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STABILIZATION OF WAVE EQUATIONS WITH VARIABLE COEFFICIENT AND DELAY IN THE DYNAMICAL BOUNDARY FEEDBACK

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Abstract. In this article we consider the boundary stabilization of a wave equation with variable coefficients. This equation has an acceleration term and a delayed velocity term on the boundary. Under suitable geometric conditions, we obtain the exponential decay for the solutions. Our proof relies on the geometric multiplier method and the Lyapunov approach.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^N (N \geq 2)$ with smooth boundary Γ. We assume $\Gamma = \Gamma_0 \cup \Gamma_1$ with $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$. We consider the following wave equation with dynamical Neumann boundary condition

$$
\begin{gathered}
u_{tt} - \operatorname{div} A(x)\nabla u = 0, \quad \text{in } \Omega \times (0,\infty), \\
u(x,t) = 0, \quad \text{on } \Gamma_0 \times (0,\infty), \\
m(x)u_{tt}(x,t) + \partial_{\nu_A} u(x,t) = C(t), \quad \text{on } \Gamma_1 \times (0,\infty), \\
u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), \quad \text{in } \Omega,
\end{gathered}
\tag{1.1}
$$

where $\operatorname{div} X$ denote the divergence of the vector field X in the Euclidean metric, $A(x) = (a_{ij}(x))$ are symmetric and positive definite matrices for all $x \in \mathbb{R}^N$ and $a_{ij}(x)$ are smooth functions on $\mathbb{R}^N$. $\partial_{\nu_A} u = \sum_{i,j=1}^n a_{ij}(x)\partial_{x_j} u\nu_i$, where $\nu = (\nu_1, \nu_2, \dots, \nu_n)^T$ denotes the outward unit normal vector of the boundary and $\nu_A = A\nu$. $C(t)$ is the boundary feedback control.

We call the equation (1.1) is with dynamical boundary conditions if $m(x) \neq 0$, which means the system has an acceleration term on part of the boundary. This is what happened in some physical applications when one has to take the acceleration terms into account on the boundary. Actually we need the models with dynamical boundary conditions, see [8, 10, 17] and many others. They are not only important theoretically but also have strong backgrounds for physical applications. There are numerous of these applications in the bio-medical domain [5, 20] as well as in

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applications related to noise suppression and control of elastic structures [2, 3, 4, 21]. For the above reasons in this article we assume

$$m(x) \in L^\infty(\Gamma_1) \quad \text{and} \quad m(x) > \alpha > 0, \tag{1.2}$$

where α is a positive constant. Here we denote $\epsilon_1(x)$ a nonnegative function as

$$\epsilon_1(x) = \frac{m(x)}{\alpha} - 1 > 0, \quad x \in \Gamma_1. \tag{1.3}$$

On the other hand, time delay effects arise in many practical problems in science and engineering. Most phenomena depends on not only the present state but also the history of the system in a very complicated way. For instance the practical systems often suffer from the actuator saturation and sometimes the control input delay. It is well known that delay effects might turn a well-behaved system into a wild one by inducing some instabilities, see [9, 11, 18]. Recently boundary feedbacks are designed to overcome the negative effect of time delays and stabilize the system; see [1, 15, 16, 19, 22] and the references therein.

In this article, we discuss the stabilization of wave equations subject to dynamical boundary conditions with acceleration terms(i.e. u_{tt}) and time delayed velocity terms. We shall design a collocated boundary feedback with time delay effect in the velocity input to obtain the exponential stabilization of the system. That is, $C(t)$ is a feedback law with input time delay

$$C(t) = -\alpha u_t(x,t) - \beta u_t(x, t-\tau) - \partial_{\nu_A} u_t\,, \tag{1.4}$$

where τ is the time delay, α is the positive constant given in (1.2) and the constant $\beta > 0$ denotes the effect of time delay in the velocity input. $\beta = 0$ means the absence of input delay. For the discussion of stabilization results in the case of absence of delay, see [6, 7, 8, 12, 14, 23] and our recent paper [28]. Here in this article we assume that, for $x \in \Gamma_1$, we have

$$0 < \frac{\beta}{\alpha} < \sqrt{\frac{\epsilon_1(x)}{1+\epsilon_1(x)}}, \tag{1.5}$$

$$\epsilon_1(x) \in R^+ \setminus [\frac{\beta^2}{2\alpha(\alpha+\beta)}, \frac{\beta^2}{2\alpha(\alpha-\beta)}], \tag{1.6}$$

where $\epsilon_1(x)$ is given in (1.3). If $t < \tau$, then $u_t(x, t-\tau)$ is determined by the datum in the past and we need an initial value in the past. We thus give the initial condition

$$u_t(x, t-\tau) = f_0(x, t-\tau), \quad (x,t) \in \Gamma_1 \times (0,\tau). \tag{1.7}$$

In equation (1.1) we adopt the feedback law given in (1.4) and the initial datum given in (1.7) to obtain the following closed loop system:

$$\begin{gathered}
u_{tt} - \operatorname{div} A(x)\nabla u = 0, \quad \text{in } \Omega \times (0,\infty),\\
u(x,t) = 0, \quad \text{on } \Gamma_0 \times (0,\infty),\\
m(x)u_{tt}(x,t) + \partial_{\nu_A} u(x,t) = -\alpha u_t(x,t) - \beta u_t(x,t-\tau) - \partial_{\nu_A} u_t, \quad \text{on } \Gamma_1 \times (0,\infty),\\
u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), \quad \text{in } \Omega,\\
u_t(x,t-\tau) = f_0(x,t-\tau), \quad (x,t) \in \Gamma_1 \times (0,\tau).
\end{gathered} \tag{1.8}$$

We define

$$g = A^{-1}(x), \quad x \in \Omega$$

as a Riemannian metric on Ω and consider the couple (Ω, g) as a Riemannian manifold. Let D denote the Levi-Civita connection of the metric g. For each $x \in \Omega$, the metric g induces an inner product on R^n_x by

$$\langle X, Y\rangle_g = \langle A^{-1}(x)X, Y\rangle, \quad |X|_g^2 = \langle X, X\rangle_g\,, \quad X, Y \in R^n_x\,,$$

where $\langle\cdot,\cdot\rangle$ denotes the standard metric of the Euclidean space R^n.

To obtain the stabilization of problem (1.8), the following geometric hypotheses are assumed:

There exists a vector field H on Riemannian manifold (Ω, g) such that the following properties hold:

(A1) $DH(\cdot,\cdot)$ is strictly positive definite on $\overline{\Omega}$: there exists a constant $\kappa > 0$ such that for all $x \in \overline{\Omega}$, for all $X \in M_x$(the tangent space at x):

$$DH(X, X) \equiv \langle D_X H, X\rangle_g \ge \kappa |X|^2. \tag{1.9}$$

(A2)

$$H \cdot \nu \le 0 \quad \text{on } \Gamma_0\,. \tag{1.10}$$

Remark 1.1. For any Riemannian manifold M, the existence of such a vector field H in (A1) has been proved in [24], where some examples are given. See also [26]. For the Euclidean metric, taking the vector field $H = x - x_0$ and we have $DH(X, X) = |X|^2$, which means assumption (A1) always holds with $\kappa = 1$ for the Euclidean case.

Before we go to the stabilization of the system, we should first define an energy connected with the natural energy of the hybrid system (1.1). We set

$$\eta(x, t) = m(x)u_t(x, t) + \partial_{\nu_A} u,\ x \in \Gamma_1\,. \tag{1.11}$$

Let u be a regular solution of system (1.8). Then we associate with system (1.8) the energy functional

$$E(t) = \int_\Omega \big(u_t^2 + |\nabla_g u|_g^2\big)dx + \int_{\Gamma_1} \frac{1}{m(x) - \alpha}\eta^2 d\Gamma \qquad + \xi \int_0^1 \int_{\Gamma_1} u_t^2(x, t - \rho\tau) d\Gamma d\rho\,, \tag{1.12}$$

for some constant $\xi > 0$ which will be determined later.

The main result of this paper is the following:

Theorem 1.2. *Let the geometric assumptions* (A1) *and* (A2) *hold. Then there exist constants $C > 0$ and $\omega > 0$ such that*

$$E(t) \le Ce^{-\omega t}E(0), \quad t \ge 0\,. \tag{1.13}$$

This article is organized as follows. In the next section, we discuss the well-posedness of the nonlinear close-loop system by semigroup theory. Section 3 devotes to the proof of the exponential stability. We construct an appropriate Lyapunov functional to obtain the main result.

2. Well-posedness of the closed loop system

In this section, we shall study well-posedness results for system (1.8) using semigroup theory. Let

$$z(x, \rho, t) = u_t(x, t - \rho\tau), \quad x \in \Gamma_1, \rho \in (0, 1), \quad t > 0\,.$$

Then the closed loop system (1.8) is equivalent to the system

$$
\begin{gathered}
u_{tt} - \operatorname{div} A(x)\nabla u = 0 \quad \text{in } \Omega \times (0,\infty), \\
u(x,t) = 0, \quad \text{on } \Gamma_0 \times (0,\infty), \\
\tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0, \quad \text{in } \Gamma_1 \times (0,1) \times (0,\infty), \\
\eta_t(x,t) = -\eta + (m-\alpha)u_t(x,t) - \beta u_t(x,t-\tau), \quad \text{in } \Gamma_1 \times (0,\infty), \\
u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega, \\
\eta(x,0) = mu_1(x) + \partial_{\nu_A} u_0(x), \quad \text{on } \Gamma_1\,, \\
z(x,0,t) = u_t(x,t), \quad \text{on } \Gamma_1 \times (0,\infty), \\
z(x,\rho,0) = f_0(x,-\rho\tau), \quad \text{on } \Gamma_1 \times (0,1)\,,
\end{gathered}
\tag{2.1}
$$

where η is given by (1.11). We consider the unknown

$$
U = \big(u, w = u_t|_\Omega, \eta, z\big)^T\,,
$$

in the state space, denoted by

$$
\Upsilon := H^1(\Omega) \times L^2(\Omega) \times L^2(\Gamma_1) \times L^2(\Gamma_1 \times (0,1))\,, \tag{2.2}
$$

with the norm defined by

$$
\begin{aligned}
&\|U\|^2 \\
&= \|(u,w,\eta,z)^T\|^2 \\
&= \int_\Omega \big(\langle A(x)\nabla u, \nabla u\rangle + w^2\big)dx + \int_{\Gamma_1} \frac{1}{m(x)-\alpha}\eta^2 d\Gamma + \xi \int_0^1 \int_{\Gamma_1} z^2 d\rho d\Gamma,
\end{aligned}
\tag{2.3}
$$

where $\xi > 0$ is the constant in (1.12).

The system (2.1) can be rewritten in the abstract form

$$
\begin{gathered}
U' = \mathcal{A}U, \\
U_0 = \big(u_0, u_1, \eta_0, f_0(\cdot, -\cdot\tau)\big)^T,
\end{gathered}
\tag{2.4}
$$

where the operator $\mathcal{A}$ is defined by

$$
\mathcal{A}\begin{pmatrix} u \\ w \\ \eta \\ z \end{pmatrix} = \begin{pmatrix} w \\ \operatorname{div} A(x)\nabla u \\ -\eta + \big(m(x)-\alpha\big)w(x,t) - \beta z(x,1,t) \\ -\tau^{-1} z_\rho \end{pmatrix}
$$

with domain

$$
\begin{aligned}
D(\mathcal{A}) := \Big\{ &(u,w,\eta,z)^T \in H^1(\Omega) \times L^2(\Omega) \times L^2(\Gamma_1) \times L^2(\Gamma_1; H^1(0,1)) : \\
&\operatorname{div} A(x)\nabla u \in L^2(\Omega), \eta = m(x)w|_{\Gamma_1} + \partial_{\nu_A} u, z(x,0,t) = w(x,t) \\
&\text{on } \Gamma_1 \times (0,\infty) \Big\}.
\end{aligned}
\tag{2.5}
$$

We will show that $\mathcal{A}$ generates a C_0 semigroup on Υ under the assumption (1.5). Now we state the well-posedness result.

Theorem 2.1. *For any initial datum $U_0 \in \Upsilon$, there exists a unique solution $U \in C([0,\infty),\Upsilon)$ of system* (2.4). *Moreover, if $U_0 \in D(\mathcal{A})$, then $U \in C([0,\infty), D(\mathcal{A})) \cap C^1([0,\infty), D(\mathcal{A}))$.*

Proof. **Step 1**. We prove that $\mathcal{A}$ is dissipative. We know Υ is a Hilbert space equipped with the adequate scalar product $\langle\cdot,\cdot\rangle_\Upsilon$ and norm $\|U\|$ defined by (2.3). For $U\in D(\mathcal{A})$, a simple computation leads to

$$
\begin{aligned}
&\langle \mathcal{A}U,U\rangle_\Upsilon\\
&=\frac{1}{2}\frac{d}{dt}\|U\|^2\\
&=\int_\Gamma u_t\partial_{\nu_A}u d\Gamma+\int_{\Gamma_1}\frac{1}{m(x)-\alpha}\eta\eta_t d\Gamma+\xi\int_0^1\int_{\Gamma_1}z(x,\rho,t)z_t(x,\rho,t)d\rho d\Gamma\\
&=\int_{\Gamma_1} u_t\partial_{\nu_A}u d\Gamma-\int_{\Gamma_1}\frac{1}{m(x)-\alpha}\eta^2 d\Gamma+\int_{\Gamma_1}\eta u_t(x,t)d\Gamma\\
&\quad-\int_{\Gamma_1}\frac{\beta}{m(x)-\alpha}z(x,1,t)\eta d\Gamma+\xi\int_0^1\int_{\Gamma_1}z(x,\rho,t)z_t(x,\rho,t)d\rho d\Gamma\\
&=\int_{\Gamma_1} u_t\partial_{\nu_A}u d\Gamma-\int_{\Gamma_1}\frac{1}{2m(x)}\eta^2 d\Gamma\\
&\quad-\int_{\Gamma_1}\big(\frac{1}{m(x)-\alpha}-\frac{1}{2m(x)}\big)\eta^2 d\Gamma+\int_{\Gamma_1}\eta u_t(x,t)d\Gamma\\
&\quad-\int_{\Gamma_1}\frac{\beta}{m(x)-\alpha}z(x,1,t)\eta d\Gamma+\xi\int_0^1\int_{\Gamma_1}z(x,\rho,t)z_t(x,\rho,t)d\rho d\Gamma\\
&=-\int_{\Gamma_1}\big(\frac{m(x)}{2}u_t^2+\frac{1}{2m(x)}\partial_{\nu_A}^2u\big)d\Gamma\\
&\quad-\int_{\Gamma_1}\big(\frac{1}{m(x)-\alpha}-\frac{1}{2m(x)}\big)\eta^2 d\Gamma+\int_{\Gamma_1}\eta u_t(x,t)d\Gamma\\
&\quad-\int_{\Gamma_1}\frac{\beta}{m(x)-\alpha}z(x,1,t)\eta d\Gamma+\frac{\xi}{2\tau}\int_{\Gamma_1}\big(u_t^2(x,t)-z^2(x,1,t)\big)d\Gamma\,,
\end{aligned}
\tag{2.6}
$$

where we used that

$$
\begin{aligned}
&\xi\int_0^1\int_{\Gamma_1}z(x,\rho,t)z_t(x,\rho,t)d\rho d\Gamma\\
&=-\tau^{-1}\xi\int_0^1\int_{\Gamma_1}z_\rho(x,\rho,t)z(x,\rho,t)d\rho d\Gamma\\
&=-\frac{1}{2}\tau^{-1}\xi\int_{\Gamma_1}\int_0^1\frac{dz^2}{d\rho}d\rho d\Gamma\\
&=\frac{\xi}{2\tau}\int_{\Gamma_1}\big(u_t^2-z^2(x,1,t)\big)d\Gamma\,.
\end{aligned}
\tag{2.7}
$$

Now we handle the items in (2.6) by applying Hölder's inequality

$$
\int_{\Gamma_1}\eta u_t(x,t)d\Gamma\le\frac{1}{k_1}\int_{\Gamma_1}\frac{\eta^2}{m(x)}d\Gamma+\frac{k_1}{4}\int_{\Gamma_1}m(x)u_t^2 d\Gamma\,,\tag{2.8}
$$

$$
-\int_{\Gamma_1}\frac{\beta}{m(x)-\alpha}z(x,1,t)\eta d\Gamma\le\int_{\Gamma_1}\frac{\beta}{m(x)-\alpha}\big(\frac{1}{2k_2}\eta^2+\frac{k_2}{2}z^2(x,1,t)\big)d\Gamma\,,\tag{2.9}
$$

where $k_1 > 0$ and $k_2 > 0$ can be chosen later. Substituting (2.7), (2.8) and (2.9) in (2.6) yields

$$\begin{aligned} \langle \mathcal{A}U, U\rangle_\Upsilon \le &-\int_{\Gamma_1} \Big(\frac{m(x)}{2} - \frac{k_1 m(x)}{4} - \frac{\xi}{2\tau}\Big) u_t^2 d\Gamma \\ &- \int_{\Gamma_1} \Big(\frac{1}{m(x)-\alpha}\big(1 - \frac{\beta}{2k_2}\big) - \frac{1}{2m(x)} - \frac{1}{k_1 m(x)}\Big)\eta^2 d\Gamma \\ &+ \int_{\Gamma_1} \Big(\frac{\beta}{m(x)-\alpha}\frac{k_2}{2} - \frac{\xi}{2\tau}\Big) z^2(x,1,t) d\Gamma - \int_{\Gamma_1} \frac{1}{2m(x)} \partial_{\nu_A}^2 u d\Gamma. \end{aligned} \tag{2.10}$$

Next we find positive numbers k_1, k_2 and ξ that guarantee the negativity of $\langle \mathcal{A}U, U\rangle_\Upsilon$. That is, we need, for all $x \in \Gamma_1$,

$$\frac{m(x)}{2} - \frac{k_1 m(x)}{4} - \frac{\xi}{2\tau} > 0\,, \tag{2.11}$$

$$\frac{1}{m(x)-\alpha}\big(1 - \frac{\beta}{2k_2}\big) - \frac{1}{2m} - \frac{1}{k_1 m(x)} > 0\,, \tag{2.12}$$

$$\frac{\beta}{m(x)-\alpha}\big(\frac{k_2}{2}\big) - \frac{\xi}{2\tau} < 0\,. \tag{2.13}$$

Inequalities (2.11) and (2.13) imply

$$\frac{\beta}{m(x)-\alpha}\big(\frac{k_2}{2}\big) < \frac{m(x)}{2} - \frac{k_1 m(x)}{4}\,, \tag{2.14}$$

that is,

$$k_2 < \frac{(2-k_1)m(x)}{2}\frac{m(x)-\alpha}{\beta}\,. \tag{2.15}$$

We can easily show that inequality (2.12) is equivalent to

$$k_2\Big(\frac{1}{m(x)-\alpha} - \frac{k_1+2}{2m(x)k_1}\Big) - \frac{\beta}{2(m(x)-\alpha)} > 0\,, \tag{2.16}$$

from which we find that

$$2m(x)k_1 - (k_1+2)\big(m(x)-\alpha\big) > 0\,, \tag{2.17}$$

that is,

$$k_1 > 2\frac{m(x)-\alpha}{m(x)+\alpha}\,, \tag{2.18}$$

and

$$k_2 > \frac{\beta m(x) k_1}{2m(x)k_1 - (k_1+2)\big(m(x)-\alpha\big)}\,, \tag{2.19}$$

for all $x \in \Gamma_1$.

Now we can determine the constants k_1 and k_2 according to inequalities (2.15), (2.18) and (2.19).

Firstly we aim to get k_1. The inequalties (2.15) and (2.19) yield

$$\frac{\beta m(x) k_1}{2m(x)k_1 - (k_1+2)\big(m(x)-\alpha\big)} < \frac{(2-k_1)m(x)\big(m(x)-\alpha\big)}{2\beta}\,,$$

that is,

$$\begin{aligned} &k_1^2\Big(m(x)\big(m(x)-\alpha\big)\big(m(x)+\alpha\big)\Big) + k_1(2m(x))\Big(\beta^2 - 2m(x)\big(m(x)-\alpha\big)\Big) \\ &+ 4m(x)\big(m(x)-\alpha\big)^2 < 0\,. \end{aligned}$$

For the quadratic form of k_1, we compute the discriminant

$$\begin{aligned}\Delta(x) &= 4m^2\big(\beta^2 - 2m(m-\alpha)\big)^2 - 16m^2(m-\alpha)^2(m^2-\alpha^2)\\ &= 4m^2\big(\beta^4 + 4(m-\alpha)(m\alpha^2 - \alpha^3 - m\beta^2)\big) > 0\,,\end{aligned}$$

for all $x \in \Gamma_1$, where we used assumption (1.5). Thus we get a possible candidate

$$k_1 = \min_{x\in\Gamma_1}\Big\{\frac{2m(x)}{m(x)+\alpha} - \frac{\beta^2}{m(x)^2-\alpha^2}\Big\}\,. \tag{2.20}$$

It's easy to verify that $0 < k_1 < 2$ and inequality (2.18) holds.

Secondly we should determine the constant k_2. Now inequalities (2.15) and (2.19) become

$$k_2 < \frac{m(x)\big(m(x)-\alpha\big)}{2\beta}(2-k_1) = \frac{2m(x)\alpha\big(m(x)-\alpha\big) + m(x)\beta^2}{2\beta\big(m(x)+\alpha\big)}\,, \tag{2.21}$$

and

$$k_2 > \frac{\beta m(x) k_1}{2m(x)k_1 - (k_1+2)\big(m(x)-\alpha\big)} = \frac{\beta m(x)}{m(x)+\alpha}\,\frac{2m(x)\big(m(x)-\alpha\big)-\beta^2}{2\alpha\big(m(x)-\alpha\big)-\beta^2}\,,$$

that is,

$$\frac{\beta m(x)}{m(x)+\alpha}\,\frac{2m(x)\big(m(x)-\alpha\big)-\beta^2}{2\alpha\big(m(x)-\alpha\big)-\beta^2} < k_2 < \frac{2m(x)\alpha\big(m(x)-\alpha\big)+m(x)\beta^2}{2\beta\big(m(x)+\alpha\big)}\,. \tag{2.22}$$

To obtain k_2 we only have to verify that

$$\begin{aligned}&\frac{2m(x)\alpha\big(m(x)-\alpha\big)+m(x)\beta^2}{2\beta\big(m(x)+\alpha\big)} - \frac{\beta m(x)}{m(x)+\alpha}\,\frac{2m(x)\big(m(x)-\alpha\big)-\beta^2}{2\alpha\big(m(x)-\alpha\big)-\beta^2}\\ &= \Big(\big(2m(x)\alpha\big(m(x)-\alpha\big)+m(x)\beta^2\big)\big(2\alpha\big(m(x)-\alpha\big)-\beta^2\big)\\ &\quad - 2\beta^2 m(x)\Big(2m(x)\big(m(x)-\alpha\big)-\beta^2\big)\Big)\Big/\Big(2\beta\big(m(x)+\alpha\big)\big(2\alpha\big(m(x)-\alpha\big)-\beta^2\big)\Big)\\ &> 0\,.\end{aligned}$$

In fact, we have

$$\begin{aligned}&\big(2m\alpha(m-\alpha)+m\beta^2\big)\big(2\alpha(m-\alpha)-\beta^2\big) - 2\beta^2 m\big(2m(m-\alpha)-\beta^2\big)\\ &= 4m^3\alpha^2 + 4m\alpha^4 - 8m^2\alpha^3 + m\beta^4 - 4\beta^2m^3 + 4\beta^2m^2\alpha\\ &= (1+\epsilon_1)\big(4\alpha^3\epsilon_1^2(\alpha^2-\beta^2) - 4\alpha^3\beta^2\epsilon_1 + \alpha\beta^4\big) > 0\,,\end{aligned}$$

where we used assumption (1.6).

Thus by (2.22) we can take

$$\begin{aligned}k_2 = \frac{1}{2}\Big(&\min_{x\in\Gamma_1}\frac{2m(x)\alpha\big(m(x)-\alpha\big)+m(x)\beta^2}{2\beta\big(m(x)+\alpha\big)}\\ &+ \max_{x\in\Gamma_1}\frac{\beta m(x)}{m(x)+\alpha}\,\frac{2m(x)\big(m(x)-\alpha\big)-\beta^2}{2\alpha\big(m(x)-\alpha\big)-\beta^2}\Big)\,.\end{aligned} \tag{2.23}$$

Finally, we can take a constant ξ to satisfy

$$\frac{\beta k_2}{m-\alpha} < \frac{\xi}{\tau} < \frac{m(2-k_1)}{2}\,,$$

after k_1 and k_2 are given in (2.20) and (2.23). Thus we show that the operator $\mathcal{A}$ is dissipative.

Step 2. We will show that $\lambda I-\mathcal{A}$ is surjective for a fixed $\lambda>0$. Given $(\bar a,\bar b,\bar c,\bar f)^T\in\Upsilon$, we seek a solution $U=(u,w,\eta,z)^T\in D(\mathcal{A})$ of

$$(\lambda I-\mathcal{A})\begin{pmatrix}u\\ w\\ \eta\\ z\end{pmatrix}=\begin{pmatrix}\bar a\\ \bar b\\ \bar c\\ \bar f\end{pmatrix}$$

that is, satisfying

$$\begin{gathered}\lambda u-w=\bar a,\quad \text{in }\Omega,\\ \lambda w-\operatorname{div}A(x)\nabla u=\bar b,\quad \text{in }\Omega,\\ \lambda\eta+\eta-(m-\alpha)w(x,t)+\beta z(x,1,t)=\bar c,\quad \text{on }\Gamma_1,\\ \lambda z+\tau^{-1}z_\rho=\bar f,\quad \text{on }\Gamma_1\times(0,1),\\ \eta=mw|_{\Gamma_1}+\partial_{\nu_A}u(x,t),z(x,0,t)=w(x),z(x,\rho,0)=f_0(x,-\rho\tau)\quad\text{on }\Gamma_1.\end{gathered}\tag{2.24}$$

where we take t as a parameter. Suppose that we have found u with the appropriate regularity, then from equation (2.24) we have $w:=\lambda u-\bar a$. Therefore we have the initial value problem for z,

$$\begin{gathered}\lambda z+\tau^{-1}z_\rho=\bar f,\\ z(x,0)=w(x)\quad\text{on }\Gamma_1\,.\end{gathered}$$

We can easily see that

$$z(x,\rho)=w(x)e^{-\lambda\rho\tau}+\tau e^{-\lambda\rho\tau}\int_0^\rho\bar f(x,\theta)e^{\lambda\theta\tau}d\theta\,.$$

In particular we have

$$z(x,1)=(\lambda u-\bar a)e^{-\lambda\tau}+\tau e^{-\lambda\tau}\int_0^1\bar f(x,\theta)e^{\lambda\theta\tau}d\theta=:\lambda ue^{-\lambda\tau}+z_0\,,\tag{2.25}$$

where $z_0=-\bar ae^{-\lambda\tau}+\tau e^{-\lambda\tau}\int_0^1\bar f(x,\rho)e^{\lambda\rho\tau}d\rho$. So once we know u,we can get $w=\lambda u-\bar a$,and we already know $z(x,1)$, so we obtain

$$\eta=\frac{\bar c}{\lambda+1}+\frac{m-\alpha}{\lambda+1}w-\frac{\beta}{\lambda+1}z(x,1)\tag{2.26}$$

from (2.24)(3). Now we lay the emphasis on how to get u. Eliminating w we find that the function u satisfies

$$\begin{gathered}\lambda^2u=\operatorname{div}A(x)\nabla u+\lambda\bar a+\bar b,\ \text{in }\Omega\\ u=0,\quad\text{on }\Gamma_0\\ \partial_{\nu_A}u=\frac{\bar c}{\lambda+1}-\frac{\lambda m+\alpha}{\lambda+1}(\lambda u-\bar a)-\frac{\beta}{\lambda+1}z(x,1),\quad\text{in }\Gamma_1\end{gathered}\tag{2.27}$$

where $z(x,1)$ is given in (2.25). We obtain a weak formulation of system (2.27) by multiplying the equation by ψ and using Green's formula

$$\begin{aligned}&\int_\Omega(\lambda^2u\psi+\langle A(x)\nabla u,\nabla\psi\rangle)dx+\int_{\Gamma_1}u\psi\frac{\lambda(\lambda m+\alpha)}{\lambda+1}d\Gamma+\int_{\Gamma_1}u\psi\frac{\lambda\beta e^{-\lambda\tau}}{\lambda+1}d\Gamma\\ &=\int_\Omega(\lambda\bar a+\bar b)\psi dx+\int_{\Gamma_1}\psi\frac{\bar c}{\lambda+1}d\Gamma+\int_{\Gamma_1}\psi\frac{\bar a(\lambda m+\alpha)}{\lambda+1}d\Gamma-\int_{\Gamma_1}\psi\frac{\beta z_0}{\lambda+1}d\Gamma\,,\end{aligned}\tag{2.28}$$

for any $\psi \in H^1(\Omega)$, where z_0 is given in (2.25). As the left hand side of (2.28) is coercive on $H^1(\Omega)$, Lax-Milgram Theorem guarantees the existence and uniqueness of a solution $u \in H^1(\Omega)$ of (2.27).

Step 3. Finally, the well-posedness result follows from Lummer-Phillips Theorem. $\square$

3. Energy decay

In this section, we obtain the exponential stability of the system by energy perturbed approach. At first we rewrite the energy $E(t)$ defined in (1.12) as

$$E(t) = \mathcal{E}(t) + E_\tau(t)\,,$$

where $\mathcal{E}(t)$ is the standard energy regardless of the time delay

$$\mathcal{E}(t) = \int_\Omega \big(u_t^2 + |\nabla_g u|_g^2\big)dx + \int_{\Gamma_1} \frac{1}{m(x)-\alpha}\eta^2 d\Gamma\,,$$

$$E_\tau(t) = \int_0^1 \int_{\Gamma_1} u_t^2(x, t-\rho\tau) d\Gamma d\rho\,.$$

To obtain the estimate of the energy $E(t)$ we define three auxiliary functions:

$$V_1(t) = \int_\Omega H(u) u_t dx\,,$$

$$V_2(t) = \frac{1}{2}\int_\Omega (\operatorname{div} H - \kappa) u u_t dx\,,$$

$$V_3(t) = \xi \int_{\Gamma_1} \int_0^1 e^{-2\tau\rho} u_t^2(x, t-\rho\tau) d\rho d\Gamma\,.$$

We need some lemmas from [26] and [13].

Lemma 3.1 ([26, Theorem 2.1.]). *Suppose that $u(x,t)$ is a solution of the $u_{tt} - \operatorname{div} A\nabla u = 0$. We have $\dot V_1(t) = B_1 + I_1$, where the boundary term is*

$$B_1(\Gamma) = \int_\Gamma \partial_{\nu_A} u H(u) d\Gamma + \frac{1}{2}\int_\Gamma (u_t^2 - |\nabla_g u|_g^2) H\cdot\nu d\Gamma, \tag{3.1}$$

and the internal term

$$I_1 = -\int_\Omega D_g H(\nabla_g u, \nabla_g u) dx - \frac{1}{2}\int_\Omega (u_t^2 - |\nabla_g u|_g^2)\operatorname{div} H dx\,.$$

Lemma 3.2 ([26, Theorem 2.2.]). *Suppose that $u(x,t)$ is a solution of the equation $u_{tt} - \operatorname{div} A\nabla u = 0$. Then $\dot V_2(t) = B_2 + I_2$, where the boundary term is*

$$B_2(\Gamma) = -\frac{1}{4}\int_\Gamma u^2 \partial_{\nu_A}(\operatorname{div} H) d\Gamma + \frac{1}{2}\int_\Gamma (\operatorname{div} H - \kappa) u \partial_{\nu_A} u d\Gamma,$$

and the internal term

$$I_2 = \frac{1}{2}\int_\Omega (\operatorname{div} H - \kappa)(u_t^2 - |\nabla_g u|_g^2) dx + \frac{1}{4}\int_\Omega u^2 \operatorname{div} A(x)\nabla(\operatorname{div} H) dx\,.$$

Lemma 3.3 ([18, Prosition 3.1.]). *Let u solve problem (1.8). We have*

$$\dot V_3(t) = \frac{\xi}{\tau}\int_{\Gamma_1} u_t^2 d\Gamma - \frac{\xi}{\tau}\int_{\Gamma_1} e^{-2\tau} z^2(x,1,t) d\Gamma - 2\xi \int_{\Gamma_1}\int_0^1 e^{-2\rho\tau} z^2 d\rho d\Gamma\,. \tag{3.2}$$

From Lemmas 3.1, 3.2 and 3.3, we obtain the following result.

Lemma 3.4. *Suppose that the geometric assumptions* (A1) *and* (A2) *hold. Let* u *solves problem* (1.8)*. There exist constants* $c_1, c_2, c_3, c_4, c_5 > 0$ *such that*

$$\begin{aligned}
&\frac{\kappa}{2}E(t)+\dot{V}_1(t)+\dot{V}_2(t)+\frac{\kappa}{4\xi}e^{2\tau}\dot{V}_3(t)\\
&\le \frac{\kappa}{2}\int_{\Gamma_1}\frac{1}{m(x)-\alpha}\eta^2 d\Gamma + c_1\int_{\Gamma_1}(u_t^2+|\nabla_g u|_g^2)d\Gamma + c_2\int_\Omega u^2 dx\,,
\end{aligned}\tag{3.3}$$

$$|V_1(t)|\le c_3\mathcal{E}(t),\quad |V_2(t)|\le c_4\mathcal{E}(t),\quad |V_3(t)|\le c_5E_\tau(t)\,. \tag{3.4}$$

Proof. Obviously the estimate (3.4) is true. Now we prove inequality (3.3). First we estimate the boundary terms B_1, B_2 given in Lemma 3.1 and Lemma 3.2. Since $u|_{\Gamma_0}=0$, we have on Γ_0 that

$$\nabla_g u = \partial_{\nu_A}u\frac{\nu_A}{|\nu_A|_g^2}\,,$$

which implies

$$H(u)=\langle H,\nabla_g u\rangle_g = \partial_{\nu_A}u\frac{1}{|\nu_A|_g^2}H\cdot\nu. \tag{3.5}$$

Substituting equality (3.5) in (3.1) yields

$$B_1(\Gamma_0)=\frac{1}{2}\int_{\Gamma_0}(u_t^2+|\nabla_g u|_g^2)H\cdot\nu d\Gamma\le 0\,,$$

where we noticed the geometric assumption (A2) is true. It is obvious that

$$B_1(\Gamma)=B_1(\Gamma_0)+B_1(\Gamma_1)\le\frac{c_1}{4}\int_{\Gamma_1}(u_t^2+|\nabla_g u|_g^2)d\Gamma\,. \tag{3.6}$$

Since $u|_{\Gamma_0}=0$, we have $B_2(\Gamma_0)=0$. Also we have

$$B_2(\Gamma)=B_2(\Gamma_0)+B_2(\Gamma_1)\le\frac{c_1}{4}\int_{\Gamma_1}|\nabla_g u|_g^2 d\Gamma. \tag{3.7}$$

Next, we estimate the internal terms I_1 and I_2. From to the geometric assumption (A1), we have

$$I_1\le -\kappa\int_\Omega|\nabla_g u|_g^2dx-\frac{1}{2}\int_\Omega(u_t^2-|\nabla_g u|_g^2)\operatorname{div}H dx. \tag{3.8}$$

It is obvious that

$$I_1+I_2\le-\frac{\kappa}{2}\int_\Omega(u_t^2+|\nabla_g u|_g^2)dx+c_2\int_\Omega u^2dx. \tag{3.9}$$

Combining inequalities (3.6), (3.7), (3.8) and (3.9) we obtain

$$\begin{aligned}
&\frac{\kappa}{2}\mathcal{E}(t)+\dot{V}_1(t)+\dot{V}_2(t)\\
&\le\frac{\kappa}{2}\int_{\Gamma_1}\frac{1}{m-\alpha}\eta^2d\Gamma+\frac{c_1}{2}\int_{\Gamma_1}(u_t^2+|\nabla_g u|_g^2)d\Gamma+c_2\int_\Omega u^2dx\,.
\end{aligned}\tag{3.10}$$

Using Lemma 3.3 we know that

$$\begin{aligned}
&\frac{\kappa}{2}E_\tau(t)+\frac{\kappa}{4\xi}e^{2\tau}\dot{V}_3(t)\\
&\le\frac{\kappa}{2}E_\tau(t)+\frac{\kappa}{4\tau}e^{2\tau}\int_{\Gamma_1}u_t^2d\Gamma-\frac{\kappa}{2}\int_{\Gamma_1}\int_0^1 e^{2\tau-2\rho\tau}z^2d\rho d\Gamma\\
&\le\frac{\kappa}{4\tau}e^{2\tau}\int_{\Gamma_1}u_t^2d\Gamma\,.
\end{aligned}\tag{3.11}$$

Combining inequalities (3.10) and (3.11) we complete the proof. □

To elinate the tangential part of the derivative $\nabla_g u$ we need the following lemma from [13].

Lemma 3.5. p[13, Lemma 7.2.]] *Let $\varepsilon > 0$ be given small. Let u solves the problem* (1.8). *Then*

$$\begin{aligned}&\int_\varepsilon^{T-\varepsilon}\int_{\Gamma_1}|\nabla_g u|_g^2 d\Gamma dt\\ &\le C_{T,\varepsilon}\Big\{\int_0^T\int_{\Gamma_1}\big((\partial_{\nu_A}u)^2+u_t^2\big)d\Gamma dt+\|u\|_{H^{\frac12+\varepsilon}(\Omega\times(0,T))}\Big\}.\end{aligned}\tag{3.12}$$

The following is the observability inequality for the system (1.8).

Lemma 3.6. *Suppose that the geometric assumptions* (A1) *and* (A2) *hold. Let u solve problem* (1.8). *Then for any given $\varepsilon > 0$, there exists a time $T_0 > 0$ and a positive constant $C_{T,\varepsilon,\rho}$ such that*

$$\begin{aligned}E(0)\le C_{T,\varepsilon,\rho}\Big\{&\int_0^T\int_{\Gamma_1}\big(u_t^2+(\partial_{\nu_A}u)^2+z^2(x,1,t)+\eta^2\big)d\Gamma dt\\ &+\|u\|_{H^{1/2+\varepsilon}(\Omega\times(0,T))}\Big\},\end{aligned}\tag{3.13}$$

for all $T > T_0$.

Proof. For a given ε small enough, integrating inequality (3.3) on the interval $(\varepsilon, T-\varepsilon)$ yields

$$\begin{aligned}&\frac{\kappa}{2}\int_\varepsilon^{T-\varepsilon}E(t)dt+V_1(T-\varepsilon)-V_1(\varepsilon)+V_2(T-\varepsilon)-V_2(\varepsilon)\\ &+\frac{\kappa e^{2\tau}}{4\xi}V_3(T-\varepsilon)-\frac{\kappa e^{2\tau}}{4\xi}V_3(\varepsilon)\\ &\le\frac{\kappa}{2}\int_\varepsilon^{T-\varepsilon}\int_{\Gamma_1}\frac{1}{m-\alpha}\eta^2 d\Gamma+c_1\int_\varepsilon^{T-\varepsilon}\int_{\Gamma_1}(u_t^2+|\nabla_g u|_g^2)d\Gamma+c_2\int_\varepsilon^{T-\varepsilon}\int_\Omega u^2dx\,.\end{aligned}$$

Then we use inequality (3.12) in Lemma 3.5 and inequality (3.4) in Lemma 3.4 to obtain

$$\begin{aligned}\int_\varepsilon^{T-\varepsilon}E(t)dt\le C_{T,\varepsilon,\rho}\Big\{&\int_0^T\int_{\Gamma_1}(\eta^2+u_t^2+|\partial_{\nu_A}u|_g^2)d\Gamma\\ &+\|u\|_{H^{1/2+\varepsilon}(\Omega\times(0,T))}\Big\}dx+c_0\big(E(T-\varepsilon)+E(\varepsilon)\big)\,,\end{aligned}\tag{3.14}$$

where the constant $C_{T,\varepsilon,\kappa}$ depends on c_1, κ, $\frac{1}{m-\alpha}$, meas(Ω) and the constant $C_{T,\varepsilon}$ given in inequality (3.12).

We notice that

$$\begin{aligned}
&E(0)+c_0\big(E(T-\varepsilon)+E(\varepsilon)\big)\\
&=\int_\varepsilon^{2c_0+\varepsilon+1}E(t)dt+\int_\varepsilon^{2c_0+\varepsilon+1}\big(E(0)-E(t)\big)dt\\
&\quad+c_0\big(E(\varepsilon)-E(0)\big)+c_0\big(E(T-\varepsilon)-E(0)\big)\\
&=\int_\varepsilon^{2c_0+\varepsilon+1}E(t)dt-\int_\varepsilon^{2c_0+\varepsilon+1}\Big(\int_0^t\dot E(\tau)d\tau\Big)dt+c_0\int_0^\varepsilon\dot E(\tau)d\tau\\
&\quad+c_0\int_0^{T-\varepsilon}\dot E(\tau)d\tau\\
&\le\int_\varepsilon^{2c_0+\varepsilon+1}E(t)dt\\
&\quad+c\int_0^{2c_0+\varepsilon+1}\int_{\Gamma_1}\big(u_t^2+(\partial_{\nu_A}u)^2+z^2(x,1,t)+\eta^2\big)d\Gamma dt\,,
\end{aligned}\tag{3.15}$$

where we used $\dot E(t)\le 0$ and the fact which is known from (2.6) that

$$-\dot E(t)=-\langle\mathcal{A}U,U\rangle_\Upsilon\le c\int_{\Gamma_1}\big(u_t^2+(\partial_{\nu_A}u)^2+z^2(x,1,t)+\eta^2\big)d\Gamma dt\,.\tag{3.16}$$

Now we shall take $T_0=2c_0+2\varepsilon+1$ to guarantee that $T-\varepsilon>2c_0+\varepsilon+1$, for all $T>T_0$. Substituting (3.15) to inequality (3.14) completes the proof. $\square$

In what follows we use the compactness-uniqueness argument to absorb the lower order term in (3.13). We list the lemma and omit the proof, which could be found in [14, 19, 25, 27] and many others.

Lemma 3.7. *Suppose that the geometric assumptions* (A1) *and* (A2) *hold. Let u solve problem* (1.1). *Then for any $T>T_0$, there exists a positive constant C depending on T, ε, ρ,* meas(Ω) *such that*

$$E(0)\le C\int_0^T\int_{\Gamma_1}\big(u_t^2+(\partial_{\nu_A}u)^2+z^2(x,1,t)+\eta^2\big)d\Gamma dt\,.$$

Proof of Theorem 1.2. From (2.10), we have

$$\dot E\le -C_1\int_{\Gamma_1}[u_t^2+\eta^2+z(x,1,t)^2+\partial_{\nu_A}^2u]dx\,,\tag{3.17}$$

where we denote

$$\begin{aligned}
C_1=\min\Big(&(\frac{m}{2}-\frac{k_1m}{4}-\frac{\xi}{2\tau}),(\frac{1}{m-\alpha}(1-\frac{\beta}{2k_2})-\frac{1}{2m}-\frac{1}{k_1m}),\\
&-(\frac{\beta}{m-\alpha}\frac{k_2}{2}-\frac{\xi}{2\tau})\Big)\,.
\end{aligned}$$

Thus from Lemma 3.7 we have that for all $T>T_0$,

$$\begin{aligned}
E(0)&\le C\int_0^T\int_{\Gamma_1}\big(\eta^2+\partial_\nu A^2u+u_t^2+z^2(x,1,t)\big)d\Gamma dt\\
&\le -\frac{C}{C_1}\int_0^T\dot Edt=-\frac{C}{C_1}(E(T)-E(0))\,;
\end{aligned}$$

that is,

$$E(T) \le \frac{C - C_1}{C} E(0) \,,$$

from which the exponential decay result follows. $\square$

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