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APPROXIMATE SOLUTION FOR AN INVERSE PROBLEM OF MULTIDIMENSIONAL ELLIPTIC EQUATION WITH MULTIPOINT NONLOCAL AND NEUMANN BOUNDARY CONDITIONS

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Abstract. In this work, we consider an inverse elliptic problem with Bitsadze-Samarskii type multipoint nonlocal and Neumann boundary conditions. We construct the first and second order of accuracy difference schemes (ADSs) for problem considered. We stablish stability and coercive stability estimates for solutions of these difference schemes. Also, we give numerical results for overdetermined elliptic problem with multipoint Bitsadze-Samarskii type nonlocal and Neumann boundary conditions in two and three dimensional test examples. Numerical results are carried out by MATLAB program and brief explanation on the realization of algorithm is given.

1. Introduction

Theory and methods of solving inverse problems for differential and difference equations have been comprehensively studied by several researchers (see [1, 2, 5, 6, 7, 11, 12, 13, 13, 15, 16, 17, 18, 19, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 38] and the references therein). In papers [6, 11, 12, 13, 14, 15, 16, 30, 32] well-posedness of various overdetermined elliptic type differential and difference problems are studied. Dirichlet type overdetermined problems for elliptic partial differential equation (PDE) were investigated in [6, 15, 16]. Neumann type overdetermined elliptic problems were studied in papers [11, 12, 14].

In recent years, different types of elliptic nonlocal boundary value problems and generalizations of such type problems to various differential and difference equations have been extensively investigated (see [3, 8, 9, 13, 32, 34] and the bibliography therein).

In this article, we study approximation of Bitsadze-Samarskii type overdetermined elliptic differential problem with Neumann boundary conditions.

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Given an integer $q \geq 2$, we assume that the nonnegative numbers $k_1, \dots, k_q$, $\lambda_0, \lambda_1, \dots, \lambda_q$ satisfy the conditions

$$\sum_{i=1}^{q} k_i = 1, \quad k_i \geq 0, \; i = 1, \dots, q, \quad 0 < \lambda_1 < \dots < \lambda_q < 1, \quad 0 < \lambda_0 < 1. \tag{1.1}$$

Let $\Omega = (0,\ell)^n \subset R_n$ be the open cube with boundary S, $\overline{\Omega} = \Omega \cup S$. In $[0,T] \times \Omega$, we consider the inverse problem of finding function $u(t,x)$ and function $p(x)$ in Ω for the following multidimensional elliptic PDE with multipoint nonlocal and Neumann boundary conditions

$$\begin{gathered}
-v_{tt}(t,x) - \sum_{r=1}^{n} (a_r(x) v_{x_r})_{x_r} + \sigma v(t,x) = g(t,x) + p(x), \\
x = (x_1, \dots, x_n) \in \Omega, \quad 0 < t < T; \\
v(T,x) - \sum_{i=1}^{q} k_i v(\lambda_i, x) = \eta(x), \quad v(0,x) = \phi(x), \quad v(\lambda_0, x) = \zeta(x), \quad x \in \overline{\Omega}, \\
\frac{\partial v(t,x)}{\partial \overrightarrow{n}} = 0, \quad x \in S, \quad 0 \leq t \leq T.
\end{gathered} \tag{1.2}$$

Here, $\overrightarrow{n}$ is the normal vector to S; a_r, φ, ψ, ξ, and g are given smooth functions, $a_r(x) \geq a > 0$ for all $x \in \Omega$.

Well-posedness of problem (1.2) was established in [13]. In this article, we apply a finite difference method to approximate the solution of problem (1.2). Namely, we construct the first and second order of ADSs with respect to t and second order of ADS with respect to x for the approximate solution of problem. Stability and coercive stability estimates for solutions of both difference schemes are established. Later, we give two and three dimensional numerical examples with brief explanation on the realization for inverse elliptic problem with multipoint Bitsadze-Samarskii type nonlocal and Neumann boundary conditions.

The differential operator [10]

$$A^x v(x) = -\sum_{r=1}^{n} (a_r(x) v_{x_r})_{x_r} + \sigma v(x) \tag{1.3}$$

is a self-adjoint positive definite (SAPD) operator $A = A^x$ acting on Hilbert space $H = L_2(\overline{\Omega})$ with the domain $D(A^x)=\{v(x) \in W_2^2(\overline{\Omega}), \ \frac{\partial v}{\partial \overrightarrow{n}} = 0 \text{ on } S\}$.

Therefore, primal problem (1.2) corresponds to the following Bitsadze-Samarskii type inverse elliptic problem of finding an element $p \in H$ and a function $v \in C([0,T], D(A)) \cap C^2([0,T], H)$:

$$\begin{gathered}
-v_{tt}(t) + Av(t) = g(t) + p, \quad t \in (0,T), \\
v(0) = \phi, \quad v(\lambda_0) = \zeta, \quad v(T) = \sum_{i=1}^{q} \alpha_i v(\lambda_i) + \eta.
\end{gathered} \tag{1.4}$$

Let $[0,T]_\tau = \{t_k = k\tau, \ k = \overline{0,N}, \ N\tau = T\}$ be the set of grid points. Introduce the notation

$$\begin{gathered}
C = \frac{1}{2}(\tau A + \sqrt{4A + \tau^2 A^2}), \quad R = (I + \tau C)^{-1}, \\
P = (I - R^{2N})^{-1}, \quad D = (I + \tau C)(2I + \tau C)^{-1} C^{-1},
\end{gathered}$$

where I is the identity operator. It is known that $A > \delta I$ ($\delta > 0$), C is SAPD operator and the bounded operator R is defined on the whole space H [10, 30].

Lemma 1.1 ([10])**.** *The following estimates hold:*

$$\|R^k\|_{H\to H} \le M(\delta)(1+\delta^{\frac{1}{2}}\tau)^{-k}, \|CR^k\|_{H\to H} \le \frac{M(\delta)}{k\tau},$$
$$k \ge 1, \|P\|_{H\to H} \le M(\delta), \quad \delta > 0.$$

The remainder of this article is organized as follows: In Section 2, we present two difference schemes for approximate solution of inverse elliptic problem (1.2) with Bitsadze-Samarskii type multipoint nonlocal and Neumann boundary conditions. In Section 3, we obtain the stability and coercive stability estimates for the solution of both presented difference schemes. Numerical results for two dimensional and three dimensional elliptic equations are presented in Section 4. Finally, the conclusion is given in Section 5.

2. Difference problems

The approximation of problem (1.2) is carried out in two steps. In the first step, we define the grid spaces

$$\widetilde{\Omega}_h = \Big\{x : x = x_m = (h_1m_1, \dots, h_nm_n),\ m = (m_1, \dots, m_n),$$
$$0 \le m_r \le M_r,\ h_rM_r = \ell,\ r = 1, \dots, n\Big\},$$
$$\Omega_h = \widetilde{\Omega}_h \cap \Omega, \quad S_h = \widetilde{\Omega}_h \cap S, \quad h = (h_1, \dots, h_n),$$

and assign the difference operator A_h^x to operator A^x (1.3) by the formula

$$A_h^x v^h(x) = -\sum_{r=1}^{n} (a_r(x)v^h_{x_r})_{x_r,m_r} + \sigma v^h(x),$$

acting in the space of grid functions $v^h(x)$, satisfying the condition $D^h v^h(x) = 0$ for all $x \in S_h$. Here and in future D^h is the approximation of operator $\frac{\partial}{\partial \vec{n}}$. It is known that A_h^x is a SAPD operator (see [36, 37]).

By using A_h^x, the overdetermined problem (1.2) is reduced to the boundary value problem for the system of ordinary differential equations

$$\begin{gathered} -\frac{d^2v^h(t,x)}{dt^2} + A_h^x v^h(t,x) = g^h(t,x) + p^h(x), \quad t \in (0,T),\ x \in \Omega_h, \\ v^h(0,x) = \phi(x), \quad v^h(\lambda_0,x) = \zeta^h(x), \\ v^h(T,x) - \sum_{i=1}^{q} k_i v^h(\lambda_i,x) = \eta^h(x), x \in \widetilde{\Omega}_h. \end{gathered} \tag{2.1}$$

Denote

$$l_i = [\frac{\lambda_i}{\tau}], \quad \mu_i = \frac{\lambda_i}{\tau} - l_i, \quad i = 0, 1, \dots, q,$$

where $[\cdot]$ is standard notation for greatest integer function.

Let $v_k^h(x) = v^h(t_k,x)$, $g_k^h(x) = g^h(t_k,x), k = \overline{0,N}$.

In the second step, we apply the following approximation formulas

$$v^h(\lambda_i,x) = v^h_{l_i}(x) + o(\tau),$$
$$v^h(\lambda_i,x) = v^h_{l_i}(x) + \mu_i(v^h_{l_i+1}(x) - v^h_{l_i}(x)) + o(\tau^2)$$

for $v^h(\lambda_i,x)$, $i=0,1,\dots,q$. Then problem (2.1) is replaced by

$$\begin{gathered}
-\tau^{-2}\left[v_{k+1}^h(x)-2v_k^h(x)+v_{k-1}^h(x)\right]+A_h^x v_k^h(x)=g_k^h(x)+p^h(x),\\
1\le k\le N-1,\quad x\in\Omega_h,\\
v_N^h(x)=\sum_{i=1}^{q}k_i v_{l_i}^h(x)+\eta^h(x),\\
v_{l_0}^h(x)=\zeta^h(x),\quad v_0^h(x)=\phi^h(x),\quad x\in\widetilde{\Omega}_h,
\end{gathered}\tag{2.2}$$

and

$$\begin{gathered}
-\tau^{-2}\left[v_{k+1}^h(x)-2v_k^h(x)+v_{k-1}^h(x)\right]+A_h^x v_k^h(x)=g_k^h(x)+p^h(x),\\
1\le k\le N-1,\quad x\in\Omega_h,\\
v_N^h(x)=\sum_{i=1}^{q}k_i(v_{l_i}^h(x)+\mu_i(v_{l_i+1}^h(x)-v_{l_i}^h(x)))+\eta^h(x),\\
v_{l_0}^h(x)+\mu_0(v_{l_0+1}^h(x)-v_{l_0}^h(x))=\zeta^h(x),\quad v_0^h(x)=\phi^h(x),\quad x\in\widetilde{\Omega}_h,
\end{gathered}\tag{2.3}$$

respectively.

By substituting

$$v_k^h(x)=u_k^h(x)+(A_h^x)^{-1}p^h(x),\quad x\in\widetilde{\Omega}_h,\quad 1\le k\le N-1,\tag{2.4}$$

difference scheme (2.2) is reduced to the auxiliary difference scheme

$$\begin{gathered}
-\tau^{-2}\left[u_{k+1}^h(x)-2u_k^h(x)+u_{k-1}^h(x)\right]+A_h^x u_k^h(x)=g_k^h(x),\\
1\le k\le N-1,\quad x\in\Omega_h,\\
u_0^h(x)-u_{l_0}^h(x)=\phi^h(x)-\zeta^h(x),\\
u_N^h(x)=\sum_{i=1}^{q}k_i u_{l_i}^h(x)+\eta^h(x),\quad x\in\widetilde{\Omega}_h.
\end{gathered}\tag{2.5}$$

The solution of system (2.5) is defined by the formula

$$\begin{aligned}
u_k^h(x)&=P\left[(R^k-R^{2N-k})u_0^h(x)+(R^{N-k}-R^{N+k})\right]u_N^h(x)\\
&\quad-P(R^{N-k}-R^{N+k})D\sum_{j=1}^{N-1}(R^{N-j}-R^{N+j})g_j^h(x)\tau\\
&\quad+D\sum_{j=1}^{N-1}(R^{|k-j|}-R^{k+j})g_j^h(x)\tau,\quad k=\overline{1,N-1},
\end{aligned}\tag{2.6}$$

where

$$
\begin{aligned}
u_0^h(x) &= F_1^{-1}\Big[\Big(I - R^{2N} - \sum_{i=1}^{q} k_i(R^{N-l_i} - R^{N+l_i})\Big)G_1^h(x) \\
&\quad + (R^{N-s} - R^{N+s})G_2^h(x)\Big], u_N^h(x) \\
&= \Delta_1^{-1}\Big[(I - R^{2N} - R^s + R^{2N-s})G_2^h(x) + \sum_{i=1}^{q} k_i(R^{l_i} - R^{2N-l_i})G_1^h(x)\Big], \\
F_1 &= (I - R^{2N})(I - R^{l_0})\Big(I - \sum_{i=1}^{q} k_i R^{N-l_i}\Big)\Big(I - \sum_{i=1}^{q} k_i R^{N-(l_0-l_i)}\Big), \\
G_1^h(x) &= P^{-1}(\phi^h(x) - \zeta^h(x)) + (R^{N-s} - R^{N+s}) \\
&\quad \times D\sum_{j=1}^{N-1}(R^{N-j-1} - R^{N+j-1})g_j^h(x)\tau \\
&\quad - P^{-1}D\sum_{j=1}^{N-1}(R^{|s-j|-1} - R^{s+j-1})g_j^h(x)\tau, \\
G_2^h(x) &= k\Big\{(R^{N-l_i} - R^{N+l_i})D\sum_{j=1}^{N-1}(R^{N-j-1} - R^{N+j-1})g_j^h(x)\tau \\
&\quad - P^{-1}D\sum_{j=1}^{N-1}(R^{|l_i-j|-1} - R^{l_i+j-1})g_j^h(x)\tau\Big\} + P^{-1}\eta^h(x).
\end{aligned}
\tag{2.7}
$$

Using (2.4), difference scheme (2.3) can be reduced to the auxiliary difference scheme

$$
\begin{gathered}
-\tau^{-2}\left[u_{k+1}^h(x) - 2u_k^h(x) + u_{k-1}^h(x)\right] + A_h^x u_k^h(x) = g_k^h(x), \\
1 \le k \le N-1, \quad x \in \Omega_h, \\
u_0^h(x) + (\mu_0 - 1)u_{l_0}^h(x) - \mu_0 u_{l_0+1}^h(x) = \phi^h(x) - \zeta^h(x), \\
u_N^h(x) + \sum_{i=1}^{q} k_i\left[(\mu_i - 1)u_{l_i}^h(x) - \mu_i u_{l_i+1}^h(x)\right] = \eta^h(x), \quad x \in \widetilde{\Omega}_h.
\end{gathered}
\tag{2.8}
$$

The solution of system (2.8) is defined by formula (2.6), where

$$
\begin{aligned}
u_0^h(x) &= F_2^{-1}\Big\{\Big[I - R^{2N} + \sum_{i=1}^{q} k_i(\mu_i - 1)(R^{N-l_i} - R_i^{N+l}) \\
&\quad - \sum_{i=1}^{q} k_i\mu_i(R^{N-l_i-1} - R^{N+l_i+1})\Big]G_3^h(x) \\
&\quad - [(\mu_0 - 1)(R^{N-l_0} - R^{N+l_0}) - \mu_0(R^{N-l_0-1} - R^{N+l_0+1})]G_4^h(x)\Big\}, \\
u_N^h(x) &= F_2^{-1}\Big\{\Big[I - R^{2N} + (\mu_0 - 1)(R^{l_0} - R^{2N-l_0}) \\
&\quad - \mu_0(R^{l_0+1} - R^{2N-l_0-1})\Big]G_4^h(x) - \Big[\sum_{i=1}^{q} k_i(\mu_i - 1)(R^{l_i} - R^{2N-l_i})
\end{aligned}
$$

$$
\begin{aligned}
&- \sum_{i=1}^{q} k_i\mu_i (R^{l_i+1} - R^{2N-l_i-1})\Big] G_3^h(x)\Big\},\\
F_2 &= [I - R^{2N} + (\mu_0 - 1)(R^{l_0} - R^{2N-l_0}) - \mu_0(R^{l_0+1} - R^{2N-l_0-1})]\\
&\quad \times \Big[I - R^{2N} + \sum_{i=1}^{q} k_i(\mu_i - 1)(R^{N-l_i} - R_i^{N+l})\\
&\quad - \sum_{i=1}^{q} k_i\mu_i(R^{N-l_i-1} - R^{N+l_i+1})\Big] \\
&\quad - [(\mu_0 - 1)(R^{N-l_0} - R^{N+l_0}) - \mu_0(R^{N-l_0-1} - R^{N+l_0+1})]\\
&\quad \times \Big[\sum_{i=1}^{q} k_i(\mu_i - 1)(R^{l_i} - R^{2N-l_i}) - \sum_{i=1}^{q} k_i\mu_i(R^{l_i+1} - R^{2N-l_i-1})\Big].
\end{aligned} \tag{2.9}
$$

$$
\begin{aligned}
G_3^h(x) &= P^{-1}(\phi^h(x) - \zeta^h(x))\\
&\quad + \big[(\mu_0 - 1)(R^{N-l_0} - R^{N+l_0}) - \mu_0(R^{N-l_0-1} - R^{N+l_0+1})\big]\\
&\quad \times D\sum_{j=1}^{N-1}(R^{N-j} - R^{N+j})g_j\tau - P^{-1}D\\
&\quad \times \sum_{j=1}^{N-1}\Big[(\mu_0 - 1)(R^{|l_0-j|} - R^{l_0+j}) - \mu_0(R^{|l_0+1-j|} - R^{l_0+j+1})\Big]\, g_j^h(x)\tau,
\end{aligned}
$$

$$
\begin{aligned}
G_4^h(x) &= \sum_{i=1}^{q} k_i\,\big[(\mu_i - 1)(R^{N-l_i} - R^{N+l_i}) - \mu_i(R^{N-l_i-1} - R^{N+l_i+1})\big]\\
&\quad \times D\sum_{j=1}^{N-1}(R^{N-j} - R^{N+j})g_j^h(x)\tau + P^{-1}\eta^h(x) - P^{-1}D\\
&\quad \times \sum_{j=1}^{N-1}\sum_{i=1}^{q} k_i\Big[(\mu_i - 1)(R^{|l_i-j|} - R^{l_i+j})\\
&\quad - \mu_0(R^{|l_i+1-j|} - R^{l_i+j+1})\Big]g_j^h(x)\tau.
\end{aligned}
$$

So, to find an approximate solution of (1.2), we consider the algorithm which contains three stages. We find $\{u_k^h(x)\}_0^N$ as solution of (2.5) or (2.8) in the first stage. Putting $k = l_0$ and $k = l_0 + 1$, we get $u_{l_0}^h(x)$ and $u_{l_0+1}^h(x)$, respectively. In the second stage, we obtain $p^h(x)$ by

$$
p^h(x) = A_h^x\zeta^h(x) - A_h^x u_{l_0}^h(x), \quad x \in \widetilde{\Omega}_h, \tag{2.10}
$$

for (2.2), and

$$
p^h(x) = A_h^x\zeta^h(x) - A_h^x\left[(1 - \mu_0)u_{l_0}^h(x) + \mu_0 u_{l_0+1}^h(x)\right], \quad x \in \widetilde{\Omega}_h, \tag{2.11}
$$

for (2.3).

In the third stage, we use formulas

$$
v_k^h(x) = u_k^h(x) + \zeta^h(x) - u_{l_0}^h(x), \quad x \in \widetilde{\Omega}_h,\ 1 \le k \le N - 1, \tag{2.12}
$$

and

$$v_k^h(x) = u_k^h(x) + \zeta^h(x) - \left[(1-\mu_0)u_{l_0}^h(x) + \mu_0 u_{l_0+1}^h(x)\right], \tag{2.13}$$

for $x \in \widetilde{\Omega}_h$, $1 \le k \le N-1$, to obtain the solution $\{v_k^h(x)\}_0^N$ of corresponding difference problems (2.2) and (2.3).

3. Stability and coercive stability estimates

Let $L_{2h} = L_2(\widetilde{\Omega}_h)$ and $W_{2h}^2 = W_2^2(\widetilde{\Omega}_h)$ be Banach spaces of the grid functions $f^h(x) = \{f(h_1 m_1, \dots, h_n m_n)\}$ defined on $\widetilde{\Omega}_h$, equipped with the following norms

$$\|f^h\|_{L_{2h}} = \Big(\sum_{x\in\widetilde{\Omega}_h} |f^h(x)|^2 h_1 \dots h_n\Big)^{1/2},$$

$$\|f^h\|_{W_{2h}^2} = \|f^h\|_{L_{2h}} + \Big[\sum_{x\in\widetilde{\Omega}_h}\sum_{r=1}^n |(f^h)_{x_r}|^2 h_1 \dots h_n\Big]^{1/2} + \Big[\sum_{x\in\widetilde{\Omega}_h}\sum_{r=1}^n |(f^h(x))_{x_r\overline{x_r},m_r}|^2 h_1 \dots h_n)\Big]^{1/2},$$

respectively. Denote by $C_\tau(H)$ and $C_\tau^{\alpha,\alpha}(H)$, the corresponding Banach spaces of H-valued mesh functions $\varphi_\tau^h = \{\varphi_k^h\}_1^N$ on $[0,T]_\tau$ with the following norms

$$\|\varphi_\tau^h\|_{C_\tau(H)} = \max_{1\le t\le N-1} \|\varphi_k^h\|_H,$$

$$\|\varphi_\tau^h\|_{C_\tau^{\alpha,\alpha}(H)} = \|\varphi_\tau^h\|_{C_\tau(H)} + \sup_{1\le k\le k+s\le N-1} \frac{((N-s)\tau)^\alpha((k+s)\tau)^\alpha}{(s\tau)^\alpha}\|\varphi_{k+s}^h - \varphi_k^h\|_H.$$

Let τ and $|h| = \sqrt{h_1^2 + \dots + h_n^2}$ be sufficiently small positive numbers.

Theorem 3.1. *Under conditions* (1.1)*, for the solution of difference problems* (2.2) *and* (2.3) *the next stability inequalities hold:*

$$\|\{v_k^h\}_1^{N-1}\|_{\mathcal{C}_\tau(L_{2h})} \le M(\delta,\lambda_1,\dots,\lambda_q)\Big[\|\phi^h\|_{L_{2h}} + \|\zeta^h\|_{L_{2h}} + \|\eta^h\|_{L_{2h}} + \|\{g_k^h\}_1^{N-1}\|_{C_\tau(L_{2h})}\Big],$$

$$\|p^h\|_{L_{2h}} \le M(\delta,\lambda_1,\dots,\lambda_q)\Big[\|\phi^h\|_{W_{2h}^2} + \|\zeta^h\|_{W_{2h}^2} + \|\eta^h\|_{W_{2h}^2} + \frac{1}{\alpha(1-\alpha)}\|\{g_k^h\}_1^{N-1}\|_{C_\tau^{\alpha,\alpha}(L_{2h})}\Big],$$

where $M(\delta,\lambda_1,\dots,\lambda_q)$ *does not depend on* $\tau,\alpha,h,\phi^h(x),\zeta^h(x)$, $\eta^h(x)$ *and* $\{g_k^h(x)\}_1^{N-1}$.

Theorem 3.2. *Under conditions* (1.1)*, for the solution of difference problems* (2.2) *and* (2.3) *the coercive stability inequality holds:*

$$\begin{aligned}&\|\{\frac{v_{k+1}^h - 2v_k^h + v_{k-1}^h}{\tau^2})\}_1^{N-1}\|_{C_\tau^{\alpha,\alpha}(L_{2h})} + \|\{v_k^h\}_1^{N-1}\|_{C_\tau^{\alpha,\alpha}(W_{2h}^2)}\\ &\le M(\delta,\lambda_1,\dots,\lambda_q)[\|\phi^h\|_{W_{2h}^2} + \|\zeta^h\|_{W_{2h}^2} + \|\eta^h\|_{W_{2h}^2} + \frac{1}{\alpha(1-\alpha)}\|\{g_k^h\}_1^N\|_{C_\tau^{\alpha,\alpha}(L_{2h})}],\end{aligned}$$

where $M(\delta,\lambda_1,\dots,\lambda_q)$ *does not depend on* $\tau,\alpha,h,\phi^h(x),\eta^h(x)$, $\zeta^h(x)$, *or* $\{g_k^h(x)\}_1^{N-1}$.

The proofs of Theorems 3.1 and 3.2 are based on the symmetry property of operator A_h^x in L_{2h}, the formulas (2.6), (2.7), (2.9), (2.10), (2.11), (2.12),(2.13) for solution of corresponding difference schemes and the following theorem on well-posedness of the elliptic difference problem.

Theorem 3.3. [35] *For the solution of the elliptic difference problem*

$$\begin{gathered} A_h^x u^h(x) = \omega^h(x), \quad x \in \widetilde{\Omega}_h, \\ D^h u^h(x) = 0, \quad x \in S_h, \end{gathered}$$

the following coercivity inequality holds:

$$\sum_{q=1}^{n} \|(u^h)_{\overline{x}_q x_q, j_q}\|_{L_{2h}} \le M \|\omega^h\|_{L_{2h}},$$

here M does not depend on h and ω^h.

4. Numerical Examples

Now, we give two and three dimensional numerical examples with brief explanation on the realization for Bitsadze-Samarskii type inverse elliptic multipoint NBVP. These numerical results are carried out by using MATLAB program.

4.1. **Two dimensional example.** Consider the following two dimensional Bitsadze-Samarskii type overdetermined problem with three point nonlocal boundary conditions,

$$\begin{gathered} -\frac{\partial^2 v(t,x)}{\partial t^2} - \frac{\partial}{\partial x}((3+\sin(\pi x))\frac{\partial v(t,x)}{\partial x}) + v(t,x) = g(t,x) + p(x), \\ t, x \in (0,1), \quad v(0,x) = \phi(x), \quad v(0.1,x) = \zeta(x), \\ v(1,x) - \frac{1}{10} \quad v(0.3,x) - \frac{1}{5} v(0.7,x) - \frac{7}{10} v(0.8,x) = \eta(x), \\ x \in [0,1], \quad v(t,0) = 0, \quad v(t,1) = 0, \quad t \in [0,1], \end{gathered} \tag{4.1}$$

where

$$\begin{gathered} g(t,x) = \big[(1+4\pi^2)\cos(\pi t) + (3\pi^2+1)t\big]\sin(\pi x) - \pi^2(\cos(\pi t) + t)\cos(2\pi x), \\ \phi(x) = 2\sin(\pi x), \quad \zeta(x) = (\cos(\frac{\pi}{10}) + \frac{\pi}{10} + 1)\sin(\pi x), \\ \eta(x) = -\Big(\frac{1}{10}\cos(\frac{3\pi}{10}) + \frac{1}{5}\cos(\frac{7\pi}{10}) + \frac{7}{10}\cos(\frac{4\pi}{5}) + \frac{73}{100}\Big)\sin(\pi x), \quad x \in [0,1]. \end{gathered}$$

It is easy to show that exact solution of problem (4.1) is the pair of functions $v(t,x) = (\cos(\pi t) + t + 1)\sin(\pi x)$ and $p(x) = (3\pi^2+1)\sin(\pi x) - \pi^2\cos(2\pi x)$.

Denote by $[0,1]_\tau \times [0,1]_h$ set of grid points

$$[0,1]_\tau \times [0,1]_h = \{(t_k, x_n) : t_k = k\tau, \quad k = \overline{0,N};\ x_n = nh,\ n = \overline{0,M}\},$$

where τ and h such that $N\tau = 1$, $Mh = 1$. Moreover,

$$\begin{gathered} \lambda_0 = \frac{1}{10}, \quad \lambda_1 = \frac{1}{10}, \quad \lambda_2 = \frac{1}{5}, \quad \lambda_3 = \frac{7}{10}, \quad l_i = [\frac{\lambda_i}{\tau}], \quad \mu_i = \frac{\lambda_i}{\tau} - l_i, \\ i = 0,1,2,3; \quad \phi_n = \phi(x_n), \quad \zeta_n = \zeta(x_n), \quad \eta_n = \eta(x_n), \\ p_n = p(x_n), \quad n = \overline{0,M}, \quad g_n^k = g(t_k, x_n), \quad k = 0,\dots,N,\ n = 0,\dots M. \end{gathered}$$

The algorithm for solving (4.1) contains three corresponding stages. In the first stage, we find numerical solutions $\{u_n^k : n=\overline{1,M-1}, k=\overline{1,N-1}\}$ of corresponding the first and second order of ADSs for auxiliary problem

$$\begin{gathered}
\frac{u_n^{k+1}-2u_n^k+u_n^{k-1}}{\tau^2}+(3+\sin(\pi x_n))\frac{u_{n+1}^k-2u_n^k+u_{n-1}^k}{h^2} \\
+\frac{u_{n+1}^k-u_{n-1}^k}{2h}=-g_n^k, \quad n=\overline{1,M-1}, k=\overline{1,N-1}; \\
u_0^k=u_1^k, \quad u_M^k=u_{M-1}^k, \quad k=\overline{0,N}; \\
u_n^0-u_n^{l_0}=\phi_n-\zeta_n, \quad u_n^N-\frac{1}{10}u_n^{l_1}-\frac{1}{5}u_n^{l_2}-\frac{7}{10}u_n^{l_3}=\eta_n, \quad n=\overline{0,M}
\end{gathered} \tag{4.2}$$

and

$$\begin{gathered}
\frac{u_n^{k+1}-2u_n^k+u_n^{k-1}}{\tau^2}+(3+\sin(\pi x_n))\frac{u_{n+1}^k-2u_n^k+u_{n-1}^k}{h^2} \\
+\frac{u_{n+1}^k-u_{n-1}^k}{2h}=-g_n^k, \quad n=\overline{1,M-1},\ k=\overline{1,N-1}; \\
3u_0^k-4u_1^k+u_2^k=0, \quad 3u_M^k-4u_{M-1}^k+u_{M-2}^k=0, \quad k=\overline{0,N}; \\
u_n^0+(\mu_0-1)u_n^{l_0}-\mu_0u_n^{l_0+1}=\phi_n-\zeta_n, \\
u_n^N+\frac{1}{10}\left[(\mu_1-1)u_n^{l_1}-\mu_1u_n^{l_1+1}\right]+\frac{1}{5}\left[(\mu_2-1)u_n^{l_2}-\mu_2u_n^{l_2+1}\right] \\
+\frac{7}{10}\left[(\mu_3-1)u_n^{l_3}-\mu_3u_n^{l_3+1}\right] \\
=\eta_n, \quad n=\overline{0,M}.
\end{gathered} \tag{4.3}$$

Difference schemes (4.2) and (4.3) can be presented in the matrix form

$$\begin{gathered}
A^{(n)}u_{n+1}+B^{(n)}u_n+C^{(n)}u_{n-1}=Ig_n, \quad n=1,\dots,M-1, \\
u_0-u_1=\overrightarrow{0}, \quad u_M-u_{M-1}=\overrightarrow{0},
\end{gathered} \tag{4.4}$$

and

$$\begin{gathered}
A^{(n)}u_{n+1}+B^{(n)}u_n+C^{(n)}u_{n-1}=Ig_n, \quad n=1,\dots,M-1, \\
3u_0-4u_1+u_1=\overrightarrow{0}, \quad 3u_M-4u_{M-1}+u_{M-1}=\overrightarrow{0},
\end{gathered} \tag{4.5}$$

respectively. Here, $A^{(n)}, B^{(n)}, C^{(n)}$, and I are $(N+1)\times(N+1)$ matrices. Moreover, I is identity matrix, $g_s=[g_s^0 \ \dots \ g_s^N]^t$ and $u_s=[u_s^0 \ \dots \ u_s^N]^t$, $(s=n-1,n,n+1)$ are $(N+1)\times 1$ column matrices. Let

$$\begin{gathered}
a^{(n)}=(3+\sin(\pi x_n))h^{-2}+h^{-1}/2, c^{(n)}=(3+\sin(\pi x_n))h^{-2}-h^{-1}/2, \\
z^{(n)}=-2\tau^{-2}-2(3+\sin(\pi x_n))h^{-2}, r=\tau^{-2}.
\end{gathered}$$

Then, we have

$$\begin{gathered}
A^{(n)}=\operatorname{diag}\{0,a^{(n)},a^{(n)},\dots,a^{(n)},0\}, \\
C^{(n)}=\operatorname{diag}\{0,c^{(n)},c^{(n)},\dots,c^{(n)},0\}, \\
g_n^0=\phi_n-\zeta_n, \quad g_n^N=\eta_n, \quad n=\overline{1,M-1}
\end{gathered}$$

for both schemes (4.2) and (4.3). The elements $b_{i,j}^{(n)}$ of matrix $B^{(n)}$ are defined by

$$b_{i,i}^{(n)}=z^{(n)}, \quad b_{i-1,i}^{(n)}=b_{i,i-1}^{(n)}=r, \quad i=\overline{2,N}; b_{1,1}^{(n)}=1, b_{1,l_0}^{(n)}=-1, \quad b_{N+1,N+1}^{(n)}=1,$$

$$b^{(n)}_{N+1,l_1} = -\frac{1}{5}, \quad b^{(n)}_{N+1,l_2} = -\frac{3}{10}, \quad b^{(n)}_{N+1,l_3} = -\frac{1}{2}, \quad b^{(n)}_{N+1,l_3+1} = \frac{1}{4},$$
$$b^{(n)}_{i,j} = 0 \quad \text{in other cases}$$

for problem (4.2), and

$$b^{(n)}_{i,i} = z^{(n)}, \quad b^{(n)}_{i-1,i} = b^{(n)}_{i,i-1} = r, \quad i = \overline{2,N}; \quad b^{(n)}_{1,1} = 1, \quad b^{(n)}_{1,l_0} = \mu_0 - 1,$$
$$b^{(n)}_{1,l_0+1} = -\mu_0, \quad b^{(n)}_{N+1,N+1} = 1, b^{(n)}_{N+1,l_1+1} = -\frac{\mu_1}{5}, \quad b^{(n)}_{N+1,l_1} = \frac{\mu_1 - 1}{5},$$
$$b^{(n)}_{N+1,l_2+1} = -\frac{3\mu_2}{10}, \quad b^{(n)}_{N+1,l_2} = \frac{3(\mu_2 - 1)}{10}, b^{(n)}_{N+1,l_3+1} = -\frac{\mu_3}{2}, \quad b^{(n)}_{N+1,l_3} = \frac{\mu_3 - 1}{2},$$
$$b^{(n)}_{i,j} = 0 \quad \text{in other cases}$$

for problem (4.3).

In the second stage, we find $\{p_n\}$ by (2.10) and (2.11), respectively.

In the third stage, $\{v^k_n\}$ are calculated by $v^k_n = u^k_n + \zeta_n - v^{l_0}_n$, and $v^k_n = v^k_n + \zeta_n - (\mu_0 u^{l_0+1}_n - (\mu_0 - 1)u^{l_0}_n)$, for the first and second order of approximations, respectively.

By using MATLAB program and modified Gauss method ([33]), numerical calculations are carried out for $N = M = 20, 40, 80, 160$. In the Tables 1–3, we give error of numerical solution for inverse problem (4.1) and auxiliary NBVP. Table 1 contains error between exact solution of NBVP and solutions derived by difference schemes (4.2) and (4.3) . Table 2 and Table 3 contain error between exact and approximately solution of overdetermined problem (4.1) for p and u, respectively. Tables 1–3 show that the second order of ADS is more accurate comparing with the first order of ADS.

Table 1. Error for NBVP

order of ADS	$N = M = 20$	$N = M = 40$	$N = M = 80$	$N = M = 160$
first	0.65402	0.31258	0.1528	7.55×10^{-2}
second	0.10305	1.37×10^{-2}	1.98×10^{-3}	3.50×10^{-4}

Table 2. Error of p for problem (4.1)

order of ADS	$N = M = 20$	$N = M = 40$	$N = M = 80$	$N = M = 160$
first	0.70016	0.35855	0.18181	9.15×10^{-2}
second	0.13998	2.32×10^{-2}	4.78×10^{-3}	1.13×10^{-3}

Table 3. Error of v for problem (4.1)

order of ADS	$N = M = 20$	$N = M = 40$	$N = M = 80$	$N = M = 160$
first	5.31×10^{-2}	2.40×10^{-2}	1.16×10^{-2}	5.69×10^{-3}
second	5.45×10^{-3}	6.45×10^{-4}	8.51×10^{-5}	1.49×10^{-5}

4.2. **Three dimensional example.** Consider the three dimensional overdetermined elliptic two point NBVP

$$
\begin{gathered}
-\frac{\partial^2 v}{\partial t^2}(t,x,y)-\frac{\partial^2 v}{\partial x^2}(t,x,y)-\frac{\partial^2 v}{\partial y^2}(t,x,y)+v(t,x,y) \\
=g(t,x,y)+p(x,y), \quad 0<x<1, \quad 0<y<1, \quad 0<t<1, \\
v(0,x,y)=\phi(x,y), \quad v(0.26,x,y)=\zeta(x,y), \\
v(1,x,y)-\frac{1}{2}v(0.38,x,y)-\frac{1}{2}v(0.88,x,y)=\eta(x,y) \\
0\le x\le 1, \quad 0\le y\le 1, \\
v_x(t,0,y)=v_x(t,1,y)=0, \quad 0\le y\le 1, 0<t<1, \\
v_y(t,x,0)=v_y(t,x,1)=0, \quad 0\le x\le 1, 0<t<1,
\end{gathered} \tag{4.6}
$$

where

$$
\begin{gathered}
g(t,x,y)=2\pi^2 e^{-t}\cos(\pi x)\cos(\pi y), \quad \phi(x,y)=2\cos(\pi x)\cos(\pi y), \\
\zeta(x,y)=(e^{-0.26}+1)\cos(\pi x)\cos(\pi y), \\
\eta(x,y)=(e^{-1}-\frac{1}{2}e^{-0.38}-\frac{1}{2}e^{-0.88})\cos(\pi x)\cos(\pi y).
\end{gathered}
$$

The pair of functions

$$
p(x,y)=(2\pi^2+1)\cos(\pi x)\cos(\pi y), \quad v(t,x,y)=(e^{-t}+1)\cos(\pi x)\cos(\pi y)
$$

is an exact solution of (4.6).

We use the notation $[0,1]_\tau\times[0,1]_h^2$ for set of grid points depending on the small parameters τ and h

$$
\begin{gathered}
[0,1]_\tau\times[0,1]_h^2=\{(t_k,x_n,y_m): t_k=k\tau, \quad k=0,\dots,N, \\
x_n=nh, \quad y_m=mh, \quad n,m=0,\dots,M,\ N\tau=1, Mh=1\}.
\end{gathered}
$$

Also suppose that

$$
\begin{gathered}
\lambda_0=0.26, \quad \lambda_1=0.38, \lambda_2=0.88, \quad l_i=[\frac{\lambda_i}{\tau}], \quad \mu_i=-l_i+\frac{\lambda_i}{\tau}, \quad i=0,1,2; \\
\varphi_{m,n}=\varphi(x_n,y_m), \quad \psi_{m,n}=\psi(x_n,y_m), \quad \zeta_{m,n}=\xi(x_n,y_m), \quad n,m=\overline{0,M}; \\
g_{m,n}^k=g(t_k,x_n,y_m), \quad k=\overline{0,N},\ n,m=\overline{0,M}.
\end{gathered}
$$

In the first stage, we can write the first and order of ADSs for approximately solution of corresponding NBVP in the following forms:

$$
\begin{gathered}
-\frac{u_{m,n}^{k+1}-2u_{m,n}^k+u_{m,n}^{k-1}}{\tau^2}-\frac{u_{m,n+1}^k-2u_{m,n}^k+u_{m,n-1}^k}{h^2} \\
-\frac{u_{m+1,n}^k-2u_{m,n}^k+u_{m-1,n}^k}{h^2}+u_{m,n}^k \\
=g_{m,n}^k, \quad k=\overline{1,N-1},\ m,n=\overline{1,M-1}, \\
u_{0,n}^k-u_{1,n}^k=0, \quad u_{M,n}^k-u_{M-1,n}^k=0, \quad k=\overline{1,N-1},\ n=\overline{1,M-1}, \\
u_{m,0}^k-u_{m,1}^k=0, \quad u_{m,M}^k-u_{m,M-1}^k=0, \quad k=\overline{1,N-1},\ m=\overline{1,M-1}, \\
u_{m,n}^1-u_{m,n}^0=\tau\varphi_{m,n}, \quad u_{m,n}^N-u_{m,n}^{N-1}-\frac{1}{2}(u_{m,n}^{l_1+1}-u_{m,n}^{l_1}) \\
-\frac{1}{2}(u_{m,n}^{l_2+1}-u_n^{l_2})=\psi_{m,n}, \quad m,n=\overline{1,M-1},
\end{gathered} \tag{4.7}
$$

and

$$\begin{aligned}
&-\frac{u_{m,n}^{k+1}-2u_{m,n}^{k}+u_{m,n}^{k-1}}{\tau^2}-\frac{u_{m,n+1}^{k}-2u_{m,n}^{k}+u_{m,n-1}^{k}}{h^2}\\
&-\frac{u_{m+1,n}^{k}-2u_{m,n}^{k}+u_{m-1,n}^{k}}{h^2}+u_{m,n}^{k}=g_{m,n}^{k},\quad k=\overline{1,N-1},\ m,n=\overline{1,M-1},\\
&3u_{0,n}^{k}-4u_{1,n}^{k}+u_{2,n}^{k}=0,\quad 3u_{M,n}^{k}-4u_{M-1,n}^{k}+u_{M-2,n}^{k}=0,\\
&k=\overline{1,N-1},n=\overline{1,M-1},\\
&3u_{m,0}^{k}-4u_{m,1}^{k}+u_{m,2}^{k}=0,\quad 3u_{m,M}^{k}-4u_{m,M-1}^{k}+u_{m,M-2}^{k}=0,\\
&k=\overline{1,N-1},\ m=\overline{1,M-1},\\
&-3u_{m,n}^{0}+4u_{m,n}^{1}-u_{m,n}^{2}=2\tau\varphi_{m,n},\\
&3u_{m,n}^{N}-4u_{m,n}^{N-1}+u_{m,n}^{N-2}-\frac{1}{2}\Big[(3+2\mu_1)u_{m,n}^{l_1+1}-(4+4\mu_1)u_{m,n}^{l_1}\\
&+(1+2\mu_1)u_{m,n}^{l_1}\Big]-\frac{1}{2}\left[(3+2\mu_2)u_{m,n}^{l_2+1}-(4+4\mu_2)u_{m,n}^{l_2}+(1+2\mu_2)u_{m,n}^{l_2}\right]\\
&=2\tau\psi_{m,n},\quad m,n=\overline{1,M-1},
\end{aligned}\tag{4.8}$$

respectively.

In the second stage, $p_{m,n}$ is calculated by formulas by (2.10) and (2.11), respectively.

In the last stage, calculation of $\{v_n^k\}$ is carried out by

$$v_{m,n}^{k}=u_{m,n}^{k}+\zeta_n-u_{m,n}^{l_0},v_{m,n}^{k}=u_{m,n}^{k}+\zeta_{m,n}-(\mu_0u_{m,n}^{l_0+1}-(\mu_0-1)u_{m,n}^{l_0})$$

in the cases corresponding to first and second order approximations.

Problems (4.7) and (4.8) can be presented in the matrix form

$$\begin{gathered}
Au_{n+1}+Bu_n+Cu_{n-1}=Ig_n,\quad n=\overline{1,M-1},\\
u_0-u_1=\overrightarrow{0},\quad u_M-u_{M-1}=\overrightarrow{0},
\end{gathered}\tag{4.9}$$

and

$$\begin{gathered}
Au_{n+1}+Bu_n+Cu_{n-1}=Ig_n,\quad n=\overline{1,M-1},\\
3u_0-4u_1+u_1=\overrightarrow{0},\quad 3u_M-4u_{M-1}+u_{M-1}=\overrightarrow{0},
\end{gathered}\tag{4.10}$$

respectively.

Note that A,B,C,I are square matrices with $(N+1)^2(M+1)^2$ elements, and I is the identity matrix, g_s and u_s $(s=n-1,n,n+1)$ are the column matrices with $(N+1)(M+1)$ elements such that

$$\begin{aligned}
u_s&=\begin{bmatrix}u_{0,s}^{0} & \dots & u_{0,s}^{N} & u_{1,s}^{0} & \dots & u_{1,s}^{N} & \dots & u_{M,s}^{0} & \dots & v_{M,s}^{N}\end{bmatrix}^t,\\
g_s&=\begin{bmatrix}g_{0,s}^{0} & \dots & g_{0,s}^{N} & g_{1,s}^{0} & \dots & g_{1,s}^{N} & \dots & g_{M,s}^{0} & \dots & g_{M,s}^{N}\end{bmatrix}^t.
\end{aligned}$$

Denote

$$\begin{gathered}
a=\frac{1}{h^2},b=1+\frac{2}{\tau^2}+\frac{4}{h^2},r=\frac{1}{\tau^2},\\
E=\operatorname{diag}(0,a,a,\dots,a,0),\quad O=O_{(N+1)\times(N+1)}.
\end{gathered}$$

Then

$$A = C = \begin{bmatrix} O & O & \dots & O & O \\ O & E & \dots & O & O \\ \dots & \dots & \ddots & \dots & \dots \\ O & O & \dots & E & \\ O & O & \dots & O & O \end{bmatrix},$$

$$B = \begin{bmatrix} Q & W & Z & \dots & O & O & O \\ O & D & O & \dots & O & O & O \\ O & O & D & \dots & O & O & O \\ \dots & \dots & \dots & \ddots & \dots & \dots & \dots \\ O & O & O & \dots & O & O & O \\ O & O & O & \dots & O & D & O \\ O & O & O & \dots & Z & W & Q \end{bmatrix},$$

$$Q = I_{(N+1)\times(N+1)}, \quad W = -I_{(N+1)\times(N+1)}, \quad Z = O,$$

$$d_{i,i} = b, \quad d_{i-1,i} = r, \quad d_{i,i-1} = r, \quad i = \overline{2, N}; \quad d_{1,1} = -1, \quad d_{1,2} = 1,$$

$$d_{N+1,N+1} = 1, \quad d_{N+1,N} = -1, \quad d_{N+1,l_1} = -\frac{1}{2}, \quad d_{N+1,l_2} = -\frac{1}{2},$$

$$d_{N+1,l_1+1} = \frac{1}{2}, \quad d_{N+1,l_2+1} = \frac{1}{2},$$

$$d_{i,j} = 0, \quad \text{for other cases},$$

$$g^0_{m,n} = \tau\varphi_{m,n}, \quad g^N_{m,n} = \tau\psi_{m,n}, \quad n, m = 1, \dots, M-1$$

for first order of ADS, and

$$Q = 3I_{(N+1)\times(N+1)}, W = -4I_{(N+1)\times(N+1)}, \quad Z = I_{(N+1)\times(N+1)},$$

$$d_{i,i} = b, \quad d_{i-1,i} = r, \quad d_{i,i-1} = r, \quad i = \overline{2, N}; \quad d_{1,1} = -3,$$

$$d_{1,2} = 4, \quad d_{1,3} = -1, \quad d_{N+1,N+1} = 3, \quad d_{N+1,N} = -4, \quad d_{N+1,N-1} = -1,$$

$$d_{N+1,l_1+1} = -\frac{1}{2}(3 + 2\mu_1), \quad d_{N+1,l_1} = 2 + 2\mu_1,$$

$$d_{N+1,l_1-1} = -\frac{1}{2}(1 + 2\mu_1), \quad d_{N+1,l_2+1} = -\frac{1}{2}(3 + 2\mu_2),$$

$$d_{N+1,l_2} = 2 + 2\mu_2, \quad d_{N+1,l_2-1} = -\frac{1}{2}(1 + 2\mu_2),$$

$$d_{i,j} = 0, \quad \text{for other } i \text{ and } j;$$

$$g^0_{m,n} = 2\tau\varphi_{m,n}, \quad g^N_{m,n} = 2\tau\psi_{m,n}, \quad n, m = \overline{1, M-1}$$

for second order of ADS.

Numerical calculations are carried out by using MATLAB program and modified Gauss method [33] for $N = M = 10, 20, 40$. In Tables 4–6, the numerical results for both order of ADSs are given. Table 4 contains error between exact and approximately solutions of NBVP. Table 5 presents error for u. Tables 6 includes error for p. These tables show that the second order of ADS is more accurate comparing to the first order of ADS.

Conclusion. In this research work, inverse elliptic problem with Bitsadze-Samarskii type multipoint nonlocal and Neumann boundary conditions are discussed. First and second order of accuracy difference schemes for this problem are presented.

Table 4. Error analysis for NBVP

Difference scheme	$N=M=10$	$N=M=20$	$N=M=40$
First order of ADS	0.0822	0.0392	0.0169
Second order of ADS	0.0226	$2.02{\times}10^{-3}$	$1.33{\times}10^{-4}$

Table 5. Error analysis for p in example (4.6)

Difference scheme	$N=M=10$	$N=M=20$	$N=M=40$
First order of ADS	0.8207	0.1693	0.1029
Second order of ADS	0.3266	0.0592	0.0106

Table 6. Error analysis for v in example (4.6)

Difference scheme	N=10,M=10	N=20,M=20	N=40,M=40
First order of ADS	0.0291	0.0135	$4.06{\times}10^{-3}$
Second order of ADS	0.0053	$4.68{\times}10^{-4}$	$3.03{\times}10^{-5}$

Stability and coercive stability estimates for solutions of corresponding difference schemes are established. Then, numerical results for inverse elliptic problem with multipoint Bitsadze-Samarskii type nonlocal and Neumann boundary conditions in two and three dimensional test examples are illustrated. Numerical results are carried out by MATLAB program and short explanation on the realization of algorithm is given.

Moreover, applying the results of papers [4, 12, 20] the high order of ADSs for the numerical solution to the Bitsadze-Samarskii type overdetermined elliptic problem with Neumann conditions can be presented.

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