frac{1}{\frac{1}{p-1}-\frac{n}{2}}} \leq T \leq C\epsilon^{-\frac{1}{\frac{1}{p-1}-\frac{n}{2}}},$$

where $\epsilon > 0$ is sufficiently small and u_0 is replaced by $u_0 = \epsilon\widetilde{u}_0$. We note that they used the comparison principle for heat equations and introduced blow-up sub-solutions and super-solutions. We remark that the exponent $(\frac{1}{p-1} - \frac{2}{n})^{-1}$ is sharp with respect to the size of the initial data. For more information about (1.3), see [9, 16, 29, 31] and the references therein.

On the other hand, Zhang [38] studied non-existence of global weak solutions of a semilinear parabolic equations for some initial data and a nonlinearity of a variable coefficient by using the so-called test function method, which is originally developed by Baras and Pierre [2] to study the existence and non-existence for semilinear elliptic and parabolic equations. His method is based on a contradiction argument with a weak form. If we apply his method to (1.3), the weak form is

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}^n} u(t,x)(-\partial_t \phi(t,x) - \Delta\phi(t,x))\,dx\,dt \\
&= \int_{\mathbb{R}^n} u_0(x)\phi(x) + \int_0^T \int_{\mathbb{R}^n} |u(t,x)|^p \phi(t,x)\,dx\,dt,
\end{aligned}$$

where ϕ is a smooth, non-negative, compact supported function (see also [30]). By modifying his argument, Kuiper [26] gave an estimate of lifespan for some parabolic equations for some slowly decreasing initial datum. His argument has been applied to semilinear Schrödinger and damped wave equations. However, the test function method does not give blow-up rate since, roughly speaking, unknown functions are canceled out in this argument.

Next we recall several previous results of blow-up for single complex Ginzburg-Landau equations,

$$\begin{gathered}
\partial_t u - \alpha\Delta u = F(u), \quad t \in [0,T), \quad x \in \mathbb{X}, \\
u(0,x) = u_0(x), \quad x \in \mathbb{X},
\end{gathered} \tag{1.6}$$

where u is complex-valued unknown function, $\alpha \in \mathbb{C}\backslash\{0\}$, and F is the nonlinearity. Ogawa and Tsutsumi [33] studied blow-up for the case where $i\alpha < 0$, $F(u) = i|u|^4 u$, $\mathbb{X} = \mathbb{T}$. Later, Ozawa and Yamazaki [35] studied (1.6) in the case where $\operatorname{Re}\alpha \geq 0$, $F(u) = (\kappa + i\beta)|u|^p + \gamma u$ with $\beta > 0$, $\kappa, \gamma \in \mathbb{R}$, and

$$\operatorname{Im} \int_{\mathbb{T}} u_0(x)dx > 0.$$

They introduced

$$M(t) = \frac{e^{-\gamma t}}{2\pi} \operatorname{Im} \int_{\mathbb{T}} u(t,x)dx, \quad 0 \leq t < T$$

and showed that M is positive and satisfies the ordinary differential inequality (ODI)

$$\frac{d}{dt} M(t) \geq \frac{e^{(p-1)\gamma t}}{2\pi} M(t)^p, \quad 0 \leq t < T, \tag{1.7}$$

which implies that M blows up at a finite positive time by a comparison principle. We remark that in their argument, the embedding $L^p(\mathbb{T}^n) \hookrightarrow L^1(\mathbb{T}^n)$ and identity

$$\int_{\mathbb{T}^n} \Delta u(t,x)dx = 0$$

for $n \geq 1$ play a crucial role to obtain the ODI (1.7). Oh [34] studied blow-up of (1.6) in the case where $\alpha \in \mathbb{C}\backslash\{0\}$, $F(u) = \lambda|u|^p$, and $\mathbb{X} = \mathbb{T}^n$ with $n \geq 1$ by the test function method. He showed that if

$$\operatorname{Re}\lambda \operatorname{Im} \int_{\mathbb{T}^n} u_0(x)dx < 0, \quad \text{or} \quad \operatorname{Im}\lambda \operatorname{Re} \int_{\mathbb{T}^n} u_0(x)dx > 0, \tag{1.8}$$

then there is no global weak solutions. The second and third authors [21] applied the test function method to study blow-up of (1.6) in the case where $\alpha = i$, $F(u) = \lambda|u|^p$, and $\mathbb{X} = \mathbb{R}^n$ with $n \geq 1$. They showed if $1 < p \leq 1 + \frac{2}{n}$ and $u_0 \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ satisfies

$$\operatorname{Re}\lambda \operatorname{Im} \int_{\mathbb{R}^n} u_0(x)dx < 0, \quad \text{or} \quad \operatorname{Im}\lambda \operatorname{Re} \int_{\mathbb{R}^n} u_0(x)dx > 0,$$

then the solution blows up in a finite time. Moreover, by modifying the test function method of Kuiper, the second author and Inui [19, 20] obtained an upper bound of lifespan of solutions for the same problem with $1 < p < 1 + \frac{4}{n}$ for some initial datum which decay slowly or have singularity at the origin, where $1 + \frac{4}{n}$ is the scaling critical exponent of $L^2(\mathbb{R}^n)$. Recently, the first author and Ozawa [12, 13] extended the results of Oh and the second author and Inui [19, 20]. They showed blow-up phenomena (blow-up rate and upper bounds of lifespan of solutions) by an ODE argument connected with the test function method. Indeed, in the torus case where $\mathbb{X} = \mathbb{T}^n$ and $p > 1$, for $L^1(\mathbb{T}^n)$-initial data, let

$$\widetilde{M}(t) = \operatorname{Re}\Big(\overline{\lambda} \int_{\mathbb{T}^n} u(t,x)dx\Big), \quad t \in [0,T).$$

Then, as with the approach of Ozawa and Yamazaki [35], they showed that, if

$$\widetilde{M}(0) = \operatorname{Re}\Big(\overline{\lambda} \int_{\mathbb{T}^n} u_0(x)dx\Big) > 0, \tag{1.9}$$

then $\widetilde{M}$ satisfies

$$\begin{aligned} \frac{d}{dt}\widetilde{M}(t) &= |\lambda|^2 \int_{\mathbb{T}^n} |u(t,x)|^p dx \\ &\geq |\mathbb{T}^n|^{-(p-1)}|\lambda|^{2-p}\widetilde{M}(t)^p, \quad t \in [0,T) \end{aligned}$$

and this ODI implies that $\widetilde{M}$ blows up at a finite time. We remark that the condition (1.9) includes the condition (1.8). Moreover, in the Euclidean case where $\mathbb{X} = \mathbb{R}^n$ and $1 < p < p_F$, for $L^1(\mathbb{T}^n)$-initial data, if $u_0 \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, then solutions for (1.6) satisfy the ODI,

$$\frac{d}{dt}\Big(\int_{\mathbb{R}^n} u(t,x)\phi(x)dx - \frac{1}{2}U_0\Big) \geq CU_0^{\frac{n(p-1)}{\frac{2}{p-1}-n}}\Big(\int_{\mathbb{R}^n} u(t,x)\phi(x)dx - \frac{1}{2}U_0\Big)^p \tag{1.10}$$

for $0 < t < T$, where C is some positive constant and

$$U_0 = \int_{\mathbb{R}^n} u_0(x)\phi(x)dx$$

with a smooth test function ϕ. Therefore, $\int_{\mathbb{R}^n} u(t,x)\phi(x)dx - \frac{1}{2}U_0$ is a super-solution of ODE (1.4) with a constant. Thus the comparison principle implies that the inequality,

$$\begin{aligned} \int_{\mathbb{R}^n} u(t,x)\phi(x)dx &\geq C_1\Big(U_0^{-p+1} - C_2U_0^{\frac{n(p-1)}{\frac{2}{p-1}-n}}t\Big)^{-\frac{1}{p-1}} \\ &= C_1U_0^{-\frac{n}{\frac{2}{p-1}-n}}\Big(U_0^{-\frac{1}{\frac{1}{p-1}-\frac{n}{2}}} - C_2t\Big)^{-\frac{1}{p-1}} \end{aligned}$$

holds with some positive constants C_1 and C_2 for any $0 < t < T$. In their argument, the condition $p < p_F$ plays a crucial role to obtain ODI (1.10) by a scaling transformation of ϕ. For related subjects, we refer the reader to [23].

Let us recall several previous results for blow-up of weakly coupled heat systems (1.2). Escobedo and Herrero [4] showed that if $p, q > 0$ and $pq > 1$, then all positive solutions blow up if

$$\max\Big(\frac{p+1}{pq-1}, \frac{q+1}{pq-1}\Big) \geq \frac{n}{2} \tag{1.11}$$

and global solutions with small data exist if

$$\max\Big(\frac{p+1}{pq-1}, \frac{q+1}{pq-1}\Big) < \frac{n}{2}.$$

We remark that if $p = q$, then (1.11) coincides with $p \leq p_F$. In this sense, their results correspond to the case of single heat equation (1.3). We remark that it is shown in [3] that even in the case where $\min(p, q) < 1$, (1.2) has local solutions. Moreover, Andreucci, Herrero, and Velázquez [1] showed that when $\mathbb{X} = \mathbb{R}^n$, the solution (u, v) to (1.2) satisfies

$$u(t,x) \leq C(T-t)^{-\frac{p+1}{pq-1}}, \quad v(t,x) \leq C(T-t)^{-\frac{q+1}{pq-1}} \tag{1.12}$$

for $0 < t < T$ with a positive constant C (see [5] for bounded domain cases). After that, for general unbounded domain $\mathbb{X} = \Omega$, Fila and Souplet [8] obtained the estimates (1.12), provided that solution (u, v) satisfies the stronger condition below than (1.2),

$$\max\Big(\frac{p+1}{pq-1}, \frac{q+1}{pq-1}\Big) \geq \frac{n+1}{2}. \tag{1.13}$$

We expect that when $\mathbb{X} = \Omega$ is an exterior domain, the condition (1.13) above can be relaxed to the condition (1.11). For corresponding results of general component cases, we refer the readers to [7] and the references therein. Next, Mochizuki [32] obtained the sharp estimate of lifespan of the same problem (1.2) with $p, q > 1$ for non-negative initial data. In order to obtain the upper bound of lifespan, he used the following ODE systems:

$$\begin{gathered} \frac{d}{dt}f(t) + C_1 f(t) = C_2 g(t)^p, \quad t \in [0, T), \\ \frac{d}{dt}g(t) + C_1 g(t) = C_2 f(t)^q, \quad t \in [0, T), \\ f(0) = f_0, \quad g(0) = g_0 \end{gathered} \tag{1.14}$$

with positive constants C_1, C_2 and positive numbers f_0, g_0. He showed that the product $f(t)g(t)$ enjoys the following ODI:

$$\frac{d}{dt}(f(t)g(t)) + C_1 f(t)g(t) \geq C_3 (f(t)g(t))^{\frac{(p+1)(q+1)}{p+q+2}}, t \in [0, T) \tag{1.15}$$

with some positive constant C_3, which implies that solutions of (1.14) blow up at a finite time. Since with small $\epsilon > 0$ and a solution (u, v) for (1.2),

$$\Big(\int_{\mathbb{R}^n} u(t,x)e^{-\epsilon|x|^2}dx, \quad \int_{\mathbb{R}^n} v(t,x)e^{-\epsilon|x|^2}dx\Big)$$

is a super-solution of (1.14), the solution of (1.2) blows up at a finite time and the lifespan is estimated from above by that of solutions of (1.14). Moreover, he composed super-solutions of (1.2) which blow up in a finite time with the blow-up solutions of

$$\begin{gathered} \frac{d}{dt}f(t) = g(t)^p, \quad t \in [0,T), \\ \frac{d}{dt}g(t) = f(t)^q, \quad t \in [0,T), \\ f(0) = f_0, \quad g(0) = g_0. \end{gathered} \tag{1.16}$$

We note that f and g satisfy the ODE

$$\frac{1}{q+1}\frac{d}{dt}f(t)^{q+1} = f(t)^p g(t)^q = \frac{1}{p+1}\frac{d}{dt}g(t)^{p+1}, \quad t \in [0,T). \tag{1.17}$$

This identity divides ODE systems (1.14) into two single ODEs. The lifespan of solutions to the divided ODEs can be estimated. In conclusion, if $p > q > 1$, for for the initial data

$$(u_0(x), v_0(x)) = (\epsilon\langle x\rangle^{-a}, \epsilon^{\frac{2(q+1)-\min(b,n)(pq-1)}{2(p+1)-\min(a,n)(pq-1)}}\langle x\rangle^{-b}),$$

with small $\epsilon > 0$ and some $a, b \neq n$ satisfying $0 < \min(a,n) < \frac{2(p+1)}{pq-1}$, $0 < \min(b,n) < \frac{2(q+1)}{pq-1}$, the lifespan is estimated by

$$c\epsilon^{-\frac{1}{\frac{p+1}{pq-1}-\frac{n}{2}}} \le T \le C\epsilon^{-\frac{1}{\frac{p+1}{pq-1}-\frac{n}{2}}}$$

with some positive constants c and C independent of ϵ. See also [6, 25, 37] and reference therein.

In this article, we introduce an unified ODE approach to show the blow-up results (blow-up rate and upper bound of lifespan) for weakly coupled systems. As an application, we show the blow-up rate and upper bound of lifespan for the complex Ginzburg-Landau systems on both of torus and Euclidean spaces.

Theorem 1.1. *Let $\mathbb{X} = \mathbb{T}^n$. Let $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{C}\backslash\{0\}$. Let $p, q > 0$ satisfying $pq > 1$ and $p \ge q$. Let (u_0, v_0) be not $(0,0)$ but $L^1(\mathbb{T}^n) \times L^1(\mathbb{T}^n)$ function. Assume that there exists a solution $(u, v) \in [C^2((0,T) \times \mathbb{T}^n)]^2$ to (1.1) with the initial data (u_0, v_0). Then (u, v) blows up in a finite time. In particular, the estimates*

$$T \le T_0 := \Big(\int_{\mathbb{T}^n} u_0(x)dx\Big)^{-\frac{pq-1}{p+1}} \tag{1.18}$$

and

$$\int_{\mathbb{T}^n} u(t,x)dx \ge C(T_0 - t)^{-\frac{p+1}{pq-1}}, \tag{1.19}$$

$$\int_{\mathbb{T}^n} v(t,x)dx \ge C(T_0 - t)^{-\frac{q+1}{pq-1}}, \tag{1.20}$$

hold for $t \in [0,T)$ with some positive constant C independent of t and T.

Theorem 1.2. *Let $\mathbb{X} = \mathbb{R}^n$. Let $\alpha_1, \alpha_2 < 0$ and $\beta_1, \beta_2 \in \mathbb{C}\backslash\{0\}$. Let $p, q > 0$ satisfying $pq > 1$ and $p \ge q$. Let (u_0, v_0) be not $(0,0)$and be $L^1_{\rm loc}(\mathbb{R}^n) \times L^1_{\rm loc}(\mathbb{R}^n)$ function satisfying that $\overline{\beta}_1 u_0$ and $\overline{\beta}_2 v_0$ are non-negative valued. Assume that there exists a classical solution $(u, v) \in [C^2((0,T) \times \mathbb{R}^n)]^2$ to (1.1) with the initial data (u_0, v_0). Then (u, v) blows up in a finite time. In particular, there is a compactly supported non-negative function $\phi \in C^2(\mathbb{R}^n)$ such that the estimates*

$$T \le T_1 := C\Big(\int_{\mathbb{R}^n} u_0(x)\phi(x)dx\Big)^{-\frac{1}{\frac{p+1}{pq-1}-\frac{n}{2}}} \tag{1.21}$$

and

$$\overline{\beta}_1 \int_{\mathbb{R}^n} u(t,x)\phi(x)dx \geq C(T_1 - t)^{-\frac{p+1}{pq-1}}, \tag{1.22}$$

$$\overline{\beta}_2 \int_{\mathbb{R}^n} v(t,x)\phi(x)dx \geq C(T_1 - t)^{-\frac{q+1}{pq-1}}, \tag{1.23}$$

hold for $t \in [0,T)$ *with some positive constant* C *independent of* t *and* T.

We collect estimates of ODE systems (1.14) and (1.16) in Section 2. Then we give a proof of Propositions 1.1 and 1.2 in Sections 4 and 3, respectively.

2. Preparation: ODE Arguments

In this section, we collect sub-solutions for some weakly coupled ODE systems connected to (1.1). In particular, we show that solutions of weakly coupled systems of two ODEs are larger than solutions of single ODEs with a modified nonlinearity.

Here, the following solutions for single ODEs give the basis to obtain the explicit sub-solutions of ODE systems.

Lemma 2.1. *Let* $\rho > 1$, $\mu > 0$, *and* $f_0 > 0$. *Set*

$$T_f := \mu^{-1}(\rho - 1)^{-1} f_0^{1-\rho}$$

$$f(t) := \{f_0^{1-\rho} - (\rho - 1)\mu t\}^{-\frac{1}{\rho-1}}, \quad \text{for } t \in [0, T_f).$$

Then $f \in C^\infty((0,T_f))$ *is the unique solution to the Cauchy problem for the ODE*

$$f'(t) = \mu f(t)^\rho,$$

$$f(0) = f_0.$$

It is well known that we can compare super-solutions and sub-solutions for single ODEs with appropriate initial datum. Since (1.15) holds, the comparison principle for singe ODE implies blow-up of (1.14) Moreover, Kamke [22] showed that the comparison principle holds for weakly coupled ODE systems. In particular, the following Lemma 2.2 holds and we can obtain lower bounds of each solutions instead of sum of product of solutions. We also refer the reader the text book of Hsu [18] and remark that their statements are more general than Lemma 2.2.

Lemma 2.2 ([18, Theorem 2.6.3][22, Satz 6])**.** *Let* $C \geq 0$, $T > 0$, *and let* $p, q > 0$ *satisfy* $pq > 1$. *Let* $\mathcal{F}, \mathcal{G}$ *be non-decreasing functions on* $\mathbb{R}$. *Let* $f_1, f_2, g_1, g_2 \in C^1([0,T))$ *satisfy ODIs,*

$$\frac{d}{dt} f_2(t) + C f_2(t) \geq \mathcal{G}(g_2(t)),$$

$$\frac{d}{dt} g_2(t) + C g_2(t) \geq \mathcal{F}(f_2(t)),$$

and

$$\frac{d}{dt} f_1(t) + C f_1(t) \leq \mathcal{G}(g_1(t)),$$

$$\frac{d}{dt} g_1(t) + C g_1(t) \leq \mathcal{F}(f_1(t))$$

for $0 \leq t < T$ *with the initial condition* $f_2(0) > f_1(0)$ *and* $g_2(0) > g_1(0)$. *Then* $f_2(t) > f_1(t)$ *and* $g_2(t) > g_1(t)$ *for* $0 \leq t < T$.

Proof. On the contrary, we assume that there exists $t_0 \in (0,T)$ such that $f_2(t_0) = f_1(t_0)$. Then, we can define

$$T_f = \min\{0 < t \le t_0;\ f_1(t) = f_2(t)\}.$$

Thus, the function $\Phi(t) := e^{Ct}(f_2(t) - f_1(t)) \in C^1((0,T))$ satisfies $\Phi(t) > 0$ for $t \in [0, T_f)$ and $\Phi(T_f) = 0$. Hence, there exists $T_f' \in (0, T_f)$ such that $\Phi'(T_f') < 0$. On the other hand, by the ODIs, we see that

$$\Phi'(t) = \frac{d}{dt}\left\{e^{Ct}(f_2(t) - f_1(t))\right\} \ge e^{Ct}\{\mathcal{G}(g_2(t)) - \mathcal{G}(g_1(t))\}, \text{ for } t \in [0,T),$$

which leads to $e^{CT_f'}\{\mathcal{G}(g_2(T_f')) - \mathcal{G}(g_1(T_f'))\} < 0$. This estimate and the monotonicity of $\mathcal{G}$ imply $g_2(T_f') < g_1(T_f')$. Noting that $g_2(0) > g_1(0)$ and g_2 and g_1 are continuous on $[0,T)$, there exists $t_1 \in (0, T_f')$ such that $g_1(t_1) = g_2(t_1)$. Therefore we can also define

$$T_g = \min\{0 < t \le t_1;\ g_1(t) = g_2(t)\}.$$

By the definitions of T_f, T_f' and T_g, we obtain $T_f > T_f' > T_g$. However, $T_g > T_f$ holds by the same argument, which leads to a contradiction. Therefore for any $t \in (0,T)$, $f_2(t) \ne f_1(t)$. By $f_2(0) > f_1(0)$ and the continuity of the functions f_1 and f_2, we have $f_2(t) > f_1(t)$ for $t \in [0,T)$. In the similar manner, we can prove $g_2(t) > g_1(t)$ for $t \in [0,T)$, which completes the proof of the lemma. □

In the following proposition, we show explicit solutions for weakly coupled systems of ordinary differential equations, which will be used to prove blow-up of solutions for the weakly coupled systems of parabolic equations on the n-dimensional Euclidean space $\mathbb{R}^n$ (see Proposition 4.1).

Proposition 2.3 ([32, Lemma 4.1],[36])**.** *Let $f_0, g_0 > 0$ and $C_p, C_q, T > 0$, and let $p, q > 0$ satisfy $pq > 1$. Let $(f,g) \in (C^1([0,T)))^2$ be non-negative and a solution of*

$$\begin{gathered}
\frac{d}{dt}f(t) = (p+1)C_p g(t)^p, \quad t \in [0,T),\\
\frac{d}{dt}g(t) = (q+1)C_q f(t)^q, \quad t \in [0,T),\\
f(0) = f_0, \quad g(0) = g_0.
\end{gathered}$$

Then the equality

$$C_q f(t)^{q+1} - C_q f_0^{q+1} = C_p g(t)^{p+1} - C_p g_0^{p+1} \tag{2.1}$$

holds for any $0 \le t < T$. Moreover, if $C_q f_0^{q+1} \ge C_p g_0^{p+1}$, then the following estimates hold:

$$\begin{aligned}
g(t) \ge &\left\{C_p^{\frac{pq-1}{(p+1)(q+1)}} C_q^{-\frac{pq-1}{(p+1)(q+1)}} f_0^{-\frac{pq-1}{p+1}} - 2^{\frac{-pq}{q+1}}(pq-1) C_p^{\frac{q}{q+1}} C_q^{\frac{1}{q+1}} t\right\}^{-\frac{q+1}{pq-1}}\\
&- \left(C_p^{-1} C_q f_0^{q+1} - g_0^{p+1}\right)^{\frac{1}{p+1}} \quad \text{for } 0 < t < T,
\end{aligned} \tag{2.2}$$

$$T \le \frac{2^{\frac{pq}{q+1}}}{pq-1} C_p^{-\frac{1}{p+1}} C_q^{-\frac{p}{p+1}} f_0^{-\frac{pq-1}{p+1}}. \tag{2.3}$$

We note that the right-hand side of (2.3) *is the blow-up time of the right-hand side of* (2.2).

Proof. Set $F := C_q f^{q+1} \in C^1([0,T))$ and $G := C_p g^{p+1} \in C^1([0,T))$. Then

$$F'(t) = (q+1)C_q f(t)^q f'(t) = (q+1)(p+1)C_q C_p f(t)^q g(t)^p = G'(t)$$

for $t \in [0,T)$, which implies that the identity $\{F(t) - G(t)\}' = 0$ for $t \in [0,T)$. Thus by integrating it with respect to time over $[0,t)$, we have ((2.1). Therefore the identities

$$\begin{aligned}
f(t) &= \left\{ C_q^{-1} C_p g(t)^{p+1} - C_q^{-1} C_p g_0^{p+1} + f_0^{q+1} \right\}^{\frac{1}{q+1}}, \\
g(t) &= \left\{ C_p^{-1} C_q f(t)^{q+1} - C_p^{-1} C_q f_0^{q+1} + g_0^{p+1} \right\}^{\frac{1}{p+1}}, \\
f'(t) &= (p+1)C_p \left\{ C_p^{-1} C_q f(t)^{q+1} - C_p^{-1} C_q f_0^{q+1} + g_0^{p+1} \right\}^{\frac{p}{p+1}}, \\
g'(t) &= (q+1)C_q \left\{ C_q^{-1} C_p g(t)^{p+1} - C_q^{-1} C_p g_0^{p+1} + f_0^{q+1} \right\}^{\frac{q}{q+1}}.
\end{aligned}$$

hold for any $t \in [0,T)$. Since $F(0) \geq G(0)$ by the assumption, the estimate

$$a^{p+1} + b^{p+1} \geq 2^{-p}(a+b)^{p+1} \quad \text{for } a, b \geq 0,$$

implies

$$\begin{aligned}
g'(t) &= (q+1)C_q \left\{ C_q^{-1} C_p g(t)^{p+1} - C_q^{-1} C_p g_0^{p+1} + f_0^{q+1} \right\}^{\frac{q}{q+1}} \\
&= (q+1) C_p^{\frac{q}{q+1}} C_q^{\frac{1}{q+1}} \left\{ g(t)^{p+1} + C_p^{-1} C_q f_0^{q+1} - g_0^{p+1} \right\}^{\frac{q}{q+1}} \\
&\geq 2^{\frac{-pq}{q+1}} (q+1) C_p^{\frac{q}{q+1}} C_q^{\frac{1}{q+1}} \left\{ g(t) + \left(C_p^{-1} C_q f_0^{q+1} - g_0^{p+1} \right)^{\frac{1}{p+1}} \right\}^{\frac{p+1}{q+1} q},
\end{aligned}$$

for $t \in [0,T)$. Then by solving the ODI and the triangle inequality, we obtain

$$\begin{aligned}
& g(t) + \left(C_p^{-1} C_q f_0^{q+1} - g_0^{p+1} \right)^{\frac{1}{p+1}} \\
&\geq \left[\{ g_0 + \left(C_p^{-1} C_q f_0^{q+1} - g_0^{p+1} \right)^{\frac{1}{p+1}} \}^{-\frac{pq-1}{q+1}} - 2^{\frac{-pq}{q+1}} (pq-1) C_p^{\frac{q}{q+1}} C_q^{\frac{1}{q+1}} t \right]^{-\frac{q+1}{pq-1}} \\
&\geq \left\{ C_p^{\frac{pq-1}{(p+1)(q+1)}} C_q^{-\frac{pq-1}{(p+1)(q+1)}} f_0^{-\frac{pq-1}{p+1}} - 2^{\frac{-pq}{q+1}} (pq-1) C_p^{\frac{q}{q+1}} C_q^{\frac{1}{q+1}} t \right\}^{-\frac{q+1}{pq-1}},
\end{aligned}$$

for $t \in [0,T)$, where we have used the identity

$$\frac{p+1}{q+1} q - 1 = \frac{pq-1}{q+1}.$$

From the right hand side of (2.4), we see that the maximal existence time T of the function (f,g) is estimated by

$$\begin{aligned}
T &\leq \frac{2^{\frac{pq}{q+1}}}{pq-1} C_p^{-\frac{q}{q+1}} C_q^{-\frac{1}{q+1}} C_p^{\frac{pq-1}{(p+1)(q+1)}} C_q^{-\frac{pq-1}{(p+1)(q+1)}} f_0^{\frac{1-pq}{p+1}} \\
&= \frac{2^{\frac{pq}{q+1}}}{pq-1} C_p^{-\frac{1}{p+1}} C_q^{-\frac{p}{p+1}} f_0^{\frac{1-pq}{p+1}},
\end{aligned}$$

where we have used the identities

$$\begin{gathered}
\frac{pq-1}{(p+1)(q+1)} - \frac{q}{q+1} = -\frac{q+1}{(p+1)(q+1)} = -\frac{1}{p+1}, \\
-\frac{pq-1}{(p+1)(q+1)} - \frac{1}{q+1} = -\frac{pq-1+p+1}{(p+1)(q+1)} = -\frac{p}{p+1}.
\end{gathered}$$

This completes the proof. □

Proposition 2.3 gives the main idea of the proof of Theorem 1.1 and Theorem 1.2. However, unlike the single heat equation with the power-type nonlinearity, localized average of solutions to (1.1) with $\mathbb{X} = \mathbb{R}^n$ does not enjoy the ODI of Proposition 2.3 because we are considering the weak coupled nonlinearity. More precisely, in the case of single heat equation (1.3), the localized average of solution to (1.3) is a super-solution for the ODI (1.4), since the localized average of $-\Delta u$ can be absorbed into the nonlinearity u^p. On the other hand, in the case of heat system (1.1) with $\mathbb{X} = \mathbb{R}^n$, since the nonlinearity is weakly coupled, the localized averages of each component of the solution with the Laplacian can not be absorbed. Thus we use the following modified ODE system in order to treat the heat systems with $\mathbb{X} = \mathbb{R}^n$ and the weakly coupled nonlinearity (see Section 4).

Corollary 2.4. *Let $f_0, g_0, \omega > 0$, $p, q > 0$ with $pq > 1$ and $C_p, C_q > 0$. Let $(f, g) \in \big(C^1([0,T))\big)^2$ be non-negative and a solution for*

$$
\begin{gathered}
\frac{d}{dt} f(t) + \frac{\omega}{q+1} f(t) = (p+1) C_p g(t)^p, \quad t \in [0,T), \\
\frac{d}{dt} g(t) + \frac{\omega}{p+1} g(t) = (q+1) C_q f(t)^q, \quad t \in [0,T), \\
f_0 = f_0, \quad g_0 = g_0.
\end{gathered}
$$

Then

$$
C_q f(t)^{q+1} e^{\omega t} - C_q f_0^{q+1} = C_p g(t)^{p+1} e^{\omega t} - C_p g_0^{p+1}.
$$

holds for any $0 \le t < T$. Moreover, if

$$
\begin{aligned}
\max\Big(& 2^{\frac{p+1}{q+1}\frac{pq}{pq-1}} (q+1)^{-\frac{p+1}{pq-1}} (p+1)^{-\frac{p+1}{pq-1}} \omega^{\frac{p+1}{pq-1}} C_p^{-\frac{1}{pq-1}} C_q^{-\frac{p}{pq-1}}, \\
& C_q^{-\frac{1}{q+1}} C_p^{\frac{1}{q+1}} g_0^{\frac{p+1}{q+1}}\Big) < f_0,
\end{aligned}
\tag{2.4}
$$

then, for $0 \le t < T$,

$$
\begin{aligned}
e^{\frac{\omega}{p+1} t} g(t) \ge \Big\{ &\Big(\frac{C_p}{C_q}\Big)^{\frac{pq-1}{(p+1)(q+1)}} f_0^{-\frac{pq-1}{p+1}} \\
&- 2^{-\frac{pq}{q+1}} (q+1)(p+1) \omega^{-1} C_q^{\frac{1}{q+1}} C_p^{\frac{q}{q+1}} \Big(1 - e^{-\frac{\omega(pq-1)}{(p+1)(q+1)} t}\Big) \Big\}^{-\frac{q+1}{pq-1}} \\
&- (C_q C_p^{-1} f_0^{q+1} - g_0^{p+1})^{\frac{1}{p+1}}
\end{aligned}
$$

and

$$
T \le -\frac{(q+1)(p+1)}{\omega(pq-1)} \log\Big(1 - \frac{2^{\frac{pq}{q+1}}}{(q+1)(p+1)} \omega C_q^{-\frac{p}{p+1}} C_p^{-\frac{1}{p+1}} f_0^{-\frac{pq-1}{p+1}}\Big).
$$

Proof. Set $F = C_q f^{q+1} \in C^1([0,T))$ and $G = C_p g^{p+1} \in C^1([0,T))$. Then by the equation, we have

$$
\begin{aligned}
F'(t) + \omega F(t) &= (q+1) C_q f(t)^q \Big(f'(t) + \frac{\omega}{q+1} f\Big) \\
&= (q+1)(p+1) C_q C_p f(t)^q g(t)^p \\
&= G'(t) + \omega G(t),
\end{aligned}
$$

for any $t \in [0,T)$, which implies that

$$F(t) - G(t) = (F(0) - G(0))e^{-\omega t},$$

for any $t \in [0,T)$. Moreover we obtain

$$\begin{aligned}
f(t) &= \Big\{C_q^{-1}C_p g(t)^{p+1} + \big(f_0^{q+1} - C_q^{-1}C_p g_0^{p+1}\big)e^{-\omega t}\Big\}^{\frac{1}{q+1}},\\
g(t) &= \Big\{C_p^{-1}C_q f(t)^{q+1} + \big(g_0^{p+1} - C_p^{-1}C_q f_0^{q+1}\big)e^{-\omega t}\Big\}^{\frac{1}{p+1}},\\
f'(t) + \frac{\omega}{q+1}f(t) &= (p+1)C_p\Big\{C_p^{-1}C_q f(t)^{q+1} + (g_0^{p+1} - C_p^{-1}C_q f_0^{q+1})e^{-\omega t}\Big\}^{\frac{p}{p+1}},\\
g'(t) + \frac{\omega}{p+1}g(t) &= (q+1)C_q\Big\{C_q^{-1}C_p g(t)^{p+1} + (f_0^{q+1} - C_q^{-1}C_p g_0^{p+1})e^{-\omega t}\Big\}^{\frac{q}{q+1}},
\end{aligned}$$

for $t \in [0,T)$. Since we have $F(0) \ge G(0)$ by the assumption of f_0, by the estimate $a^{p+1} + b^{p+1} \ge 2^{-p}(a+b)^{p+1}$ for $a, b \ge 0$, we have

$$\begin{aligned}
&g'(t) + \frac{\omega}{p+1}g(t)\\
&= (q+1)C_q\Big\{C_q^{-1}C_p g(t)^{p+1} + \big(f_0^{q+1} - C_q^{-1}C_p g_0^{p+1}\big)e^{-\omega t}\Big\}^{\frac{q}{q+1}}\\
&= (q+1)C_q^{\frac{1}{q+1}}C_p^{\frac{q}{q+1}}e^{-\frac{\omega q}{q+1}t}\big(e^{\omega t}g(t)^{p+1} + C_qC_p^{-1}f_0^{q+1} - g_0^{p+1}\big)^{\frac{q}{q+1}}\\
&\ge 2^{-\frac{pq}{q+1}}(q+1)C_q^{\frac{1}{q+1}}C_p^{\frac{q}{q+1}}e^{-\frac{\omega q}{q+1}t}\Big\{e^{\frac{\omega}{p+1}t}g(t) + (C_qC_p^{-1}f_0^{q+1} - g_0^{p+1})^{\frac{1}{p+1}}\Big\}^{\frac{p+1}{q+1}q},
\end{aligned}$$

for any $t \in [0,T)$. Let

$$\widetilde{g}(t) = e^{\frac{\omega}{p+1}t}g(t) + (C_qC_p^{-1}f_0^{q+1} - g_0^{p+1})^{\frac{1}{p+1}}.$$

Then

$$\begin{aligned}
\widetilde{g}(t)' &\ge 2^{-\frac{pq}{q+1}}(q+1)C_q^{\frac{1}{q+1}}C_p^{\frac{q}{q+1}}e^{(\frac{1}{p+1} - \frac{q}{q+1})\omega t}\widetilde{g}(t)^{\frac{p+1}{q+1}q}\\
&= 2^{-\frac{pq}{q+1}}(q+1)C_q^{\frac{1}{q+1}}C_p^{\frac{q}{q+1}}e^{-\frac{pq-1}{(p+1)(q+1)}\omega t}\widetilde{g}(t)^{\frac{p+1}{q+1}q}
\end{aligned}$$

and therefore

$$\widetilde{g}(t) \ge \Big(\widetilde{g}_0^{-\frac{pq-1}{q+1}} - 2^{-\frac{pq}{q+1}}(p+1)(q+1)\omega^{-1}C_q^{\frac{1}{q+1}}C_p^{\frac{q}{q+1}}(1 - e^{-\frac{pq-1}{(p+1)(q+1)}\omega t})\Big)^{-\frac{q+1}{pq-1}},$$

where $\tilde{g}_0 = \tilde{g}(0)$. We remark that the right hand side of this estimate blows-up at a certain time because we assume f_0 is sufficiently large. Since

$$\widetilde{g}_0 \ge C_q^{\frac{1}{p+1}}C_p^{-\frac{1}{p+1}}f_0^{\frac{q+1}{p+1}},$$

the lifespan is estimated by

$$\begin{aligned}
T &\le -\frac{(p+1)(q+1)}{\omega(pq-1)}\log\Big(1 - \frac{2^{\frac{pq}{q+1}}}{(q+1)(p+1)}\omega C_q^{-\frac{1}{q+1}}C_p^{-\frac{q}{q+1}}\widetilde{g}_0^{-\frac{pq-1}{q+1}}\Big)\\
&\le -\frac{(q+1)(p+1)}{\omega(pq-1)}\log\Big(1 - \frac{2^{\frac{pq}{q+1}}}{(q+1)(p+1)}\omega C_q^{-\frac{p}{p+1}}C_p^{-\frac{1}{p+1}}f_0^{-\frac{pq-1}{p+1}}\Big).
\end{aligned}$$

$\square$

3. Blow-up of a Weakly Coupled Systems of Complex Ginzburg-Landau Equations on the Torus

In this section, we show Theorem 1.1. Namely, we consider the Cauchy problem for weakly coupled systems of complex Ginzburg-Landau equations on the torus:

$$\begin{gathered} \partial_t u+\alpha_1\Delta u=\beta_1|v|^p, \quad t\in[0,T), \quad x\in\mathbb{T}^n,\\ \partial_t v+\alpha_2\Delta v=\beta_2|u|^q, \quad t\in[0,T), \quad x\in\mathbb{T}^n,\\ u(0,x)=u_0(x), \quad v(0,x)=v_0(x), \quad x\in\mathbb{T}^n, \end{gathered} \tag{3.1}$$

where $p,q>0$ with $pq\geq 1$ and $\alpha_1,\alpha_2,\beta_1,\beta_2\in\mathbb{C}\backslash\{0\}$. Here, u_0,v_0 are non-zero complex-valued $L^1(\mathbb{T}^n)$-functions and u,v are complex-valued unknown functions of (t,x). We remark that if $\alpha_1,\alpha_2\in -i\mathbb{R}$, (3.1) is the weakly coupled systems of Schrödinger equations and if $\alpha_1,\alpha_2<0$, (3.1) is the weakly coupled systems of heat equations. Then Proposition 2.3 gives an a priori upper bound of lifespan and lower bounds of blow-up solution. The precise statement of Theorem 1.1 is the following:

Proposition 3.1. *Let $u_0,v_0\in L^1(\mathbb{T}^n)$ satisfy*

$$\begin{aligned} &\frac{|\beta_2|^{2+p}}{q+1}(2\pi)^{-n(p-1)}\Big\{\operatorname{Re}\Big(\overline{\beta}_1\int_{\mathbb{T}^n}u_0(x)dx\Big)\Big\}^{q+1}\\ &\geq \frac{|\beta_1|^{2+q}}{p+1}(2\pi)^{-n(q-1)}\Big\{\operatorname{Re}\Big(\overline{\beta}_2\int_{\mathbb{T}^n}v_0(x)dx\Big)\Big\}^{p+1}>0. \end{aligned}$$

If there is a solution $(u,v)\in C^1((0,T),H^2(\mathbb{T}^n))$ for (3.1) with the initial datum (u_0,v_0). Then (u,v) blows up in a finite time. Moreover, let T be the lifespan. Then with some positive constants C_1,C_2,C_3, the estimates

$$\begin{gathered} T\leq T_3:=C_1\Big\{\operatorname{Re}\Big(\overline{\beta_1}\int_{\mathbb{T}^n}u_0(x)dx\Big)\Big\}^{\frac{pq-1}{p+1}},\\ \operatorname{Re}\Big(\overline{\beta_1}\int_{\mathbb{T}^n}u(t,x)dx\Big)\geq C_2(T_3-t)^{-\frac{p+1}{pq-1}}, \quad t\in[0,T)\\ \operatorname{Re}\Big(\overline{\beta_2}\int_{\mathbb{T}^n}v(t,x)dx\Big)\geq C_3(T_3-t)^{-\frac{q+1}{pq-1}}, \quad t\in[0,T). \end{gathered}$$

Proof. Let

$$U(t)=\operatorname{Re}\Big(\overline{\beta}_1\int_{\mathbb{T}^n}u(t,x)dx\Big), \quad V(t)=\operatorname{Re}\Big(\overline{\beta}_2\int_{\mathbb{T}^n}v(t,x)dx\Big).$$

Multiplying $\overline{\beta}_1$ by the both sides of the first equation of (3.1), integrating over $\mathbb{T}^n$, taking real part, and by the Hölder inequality,

$$\frac{d}{dt}U(t)=|\beta_1|^2|\beta_2|^{-p}\int_{\mathbb{T}^n}|\overline{\beta}_2 v(t,x)|^p dx\geq |\beta_1|^2|\beta_2|^{-p}(2\pi)^{-n(p-1)}V(t)^p.$$

Here we have used the condition that $p\geq 1$ and the fact that

$$\int_{\mathbb{T}^n}\Delta f(x)dx=0$$

for $f\in H^2(\mathbb{T}^n)$. Similarly,

$$\frac{d}{dt}V(T)\geq |\beta_2|^2|\beta_1|^{-q}(2\pi)^{-n(q-1)}U(t)^q.$$

Then by Proposition 2.3 with

$$C_p = \frac{|\beta_1|^2|\beta_2|^{-p}(2\pi)^{-n(p-1)}}{p+1}, \quad C_q = \frac{|\beta_2|^2|\beta_1|^{-q}(2\pi)^{-n(q-1)}}{q+1},$$

we obtain Proposition 3.1. ☐

4. Blow-up of weakly coupled systems of complex Ginzburg-Landau equations on the Euclidean space

In this section, we show Theorem 1.2. To state Theorem 1.2, we introduce some notation. Let $B(r) \subset \mathbb{R}^n$ be the open ball centered at the origin with radius r. Let $\phi \in C^2(B(1))$ be a non-negative function satisfying

$$\begin{gathered} -\Delta\phi \le \lambda\phi, \quad \text{in } B(1), \\ \phi = 0, \quad \text{on } \partial B(1), \\ \nabla\phi = 0, \quad \text{on } \partial B(1) \end{gathered}$$

for some $\lambda > 0$. We remark that such ϕ exists. For example, let ψ be an eigenfunction of $-\Delta$ on $B(1)$ with the Dirichlet condition with a positive eigenvalue λ. Then ψ^2 also satisfies the Neumann condition and

$$-\Delta(\psi^2) = 2\psi(-\Delta\psi) - 2|\nabla\psi|^2 \le 2\lambda\psi^2.$$

Let $p, q > 0$, $pq > 1$ and $U_0, V_0 > 0$. Put

$$\begin{aligned} R_0 &= R_0(p,q,\phi,\lambda,\beta_1,\beta_2,U_0,V_0) = \max(R_1,R_2), \\ R_1 &= R_1(p,q,\phi,\lambda,\beta_1,\beta_2,U_0) \\ &= 2^{\frac{pq}{q+1}\frac{1}{2-n\frac{pq-1}{p+1}}} \Big(\frac{p+1}{q+1}\Big)^{\frac{1}{pq-1}\frac{1}{2\frac{p+1}{pq-1}-n}} (\max(|\alpha_1|,|\alpha_2|)\lambda)^{\frac{1}{2-n\frac{pq-1}{p+1}}} \\ &\quad \times |\beta_1|^{\frac{pq-2}{2(p+1)-n(pq-1)}} |\beta_2|^{-\frac{p}{2(p+1)-n(pq-1)}} U_0^{-\frac{1}{2\frac{p+1}{pq-1}-n}}, \\ R_2 &= R_2(p,q,\phi,\beta_1,\beta_2,U_0,V_0) \\ &= \Big(\frac{q+1}{p+1}\Big)^{\frac{1}{n(p-q)}} |\beta_1|^{\frac{q+2}{n(p-q)}} |\beta_2|^{-\frac{2+p}{n(p-q)}} \|\phi\|_{L^1(B(1))}^{-\frac{1}{n}} V_0^{\frac{p+1}{n(p-q)}} U_0^{-\frac{q+1}{n(p-q)}}. \end{aligned}$$

Moreover, we define positive constants C_1, C_2, C_3 as follows:

$$\begin{aligned} C_1 &= C_1(p,q,\phi,\beta_1,\beta_2) \\ &= \Big(\frac{q+1}{p+1}\Big)^{\frac{pq-1}{(p+1)(q+1)}} \Big(\frac{|\beta_1|^{2+q}}{|\beta_2|^{2+p}}\Big)^{\frac{pq-1}{(p+1)(q+1)}} \|\phi\|_{L^1(B(1))}^{-\frac{(pq-1)(p-q)}{(p+1)(q+1)}}, \end{aligned} \tag{4.1}$$

$$\begin{aligned} C_2 &= C_2(p,q,\phi,\beta_1,\beta_2) \\ &= 2^{-\frac{pq}{q+1}} \Big(\frac{q+1}{p+1}\Big)^{\frac{q}{q+1}} |\beta_1|^{\frac{q}{q+1}} |\beta_2|^{\frac{2-pq}{q+1}} \|\phi\|_{L^1(B(1))}^{-\frac{pq-1}{q+1}}, \end{aligned} \tag{4.2}$$

$$\begin{aligned} C_3 &= C_3(p,q,\beta_1,\beta_2) \\ &= 2^{\frac{pq}{q+1}\frac{1}{1-\frac{n}{2}\frac{pq-1}{p+1}}} \frac{q+1}{pq-1} \Big(\frac{p+1}{q+1}\Big)^{\frac{1}{p+1}\frac{1}{1-\frac{n}{2}\frac{pq-1}{p+1}}} \\ &\quad \times \max(|\alpha_1|,|\alpha_2|)^{\frac{n}{2}\frac{1}{\frac{p+1}{pq-1}-\frac{n}{2}}} |\beta_1|^{-\frac{2(2-pq)}{2(p+1)-n(pq-1)}} |\beta_2|^{-\frac{2p}{2(p+1)-n(pq-1)}}. \end{aligned}$$

Now we are in a position to state our main statement.

Proposition 4.1. *Let $\alpha_1, \alpha_2 < 0$, $\beta_1, \beta_2 \in \mathbb{C}\backslash\{0\}$. Let $p, q \geq 1$ satisfy $p \geq q$ and $\frac{p+1}{pq-1} > \frac{n}{2}$. Let ϕ be the function defined above. Let U_0, V_0 be positive numbers. Let u_0 and v_0 be $L^1_{\rm loc}$-functions satisfying*

$$U_0 \leq \mathrm{Re}\Big(\overline{\beta}\int_{B(R)} u_0(x)\phi\Big(\frac{x}{R}\Big)dx\Big), \quad V_0 \leq \mathrm{Re}\Big(\overline{\beta}\int_{B(R)} v_0(x)\phi\Big(\frac{x}{R}\Big)dx\Big)$$

for $R > R_0$ with R_0 as above and

$$\overline{\beta}_1 u_0(x), \quad \overline{\beta}_2 v_0(x) > 0, \quad a.e.\ x \in \mathbb{R}^n$$

If there exists a solution $(u, v) \in (C^2((0,T)\times\mathbb{R}^n))^2$ for (1.1), then

$$\begin{aligned}
&\int_{B(R)} v(t,x)\phi\Big(\frac{x}{R}\Big)dx + \Big(\frac{p+1}{q+1}\|\phi\|^{p-q}_{L^1(B(1))}R^{n(p-q)}U_0^{q+1} - V_0^{p+1}\Big)^{\frac{1}{p+1}}\\
&\geq \Big\{C_1 R^{-\frac{n(pq-1)(p-q)}{(p+1)(q+1)}}U_0^{-\frac{pq-1}{p+1}} - C_2\lambda^{-1}R^{2-n\frac{pq-1}{q+1}}(1-e^{-\frac{pq-1}{q+1}\lambda R^{-2}t})\Big\}^{-\frac{q+1}{pq-1}},
\end{aligned}$$

for $R > R_0$ and $0 \leq t < T$. Moreover, the lifespan T is estimated by

$$\begin{aligned}
T \leq C_3 \min\Big\{ &- \widetilde{R}^2\log(1-\widetilde{R}^{-2+n\frac{pq-1}{p+1}}),\ \widetilde{R} \geq \frac{R_0}{R_1}\Big\}\\
&\times \lambda^{\frac{n}{2}\frac{1}{\frac{p+1}{pq-1}-\frac{n}{2}}}\|\phi\|^{\frac{1}{\frac{p+1}{pq-1}-\frac{n}{2}}}U_0^{-\frac{1}{\frac{p+1}{pq-1}-\frac{n}{2}}},
\end{aligned}$$

Proof. Since (u, v) is a solution, by letting

$$\begin{aligned}
U(t) &= \mathrm{Re}\Big(\overline{\beta}_1\int_{\mathbb{R}^n} u(t,x)\phi(\frac{x}{R})dx\Big),\\
V(t) &= \mathrm{Re}\Big(\overline{\beta}_2\int_{\mathbb{R}^n} v(t,x)\phi(\frac{x}{R})dx\Big),
\end{aligned}$$

the following ODI holds as long as $U(t), V(t) \geq 0$:

$$\begin{aligned}
&U'(t) + \max(|\alpha_1|, |\alpha_2|)\frac{p+1}{q+1}\lambda R^{-2}U(t)\\
&\qquad \geq U'(t) + |\alpha_1|\lambda R^{-2}U(t)\\
&\qquad \geq R^{-n(p-1)}\|\phi\|^{-p+1}_{L^1(\mathbb{R}^n)}|\beta_1|^2|\beta_2|^{-p}V(t)^p,\\
&V'(t) + \max(|\alpha_1|, |\alpha_2|)\frac{p+1}{p+1}\lambda R^{-2}V(t) \geq V'(t) + |\alpha_2|\lambda R^{-2}V(t)\\
&\qquad \geq R^{-n(q-1)}\|\phi\|^{-q+1}_{L^1(\mathbb{R}^n)}|\beta_2|^2|\beta_1|^{-q}U(t)^q,
\end{aligned}$$

where α_1 and $\alpha_2 \leq 0$. Let $\widetilde{\lambda} = \max(|\alpha_1|, |\alpha_2|)\lambda$. By applying Corollary 2.4 with $\omega = (p+1)\widetilde{\lambda}R^{-2}$ and

$$C_p = \frac{\|\phi\|^{-p+1}_{L^1(B(1))}}{p+1}|\beta_1|^2|\beta_2|^{-p}R^{-n(p-1)}, \quad C_q = \frac{\|\phi\|^{-q+1}_{L^1(B(1))}}{q+1}|\beta_2|^2|\beta_1|^{-q}R^{-n(q-1)},$$

U and V blow up if (2.4) is satisfied. (2.4) is rewritten as

$$U_0 > \max\Big(\mu_1 R^{-2\frac{p+1}{pq-1}+n},\ \mu_2 R^{-n\frac{p-q}{q+1}}V_0^{\frac{p+1}{q+1}}\Big),$$

where

$$
\begin{aligned}
\mu_1 &= \mu_1(\beta_1,\beta_2,p,q,\widetilde{\lambda},\phi)\\
&= 2^{\frac{p+1}{q+1}\frac{pq}{pq-1}}\Big(\frac{p+1}{q+1}\Big)^{\frac{1}{pq-1}}\widetilde{\lambda}^{\frac{p+1}{pq-1}}|\beta_1|^{1-\frac{1}{pq-1}}|\beta_2|^{-\frac{p}{pq-1}}\|\phi\|_{L^1(B(1))},\\
\mu_2 &= \mu_2(\beta_1,\beta_2,p,q,\phi)\\
&= \Big(\frac{q+1}{p+1}\Big)^{\frac{1}{q+1}}|\beta_1|^{1+\frac{1}{q+1}}|\beta_2|^{-\frac{2+p}{q+1}}\|\phi\|_{L^1(B(1))}^{-\frac{p-q}{q+1}}.
\end{aligned}
$$

Here

$$
U_0 > \mu_1 R^{-2\frac{p+1}{pq-1}+n}
$$

for $R > R_1$, since $-2\frac{p+1}{pq-1}+n<0$ and

$$
\begin{aligned}
&\mu_1(p,q,\widetilde{\lambda},\phi)^{\frac{1}{2\frac{p+1}{pq-1}-n}}U_0^{-\frac{1}{2\frac{p+1}{pq-1}-n}}\\
&= 2^{\frac{pq}{q+1}\frac{1}{2-n\frac{pq-1}{p+1}}}\Big(\frac{p+1}{q+1}\Big)^{\frac{1}{pq-1}\frac{1}{2\frac{p+1}{pq-1}-n}}\widetilde{\lambda}^{\frac{1}{2-n\frac{pq-1}{p+1}}}\\
&\quad\cdot|\beta_1|^{\frac{pq-2}{2(p+1)-n(pq-1)}}|\beta_2|^{-\frac{p}{2(p+1)-n(pq-1)}}U_0^{-\frac{1}{2\frac{p+1}{pq-1}-n}}=R_1.
\end{aligned}
$$

Similarly,

$$
U_0 > \mu_2 R^{-n\frac{p-q}{q+1}}V_0^{\frac{p+1}{q+1}}
$$

for $R > R_2$, since

$$
\begin{aligned}
&\mu_2(p,q,\phi)^{\frac{q+1}{n(p-q)}}V_0^{\frac{p+1}{n(p-q)}}U_0^{-\frac{q+1}{n(p-q)}}\\
&= \Big(\frac{q+1}{p+1}\Big)^{\frac{1}{n(p-q)}}|\beta_1|^{\frac{q+2}{n(p-q)}}|\beta_2|^{-\frac{2+p}{n(p-q)}}\|\phi\|_{L^1(B(1))}^{-\frac{1}{n}}V_0^{\frac{p+1}{n(p-q)}}U_0^{-\frac{q+1}{n(p-q)}}=R_2.
\end{aligned}
$$

(4.1) and (4.2) are calculated as

$$
\begin{aligned}
&C_q^{-\frac{pq-1}{(p+1)(q+1)}}C_p^{\frac{pq-1}{(p+1)(q+1)}}U_0^{-\frac{pq-1}{p+1}},\\
&= \Big(\frac{q+1}{p+1}\Big)^{\frac{pq-1}{(p+1)(q+1)}}\|\phi\|_{L^1(B(1))}^{-\frac{(pq-1)(p-q)}{(p+1)(q+1)}}\Big(\frac{|\beta_1|^{2+q}}{|\beta_2|^{2+p}}\Big)^{\frac{pq-1}{(p+1)(q+1)}}R^{-\frac{n(pq-1)(p-q)}{(p+1)(q+1)}}U_0^{-\frac{pq-1}{p+1}}\\
&= C_1 R^{-\frac{n(pq-1)(p-q)}{(p+1)(q+1)}}U_0^{-\frac{pq-1}{p+1}},
\end{aligned}
$$

$$
\begin{aligned}
&2^{-\frac{pq}{q+1}}(q+1)(p+1)\omega^{-1}C_q^{\frac{1}{q+1}}C_p^{\frac{q}{q+1}}\\
&= 2^{-\frac{pq}{q+1}}\Big(\frac{q+1}{p+1}\Big)^{\frac{q}{q+1}}|\beta_1|^{\frac{q}{q+1}}|\beta_2|^{\frac{2-pq}{q+1}}\|\phi\|_{L^1(B(1))}^{-\frac{pq-1}{q+1}}\widetilde{\lambda}^{-1}R^{2-n\frac{pq-1}{q+1}},\\
&= C_2\widetilde{\lambda}^{-1}R^{2-n\frac{pq-1}{q+1}}.
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
T &\le -\min\Big\{\frac{(q+1)(p+1)}{\omega(pq-1)}\log\Big(1-\frac{2^{\frac{pq}{q+1}}}{(q+1)(p+1)}\omega C_q^{-\frac{p}{p+1}}C_p^{-\frac{1}{p+1}}U_0^{-\frac{pq-1}{p+1}}\Big),\\
&\qquad R>R_0\Big\}\\
&= -\min\Big\{\frac{(q+1)}{\widetilde{\lambda}(pq-1)}R^2\log\Big(1-\Big(\frac{R}{R_1}\Big)^{-2+n\frac{pq-1}{p+1}}\Big),\ R>R_0\Big\}
\end{aligned}
$$

Let $\widetilde{R} = R/R_1$. Then

$$
\begin{aligned}
T &\le 2^{\frac{pq}{q+1}\frac{1}{1-\frac{n}{2}\frac{pq-1}{p+1}}} \frac{q+1}{pq-1}\Big(\frac{p+1}{q+1}\Big)^{\frac{1}{p+1}\frac{1}{1-\frac{n}{2}\frac{pq-1}{p+1}}} \widetilde{\lambda}^{\frac{1}{1-\frac{n}{2}\frac{pq-1}{p+1}}-1} \|\phi\|^{\frac{1}{\frac{p+1}{pq-1}-\frac{n}{2}}} U_0^{-\frac{1}{\frac{p+1}{pq-1}-\frac{n}{2}}} \\
&\quad \times |\beta_1|^{-\frac{2(2-pq)}{2(p+1)-n(pq-1)}} |\beta_2|^{-\frac{2p}{2(p+1)-n(pq-1)}} \\
&\quad \times \min\Big\{ -\widetilde{R}^2 \log(1-\widetilde{R}^{-2+n\frac{pq-1}{p+1}}),\ \widetilde{R} \ge \frac{R_0}{R_1}\Big\} \\
&\le C_3 \lambda^{\frac{n}{2}\frac{1}{\frac{p+1}{pq-1}-\frac{n}{2}}} \|\phi\|^{\frac{1}{\frac{p+1}{pq-1}-\frac{n}{2}}} U_0^{-\frac{1}{\frac{p+1}{pq-1}-\frac{n}{2}}} \\
&\quad \times \min\Big\{ -\widetilde{R}^2 \log(1-\widetilde{R}^{-2+n\frac{pq-1}{p+1}}),\ \widetilde{R} \ge \frac{R_0}{R_1}\Big\}.
\end{aligned}
$$

Here, we have used the fact that $f(x) = -x\log(1-x^{-\theta})$ for $0 < \theta < 1$ attains the minimum on $[R_0/R_1, \infty) \subset [1,\infty)$. Indeed,

$$
f'(x) = -\log(1-x^{-\theta}) - \theta\frac{x^{-\theta}}{1-x^{-\theta}}. = \alpha - \log(1-x^{-\theta}) - \theta\frac{1}{1-x^{-\theta}}.
$$

by putting $y = -\log(1-x^{-\theta}) \in (0,\infty)$, $f'(x) = g(y) = \theta + y - \theta e^y$ and therefore f attains the minimum. □

Acknowledgments. K. Fujiwara was partly supported by the Japan Society for the Promotion of Science, Grant-in-Aid for JSPS Fellows no 16J30008. M. Ikeda was partly supported by the Japan Society for the Promotion of Science, Grant-in-Aid for Young Scientists. Research (B), No. 15K17571. Y. Wakasugi was partly supported by the Japan Society for the Promotion of Science, Grant-in-Aid for Young Scientists Research (B), No. 16K17625.

The authors are grateful to Professor Yūki Naito for giving us useful references. The authors also are grateful to the anonymous referee for the helpful comments.

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